"Sur la propagation par ondes et sur le problème de Mayer," J. de Math. (6) **9** (1913), 39-76.

On the propagation of waves and the Mayer problem

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1. The following pages are attached to the articles ([1]) that I have published on the analytical consequences of Huygens's principle, when it is considered to define the infinitesimal propagation of waves in a medium of an arbitrary number of dimensions and an arbitrary nature, and, in particular, on the relationships that this propagation of waves has with the theory of first order partial differential equations and canonical systems, the calculus of variations, and analytical mechanics.

The mode of propagation is defined when one gives the limiting form that is approached by a wave that is emitted by a disturbance at an arbitrary point when the duration of the propagation tends to zero: This is what we call the *wave multiplicity*. We call the *elementary wave* the homothetic image of that wave, the center of the homothety being the disturbed point and the homothety ratio being the infinitely small duration *dt* of the propagation. In the general case where the regime of propagation is variable these wave multiplicities and elementary waves depend upon the instant *t* of their emission.

In first two of the articles recalled, I studied the case where the elementary waves have ∞^{n-1} points and ∞^{n-1} tangent planes. In the third, the study of the general isoperimetric problem, which is called the *Lagrange* problem, led me to consider the case where the elementary waves have $\infty^{n-\alpha-1}$ points, while still having ∞^{n-1} tangent planes, and I have stated only the results relating to this case that I had previously obtained. This case is the most general one since, as I have already indicated and I will show here incidentally, if the elementary waves have at least ∞^{n-1} tangent planes then one will no longer be dealing with a medium in which an *arbitrary* wave may propagate.

2. In the first part of the present work, I recall the analysis in the general case: We will then be concerned with the variable regime here, whereas the Lagrange problem is attached to the case of a permanent regime.

The system of elementary waves is defined, from the pointlike viewpoint, by a system of (a + 1) equations, which one may express in the form:

$$(1) \qquad \begin{cases} F(t \mid x_1, \cdots, x_n \mid dx_1, \cdots, dx_n) = dt, \\ F_h(t \mid x_1, \cdots, x_n \mid dx_1, \cdots, dx_n) = 0 \qquad (h = 1, 2, \cdots, \alpha), \end{cases}$$

([1]) *Sur l'interpretation mécaniaue des transformations de contact infinitésimales* (*Bull. Soc. Math. de France*, t. XXXIV, 1906); *Essai sur la propagation par ondes* (*Ann. Ec. Norm. Sup.*, 3rd series, t. XXVI, 1909); *Sur la théorie des multiplicités et le Calcul des variations* (*Bull. Soc. Math. de France*, t. XL, 1912).

the left-hand sides being homogeneous of degree one with respect to the differentials. The origin of emission of the elementary wave has the coordinates $x_1, \ldots, x_n$; the instant – or *date* ([1]) – of emission is t. A current point of the elementary wave has the coordinates:

$$x_1 + dx_1, \ldots, x_n + dx_n .$$

From the tangential viewpoint, the corresponding wave multiplicity, when considered to be the envelope of the variable plane:

$$\sum_{i=1}^{n} q_i (X_i - x_i) = 1, \tag{2}$$

is defined by only one equation:

$$G(t \mid x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 1, \tag{3}$$

and this is what makes the present results analogous to the ones that I have obtained elsewhere.

The *family of waves*, i.e., the families of multiplicities of the form:

$$t = V(x_1, \ldots, x_n), \tag{4}$$

which are composed of the successive states through which an arbitrary wave passes in its propagation, are furnished by the solutions of equation (3), where one interprets $q_1, \ldots, q_n$ as the partial derivatives of V:

$$q_i = \frac{\partial t}{\partial x_i} \qquad (i = 1, 2, \ldots, n). \tag{5}$$

The waves propagate by contact elements, individually considered; each contact element propagates in the same manner, starting at a given instant, no what the initial wave to which it belongs at that instant. The set of successive positions that are thus taken by an arbitrary contact element, with the dates of these successive positions, corresponds to an arbitrary *characteristic* of the partial differential equation (3). The variables $x_1, \ldots, x_n$; $q_1, \ldots, q_n$ are then interpreted as the homogeneous coordinates of the contact element comprised of the point $(x_1, \ldots, x_n)$ and the plane (2).

The differential system of the characteristics:

$$dx_i = \frac{\partial G}{\partial q_i} dt, \qquad dq_i = -\left(\frac{\partial G}{\partial x_i} + q_i \frac{\partial G}{\partial t}\right) dt \qquad (i = 1, 2, \ldots, n) \tag{6}$$

([1]) We will often make use of the word "date," whose use was proposed by Fontené (*Géométrie dirigée*, Paris, Nony, 1897, pp. 75; *Bull. des Sc. math. et phys.*, December 1906), and which is convenient when one wishes to distinguish the two senses of the word "time," namely, an instant and a duration.

presents itself as defining the infinitesimal transformation that corresponds to the propagation during the infinitely small time dt, starting at the instant t, a transformation whose symbol is written, with the Poisson bracket notation:

$$T\mathcal{F} = \frac{\partial \mathcal{F}}{\partial t} + [G, \mathcal{F}]. \tag{7}$$

As for the *finite propagation* between two arbitrary instants t_0 and t, it has its expression in a *contact transformation* in such a way that the principle of enveloping waves may be applied to the propagation in the finite sense of the word, in the most general form, and not only in its infinitesimal sense, as this is true by hypothesis. This results implies, as a particular case, the theory of the integration of a first-order partial differential equation by means of complete integrals.

In the case of a permanent regime, the contact transformation in question depends only upon the duration $(t - t_0)$, and is the general transformation of a one-parameter group.

In the general case, it will be defined, according to the theory of Lie, by relations between $x_1, \ldots, x_n ; X_1, \ldots, X_n$, which represent the wave emitted by a disturbance that is produced at the instant t_0 at the unique point $(x_1, \ldots, x_n)$, and in the state of propagation that it is found in at the instant t. This wave, whose elementary wave is the limiting form for $(t - t_0)$ infinitely small may have more dimensions than the elementary wave, and likewise has ∞^{n-1} points, in general. It is only in the case where the partial differential equation (3) is, as Lie said, *semi-linear*, or *pseudo-linear* – i.e., where the curves that serve to support the characteristics depend only upon $(2n - 1 - \gamma)$ essential arbitrary parameters – that the waves issuing from the points are multiplicities of $(n - 1 - \gamma)$ dimensions.

5. Instead of allowing the disturbance that is produced at a point to propagate freely in all directions around this point, one may imagine that it is guided in its propagation – for example, by means of a curvilinear tube of infinitely small section – where one supposes that the wall eliminates any propagation except in the sense of the axis of the tube. One thus has what one may call *propagation along a curve*. However, if $\alpha > 0$ then one may choose arbitrarily neither the curve nor the instant where the disturbance passes through an arbitrary point of the curve, because the curve and the date at which an arbitrary one of its points is found to be disturbed must satisfy the *Monge system* (1), which may be arbitrary.

Among the solutions of this system figure the ones that constitute the *trajectories of propagation* – i.e., by the locus that is described by the point of an arbitrary contact element and the date that is associated with each point of this locus. It is found that these trajectories correspond to the *minimum duration of propagation along a curve* between two points of that curve, the disturbance starting from the first of these points at a given instant.

It is the study of this minimum problem, which is only a physical statement of a general problem in the calculus of variations relating to one independent variable, which is generally denoted by the name of *Mayer problem*, to which the second part of our article is dedicated.

In order to put this into the form of equations, we have followed the method that was employed in our article on the Lagrange problem: It is founded on the parametric representation:

$$dx_i = \frac{\partial G}{\partial q_i} dt \qquad (i = 1, 2, \ldots, n) \tag{6}$$

of the elementary wave. One knows that the classical method of integration by parts has the advantage of not giving rise to the objection of Du Bois Reymond relating to the unjustified introduction of second derivatives. It therefore also gives a reason for the intervention of Lagrange multipliers: The propagation along a curve in the case of the minimum corresponds to the propagation of a contact element of a wave whose orientation is found to be defined precisely by these multipliers. The formulation in terms of equations may be done independently of these multipliers, moreover.

In order to establish the sufficient conditions for the minimum, we make use of the method – which is equivalent to the Weierstrass method – that was already used in our preceding articles. The extremal field Weierstrass and its property of corresponding to a *Unabhängigkeit Satz* analogous to that of Hilbert presents itself when one considers the contact element of the wave that propagates along the trajectory considered as being part of an finitely-extended wave: The trajectories correspond to the various elements of that wave constituting the field, and the date that corresponds to an arbitrary point of one of these trajectories being one solution of the partial differential equation (3), is obtained by a quadrature of the total differential that may be carried out along any other curve having the same origin (and likewise, date), and the same extremal as that trajectory.

One is thus led to compare the integrals of two differential equations of the form:

$$dt = \mathcal{F}(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n), \tag{7}$$

when taken along the same curve situated in the field (and close to the trajectory being studied) and with the same initial value. In our essay on the propagation of waves, we have introduced the hypothesis that one deals with analytic functions: Here, we give a method that introduced only the hypotheses of continuity and differentiability that are inherent to the problem itself.

