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# The electron hypothesis and the theory of magnetism

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The goal of the present investigation is to resolve the question of whether an explanation for the phenomena of magnetization and diamagnetism can be obtained on the basis of the existence of electrons that has been established in optics. In general, the reigning opinion seems to be that the hypotheses of the orientation of the molecular currents that are present and the induction of new currents in a magnetic field, which were employed for the derivation of the facts for some time, were not just *consistent* with the theory of electrons, but were even *based* in it in a simple way. Meanwhile, the situation is somewhat otherwise, and it can be inferred from the following argument that the electron hypothesis makes an explanation for the paramagnetic and diamagnetic influences seen possible, but that the mechanism for the process that is required by that hypothesis deviates from the older presentation at some essential points.

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**1.** As long as the velocity $G$ of a moving electrical charge is small in comparison to the speed of light (which we can assume in our present case), we can derive the elementary law of its magnetic effect easily from the so-called **Biot-Savart** formula, as long as the components $\delta X_1$, $\delta X_2$, $\delta X_3$ of the effect of the field of a line current element $J\,\delta s$ at the location $x$, $y$, $z$ on a unit pole at $x_1$, $y_1$, $z_1$ satisfy the equations:

$$\delta X_1 = \frac{\delta s}{\omega E_1^3}[V\,(z_1 - z) - W\,(y_1 - y)], \quad \text{etc.},$$

in which, $E_1$ is understood to mean the distance from the element to the "reference point" $x_1$, $y_1$, $z_1$, and $U$, $V$, $W$ are the components of $J$, and with the assumption that the electrostatic-magnetic system of units is used.

Now, if $\alpha$ charge elements $e$ go through the cross-section of the linear conductor per unit time then $J = \alpha\, e$, and if one chooses $\delta s$ to be the path that is traversed in the time element of $\delta t = 1 / \alpha$ seconds then $\alpha\,\delta s$ will be equal to the velocity of motion $G$ of the charges, and one will also have $J\,\delta s = G\,e$. Since, with the assumption that was made, only *one* electron is present in the element $\delta s$ at each point in time, moreover:

$$(1)\qquad X_1 = \frac{e}{\omega E_1^3}[v\,(z_1 - z) - w\,(y_1 - y)], \qquad \text{etc.},$$

in which $u$, $v$, $w$ are understood to mean the components of $G$, will represent the field components of the moving charge element.

If the charge element is a volume element $d\kappa$ that is provided with an electrical density $\varepsilon$ in a homogeneously-charged body that rotates around its center-of-mass (which lies at the coordinate origin) then one will find the following expressions:

$$(X_1) = \frac{\varepsilon}{\omega}\int [v(z_1 - z) - w(y_1 - y)]\frac{d\kappa}{E_1^3}$$
$$= \frac{\varepsilon}{\omega}\int [(xr - zp)(z_1 - z) - (yp - xq)(y_1 - y)]\frac{d\kappa}{E_1^3}, \text{ etc.},$$

for the components of the total magnetic field that emanates from it, relative to the *absolute fixed* coordinate axes $X$, $Y$, $Z$, in which, $p$, $q$, $r$ represent the components of the angular velocity with respect to the coordinate axes.

Here, $x$, $y$, $z$ might be small compared to $E_1$ ; one might then consider the effect of the rotating body on a *distant* point. $1/E_1^3$ can then be developed in powers of $x$, $y$, $z$ and one can at least keep the terms of first order. One will then obtain, after a simple calculation, for the absolutely-fixed coordinate system, and in the event that $E_0^2 = x_1^2 + y_1^2 + z_1^2$:

$$(2)\qquad \begin{cases} (X_1) = \dfrac{\varepsilon}{\omega}\Big\{\dfrac{1}{E_0^3}[p\Xi + q\mathrm{Z}' + r\mathrm{H}'] \\ \quad + \dfrac{1}{E_0^3}[p(2y_1z_1\,\Xi' + z_1x_1\,\mathrm{H}' + x_1y_1\,\mathrm{Z}' - y_1^2\mathfrak{Y} - z_1^2\mathfrak{Z}) \\ \quad - q(y_1z_1\,\mathrm{H}' + y_1^2\,\mathrm{Z}' - x_1y_1\,\mathfrak{X}) \\ \quad - r(y_1z_1\,\mathrm{Z}' + z_1^2\,\mathrm{H}' - x_1z_1\,\mathfrak{X})]\Big\}, \text{etc.} \end{cases}$$

In this, Ξ, H, Z represent the moments of inertia, and Ξ′, H′, Z′ represent the moments of deviation of the *volume* of the body about the absolutely-fixed axes $X$, $Y$, $Z$; one will then have:

$$(3)\qquad \Xi = \int (y^2 + z^2)\,d\kappa;\ \ldots,\ \Xi' = -\int y\,z\,d\kappa;\ \ldots$$

In addition, one sets:

$$(4)\qquad \int x^2\,d\kappa = \tfrac{1}{2}(\mathrm{H} + \mathrm{Z} - \Xi) = \mathfrak{X},\ \ldots$$

For a coordinate system *A*, *B*, *C* that is *fixed in the body*, whose origin lies at the point of rotation (which is identical with the center-of-mass), and whose axes point along the principal axes of the body, one will have more simply, upon introducing the angular velocities *f*, *g*, *h* and the quantities Α, Β, Γ, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ that correspond to Ξ, Η, Ζ, $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, resp.:

(5) $$(A_1) = \frac{\varepsilon}{\omega}\left\{\frac{1}{E_0^2} f \mathrm{A} - \frac{3}{E_0^5}[f(a_1^2\mathfrak{A} + b_1^2\mathfrak{B} + c_1^2\mathfrak{C}) - (fa_1 + gb_1 + hc_1)\, a_1\mathfrak{A}]\right\}, \text{ etc.}$$