The condition is further expressed by the concavity of the elementary wave that has its origin at an arbitrary point of the trajectory in the domain for which its contact elements are parallel to the contact element along the trajectory: That element of the elementary wave has, moreover, for its contact point the point of the trajectory that is infinitely close to the point considered in the sense of propagation.

Relative to the Mayer problem, our exposition supposes that the function F in equations (1) is essentially positive on the curves that one considers. However, one may waive this restriction by adding to F a conveniently-chosen total differential, as we have done in the study of the Lagrange problem.

I. – Fundamental properties of the propagation of waves.

1. Imagine an elastic medium whose properties vary with time t, while it fills a space of n dimensions whose coordinates are $x_1, \ldots, x_n$, and assume that this medium can propagate disturbances of a well-defined nature. The disturbance produced at the instant t and the point $(x_1, \ldots, x_n)$ is transmitted at the instant $(t + dt)$ to an infinitude of points $(x_1 + \Delta x_1, \ldots, x_n + \Delta x_n)$. Take the homothetic images of these points with respect to the origin $(x_1, \ldots, x_n)$ and the homothety ratio $(1/dt)$, we obtain *in the limit* when dt goes to zero, the *wave multiplicity* at the origin $(x_1, \ldots, x_n)$ relative to the instant t.

Let M be the origin $(x_1, \ldots, x_n)$, $\overline{MA}$ an arbitrary vector with components $a_1, \ldots, a_n$ that issues from M. Upon separating, if need be, the wave multiplicity into arcs or nappes one may assume that along the direction $\overrightarrow{MA}$ there is *at most* one point P on that multiplicity, which will be defined by the positive ratio $\rho = MP/MA$ and given by an equation:

$$\rho = F(t \mid x_1, \ldots, x_n \mid a_1, \ldots, a_n), \tag{1}$$

the quantities $a_1, \ldots, a_n$ being coupled, if the wave multiplicity contains $\infty^{n-1-\alpha}$ points, by the equations of condition:

$$F_h(t \mid x_1, \ldots, x_n \mid a_1, \ldots, a_n) = 0 \qquad (h = 1, 2, \ldots, \alpha). \tag{2}$$

The formulas must persist if one changes the vector $\overline{MA}$ without changing its direction, i.e., if one replaces $a_1, \ldots, a_n$ and ρ by $ma_1, \ldots, ma_n$ and $m\rho$, is an arbitrary *positive* number. Therefore, F is a positive function ([1]) that is *positively homogeneous* with respect to its arguments $a_1, \ldots, a_n$, and the functions F_h are positively homogeneous. The degree of homogeneity of F is 1, and one may suppose that the same is true for the F_h ([2]).

If one takes the point A to be the point P and lets $p_1, \ldots, p_n$ denote its coordinates in the system of coordinates that is parallel to the general system $x_1, \ldots, x_n$, which has the point M for origin, then one has $\rho = 1$, and one obtains the equations of the wave multiplicity in the form:

$$F(t \mid x_1, \ldots, x_n \mid a_1, \ldots, a_n) = 1, \tag{3}$$

$$F_h(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0 \qquad (h = 1, 2, \ldots, \alpha). \tag{4}$$

2. These equations being assumed to be given, one has, *up to infinitesimals of second order* ([3]), the locus of points:

$$(x_1 + dx_1, \ldots, x_n + dx_n)$$

([1]) At least, for the directions that one must consider.

([2]) This is not at all essential. In our article in the *Bulletin de la Société mathématique*, t. XL, 1912, we supposed that the F_h are of degree zero.

([3]) For more precision on this point, *cf.*, our article: *Essai sur la propagation par ondes* (*Ann. Ec. Norm. sup.*, 3rd series, t. XXVI, 1909, pp. 409).

to which the disturbance that is produced at $(x_1, \ldots, x_n)$ and the instant t is transmitted at the instant $(t + dt)$, if one takes the homothety ratio of the wave multiplicity to have the ratio dt with its origin at $(x_1, \ldots, x_n)$. One thus obtains the *elementary wave* that is defined by the equations:

$$(5) \qquad F(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = dt,$$
$$(6) \qquad F_h(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = 0 \qquad (h = 1, 2, \ldots, \alpha),$$

where $dx_1, \ldots, dx_n$ may be considered as current coordinates in the coordinate system whose origin is $(x_1, \ldots, x_n)$.

These equations, from the differential viewpoint, constitute a Monge system that, from the sign condition that was imposed on F and the positive character of F and the F_h, may be arbitrary, provided that they are soluble with respect to dt. Indeed, the variable t must play a special role in them.

An arbitrary solution of this system is composed of a curve:

$$(7) \qquad x_i = \psi_i(u) \qquad (i = 1, 2, \ldots, n),$$

and a correspondence between the points of that curve and the corresponding values at the time *t:*

$$(8) \qquad t = \psi(u).$$

This is what one may call a *dated* curve ([1]). We let the letter (C) denote any one of these curves.

A curve (C) being regarded as an infinitely thin tube whose wall instantaneously absorbs the disturbances considered, a disturbance that is produced at a point $u = u_0$ of that curve may propagate in this tube, i.e., *along that curve* (C), provided that it is produced at precisely the instant $t_0 = \psi(u_0)$, and formula (8) will give the numerical law of that propagation.

Meanwhile, if equations (6) are independent of t then the curve (C) will be capable of guiding a disturbance that is produced at any of its points at an arbitrary instant. In this case, the value of t for a current point of the curve is obtained by integrating the differential equation (5), and its initial value will be arbitrary when one is given the curve itself. In any other case, that value t is given without integration by one of equations (6).

Observe, moreover, that it will not be legitimate, in general, to change $dx_1, \ldots, dx_n$ into $-dx_1, \ldots, -dx_n$ in equations (5) and (6), in such a way that the disturbance may propagate along the curve (C) only in a determined sense. Analytically, it is the sense in which u must vary in formula (8) in order for $t = \psi(u)$ to be increasing. It results, moreover, from the hypothesis that was made one F (namely, that it remains positive for the displacements considered) that the function t varies effectively by increasing along (C).

([1]) These curves satisfy the differential system that is obtained by eliminating t between equations (5) and (6). In general, the system consists of $(\alpha - 1)$ Monge equations and one equation of second order. It reduces to the Monge system (6) in the particular case where t does not appear in equations (6).

3. Now suppose that at the instant t all of the points of a multiplicity (S) are simultaneously disturbed: One thus has a *wave* that propagates according to a new hypothesis that conforms to the *infinitesimal principle of enveloping waves* ([1]). We intend this to mean that the envelope (Σ') of the elementary waves that issue from the various points of (S) (at the instant t) represents, up to higher-order infinitesimals, the state (S') of the wave at the instant ($t + dt$).

The term "of the envelope" must be used here in the general sense of the theory of multiplicities, i.e., the contact elements "of the envelope" are borrowed from the contact elements being enveloped.

Let us find this envelope (Σ'). To that effect, let ($x_1, \ldots, x_n$) be an arbitrary point M of (S). To an arbitrary point:

$$x_1 + X_1 , \ldots, x_n + X_n$$

of the elementary wave that has the point M for its origin there corresponds, by way of the formulas:

$$X_i = P_i\, dt \qquad (i = 1, 2, \ldots, n), \tag{9}$$

a point ($x_1 + P_1 , \ldots, x_n + P_n$) of the wave multiplicity (3), (4). At these two homologous points of the elementary wave and of the wave multiplicity the contact elements are parallel, and one may define ([2]) their common direction by the formulas:

$$Q_i = \frac{\partial f}{\partial P_i} \qquad (i = 1, 2, \ldots, n), \tag{10}$$

where one has set:

$$\begin{aligned} f(t \mid x_1, \ldots, x_n \mid P_1, \ldots, P_n) \\ = F(t \mid x_1, \ldots, x_n \mid P_1, \ldots, P_n) + \sum_{h=1}^{\alpha} \lambda_h F_h(t \mid x_1, \cdots, x_n \mid P_1, \cdots, P_n)\,. \end{aligned} \tag{11}$$

One may thus consider ($P_1, \ldots, P_n$; $Q_1, \ldots, Q_n$) to be the coordinates of an arbitrary contact element of the elementary wave, and these coordinates satisfy the equation:

$$\sum_{i=1}^{n} P_i Q_i = 1. \tag{12}$$

At least one of these contact elements belong to the envelope (Σ'), and we now reserve the notation:

$$(P_1, \ldots, P_n; Q_1, \ldots, Q_n)$$

for that element. Thus, to any variation ($\delta x_1, \ldots, \delta x_n$) of the point M on (S) there correspond variations ($\delta P_1, \ldots, \delta P_n$) such that the point with coordinates $(x_i + P_i\, dt) + \delta(x_i$

([1]) Cf., *loc. cit.*, pp. 409-412.

([2]) As far as the analytic geometry of multiplicities is concerned, we refer to our article in the *Bulletin de la Société Mathématique*, t. XL, 1912 – more especially page 78 – for the present case.