In this, $a_1$, $b_1$, $c_1$ denote the coordinates of the reference point $x_1$, $y_1$, $z_1$ relative to the system *A*, *B*, *C*. Let the relative position of the two axis-crosses be represented by the matrix:

(6)

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $x$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
| $y$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| $z$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ |

in which
$$\alpha_1 = \beta_2\gamma_3 - \beta_3\gamma_2,\quad \beta_1 = \gamma_2\alpha_3 - \gamma_3\alpha_2,\quad \gamma_1 = \alpha_2\beta_3 - \alpha_3\beta_2, \quad \text{etc.}$$

We would now like to find the component $(Z_1)$ of the force components that were given in (5) along the *Z*-axis, under the assumption that the reference point $x_1$, $y_1$, $z_1$ likewise belongs to the *Z*-axis, so one then has $a_1 = E_0\,\gamma_1$, $b_1 = E_0\,\gamma_2$, $c_1 = E_0\,\gamma_3$, and $E_0 = z_1$. One will then find very simply that:

(7) $$(Z_1) = \frac{\varepsilon}{\omega E_0^3}(\mathrm{A}\, f\gamma_1 + \mathrm{B} g\,\gamma_2 + \Gamma h\gamma_3),$$

which is a formula that gives the result that:

(8) $$(Z_1) = \frac{\varepsilon}{\omega E_0^3}[\mathrm{A}(f\gamma_1 + g\,\gamma_2) + \Gamma h\gamma_3]$$

for Α = Β, and that:

(9) $$(Z_1) = \frac{\varepsilon \mathrm{M} r}{\omega E_0^3}$$

for three equal principal moments of inertia Α = Β = Γ = Μ. If a magnetic molecule with a moment of $\mu$ along the *Z*-axis were present then one would have:

(10) $$(Z_1) = \frac{2\mu}{E_0^3}$$

under the same circumstances. A comparison of the formulas above with this expression will yield immediately how large one should set the magnetic moment of the rotating body in calculations.

**2.** The forces that an electron experiences in a magnetic field divide into two parts: The first one is determined by the *instantaneous value* of the magnetic field strength $R$, while the second one is determined by its *temporal variation*. We consider both parts only in the case for which the lines of force of the field run parallel to the fixed coordinate system.

Here, the *first* part is given for a charge element $e$ by the component values:

$$X = e\,R\,v\,/\,\omega, \qquad Y = -\,e\,R\,u\,/\,\omega, \qquad Z = 0. \tag{11}$$

It follows from this that the rotational moments around the *fixed axes* $X$, $Y$, $Z$ for a body with constant charge density $e$ that rotates around the coordinate origin will be:

$$L = \varepsilon R\,(q\,\mathfrak{Z} + r\,\Xi')\,/\,\omega, \quad M = -\,\varepsilon R\,(p\,\mathfrak{Z} + r\,\mathrm{H}')\,/\,\omega, \quad N = \varepsilon R\,(p\,\Xi' + q\,\mathrm{H}')\,/\,\omega. \tag{12}$$

By contrast, the rotational moments around the *principal moment axes* of the body through its center-of-mass read:

$$F = \varepsilon\,(qC\,\mathfrak{C} - rB\,\mathfrak{B})\,/\,\omega, \quad G = \varepsilon\,(hA\,\mathfrak{A} - fC\,\mathfrak{C})\,/\,\omega, \quad H = \varepsilon\,(fB\,\mathfrak{B} - gA\,\mathfrak{A})\,/\,\omega, \tag{13}$$

in which $A = R\gamma_1$, $B = R\,\gamma_2$, $C = R\,\gamma_3$ represent the components of $R$ along the principal moment axes.

The *second* part follows from the general **Maxwell-Hertz** formulas according to the law of temporal variation of $R$. Since we are dealing with only the *time integral* of the force over a very short time element (namely, over the time interval in which the magnetic field arises, which we can assume to be arbitrarily small in comparison to the time of flight of the electron, as it seems, without arriving at any physical impossibilities), we can, with no loss of generality in that law, arrange that the calculation should be as simple as possible.

We would accordingly like to propose that a magnetic field that is parallel to the $YZ$-plane extends through space in such a way that the field strength is equal to zero for $t - x\,/\,\omega < 0$, increases from 0 to $R$ for $0 < t - x\,/\,\omega < \tau$, and remains constantly equal to $R$ for $t - x\,/\,\omega > \tau$. In this case, the basic electrodynamical formulas will reduce to:

$$\partial X'/\,\partial x = 0, \qquad \partial Z'/\,\partial x = 0, \tag{14}$$

$$-\,\omega\,\frac{\partial Y'}{\partial x} = \frac{\partial R'}{\partial t}, \qquad -\,\omega\,\frac{\partial R'}{\partial x} = \frac{\partial Y'}{\partial t}, \tag{15}$$

if $R$ denotes the variable magnetic field strength, and $X$, $Y$, $Z$ denote the components of the variable electrical field strengths, and they will be satisfied by:

$$X' = 0, \qquad Z' = 0, \qquad Y' = R' = \varphi\,(t - x\,/\,\omega), \tag{16}$$

in which $\varphi$ denotes an arbitrary function of the argument $t - x\,/\,\omega$.

If we assume, for the sake of simplicity, that the growth in $R'$ is linear in time then we will have:

$$(17)\qquad \begin{cases} Y' = 0 & \text{for } t - x/\omega < 0, \\ Y' = R(t - x/\omega)/\tau & \text{for } 0 < t - x/\omega < \tau, \\ Y' = R & \text{for } t - x/\omega > \tau, \end{cases}$$

while $X'$ and $Z'$ vanish continually.

This Ansatz seems conceivable, insofar as it also implies a constant *electric* field strength for $t - x / \omega > \tau$, along with a constant *magnetic* one. Meanwhile, on the one hand, the latter has no influence on the magnetic processes, while on the other, one can make it vanish when one can think of the existence of the field $R$ as being produced, not by *one* wave that advances parallel to $+ X$, but by *two* successive waves of half the intensity, one of which proceeds parallel to $+ X$, and the other one, parallel to $– X$.

(17) implies the following value for the time integral over $\tau$ of the force $X = e X'$, etc., that acts upon the charge $e$, during the second period, inside of which one considers the change in position of the electron to be unnoticeable:

$$(18)\qquad \int X\,dt = 0, \qquad \int Y\,dt = \tfrac{1}{2} e\,R\,\tau, \qquad \int Z\,dt = 0,$$

so the time integral of the $Y$-component will depend upon the length of time $\tau$ that it takes to produce the magnetic field and vanishes with it.

One will obtain the value zero for the rotational moment around the coordinate axes $X$, $Y$, $Z$ that a body (whose dimensions are small in comparison to $\tau\,\omega$) that rotates (as before) around the coordinate origin suffers as a result of the forces that were introduced above during the first and third period. During the second one, (with the assumption of the vanishingly-short time intervals during which the wave enters and exists the body), one will have the values:

$$(19)\qquad L = -\frac{\varepsilon R}{\tau\omega}\,\mathrm{H}', \qquad M = 0, \qquad N = -\frac{\varepsilon R}{\tau\omega}\,\mathfrak{X}.$$

The time integral of this expression – i.e., the *moment of impulse* – will be:

$$(20)\qquad \int L\,dt = -\frac{\varepsilon R}{\omega}\,\mathrm{H}', \qquad \int M\,dt = 0, \qquad \int N\,dt = -\frac{\varepsilon R}{\omega}\,\mathfrak{X},$$

when one ignores the change in position of the body during $\tau$, so it will be independent of the time duration $\tau$.

A simple calculation will give the expressions for the rotational moment $F$, $G$, $H$ of the body about the principal moment axes $A$, $B$, $C$ through its center-of-mass:

$$(21)\qquad F = \frac{\varepsilon R}{\tau\omega}(\beta_2\alpha_3\mathfrak{C} - \beta_3\alpha_2\mathfrak{B}), \quad G = \frac{\varepsilon R}{\tau\omega}(\beta_3\alpha_1\mathfrak{A} - \beta_1\alpha_3\mathfrak{C}), \quad H = \frac{\varepsilon R}{\tau\omega}(\beta_1\alpha_2\mathfrak{B} - \beta_2\alpha_1\mathfrak{A}).$$

If the body has three equal principal moments of inertia M about the center-of-mass (so it is, e.g., a ball) then it will follow from (21), if one recalls (6) and the relation $\mathfrak{A} = \mathfrak{B} = \mathfrak{C} = \frac{1}{3}\mathrm{M}$, that:

$$(22) \qquad F = -\frac{\varepsilon R \mathrm{M}}{2\tau\omega}\gamma_1, \qquad G = -\frac{\varepsilon R \mathrm{M}}{2\tau\omega}\gamma_2, \qquad H = -\frac{\varepsilon R \mathrm{M}}{2\tau\omega}\gamma_3.$$

One will get the time integral of this rotational moment – i.e., the *moment of impulse* – from formulas (21) and (22) by simply eliminating the denominator $\tau$ when one, in turn, ignores the change of position of the body during the time that the field strengths arise.

If one is dealing with a *disappearance* of the field strength $R$, instead of the creation of one, then the moments will take the opposite signs in the expressions above.

**3.** If the lines of force of the magnetic field of strength $R$ are parallel to the $Z$-axis then the general equations of motion for a point-like electron with an electrostatically-measured charge of $e$, which the theory of dispersion is based upon, will read:

$$(23) \qquad \begin{cases} m\dfrac{d^2x}{dt^2} + kx + h\dfrac{dx}{dt} = +\dfrac{eR}{\omega}\dfrac{dy}{dt} + X, \\ m\dfrac{d^2y}{dt^2} + ky + h\dfrac{dy}{dt} = -\dfrac{eR}{\omega}\dfrac{dx}{dt} + Y, \\ m\dfrac{d^2z}{dt^2} + kz + h\dfrac{dz}{dt} = Z. \end{cases}$$

In this, $m$ is the mass of the electron, $k$ and $h$ are constants, and $X = e\,X'$, … are the components of the electric force that acts upon the electron, which, from (17), will be piece-wise non-zero only during the creation (or destruction) of the field $R$ in our case.

We would now like to direct our attention to an electron that initially moves along an arbitrary (elliptical) path in the absence of field effects, and at the time $t = 0$, its motion might change at a point $x_0$, $y_0$, $z_0$ as a result of the almost-instantaneous creation of a field. In that way, we then find that the theory of the electron replaces the old assumptions about an induction and an arrangement of molecular currents with just the consideration of the *advancing* motion.

For the very short time during which the field is created, we take, as above, for the sake of simplicity, $R$ to be a linear function of time, so $\int R\;dt = \frac{1}{2}R\;\tau$, and when we denote the velocity components at the initial (final, resp.) time by $u_0$, $v_0$, $w_0$ ($u_1$, $v_1$, $w_1$, resp.), and restrict the effect of the field $R$ to terms of first order, (20) and (15) will imply the relations:

(24)
$$\begin{cases} m(u_1 - u_0) = +\dfrac{eR\tau v_0}{2\omega}, \\ m(v_1 - v_0) = +\dfrac{eR\tau v_0}{2\omega}(\omega - u_0), \\ m(w_1 - w_0) = 0. \end{cases}$$

The electron begins the (varied) motion with the velocities $u_1$, $v_1$, $w_1$ whose magnetic effect we would like to compare to that of the original. If we then neglect the force of resistance from now on – and thus set $h = 0$ – then we will find ourselves to be in harmony with the older assumption of *resistance-free* molecular currents.