$+ P_i\ dt$) still has the contact element ($P_1, \ldots, P_n$; $Q_1, \ldots, Q_n$), i.e., they satisfy the condition:

$$\sum_{i=1}^{n} Q_i(\delta x_i + \delta P_i\, dt) = 0. \tag{13}$$

However, on the other hand, since the point ($x_1 + P_1, \ldots, x_n + P_n$) is on the wave multiplicity, it satisfies equations (3) and (4), from which one derives, by differentiation, the relations:

$$\begin{cases} \sum_{i=1}^{n}\left(\dfrac{\partial F}{\partial x_i}\delta x_i + \dfrac{\partial F}{\partial P_i}\delta P_i\right) = 0, \\ \sum_{i=1}^{n}\left(\dfrac{\partial F_h}{\partial x_i}\delta x_i + \dfrac{\partial F_h}{\partial P_i}\delta P_i\right) = 0 \quad (h = 1, 2, \cdots, \alpha), \end{cases} \tag{14}$$

and upon combining these relations one obtains:

$$\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\delta x_i + \frac{\partial f}{\partial P_i}\delta P_i\right) = 0, \tag{15}$$

which one may describe, due to formulas (10):

$$\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\delta x_i + Q_i\delta P_i\right) = 0. \tag{16}$$

Upon multiplying that equation by dt and subtracting it from (13), what results is:

$$\sum_{i=1}^{n}\left(Q_i - \frac{\partial f}{\partial x_i}dt\right)\delta x_i = 0. \tag{17}$$

Such an equation thus has meaning when the variation ($\delta x_1, \ldots, \delta x_n$) acts upon a contact element of (S) that contains the point:

$$(x_1, \ldots, x_n).$$

Let ($q_1, \ldots, q_n$) denote the direction coefficients of such an element. The result obtained is equivalent to saying that it corresponds to a contact element of (Σ') such that equation (17) is a consequence of the equation of condition:

$$\sum_{i=1}^{n} q_i\, \delta x_i = 0, \tag{18}$$

i.e., such that one has, m denoting a convenient factor:

(19) $$Q_i = \frac{\partial f}{\partial x_i} dt + mq_i \qquad (i = 1, 2, \ldots, n).$$

4. These formulas first show that if dt tends to zero then that contact element of (Σ') tends to the contact element considered on (S), because due to relation (12) one may suppose only that m becomes null. Now, that contact element of (Σ') is parallel to a contact element of the wave multiplicity that has M for its origin. Therefore, one may have propagation of the wave considered only if every contact element of (S) is parallel to a contact element of the corresponding wave multiplicity. Thus, if one desires that the original wave (S) must be arbitrary, i.e., that at each point M the orientation $(q_1, \ldots, q_n)$ of the contact element considered is not restricted by any (homogeneous) equation of condition between its coefficients $(q_1, \ldots, q_n)$, then one must have that the wave multiplicity has contact elements of arbitrary orientation.

Now, the quantities $(Q_1, \ldots, Q_n)$ are coupled by the equations that one obtains upon eliminating $(P_1, \ldots, P_n)$ between equations (10) and the equations of condition:

(20) $$\begin{cases} F(t \mid x_1, \cdots, x_n \mid P_1, \cdots, P_n) = 1, \\ F_h(t \mid x_1, \cdots, x_n \mid P_1, \cdots, P_n) = 0 \quad (h = 1, 2, \cdots, \alpha). \end{cases}$$

The equations thus obtained define the *tangential support* ([1]) of the wave multiplicity. One of them is not homogeneous and may be written ([2]):

(21) $$G(t \mid x_1, \ldots, x_n \mid Q_1, \ldots, Q_n) = 1,$$

G being homogeneous of degree 1 in $Q_1, \ldots, Q_n$. However, the others, if one desires, may be written in the homogeneous form in $Q_1, \ldots, Q_n$, and constitute a limitation on the degree of freedom in the orientation of the contact elements.

We thus conclude that *propagation is possible for an arbitrary wave only if the tangential support of the wave multiplicity is defined by just one equation; i.e., if this tangential support has* $n - 1$ *dimensions.*

This is what we shall assume from now on, and we suppose, in addition, that the coordinates $(x_1, \ldots, x_n ; q_1, \ldots, q_n)$ of any contact element, when considered at the instant t, are, by definition, coupled by the equation of condition:

(22) $$G(x_1, \ldots, x_n ; q_1, \ldots, q_n) = 1.$$

5. Therefore, when dt tends to zero, Q_i tends to q_i, and m tends to 1 in formulas (19).

Moreover, it results from equations (10) and (20) that $P_1, \ldots, P_n$ are then determined as functions of $Q_1, \ldots, Q_n$; one may likewise write the formulas that give them when one introduces equation (21). They are ([3]):

([1]) Cf., *Bull. de la Soc. math. de France*, t. XL, 1912, pp. 74.
([2]) Cf., *Ibid.*, pp. 78.
([3]) Cf., *loc. cit.*, pp. 79.

(23) $$P_i = \frac{\partial G(t \mid x_1, \cdots, x_n \mid Q_1, \cdots, Q_n)}{\partial Q_i} \qquad (i = 1, 2, \ldots, n).$$

We let $p_1, \ldots, p_n$ denote the analogous quantities:

(24) $$p_i = \frac{\partial G(t \mid x_1, \cdots, x_n \mid q_1, \cdots, q_n)}{\partial q_i} \qquad (i = 1, 2, \ldots, n).$$

One then sees that P_i tends to p_i when dt tends to zero.

By definition, we have determined a contact element on (Σ') that tends, when dt tends to zero, to the contact element:

$$(x_1, \ldots, x_n ; q_1, \ldots, q_n)$$

of (S). The corresponding infinitesimal variation is deduced from equations (9), (19) by replacing: X_i with dx_i, P_i with $p_i + dp_i$, Q_i with $q_i + dq_i$, $-m$ with $1 + dm$, and suppressing the infinitesimals of second order. One thus has the system of formulas:

(25) $$dx_i = p_i \, dt \qquad (i = 1, 2, \ldots, n),$$

(26) $$dq_i = \frac{\partial f}{\partial x_i} dt + q_i \, dm \qquad (i = 1, 2, \ldots, n),$$

where f must now denote the function:

(27) $$f = F(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) + \sum_{h=1}^{\alpha} \lambda_h F_h(t \mid x_1, \ldots, x_n \mid p_1, \ldots, p_n) .$$

In order to not complicate the notation, we have kept the letters λ_h to denote the limiting values of the quantities represented by the same letters in the formulas (10).

Finally, equations (20) and (10) give, in the limit:

(28) $$F(t \mid x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 1,$$

(29) $$F_h(t \mid x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 0, \qquad (h = 1, 2, \ldots, \alpha)$$

and:

(30) $$q_i = \frac{\partial f}{\partial p_i} \qquad (i = 1, 2, \ldots, n).$$

It only remains for us to calculate dm, which one does by differentiating the equation:

(31) $$\sum_{i=1}^{n} p_i q_i = 1,$$

which also provides (12) by the same passage to the limit. This gives:

(32) $$\sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i} dt + dm + \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} dp_i = 0,$$

upon taking into account (26) and (30). If one regards equations (25) then one infers:

(33) $$dm = - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i - \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} dp_i .$$

Now, one further infers from relations (28) and (29), upon differentiating them, the combination:

(34) $$\frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i + \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} dp_i = 0.$$

For *dm*, all that remains is simply the value:

(35) $$dm = \frac{\partial f}{\partial t} dt ,$$

which permits us to write the equations (26) in their definitive form:

(36) $$dq_i = \left(\frac{\partial f}{\partial x_i} + q_i \frac{\partial f}{\partial t} \right) dt \qquad (i = 1, 2, \ldots, n).$$

6. In summary, *if propagation is possible*, such as it is defined by Huygens's principle, which we intend in its infinitesimal sense, then this translates into a continuous variation of contact elements in space that is defined by formulas (25), (28), (29), (30), (36).

We recall these formulas here, upon eliminating the auxiliary quantities $p_1, \ldots, p_n$ and upon denoting this by f, which becomes the function (27) when one replaces the p_i in it with dx_i . We have the differential system:

(37) $$F(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = dt,$$

(38) $$F_h(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = 0 \qquad (h = 1, 2, \ldots, \alpha),$$

(39) $$q_i = \frac{\partial f}{\partial\, dx_i} \qquad (i = 1, 2, \ldots, n),$$

(40) $$dq_i = \frac{\partial f}{\partial x_i} + q_i \frac{\partial f}{\partial t} \qquad (i = 1, 2, \ldots, n),$$

with:

(41) $$f = F + \sum_{h=1}^{\alpha} \lambda_h F_h .$$

This system contains the α unknown auxiliary $\lambda_1, \ldots, \lambda_\alpha$, and the unknowns $x_1, \ldots, x_n$; $q_1, \ldots, q_n$. It is therefore over-determined, because it contains $(2n + \alpha + 1)$ equations. However, relation (20), which it entails, is verified, from the manner by which we have arrived at equations (36), from which it is found to satisfy for the initial values of $x_1, \ldots, x_n$; $q_1, \ldots, q_n$, and t. It thus indeed disappears, and from the elimination of $\lambda_1, \ldots, \lambda_\alpha$ one must obtain a system of $2n$ differential equations of first order in $2n$ unknowns.