We consider only the $Z$-component of the magnetic effect of the electron, which is sufficient for our stated problem, and indeed consider it only for a distant point on the $Z$-axis. From (1), its value is:

(25)
$$Z_1 = +\frac{e}{\omega E_1^3}(vx - uy),$$

and if we develop $1/E_1^3$ up to terms of order two:

(26)
$$Z_1 = \frac{e}{\omega E_1^3}(vx - uy)\left(1 + \frac{3z}{E_0}\right), \qquad E_0 = z_1 .$$

(23) now implies the following values for $x$, $y$, $z$ when $X$, $Y$, $Z$ vanish:

(27)
$$\begin{cases} x = a_1 \cos(p_1 t + \alpha_1) + a_2 \cos(p_2 t + \alpha_2), \\ y = a_2 \sin(p_1 t + \alpha_1) - a_2 \sin(p_2 t + \alpha_2), \\ z = b \sin(pt + \beta), \end{cases}$$

in which $a_1$, $a_2$, $b$, $\alpha_1$, $\alpha_2$, $\beta$, $p_1$, $p_2$, $p$ are constants, and in fact, one has:

(28)
$$p_1 = \sqrt{p^2 + \Pi^2} - \Pi, \qquad p_3 = \sqrt{p^2 + \Pi^2} + \Pi,$$

(29)
$$p = \sqrt{k/m}, \qquad \Pi = eR/2m\omega.$$

The expressions for $x$, $y$, $u$, $v$, $z$ that follow from (27) are now substituted in (26).

According to them, $Z_1$ varies periodically in time. The temporal mean $\bar{Z}_1$ that is characteristic of the magnetic effect is calculated very easily to be:

(30)
$$\bar{Z}_1 = \frac{e}{\omega E_1^3}(p_1 a_1^2 - p_2 a_2^2).$$

This expression can be given easily in terms of the values of the coordinates $x_1$, $y_1$, and the velocities $u_1$, $v_1$ with which the electron began the motion in question. If one sets:

(31) $$x^2 + y^2 = c^2, \qquad u^2 + v^2 = W^2$$

then one will get:

(32) $$\bar{Z}_1 = \frac{e}{\omega E_1^3} \cdot \frac{(p_1 - p_2)(W_1^2 - p_1 p_2 c_1^2) + 4 p_1 p_2 (v_1 x_1 - u_1 y_1)}{(p_1 + p_2)^2}.$$

Now, the coordinates $x_1$, $y_1$, and the velocities $u_1$, $v_1$ have a simple relationship to the ones $x_0$, $y_0$, and $u_0$, $v_0$ , resp., *before* the magnetic field was created. Namely, with the restriction to terms of first order in $\Pi$, one will have:

(33) $$x_1 = x_0, \qquad y_1 = y_0, \qquad u_1 = u_0 + \Pi\,\tau v_0, \qquad v_1 = v_0 + \Pi\,\tau(\omega - u_0),$$

and then also:

(34) $$c_1 = c_0, \qquad W_1^2 = W_0^2 + 2\Pi\,\omega\tau v_0\,;$$

in the same approximation, one will have:

(35) $$p_1 = p - \Pi, \qquad p_2 = p + \Pi.$$

The effects sum very simply to $\bar{Z}_1$ with the restriction to terms of first order in $\Pi$ that we introduced. We can then simplify matters when we *temporarily drop the terms that are multiplied by the length of time during which the field was created*, and thus consider the case of an *instantaneous* creation of the field, to a certain extent.

The introduction of the above values into (32) then directly gives:

(36) $$\bar{Z}_1 = \frac{e}{\omega E_0^3}\left[(v_0 x_0 - u_0 y_0) - \frac{\Pi}{2p^2}(W_0^2 - p^2 c_0^2)\right].$$

In this, the first term represents the (temporarily constant in time) value $\bar{Z}_0$ of $\bar{Z}_1$ *before* the excitation of the field; we can then write:

(37) $$\bar{Z}_1 - \bar{Z}_0 = -\frac{e\Pi}{2\omega E_0^3 p^2}(W_0^2 - p^2 c_0^2)$$

for the change in the magnetic $Z$-component that emanates from the electron that is provoked by the field, or if one recalls the relations $\Pi = e\,R / 2m\omega$ and $p^2 = k / m$:

(38) $$\bar{Z}_1 - \bar{Z}_0 = -\frac{e^2 R}{4m^2 \omega E_0^3 p^2}(mW_0^2 - kc_0^2).$$

In order to compute the mean value of the expression in parentheses for a system of electrons that is found in a state of disordered motion at the outset, we start with the original elliptical motion of an electron whose projection onto the *XY*-plane might be given by:

(39) $$x = \alpha \cos p\, t, \qquad y = \beta \sin p\, t,$$

so

(40) $$u = -\alpha p \sin p\, t, \qquad v = +\beta p \cos p\, t.$$

It will then follow from this, if one recalls (31), that:

(41) $$W^2 - p^2 c^2 = p\,(\beta^2 - \alpha^2) \cos 2p\, t.$$

The expression [which also appears in (38)] on the left-hand side then changes its value steadily, and in particular, will become positive twice and negative twice in equal time intervals during any orbit. Depending upon the moment at which the excitation of the external field takes place, $W_0^2 - p^2 c_0^2$ will then possess different magnitudes and different signs in (38). In order to compute its mean value for a very large number of electrons whose paths all yield projection ellipses onto the *XY*-plane that have the same form, one must observe how the probability of excitation of the field is distributed along the various parts of the path ellipse.

It is clear that when one divides the path into a large number of elements that all require the same time to traverse, the density of endpoints of the elements will give a measure for the probability that the electron will take a certain position at the moment that the field is excited. However, that density is proportional to 1 / *W*, where *W*, in turn, is understood to be the velocity normal to the *Z*-axis. Accordingly, we will obtain the mean value $W_0^2 - p^2 c_0^2$ for the type of ellipse considered when we define:

(42) $$\overline{W}_0^2 - p^2 \overline{c}_0^2 = \frac{\overline{W}}{\Sigma} \int \frac{W^2 - pc^2}{W}\, ds\,,$$

in which Σ denotes the length of its periphery, and $\overline{W}$, the mean value of *W*, and the integral must be taken around the ellipse. However, one has *W* = *ds* / *dt*, and we will then come to the result:

(43) $$\overline{W}_0^2 - p^2 \overline{c}_0^2 = \frac{\overline{W}}{\Sigma} \int (W^2 - pc^2)\, dt\,,$$

in which the integral is taken over the duration of an orbit. However, from the value (41), this expression will be equal to zero. *All particles whose paths project onto the same ellipses in the XY-plane will annihilate the part of the mean value* $\overline{Z}_1$ *of* $Z_1$ *that originates in the expression* $W_0^2 - p^2 c_0^2$.

Before we discuss this result, we shall briefly go into the part of $Z_1$ in (32) that was initially deferred above, and that contained the time $\tau$ during which the field was created as a factor. It possesses the value:

$$\frac{e\tau\Pi}{\omega E_0^3}[(\omega-u_0)x_0-v_0y_0]$$

and gives a mean value of *zero* in the manner that was applied above, so it will likewise give no contribution to the magnetic effect being examined. This is entirely consistent with experiment, in that the influence of the time during which the field is created generally does not show up as a magnetic excitation.

One then finds the value:

(44) $$(\bar{Z}_1) = 0$$

for the spatial and temporal mean of the magnetic force in the direction of the external field that arises from all of the electrons of a system that is originally in a state of disordered motion.

We thus arrive at the result:

*If one sees the analogy to the induction and alignment of molecular currents with which the older theory of magnetization operated in the change of the advancing motion of electrons with the emergence of a magnetic field then the electron hypothesis will lead to no magnetic excitation at all.*

**4.** If the path that was taken above gives no explanation for a magnetic excitation then that disappointing result suggests that one must present a modified treatment of the process that relates to magnetization. It seems that the following development will almost serve that purpose almost by itself.

In the theory of dispersion above, we have ignored the all-important *damping* of the electron oscillations and set the constant *h* of the force of resistance equal to zero in order to approach the older presentation of resistance-free molecular currents as much as possible. Meanwhile, if (as the phenomena of optics hardly make one doubt) resistance (which partially, if not totally, originates in the radiation of energy in any case) does exist then the entire discussion above for the resolution of the question that was posed will lose its definitive meaning, insofar as the motion *after* the excitation of the field would very soon become independent of the one *before* the excitation; namely, it must vanish completely in the absence of new stimuli. The presence of a stationary state in this case would then imply continually-renewing driving forces, which we would like to assume act instantaneously, for the sake of simplicity.

Naturally, an analogous situation will always exist when no damping is present, but the electrons bounce against each other and the ponderable parts very frequently.

If one is then forced to assume, or at least justified in assuming, that collisions act upon the electrons randomly in their motion then that will raise the question: *Can one distinguish motions that are thus obtained in the case of the action of a magnetic field from the ones in which it does not act, in regard to the magnetic forces?*

In the resolution of this question, we can once more overlook the damping, since it will be compensated for precisely by the ever-repeated driving forces on the electrons, and one can restrict oneself to the subsequent consideration of the motions immediately after the collisions and *before* noticeable effects of the essential resistance that appears after numerous periods (after the observation). One can easily convince oneself that the damping of the electron motion takes place independently of its sense of rotation around the $Z$-axis, and accordingly cannot cause a magnetic excitation by itself.

Furthermore, the desired answer is already implied by the developments above. From (37) or (38), *for a constant external field*, the magnetic force component $Z_1$ that acts parallel to those lines of force and emanates from a moving electron will be coupled with the force $Z_0$ that is present for the same initial state with no external field by the formula:

$$(45)\qquad \begin{cases} Z_1 - Z_0 = -\dfrac{e^2 R}{4\omega^2 m p^2 E_0^3}(W_2^1 - p^2 c_1^2) \\ \qquad\quad\; = -\dfrac{e^2 R}{4\omega^2 m p^2 E_0^3}(W_1^2 - p^2 c_1^2). \end{cases}$$

In this, $c_1$ denotes the distance from the electron to the $Z$-axis, which is constructed to be parallel to the external line of force in its rest position, and $W_1$ is its velocity normal to the $Z$-axis, and both of them relate to *the time after the collision that initiated the motion considered.*

Furthermore, $\frac{1}{2}kc_1^2$ is the initial value of the potential energy, and $\frac{1}{2}mW_1^2$ is the initial value of the kinetic energy of motion, which brings about the *projection* of the electron onto the $XY$-plane. $Z_1 - Z_0$ will then be positive or negative for the one electron according to whether it begins the new motion with an excess of potential or kinetic energy, resp., for the motion normal to the $Z$-axis.

If we now think of the impulses as being completely random then (at least, for isotropic bodies), by symmetry, the mean value of $Z_0$ must vanish inside of a volume element: No magnetic polarization can exist without an exciting field. The formula above for the mean value $\bar{Z}_1$ of $Z_1$ will then be valid after one drops $Z_0$ and will replace $W_1^2$ and $c_1^2$ with the mean values $\bar{W}_1^2$ and $\bar{c}_1^2$, resp. If one understands $\zeta$ to mean the number of electrons of the same kind in a unit volume then the force that emanates from a volume element in a body with *one* kind of electron will be:

$$(46)\qquad Z_\kappa^1 = -\frac{e^2 R\zeta\, d\kappa}{4\omega^2 m^2 p^2 E_0^3}(m\bar{W}_1^2 - k\bar{c}_1^2).$$

However, for completely disorganized motion, $m\bar{W}_1^2$ is equal to 4/3 the mean kinetic energy of the electrons of the kind considered, and $m\bar{c}_1^2$ is equal to 4/3 the mean potential energy. Thus, when one introduces the mean kinetic and potential energies $\bar{\psi}_1$ and $\bar{\varphi}_1$ after the collisions, one will also get:

$$(47)\qquad Z_\kappa = -\frac{e^2 R\zeta\, d\kappa}{3\omega^2 m^2 p^2 E_0^3}(\bar{\psi}_1 - \bar{\varphi}_1),$$

or, when one introduces the total potential and kinetic energy per unit volume:

(48) $$\zeta\,\bar{\varphi}_1 = \Phi_1, \qquad \zeta\,\bar{\psi}_1 = \Psi_1,$$

one will also get:

(49) $$Z_\kappa = \frac{e^2 R\, d\kappa}{3\omega^2 m^2 p^2 E_0^3}(\Phi_1 - \Psi_1).$$

It then follows from this that the specific moment $\mu$ will have the value:

(50) $$\mu = \frac{e^2 R}{6m^2 p^2 \omega^2}(\Phi_1 - \Psi_1),$$

and the magnetization number $\mathfrak{m}$ will have the expression:

(51) $$\mathfrak{m} = \frac{e^2}{6m^2 p^2 \omega^2}(\Phi_1 - \Psi_1),$$

or, with the introduction of the electromagnetically-measured charge $e' = e / \omega$ and the period $T = 2\pi / p$ of the motion in the absence of field effects:

(52) $$\mathfrak{m} = \left(\frac{e'}{m}\right)^2 \cdot \frac{T^2}{24\pi^2}(\Phi_1 - \Psi_1).$$

This implies the following result:

*The electrons of a body that moves in a constant magnetic field will give rise to magnetic effects when their motion (which probably acts against a resistance) is incessantly interrupted by some sort of completely randomly-distributed collisions (and therefore, which possibly lead to a constant mean energy). Thus, the body will exhibit paramagnetic or diamagnetic properties according to whether the motion of the electrons, in the mean, possess an excess of potential or kinetic energy, resp., after those collisions.*

With that law, it then seems possible to base a theory of paramagnetic and diamagnetic influences on the electron hypothesis, whose advantage over the picture that has proved to be so fruitful in optics especially lies in the fact that one does not appeal to two completely different explanations for paramagnetic and diamagnetic excitations, as one unavoidably does in the older theory. It also makes the inertia of the magnetic excitation that has been established by observations completely understandable. The excitation will then first be complete when each electron experiences a new collision after the formation of the external field.

All of this is connected with the fact that the picture that we have presented makes the variability of the magnetization number with varying behavior of the body very tangible. Any effect that the electrons suffer when the collisions are modified will also modify the magnetization number. If the collisions are found at a very small distance from the rest position – e.g., with very advanced damping and then naturally (in order to replace the energy loss) with significant strengths – then $\mathfrak{m}$ will possess a large *negative* value; if they affect only, or predominantly, those particles that have attained an especially large distance from the rest position for a very small velocity in a very extended path then $\mathfrak{m}$ will have a considerably *positive* value. If the collisions are distributed uniformly over all positions and velocities then $\mathfrak{m}$ will be *imperceptible.*

We still know too little about the mechanism that governs and preserves the motion of electrons in order to be able to assess the circumstances under which the one or the other will enter in. In particular, one must recall that our basic formulas (23) were tested only for *weakly* paramagnetic or diamagnetic bodies. Therefore, one must first of all demonstrate the *possibility* of the diamagnetic, as well as paramagnetic, excitation of a system of electrons.

We might temporarily draw attention to a singular special case.

If the quasi-elastic force, and therefore, the parameter $k$ ($p$, resp.), as well as the potential, decreases without bound then the period $T$ will increase beyond all limits, and with it, from (52), the magnetization number $\mathfrak{m}$. If one reverts to formulas (28) and (32) for a closer examination of this case then one will get $\bar{Z}_1 = -e\bar{W}_1^2 / \omega p_2 E_0^3 = -4\bar{\psi}_1 / 3RE_0^3$; the magnetic force that emanates from the electrons of the system will seem to be indirectly proportional to the external field. This impossible result then proves that formulas (30) and the following ones are not applicable to free particles. In fact, the temporal mean cannot be determined in this case by using the method of calculation that was employed for (30). *The method in question is applicable only when a large number of unperturbed orbits lies between any two collisions for each electron that exists inside of the volume element.* This rule will be illustrated very intuitively by the limiting case that was mentioned, and which is treated in the remark ([1]).

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([1]) The case of the magnetic excitation of a system of *free* electrons in a constant magnetic field, which was only touched upon above, can be dealt with very simply with the following formulas. If one sets $e R / m\omega = q$ then from formulas (23), with $h = 0$, $k = 0$, one will have:

$$vx - uy = -(u_0 x_0 + v_0 y_0) \sin qt - W_0^2 / q + (v_0 x_0 - u_0 y_0 + W_0^2 / q) \cos qt$$

for an electron, in which $x_0$, $y_0$, $u_0$, $v_0$ mean the initial values, and $W_0^2$ means the sum $u_0^2 + v_0^2$. If the time $T_0$, which is small compared to $2\pi / q$ (which represents a kind of orbital period), elapses between two successive collisions then the *free* part of the path will be noticeably rectilinear, and the expression for $T_0$ above will assume the mean value:

$$\overline{vx} - \overline{uy} = -(u_0 x_0 + v_0 y_0) \tfrac{1}{2} q\, T_0 - \frac{1}{6} W_0^2 T_0^2 q + (v_0 x_0 - u_0 y_0)(1 - \frac{1}{6} T_0^2 q)$$

in the second approximation. With the assumption of a disordered initial state, the spatial mean value of this for a volume element $d\kappa$ whose center-of-mass lies on the $Z$-axis will clearly be:

$$\overline{vx} - \overline{uy} = -\frac{1}{6} W_0^2 T_0^2 q ,$$

and the magnetic component that is exerted by that volume element on a point on the $Z$-axis at a distance $E_0$ will read:

**5.** Up to now, we have directed our attention exclusively to an *advancing* motion of a (point-like) electron, so we would now like to investigate the *rotation* of uniformly-charged, homogeneous body of any form (which might serve as an electron in the broader sense of the word) around its center of mass.