One arrives at them by using some known relations between the equations that define the same multiplicity, depending on whether one starts with its *pointlike support* or its *tangential support* ([1]). The tangential support being defined by equation (22), the formulas (28), (29), and (30) being replaced by the equation (22) and the equations:

$$p_i = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n), \tag{42}$$

and one has, moreover:

$$\frac{\partial f}{\partial x_i} = -\frac{\partial G}{\partial x_i}, \qquad \frac{\partial f}{\partial t} = -\frac{\partial G}{\partial t} \qquad (i = 1, 2, \ldots, n). \tag{43}$$

One thus obtains the desired differential system in the final form:

$$\frac{dx_i}{dt} = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n), \tag{44}$$

$$\frac{dq_i}{dt} = -\frac{\partial G}{\partial x_i} - q_i \frac{\partial G}{\partial t} \qquad (i = 1, 2, \ldots, n). \tag{45}$$

One may consider them as defining an *infinitesimal transformation* at $t, x_1, \ldots, x_n, q_1, \ldots, q_n$ that is the definitive expression of the propagation considered, namely:

$$T\mathcal{F} \equiv \frac{\partial \mathcal{F}}{\partial t} + \sum_{i=1}^{n} \left[\frac{\partial G}{\partial q_i}\frac{\partial \mathcal{F}}{\partial x_i} - \left(\frac{\partial G}{\partial x_i} + q_i \frac{\partial G}{\partial t} \right) \frac{\partial \mathcal{F}}{\partial q_i} \right]. \tag{46}$$

One immediately verifies that it leaves equation (22) invariant, because one has the identity:

$$T(G - 1) = -\frac{\partial G}{\partial t}(G - 1). \tag{47}$$

It suffices to observe that since G is homogeneous of degree *one* in $q_1, \ldots, q_n$ one may apply the Euler identity to it.

Since one must operate only on the values of the variables verifying that equation (22), one may further substitute for the transformation (46), the following one:

([1]) Cf., *Bull. de la Soc. math. de France*, t. XL, 1912, pp. 78-80.

(48) $$T\mathcal{F} = [G, \mathcal{F}] \equiv \sum_{i=1}^{n}\left[\frac{\partial G}{\partial q_i}\left(\frac{\partial \mathcal{F}}{\partial x_i} + q_i\frac{\partial \mathcal{F}}{\partial t}\right) - \left(\frac{\partial G}{\partial x_i} + q_i\frac{\partial G}{\partial t}\right)\frac{\partial \mathcal{F}}{\partial q_i}\right],$$

where the right-hand side is the Poisson bracket.

7. One proves that propagation is possible by verifying that *the transformation T is a contact transformation.* This fact results from the following identity ([1]):

(49) $$T\left(\sum_{i=1}^{n} q_i \delta x_i\right) = -\frac{\partial G}{\partial t}\sum_{i=1}^{n} q_i \delta x_i\ .$$

Moreover, it results from this fact that *the principle of enveloping waves is true, not only in the infinitesimal sense, but also in the finite sense of the word*, and in the most general form ([2]). In particular, a wave that occupies a position (S) at the instant t is, at a final instant t', the envelope of the waves that have been emitted ([3]) during the interval of time from t to t' by the various points of (S).

8. We call a *family of waves* the set of various successive states through which a wave passes in its propagation. One such family is represented by one equation:

(50) $$t = V(x_1 , \dots, x_n).$$

The contact elements of the wave are, at the same time, given by the formulas:

(51) $$q_i = q_0 \frac{\partial V}{\partial x_i} \qquad (i = 1, 2, \dots, n),$$

where q_0 is a factor that is determined by the condition (22). Set:

(52) $$\bar{G} = G\left(V \,\middle|\, x_1, \dots, x_n \,\middle|\, \frac{\partial V}{\partial x_i}, \dots, \frac{\partial V}{\partial x_n}\right),$$

and we obtain, due to the homogeneity of G, the condition:

(53) $$q_0 \bar{G} = 1.$$

([1]) The calculation of that identity is indicated in our memoir, cited above, in the *Annales de l'Écoles Normale*, pp. 422.

([2]) This is what we have explained in the memoir cited in the preceding footnote (pp. 429). We have developed the consequences of this fact from the viewpoint of the theory of integration of partial differential equations.

([3]) Observe that these latter waves, whose limiting form is given by the elementary waves, may have more than $\infty^{n-1-\alpha}$ points. Cf., *Bull. Soc. math.*, pp. 131.

Having said this, we shall express the fact that the system (50), (51) admits the transformation T; this will give us the analytical character of the family of waves (50).

Upon first applying the transformation of equation (50), we obtain the necessary condition:

$$1-\sum_{i=1}^{n}\frac{\partial G}{\partial q_i}\frac{\partial V}{\partial x_i}=0, \tag{54}$$

which, due to the homogeneity (of degree zero) of the derivatives $\partial G/\partial q_i$, may be written:

$$1-\sum_{i=1}^{n}\frac{\partial \bar{G}}{\partial \dfrac{\partial V}{\partial x_i}}\frac{\partial V}{\partial x_i}=0, \tag{55}$$

i.e., simply:

$$G\left(V \;\middle|\; x_1,\dots,x_n \;\middle|\; \frac{\partial V}{\partial x_i},\dots,\frac{\partial V}{\partial x_n}\right)=1. \tag{56}$$

One then sees, from (53), that q_0 must have the value *one*, and that equations (51) reduce to:

$$q_i=\frac{\partial V}{\partial x_i} \qquad (i = 1, 2, \dots, n). \tag{57}$$

However, if one now applies the transformation T to equations (57) then one obtains the equations:

$$\frac{\partial G}{\partial x_i}+q_i\frac{\partial G}{\partial t}+\sum_{i=1}^{n}\frac{\partial^2 V}{\partial x_i\partial x_j}\frac{\partial G}{\partial q_j}=0 \qquad (i = 1, 2, \dots, n), \tag{58}$$

which must be consequences of equations (50) and (57). This is expressed by the identities:

$$\frac{\partial \bar{G}}{\partial V}\frac{\partial V}{\partial x_i}+\frac{\partial \bar{G}}{\partial x_i}+\sum_{i=1}^{n}\frac{\partial \bar{G}}{\partial \dfrac{\partial V}{\partial x_j}}\frac{\partial \dfrac{\partial V}{\partial x_j}}{\partial x_i}=0 \qquad (i = 1, 2, \dots, n), \tag{59}$$

which are consequences of (56).

The partial differential equation (56) *is therefore the necessary and sufficient condition for equation* (50) *to be that of a family of waves*, and the coordinates of the contact elements of the waves of that family are given by formulas (57).

9. We call a *characteristic* any solution of the *canonical system* (44), (45), which also verifies the condition (22). A characteristic is composed of a dated curve (C) (*cf.*, no. **2**) to each point of which there is associated a contact element. We call a *trajectory* any dated curve that serves to support a characteristic.

The trajectories satisfy the differential system that one deduces from the system (37), (38), (39), (40) upon eliminating the q_i and the λ_h . One can easily eliminate the q_i , which gives the equations:

$$d\frac{\partial f}{\partial\, dx_i} - \frac{\partial f}{\partial\, dx_i}\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n). \tag{60}$$

This system characterizes the trajectories, among all the solutions of the system (5), (6) that was considered in no. **2**.

We remark that the determination of the motion of propagation that is defined by a given family of wave multiplicities is equivalent to the determination of the characteristics. This entails, as a consequence, the knowledge of all families of waves. This fact is equivalent to the method of integration of the partial differential equation (56) by means of characteristics.

Any characteristic comes about in the construction of an infinitude of families of waves, because it suffices for this that one of its contact elements be attached to part of the wave at the corresponding instant.

Conversely, any family of waves provides, by integrating the system:

$$\frac{dx_i}{dt} = \frac{\partial \bar{G}}{\partial \dfrac{\partial V}{\partial x_i}} \qquad (i = 1, 2, \ldots, n) \tag{61}$$

[where $\bar{G}$ represents the function (52)], a family of trajectories to which the waves of the family are called *transversal*. The formulas (57) serve to define the corresponding family of characteristics, i.e., they serve to generate that family of waves. The values of x_1 , …, x_n , t for each of these trajectories, must, moreover, satisfy equation (50), which is, due to equation (56), compatible with the system (61).

We finally point out the equation:

$$dt - \sum_{i=1}^{n} q_i dx_i = 0, \tag{62}$$

which is a combination of equations (44), when one takes into account equation (22), which the characteristics consequently satisfy.

II. – The Mayer problem.

10. One recovers the trajectories and characteristics of the propagation when one *seeks the curves* (C) – [*cf.*, no. **2**] – *along which a disturbance propagates the most rapidly*. It is this minimum problem, which is nothing but the one to which one gives the name of *Mayer problem* ([1]) in the calculus of variations, that we shall study. We shall first state it more precisely.

We consider a solution (7), (8) of the Monge system (5), (6), and we vary it in such a manner that it does not cease to be a solution of this system, in such a way that the curve (C) represented by equations (7) always pass through the same two points M_0 and M_1, with the coordinates $(x_1^0, \dots, x_n^0)$ and $(x_1^1, \dots, x_n^1)$; we may suppose that these points correspond to fixed values u_0 and u_1 of the parameter u on the different curves thus obtained. We suppose, moreover, that the function (8) keeps a constant value t_0 at the point M_0 under that variation and that t increases from M_0 to M_1. On the contrary, the value t_1, which corresponds to the point M_1, will vary in general. The difference $(t_1 - t_0)$ represents the time taken by a disturbance that is produced at M_0 and the time t_0 to propagate along the curve (C) up to M_1. This duration will be a minimum at the same time as t_1, and these are the conditions for the minimum that we seek.

We first look for the conditions that express the idea that the variation of t_1 is null under the indicated conditions.

To that effect, we set:

(63) $$\frac{dt}{du} = \omega,$$

(64) $$\frac{dx_i}{du} = \omega p_i \qquad (i = 1, 2, \dots, n),$$

in such a way that the variables $p_1, \dots, p_n$; $x_1, \dots, x_n$; t are coupled by the equations of condition:

(67) $$F(t \mid x_1, \dots, x_n \mid p_1, \dots, p_n) = 1,$$

(68) $$F_h(t \mid x_1, \dots, x_n \mid p_1, \dots, p_n) = 0, \qquad (h = 1, 2, \dots, \alpha).$$

These equations represent the wave multiplicity at $p_1, \dots, p_n$ – [*cf.* no. **1**, equations (3) and (4)] – and if we introduce the equation that represents the tangential support – [*cf.* no. **4**, equation (21) or (22)] – then we may replace them by the parametric equations – [equations (23) or (24) in no. **5**]. For the sake of clarity, set:

(69) $$G' \equiv G(t \mid x_1, \dots, x_n \mid \gamma_1, \dots, \gamma_n),$$

and these parametric equations may be written:

(70) $$p_i = \frac{\partial G'}{\partial \gamma_i} \qquad (i = 1, 2, \dots, n).$$

Since the right-hand sides are homogeneous of degree zero in $\gamma_1, \dots, \gamma_n$, one may consider these parameters to be completely independent ([2]).

([1]) *Cf.* HADAMARD, *Leçons sur le Calcul des variations,* t. I, pp. 223.

([2]) Cf., *Bulletin de la Soc. math.*, t. XL, 1912, pp. 79-80. We remark that ($p_1, \dots, p_n$; $\gamma_1, \dots, \gamma_n$) are the *homogeneous* coordinates of a contact element of the wave multiplicity, when referred to its origin at ($x_1, \dots, x_n$). The general expression of the γ_i will be given by the right-hand sides of formulas (99), where one

We thus have, by definition, some functions of u: t ; x_1 , …, x_n ; ω; γ_1 , …, γ_n , that are coupled by the differential equations:

$$\frac{dt}{du} = \omega \tag{71}$$

$$\frac{dx_i}{du} = \omega\frac{\partial G'}{\partial \gamma_i} \quad (i = 1, 2, \ldots, n). \tag{72}$$

Their variations are, in turn, defined by the linear system:

$$\frac{d\delta t}{du} = \delta\omega, \tag{73}$$

$$\frac{d\delta x_i}{du} = \omega\frac{\partial^2 G'}{\partial \gamma_i \partial t}\delta t + \omega\sum_{j=1}^{n}\frac{\partial^2 G'}{\partial \gamma_i \partial x_j}\delta x_j + \frac{\partial G'}{\partial \gamma_i}\delta\omega + \omega\sum_{j=1}^{n}\frac{\partial^2 G'}{\partial \gamma_i \partial \gamma_j}\delta\lambda_j . \tag{74}$$

In order to integrate this system, we consider the homogeneous system ([1]):

$$\begin{cases} \dfrac{du_0}{dt} = 0, \\ \dfrac{du_i}{dt} = \omega\dfrac{\partial^2 G'}{\partial \gamma_i \partial t}u_0 + \omega\displaystyle\sum_{j=1}^{n}\dfrac{\partial^2 G'}{\partial \gamma_i \partial x_j}u_j \quad (i = 1, 2, \cdots, n), \end{cases} \tag{75}$$

and, introducing $(n + 1)$ independent solutions of this system:

$$u_k = u_{l,k} \qquad (k, l = 0, 1, 2, \ldots, n), \tag{76}$$

we set:

$$\delta t = \sum_{l=0}^{n} y_l u_{l,0} , \tag{77}$$

$$\delta x_i = \sum_{l=0}^{n} y_l u_{l,i} \quad (i = 1, 2, \ldots, n). \tag{78}$$

We thus obtain the simplified linear system:

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considers the λ_0 , λ_1 , …, λ_α to be arbitrarily chosen functions of u. In the calculations that follow, one may suppose that one has made a particular *arbitrary* choice for these auxiliary functions.

([1]) The existence of the integrals of this system supposes only the continuity of the functions t, x_1, …, x_n of u and their derivatives ω, p_1 , …, p_n , as well as the continuity of the derivatives of the function G that intervenes, because this suffices for the Lipschitz conditions to be verified by the right-hand sides.

One thus does not suppose that the functions t and x have second derivatives, and the argument is not, consequently, subject to *the classical objection of Du Bois Reymond.*

On the contrary, formulas (97) show that these second derivatives necessarily exist *for the extremals.*

(79) $$\sum_{l=0}^{n} u_{l,0}\frac{dy_l}{du} = \delta\omega,$$

(80) $$\sum_{l=0}^{n} u_{l,i}\frac{dy_l}{du} = \frac{\partial G'}{\partial \gamma_i}\delta\omega + \omega\sum_{j=0}^{n}\frac{\partial^2 G'}{\partial\gamma_i\partial\gamma_j}\delta\gamma_j \qquad (i = 1, 2, \ldots, n),$$

which are solved by employing the multipliers – (*the adjoint system* of the $u_{l,k}$) – that are defined by the relations:

(81) $$\sum_{l=0}^{n} u_{l,k}v_{l.m} = \varepsilon_{k,\,m} \qquad (k, m = 0, 1, 2, \ldots, n),$$

where $\varepsilon_{k,\,m}$ is equal to 1 or 0, according to whether $k = m$ or $k \neq m$.

One thus obtains the auxiliary system:

(82) $$\frac{dy_l}{du} = Y_l \equiv \left(v_{l,0} + \sum_{i=1}^{n} v_{l,i}\frac{\partial G'}{\partial\gamma_i}\right)\delta\omega + \omega\sum_{j=1}^{n}\sum_{i=1}^{n} v_{l,i}\frac{\partial^2 G'}{\partial\gamma_i\partial\gamma_j}\delta\gamma_j\,.$$
$$(l = 0, 1, 2, \ldots, n).$$

Moreover, from the formulas (77) and (78), in which the determinant of the $u_{l,k}$ is not null, the necessary and sufficient condition for the δt, δx_1, …, δx_n to be annulled for $u = u_0$ is that the same is true for the y_l. One thus has, for the variations δt, δx_1, …, δx_n, the formulas:

(83) $$\begin{cases} \delta t = \displaystyle\sum_{l=0}^{n} u_{l,0}\int_{u_0}^{u} Y_l\,du, \\ \delta x_i = \displaystyle\sum_{l=0}^{n} u_{l,i}\int_{u_0}^{u} Y_l\,du \qquad (i = 1, 2, \ldots, n), \end{cases}$$

which becomes, for $u = u_1$, if one denotes the values that are taken by the functions of u by an index (1):

(84) $$\begin{cases} (\delta t)^{(1)} = \displaystyle\int_{u_0}^{u_1}\sum_{l=0}^{n} u_{l,0}^{(1)}Y_l\,du \\ (\delta x_i)^{(1)} = \displaystyle\int_{u_0}^{u_1}\sum_{l=0}^{n} u_{l,i}^{(1)}Y_l\,du \quad (i = 1, 2, \ldots, n). \end{cases}$$

11. We thus have to write that the first of these integrals is null for any choice of functions u: $\delta\omega$, $\delta\gamma_1$, …, $\delta\gamma_n$, for which the last n integrals are null. Since the quantities placed before the du under the integration signs are linear forms in $\delta\omega$, $\delta\gamma_1$, …, $\delta\gamma_n$ whose coefficients are known function of u, which is expressed ([1]) by an identity – (in u, $\delta\omega$, $\delta\gamma_1$, …, $\delta\gamma_n$) – with constant coefficients c_0, c_1, …, c_n, where c_0 must not be null:

([1]) Cf., *Bull. de la Soc. math.*, t. XL, 1912, pp. 120.