Such a body in a constant magnetic field of strength $R$ whose lines of force are parallel to the $Z$-axis will experience moments around the fixed $X$, $Y$, $Z$-coordinate whose values are given in (12), and when one introduces the density $\rho$ of ponderable mass, its equations of motion will read:

$$(53)\qquad \begin{cases} \dfrac{d}{dt}(p\,\Xi + q\,\mathrm{Z}' + r\,\mathrm{H}') = \dfrac{\varepsilon R}{\rho\omega}(q\mathfrak{Z} + r\,\Xi'), \\ \dfrac{d}{dt}(p\,\mathrm{Z}' + q\,\mathrm{H} + r\,\Xi') = -\dfrac{\varepsilon R}{\rho\omega}(p\mathfrak{Z} + r\,\mathrm{H}'), \\ \dfrac{d}{dt}(p\,\mathrm{H}' + q\,\Xi' + r\,\mathrm{Z}) = \dfrac{\varepsilon R}{\rho\omega}(q\,\mathrm{H}' + p\,\Xi'). \end{cases}$$

In this, one might recall that Ξ, H, Z denote the moments of inertia of the body-*volume*, and Ξ′, H′, Z′ denote its moments of deviation.

These formulas will become very simple when one considers a special instant at which the principal moment axes of the body coincide with the absolutely fixed coordinate axes. Since Ξ = A, H = B, Z = Γ are the maximum and minimum values of the moments of inertia here, and $\mathfrak{X} = \mathfrak{A}$, $\mathfrak{Y} = \mathfrak{B}$, $\mathfrak{Z} = \mathfrak{C}$, one will then have:

$$\mathrm{A}\frac{dp}{dt} + rq\,(\mathfrak{B} - \mathfrak{C}) = \frac{\varepsilon Rq}{\rho\omega}\,\mathfrak{C}, \qquad \mathrm{B}\frac{dq}{dt} + pr\,(\mathfrak{C} - \mathfrak{A}) = -\frac{\varepsilon Rq}{\rho\omega}\,\mathfrak{C},$$

$$\Gamma\frac{dr}{dt} + qp\,(\mathfrak{A} - \mathfrak{B}) = 0.$$

Thus, if only a rotation around the $X$-axis exists instantaneously, so $q = 0$, $r = 0$, then a rotation around the $Y$-axis in the negative direction will begin as a result of the magnetic

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$$\bar{Z}_1 = -\frac{RmW_0^2T_0^2}{6E_0^3}\cdot\left(\frac{e}{m\omega}\right)^2 \zeta\, d\kappa;$$

in which $\zeta$ denotes the number of electrons in a unit volume. That will correspond to a specific moment:

$$\mu = -\frac{1}{12}RmW_0^2\zeta T_0^2\left(\frac{e}{m\omega}\right)^2,$$

and a magnetization number:

$$m = -\frac{1}{12}mW_0^2\zeta T_0^2\left(\frac{e}{m\omega}\right)^2 = -\frac{1}{9}\Psi_0 T_0^2\left(\frac{e}{m\omega}\right)^2,$$

which represents a *steady diamagnetic* excitation.

field. *In this sense*, the body behaves in a manner that is completely equivalent to a permanent magnet or a solenoid with its axis parallel to the $X$-direction.

*However, the further motion proceeds according to completely different laws.*

These laws can be obtained by entirely elementary tools in the simplest case, for which *the body possesses three equal principal moments of inertia around its center-of-mass*; hence, it has the form of, e.g., a ball. One will then have:

$$\text{(54)} \qquad \begin{cases} \Xi = \mathrm{H} = \mathrm{Z} = 2\mathfrak{X} = 2\mathfrak{Y} = 2\mathfrak{X} = \mathrm{M} \\ \Xi' = 0, \quad \mathrm{H}' = 0, \quad \mathrm{Z}' = 0 \end{cases}$$

here.

We would like to *first* address this case and thus also include the effect of the *formation* of the external field in the calculation, and generally while immediately neglecting, as on pp. 5, the terms that contain the time of formation $\tau$ as a factor, which should vanish in their own right for an almost-instantaneous formation, and also, as one easily recognizes, must otherwise exert no noticeable influence on the magnetic effects that will be examined here.

Here, only the value of the moment of impulse that was given in (20) will be employed for the first period of the formation of the field, which will yield:

$$\text{(55)} \qquad \begin{cases} p_1 - p_0 = 0, \quad p_1 - p_0 = 0, \quad p_1 - p_0 = -\mathsf{P}, \\ \text{where} \\ \mathsf{P} = \varepsilon R / 2\rho\omega, \end{cases}$$

directly when one introduces the initial and final velocities $p_0$, $q_0$, $r_0$, and $p_1$, $q_1$, $r_1$, resp.

For the second period of *constant* field strength, equations (53) will assume the form:

$$\text{(56)} \qquad \frac{dp}{dt} = \mathsf{P}\, q, \qquad \frac{dq}{dt} = -\mathsf{P}\, q, \qquad \frac{dr}{dt} = 0,$$

and when we once more measure $t$ from zero onward in that period, they will integrate to the expressions:

$$\text{(57)} \qquad \begin{cases} p = \phantom{-}p_1 \cos \mathsf{P}t + q_1 \sin \mathsf{P}t, \\ q = -p_1 \sin \mathsf{P}t + q_1 \cos \mathsf{P}t, \quad r = r_1. \end{cases}$$

If the rotating electric ball were equivalent to a permanent magnet then the excitation of the magnetic field in the special initial state ($q_1 = 0$, $r_1 = 0$) that was assumed above would give rise to a pendulum motion in the $XY$-plane. By contrast, the formulas above show that one will have:

$$\text{(58)} \qquad p = p_1 \cos \mathsf{P}t, \qquad q = -p_1 \sin \mathsf{P}t, \qquad r = 0$$

in this case, so the rotational axis will rotate with constant velocity – P in the equatorial plane *XY* around the direction of the field $R = Z$, while the body would further rotate around that moving axis with the original velocity $p_1$ . If one puts the axis of instantaneous rotation in parallel with the magnetic axis then the latter would not oscillate pendulously in the *XZ*-plane (as it would for a permanent magnet) in our case, but would rotate around the *Z*-axis.