(85) $$\sum_{k=0}^{n} c_k \sum_{l=0}^{n} u_{l,k}^{(1)} Y_l = 0.$$

If one sets:

(86) $$\sum_{k=0}^{n} c_k u_{l,k}^{(1)} = c'_l \qquad (l = 0, 1, 2, \ldots, n),$$

(87) $$\sum_{k=0}^{n} c'_l v_{l,m} = v_l \qquad (m = 0, 1, 2, \ldots, n),$$

then that identity decomposes into:

(88) $$v_0 + \sum_{i=1}^{n} v_i \frac{\partial G'}{\partial \gamma_i} = 0,$$

(89) $$\sum_{i=1}^{n} v_i \frac{\partial^2 G'}{\partial \gamma_i \partial \gamma_j} = 0 \qquad (j = 1, 2, \ldots, n).$$

Furthermore, the constants c_k are calculated as functions of the constants c'_l upon employing the values that the functions $v_{l,m}$ of u take for $u = u_1$ as the multipliers. This gives, upon taking into account formulas (89):

(90) $$c_k = \sum_{k=0}^{n} c'_l v_{l,m}^{(1)} = v_k^{(1)} \qquad (k = 0, 1, 2, \ldots, n).$$

The hypothesis $c_0 \neq 0$ thus translates into $v_0^{(1)} \neq 0$.

Finally, formulas (89) express the idea that v_0 , v_1, …, v_n constitutes a solution of the *adjoint linear system* to system (75), which is:

(91) $$\frac{dv_0}{du} + \omega \sum_{i=1}^{n} \frac{\partial^2 G'}{\partial \gamma_i \partial t} v_i = 0,$$

(92) $$\frac{dv_j}{du} + \omega \sum_{i=1}^{n} \frac{\partial^2 G'}{\partial \gamma_i \partial x_j} v_i = 0 \qquad (j = 1, 2, \ldots, n).$$

We thus obtain the condition that *this adjoint system must admit a* ([1]) *solution that satisfies equations* (88), (89), *and is such that the value of* v_0 *is not null for* $u = u_1$.

12. With regard to equation (71), the system (91), (92) may be written:

([1]) It might happen that it admits an infinitude by admitting one. This is the case where the partial differential equation (56) admits at least ∞^{2n-1} characteristics. Cf., *Bull. de la Soc. math.*, t. XL, 1912, pp.110.

(93) $$\frac{dv_0}{dt} + \omega \sum_{i=1}^{n} \frac{\partial^2 G'}{\partial \gamma_i \partial t} v_i = 0$$

(94) $$\frac{dv_j}{dt} + \omega \sum_{i=1}^{n} \frac{\partial^2 G'}{\partial \gamma_i \partial x_j} v_i = 0 \qquad (j = 1, 2, \ldots, n).$$

As for equations (88) and (89), they express the idea that the plane that has the equation, in the coordinate system with its origin at $(x_1, \ldots, x_n)$:

(95) $$\sum_{i=1}^{n} v_i X_i + v_0 = 0$$

is tangent to the wave multiplicity at the point $(p_1, \ldots, p_n)$, because, due to equations (70), equation (88) expresses the idea that this plane passes through that point, and equations (89) express the idea that any displacement of that point on the wave multiplicity is parallel to that plane.

From the viewpoint of the auxiliary unknowns $v_0, v_1, \ldots, v_n$, these equations may thus be replaced by the following ones:

(96) $$\sum_{i=1}^{n} v_i p_i + v_0 = 0 \qquad (v_0^{(1)} \neq 0),$$

(97) $$p_i = \frac{\partial G''}{\partial v_i} \qquad [G'' \equiv G(t \mid x_1, \ldots, x_n \mid v_1, \ldots, v_n)] \qquad (i = 1, 2, \ldots, n).$$

One may then rid oneself of the auxiliary unknowns γ_i, because, upon comparing (70) and (97), one has the equations:

(98) $$\frac{\partial G'}{\partial \gamma_i} = \frac{\partial G''}{\partial v_i} \qquad (i = 1, 2, \ldots, n),$$

which one may regard as defining the v_i as functions of the γ_i, of $t, x_1, \ldots, x_n$, and the $(\alpha + 1)$ auxiliary variables $\lambda_0, \lambda_1, \ldots, \lambda_\alpha$, all of these variables being regarded as independent, because the general solution of equations (97) will be:

(99) $$v_i = \lambda_0 \frac{\partial F}{\partial p_i} + \sum_{h=1}^{n} \lambda_h \frac{\partial F_h}{\partial p_i} \qquad (i = 1, 2, \ldots, n),$$

and one will only have to replace the p_i with the values (70) in order to have the functions in question ([1]).

([1]) Cf., *Bull. de la Soc. math.*, t. XL, 1912, pp. 107.

We may thus differentiate equations (98) with respect to the variables t, x_1 , …, x_n under this hypothesis, and this gives:

$$\frac{\partial^2 G'}{\partial \gamma_i\,\partial t} = \frac{\partial^2 G''}{\partial v_i\,\partial t} + \sum_{k=1}^{n} \frac{\partial^2 G''}{\partial v_i\,\partial v_k}\frac{\partial v_k}{\partial t} \qquad (i = 1, 2, \ldots, n), \tag{100}$$

$$\frac{\partial^2 G'}{\partial \gamma_i\,\partial x_j} = \frac{\partial^2 G''}{\partial v_i\,\partial x_j} + \sum_{k=1}^{n} \frac{\partial^2 G''}{\partial v_i\,\partial v_k}\frac{\partial v_k}{\partial x_j} \qquad (i, j = 1, 2, \ldots, n). \tag{101}$$

If one substitutes these expressions in equations (93) and (94), and if one takes into account the Euler identities as they relate to the first and second derivatives of G'', what remains is simply:

$$\frac{dv_0}{dt} + \frac{\partial G''}{\partial t} = 0, \tag{102}$$

$$\frac{dv_i}{dt} + \frac{\partial G''}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n), \tag{103}$$

to which one must add equations (96), (97), and the equations:

$$\frac{dx_i}{dt} = p_i \qquad (i = 1, 2, \ldots, n), \tag{104}$$

which results from (63) and (64).

13. *The extremals thus defined are nothing but the trajectories of propagation,* and the auxiliary unknowns v_0 , v_1 , …, v_n correspond to the introduction of contact elements that make each trajectory the support of a *characteristic*. To confirm this, it suffices to set:

$$v_i = -\,q_i\, v_0 \qquad (i = 1, 2, \ldots, n). \tag{105}$$

This supposes that v_0 is not annulled at any point of the arc of the curve (C) that is being considered, a hypothesis that was introduced already for the extremity of that arc. Geometrically, it signifies that the plane of the contact element of the wave multiplicity (p_1 , …, p_n ; v_1 , …, v_n) must not pass through the origin, i.e., it must not contain the radius vector of the point (p_1 , …, p_n). Now, from equations (104), this radius vector is tangent to the curve (C). *The condition that we impose is therefore that the contact element that is associated with each point of the dated curve* (C), *by its preceding formulation, must never belong to that curve.*

Upon considering degree of homogeneity, one has, by the change of variables (105):

Here we have, in addition, the parameter λ_0 , because we operate on the homogeneous coordinates v_0 , v_1 , …, v_n for the general contact element of the wave multiplicity at the point (p_1 , …, p_n), which it serves to represent.

(106) $$\frac{\partial G''}{\partial v_i} = \frac{\partial G}{\partial q_i}, \qquad \frac{\partial G''}{\partial t} = -v_0 \frac{\partial G}{\partial t}, \qquad \frac{\partial G''}{\partial x_i} = -v_0 \frac{\partial G}{\partial x_i} \qquad (i = 1, 2, \ldots, n).$$

On the other hand, one has:

(107) $$\frac{dv_i}{dt} = -v_0 \frac{dq_i}{dt} - q_i \frac{dv_0}{dt} \qquad (i = 1, 2, \ldots, n),$$

and, as a result:

(108) $$v_0 \frac{\partial G}{\partial x_i} = -v_0 \frac{dq_i}{dt} + v_0 q_i \frac{dG}{dt} \qquad (i = 1, 2, \ldots, n).$$

Furthermore, as a consequence, under the hypothesis that we made, one obtains the equations:

(109) $$\frac{dq_i}{dt} = -\frac{\partial G}{\partial x_i} - q_i \frac{\partial G}{\partial t} \qquad (i = 1, 2, \ldots, n),$$

which replaces equations (102) and (103). As for equations (96) and (97), they become, due to (104):

(110) $$\frac{dx_i}{dt} = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n)$$

and:

(111) $$dt = \sum_{i=1}^{n} q_i dx_i .$$

By comparing (110) and (111), one finally recovers equations (22), namely:

(112) $$G(t \mid x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 1.$$

We thus recover precisely all of the characteristic equations [*cf.*, no. **9**].

One may remark that if $q_1, \ldots, q_n$ are assumed to have been calculated then the unknown v_0 is given by a quadrature by means of:

(113) $$\frac{dv_0}{dt} = v_0 \frac{\partial G}{\partial t},$$

and that the v_i are then given by equations (105). The unknown v_0 corresponds to the quantity m that was also introduced in the theory of propagation [*cf.*, no. **5**].

Its initial value remains arbitrary, and the linear form of equation (113) shows that it is certainly not annulled, as long as it does not produce the singular situation in which $\partial G/\partial t$ becomes infinite. Now, this singularity is already excluded implicitly in the considerations of nos. **6** and **7**.

Observe finally that if G is only *positively* homogeneous – which one may, in certain cases, be obliged to suppose – then our transformations remain legitimate provided that

v_0 is negative. Now, this may always be assumed, since we impose the condition on it that it is not annulled and that, on the other hand, $v_0, v_1, \ldots, v_n$ are defined in all of what follows only by homogeneous equations, and consequently, up to a constant factor.

14. *It remains to examine whether the trajectories effectively correspond to a minimum of the duration of propagation* [*cf.*, no. **10**]. Therefore, let (T) be one of these trajectories, M_0 and M_1 two of its points, dated t_0 and q_1 ($q_1 > t_0$), and let $\overline{T}$ denote the arc of that trajectory that falls between M_0 and M_1, considered independently of the values of t that are associated with each point of the trajectory, but assumed to run from M_0 to M_1.

On the other hand, let (C) be another dated curve of the type that was defined in no. **2** that also passes from M_0 to M_1, and let it be dated t_0 at M_0. It will be dated t_1 at M_1, and everything comes down to the study of the sign of $(t_1 - \theta_1)$ for the solutions (C) of the Monge system (5), (6) that are sufficiently close to the solution (T). We further let $\overline{C}$ represent the geometrical arc M_0M_1 of (C), considered independently of any date for its points, but assumed to run from M_0 to M_1.

The date t may be considered to be defined in the following manner: One takes the differential equation (5), i.e.:

$$dt = F(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n), \tag{114}$$

and *one integrates along* $\overline{C}$, upon taking t_0 to be the initial value. The final value that this integral takes at M_1 is t_1. The phrase "integration along $\overline{C}$" signifies that one replaces $x_1, \ldots, x_n$ and their differentials in equation (144) by means of equations (7), i.e.,:

$$x_i = \psi_i(u) \qquad (i = 1, 2, \ldots, n), \tag{115}$$

which defines the arc $\overline{C}$ when u varies by increasing from u_0 to u_1. Since du is therefore positive, one obtains the differential equation:

$$\frac{dt}{du} = F\left[t \;\middle|\; \psi_1(u), \cdots, \psi_n(u) \;\middle|\; \frac{d\psi_1(u)}{du}, \cdots, \frac{d\psi_n(u)}{du}\right], \tag{116}$$

which one must integrate with the initial condition $t = t_0$ for $u = u_0$.

Likewise, θ_1 is obtained by integrating equation (114) along $\overline{T}$, with the same initial value, because (T) is nothing but a particular dated curve (C).

One may, moreover, substitute for equation (114) an infinitude of other equations that give the same results for the calculation of t_1 and θ_1, because the dated curves considered satisfy equations (6), namely:

$$F_h(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = 0 \qquad (h = 1, 2, \ldots, \alpha). \tag{117}$$

One may thus use, in place of (114), and in the same manner, any equation:

$$dt = f(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n), \tag{118}$$

where, as above – [for example, no. **6**, equation (41)] – f denotes a linear combination of the form:

(119) $$f = F + \sum_{h=1}^{\alpha} \lambda_h F_h ,$$

the λ_h being arbitrary functions of $x_1, \ldots, x_n$ here.

15. We shall transform this result in such manner as to involve the wave multiplicities. Let M be the point with the coordinates $x_1, \ldots, x_n$ and let $\Omega_{x,t}$ be the wave multiplicity that has that point for its origin at the instant t. Suppose M is $\bar{C}$ and call P the point such that the positive direction of the tangent attached to $\bar{C}$ at M pierces $\Omega_{x,t}$. Equations (114) and (117) then express that if M has the date t on (C) then the point P has for its coordinates, when one takes M for origin, the derivatives:

(120) $$\frac{dx_i}{dt} = p_i \qquad (i = 1, 2, \ldots, n) \qquad (\textit{Cf.}, \text{ nos. } \mathbf{1} \text{ and } \mathbf{2}).$$

If one then sets:

(121) $$q_i = \frac{\partial f}{\partial\, dx_i} = \frac{\partial f(t \mid x_1, \cdots, x_n \mid p_1, \cdots, p_n)}{\partial p_i} \qquad (i = 1, 2, \ldots, n),$$

the quantities $(p_1, \ldots, p_n ; q_1, \ldots, q_n)$ are, in the same system of coordinates, the coordinates of a contact element (E) of $\Omega_{x,t}$ that is associated with that point P. These coordinates are subject to verifying the following relation of condition ([1]), which is, indeed, equivalent to (118):

(122) $$\sum_{i=1}^{n} p_i q_i = 1.$$

With these notations, equation (118) may be put into the form:

(123) $$dt = \sum_{i=1}^{n} q_i dx_i .$$

For more neatness, we denote by:

(124) $$q_i = K_i(t, u) \qquad (i = 1, 2, \ldots, n)$$

the functions that one obtains by replacing the x_i and dx_i in formulas (121) by means of formulas (115). The date t, relative to the curve (C), is thus obtained by integrating the equation:

(125) $$\frac{dt}{du} = \sum_{i=1}^{n} K_i(t, u) \frac{d\psi_i(u)}{du} \equiv K(t, u),$$

([1]) Cf., *Bull. de la Soc. math.*., t. XL, 1912, pp. 78.

with their initial value $t = t_0$ for $u = u_0$.

16. First, this trajectory serves to support at least one characteristic, which is obtained by adjoining a contact element to each point M of the trajectory whose direction coordinates $q_1, \dots, q_n$ satisfy equations (44) and (22), which we rewrite:

$$p_i = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \dots, n), \tag{126}$$

$$G(t \mid x_1, \dots, x_n \mid q_1, \dots, q_n) = 1. \tag{127}$$

These entail equation (122). Now, equation (127) is the tangential equation of $\Omega_{x,t}$, and these equations (126) give the point of contact of an arbitrary plane of this wave multiplicity. We thus have a particular contact element (E) that is found to be associated with the point P: It is parallel to one of the ones that the trajectory is capable of transporting under the mode of propagation considered. Conversely, we have the geometrical interpretation for equations (44) of the characteristics: They express the relation that we just defined between the direction of the tangent to the trajectory and that of the transported contact element, and which bears the name of *transversality* [*cf.*, no. **9**].

It results, moreover, from the auxiliary condition – ($v_0 \neq 0$) – that that we have imposed in no. **15**, that the contact element that is transversal to the trajectory, which is associated with a well-defined solution of canonical system of the characteristics, does not always pass through the tangent to the trajectory. One may thus – and this is the second stated peculiarity – introduce a family of waves that are transversal to the trajectory (T) considered – [*cf.*, no. **9**] – which fills up a space of n dimensions ($\mathcal{E}$) in which the arc $\bar{T}$ will be completely contained, in such a manner that through each point of that space ($\mathcal{E}$) there passes one and only one wave of that family ([1]). From now on, we suppose that the arc $\bar{C}$ itself is contained in that space ($\mathcal{E}$).

We thus recall the notations of no. **8**, and let:

$$t = V(x_1, \dots, x_n) \tag{128}$$

be the general equation of that family of waves. To each point M of the space ($\mathcal{E}$) there corresponds a value of t and the quantities:

$$q_i = \frac{\partial V}{\partial x_i} \qquad (i = 1, 2, \dots, n), \tag{129}$$

which satisfy equation (127). These quantities define the direction of a contact element that is transversal to the corresponding direction whose coefficients are provided by formulas (126). One may thus suppose that the functions λ_h of $x_1, \dots, x_n$ that figure in

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([1]) Of course, this nevertheless constitutes a new hypothesis on the arc $\bar{T}$, since this amounts to assuming that the *Jacobi condition* is verified.

Cf., HADAMARD, *Leçons sur le Calcul des variations*, t., pp. 360.

formula (119), which are chosen in such a manner that formulas (121), when applied to the curve (C), conversely give back the values (129), when one replaces t in them with the value (128) and $p_1, \ldots, p_n$ *by* functions of $x_1, \ldots, x_n$:

$$p_i = \varpi_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n), \tag{130}$$

which are obtained by substituting the values (128) and (129) in formulas (126). From now on, we make this hypothesis on the choice of functions λ_h .

Relative to the arc $\bar{T}$ itself, in order to obtain the date θ_1 (at which the disturbance that starts at M_0 at the instant t_0 arrives at M_1 when it propagates along $\bar{T}$) we must, from the results of no. **15**, integrate equation (123) along $\bar{T}$, where we may assume that the q_i are replaced by the expressions (129).

However, the right-hand side being the total differential dV, one will also obtain θ by doing that integration along C. That essential remark is the form in which the *Unabhängigkeit Satz* is presented here, that is exhibited in the, already classical, method of Weierstrass.

17. We have thus introduced two values of t at each point M of the arc $\bar{C}$: the integral of equation (125) and the value of the function V; henceforth, we denote the latter value by q. As a result, we must consider two wave multiplicities that have that point for origin: $\Omega_{x,t}$ and $\Omega_{x,\theta}$. The positive direction of the tangent to $\bar{C}$ at this point pierces $\Omega_{x,t}$ at the point P whose coordinates are given by formulas (120), or, more explicitly, by the formulas:

$$p_i = \frac{d\psi_i(u)}{du}\frac{1}{K(t,u)} \equiv H_i(t, u) \qquad (i = 1, 2, \ldots, n); \tag{131}$$

one obtains a contact element (E) of $\Omega_{x,t}$.

However, these formulas are absolutely independent of the fact that the value of t is a particular one. Upon leaving t absolutely arbitrary, they always give a contact element of $\Omega_{x,t}$ such that the point is on the positive direction of the tangent to $\bar{C}$, because the coordinates of such a point are *positively* proportional to $d\psi_i / du$, which intervenes only in formulas (121), and, as a result, the functions (131) and (132) verify equations (121) identically. Moreover, since they also satisfy equation (122) identically they are indeed the coordinates of a contact element of $\Omega_{x,t}$.

In particular, the quantities:

$$p_i = H_i(\theta, u), \qquad q_i = K_i(\theta, u) \qquad (i = 1, 2, \ldots, n) \tag{133}$$

are the coordinates of a contact element (H) of $\Omega_{x,t}$.

On the other hand, formulas (130) and (129) give, when one replaces the x_i in them with functions $\psi_i(u)$, some functions of u that we denote by:

$$p_i' = H_i'(u), \qquad q_i' = K_i'(u) \qquad (i = 1, 2, \ldots, n), \tag{134}$$

and which, from the explanation in no. **16**, are the coordinates of another contact element (H') of $\Omega_{x,t}$. Finally, with these latter notations, θ satisfies the differential equation:

$$\frac{d\theta}{du} = \sum_{i=1}^{n} K_i'(u) \frac{d\psi_i(u)}{du} = K'(u). \tag{135}$$

18. It is equations (125) and (135) that permit us to compare t_1 and θ_1. However, some preliminary remarks are indispensible.

We suppose that the dated curves (C) and (T) have a neighborhood of order *one*. Thus, to each point $(x_1, \ldots, x_n)$ – or M – of C and the date t to which it associated there corresponds a point $(\xi_1, \ldots, \xi_n)$ – or M' – of T and a date θ, such that the differences:

$$x_i - \xi_i, \qquad t - \theta, \qquad \frac{dx_i}{dt} - \frac{d\xi_i}{d\theta} \qquad (i = 1, \ldots, n) \tag{136}$$

are inferior in absolute value to a positive number ε. To these points M and M', from formulas (120) and (121), there correspond the contact element (E) and the contact element (E'), respectively ([1]), where the various coordinates are as small as one desires when ε is conveniently chosen.

However, it results from the explanations of no. **16** that the contact element (E') is also given by formulas (130) and (129) when one replaces the x_i with ξ_i, and it follows from this that the coordinates of the elements (E) and (H') are as close as one desires.

Finally, since (E) and (H) have coordinates as close as one desires, from the fact that $|t - \theta|$ is sufficiently small we conclude, by definition, that *the contact elements* (H) *and* (H'), which both belong to $\Omega_{x,\theta}$, *may be assumed to be as close as one desires.*

19. Having made this point, consider the difference:

$$K(\theta, u) - K'(u). \tag{137}$$

One may write it:

$$K(\theta,u)\left[1 - \sum_{i=1}^{n} K_i'(u) H_i(\theta,u)\right]. \tag{138}$$

Now, the first factor is positive, because dt/du, which is equal to $K(t, u)$, is positive along (C), and the same is true, as a result, of $K(\theta, u)$, since θ is as close to t as one wishes.

As for the other factor, it is written:

$$1 - \sum_{i=1}^{n} p_i q_i', \tag{139}$$

([1]) By changing x_i into ξ_i, and t into θ_i.

upon denoting, as in formulas (133) and (134), the coordinates of the two contact elements (H) and (H') of $\Omega_{x,\theta}$ by (p_1, …, p_n ; q_1, …, q_n) and $(p'_i, \ldots, p'_n; q'_1, \ldots, q'_n)$, resp. Its sign is therefore coupled to the concavity ([1]) of the wave multiplicities $\Omega_{x,\theta}$ in the neighborhood of the element (H), and consequently, by reason of continuity, that of the wave multiplicities $\Omega_{\xi,\theta}$ that have their origins at the various points M' of $\bar{T}$ that are in the neighborhood of the contact element (E').

If we recall – [*cf.*, no. **16**] – that these elements (E') have for coordinates the values of (p_1, …, p_n ; q_1, …, q_n) that are provided by a characteristic that has its support on the trajectory considered then we may state the following result: *The difference* (137) *may only be positive or null along* $\bar{C}$ *if, at each point of* $\bar{T}$, *and at the instant when the disturbance pass through this point, the wave multiplicity having this point for its origin is concave towards its origin in the neighborhood of the contact element that has for coordinates the values of* ($p_i = dx_i / dt$, …, $p_n = dx_n / dt$; q_1, …, q_n) *that are given by the equations of the characteristics that has provided the trajectory* (T).

We suppose that this sufficient condition is satisfied.

Observe, moreover, that the concavity being thus, the factor (141) may be annulled only if the contact elements (H) and (H') correspond to the same point of $\Omega_{x,\theta}$. Now, the points of these contact elements are situated, one of them in the positive direction of the tangent to $\bar{C}$ and the other one in the direction that is transversal to the wave of the family (128) that passes through the point of $\bar{C}$ considered, and it is possible that these two directions coincide at every point of $\bar{C}$, because if this is true then $\bar{C}$ will be one of the trajectories to which the waves of the family (128) are transversal – [*cf.* no. **9** and no. **16**]. Now, this is impossible because, from the equations (61) that define this family of trajectories, there passes one and only one of them for each point of the space ($\mathcal{E}$), and, by the point M_0, from which one starts on $\bar{C}$ already passes the trajectory $\bar{T}$, which belongs to the family considered and which, by hypothesis, is distinct from $\bar{C}$.

Therefore, *the difference* (139) *is not null at any point of* $\bar{C}$.

20. Having said this, consider the difference:

$$\Delta = t - \theta. \tag{140}$$

From the notations adopted in no. **17**, it is a function of u that is defined at all points of $\bar{C}$, i.e., in the interval from u_0 to u_1. It admits a continuous derivative in that interval ([2]) that is given by the formula:

$$\frac{d\Delta}{du} = K(t, u) - K'(u), \tag{141}$$

[1] Cf., *Bulletin de la Soc. math.*, t. XL, 1912, pp. 92.

[2] From the definition of the function K, this continuity supposes that the tangent to $\bar{C}$ varies in a continuous manner. The nature of the reasoning that follows permits us to assume that the discontinuities consist of jump variations in that direction at isolated points.

which results immediately from equations (125) and (135). Finally, it is annulled for $u = u_0$ since both t and θ then have the value t_0 .

Write formula (143) in the form:

$$\frac{d\Delta}{du} = [K(t, u) - K(\theta, u)] + [K(t, u) - K'(u)], \tag{142}$$

and observe that, from equation (125), which defines $K(t, u)$, that function possesses a partial derivative with respect to t, on just the condition of supposing that the functions F and F_h – [given in no. **1**] – have second derivatives of the type $\partial^2 / \partial p_i\, \partial t$. One may thus suppose:

$$K(t, u) - K(\theta, u) = (t - \theta)\, A, \tag{143}$$

A being a function of u that will be defined in any interval (u_0, u_1), that $(t - \theta)$ is annulled or not. Indeed, when $(t - \theta)$ is not annulled the continuity of A results from that of the function $K(t, u)$, and the functions t and θ of u. If $(t - \theta)$ is annulled then it results from the expression for A:

$$A = \frac{\partial K(\bar{\theta}, u)}{\partial \bar{\theta}}, \tag{144}$$

which furnishes the theorem of finite increases, and in which θ is between t and θ, provided that one supposes the continuity of the derivatives of F and F_h that we just assumed the existence of.

We thus write equation (142) in the form:

$$\frac{d\Delta}{dt} = A\Delta + B, \tag{145}$$

upon further letting B denote the difference (137), which is a function of u that is also continuous, due to the preceding hypotheses. Moreover, from no. **19**, B is positive or null, and is not constantly null.

From this equation, upon taking into account the fact that Δ is annulled for $u = u_0$, one derives the expression for Δ:

$$\Delta = e^{\int_{u_0}^{u} A\,du} \int_{u_0}^{u} B e^{-\int_{u_0}^{u} A\,du}\, du, \tag{146}$$

which shows that Δ is positive for $u_0 < u \leq u_1$. In particular, one has, for $u = u_1$, the consequence:

$$t_1 - \theta > 0. \tag{147}$$

It is thus proved that *under the hypothesis on the concavity of the wave multiplicity that was specified in no.* **19**, *the trajectory* $\bar{T}$ *corresponds to a minimum in the duration of propagation.*