The basis for that discrepancy clearly lies in the fact that any variation of the motion in our case will also alter the system of currents to which the rotating electric body is equivalent.

If one applies formulas (2) for the magnetic forces that emanate from a rotating, electrified body and act upon a ball then, if one recalls (54), one will have:

(59)
$$\begin{cases} (X_1) = \dfrac{\varepsilon \mathrm{M}}{\omega}\left\{\dfrac{p}{E_0^3} - \dfrac{3}{2E_0^5}[p(y_1^2 + z_1^2) - q x_1 y_1 - r x_1 z_1]\right\}, \\ (Y_1) = \dfrac{\varepsilon \mathrm{M}}{\omega}\left\{\dfrac{q}{E_0^3} - \dfrac{3}{2E_0^5}[q(z_1^2 + x_1^2) - r y_1 z_1 - p y_1 x_1]\right\} \\ (Z_1) = \dfrac{\varepsilon \mathrm{M}}{\omega}\left\{\dfrac{r}{E_0^3} - \dfrac{3}{2E_0^5}[r(x_1^2 + y_1^2) - p z_1 x_1 - q z_1 y_1]\right\}. \end{cases}$$

The expressions are then the same as the ones that follow for a magnetic molecule with the moments:

(60)
$$\alpha_1 = \frac{\varepsilon \mathrm{M}\, p}{2\omega}, \quad \beta_1 = \frac{\varepsilon \mathrm{M}\, q}{2\omega}, \quad \gamma_1 = \frac{\varepsilon \mathrm{M}\, r}{2\omega},$$

and it will follow from formulas (55) and (57) that the *mean* magnetic moment of the rotating ball in any direction that is *normal* to the lines of force of the field will vanish, while the one that is *parallel* to the direction of the lines of force will possess the value:

(61)
$$\bar{\gamma}_1 = \frac{\varepsilon \mathrm{M}}{2\omega}(r_0 - \mathrm{P}), \qquad \text{in which} \qquad \mathrm{P} = \varepsilon R / 2\,\rho\,\omega.$$

Thus, if $\zeta$ bodies of the type considered are present in the unit volume, and if their rotational axes and magnitudes of the rotational velocity are distributed randomly over all possible directions and values then the unit volume will take on a moment of:

(62)
$$\mu = -\,\frac{\varepsilon^2\, \mathrm{M}\, \zeta R}{4\rho\omega^2}$$

*as a result of the formation of the external field of strength R*, or when one introduces the charge density $\varepsilon / \omega = e'$ in electromagnetic units:

(63) $$\mu = -\frac{\varepsilon'^2 \mathrm{M} \zeta R}{4\rho}.$$

If the rotations that are modified by the formation of the field are not damped by resistance or modified by collisions then a medium with the constitution that was described will behave *diamagnetically* in a constant magnetic field with the magnetization number:

(64) $$\mathfrak{m} = -\frac{\varepsilon'^2 \mathrm{M} \zeta}{4\rho}.$$

If resistance and collisions are present that uniformly excite the latter with all possible directions and strengths of rotations then the medium will behave in a magnetically indifferent way in a constant field, and, from formulas (57), the rotations will also remain disordered in the magnetic field when they were initially disordered.

*Thus, when very many rotating, charged particles with three equal principal moment axes are present in a volume element of a medium, and their rotational axes and velocities are distributed randomly in such a way that the volume element does not possess a total magnetic moment, diamagnetism will initially be excited in the volume element by the appearance of an external magnetic field, which will, however, persist for a constant external field only when the rotation of the particles takes place without always renewing the disordered driving forces.*

**6.** One can suspect that equality of the three principal moment axes of the rotating particles that was assumed in foregoing by itself gave rise to the recently-emphasized result that a *constant* magnetic field will either induce diamagnetism in a system of initially-disordered rotating bodies or no sort of magnetic excitation at all, but that for particles of lower symmetry – e.g., rotating bodies – a different result would come about. In order to discuss this question, it will be necessary to base the equations of motion on the principal moment axes *A*, *B*, *C*, which will then read:

(65) $$\rho\left(\mathrm{A}\frac{df}{dt} + (\Gamma - \mathrm{B})gh\right) = F, \qquad \text{etc.,}$$

in the previous notation.

From (21), these imply, in the previous approximation:

(66) $$\mathrm{A}\,(f_1 - f_0) = \frac{\varepsilon R}{\rho\omega}(\beta_2\,\alpha_3\,\mathfrak{C} - \beta_3\,\alpha_2\,\mathfrak{B}), \qquad \text{etc.,}$$

for the period of *formation* of the external field, and once more $f_0$, $g_0$, $h_0$ and $f_1$, $g_1$, $h_1$ will denote the initial and final values of the angular velocities.

If one substitutes these values in the expression (7) for the magnetic *Z*-component at a point on the *Z*-axis at a distance $E_0$ then one will get:

(67) $$(Z_1) - (Z_0) = -\frac{\varepsilon^2 R}{\omega^2 \rho E_0^3}(\mathfrak{A}\alpha_1^2 + \mathfrak{B}\beta_1^2 + \mathfrak{C}\gamma_1^2),$$

so, since the expression in parentheses is always positive, $(Z_1) - (Z_0)$ will always have a *negative* value. Since $(\bar{Z}_0)$ vanishes, the mean value $(\bar{Z}_1)$ of $(Z_1)$ for all possible initial orientations of the body will be:

(68) $$(\bar{Z}_1) = -\frac{\varepsilon^2 R}{3\omega^2 \rho E_0^3}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}),$$

and the moment of the unit volume that contains $\zeta$ such bodies will be determined to be:

(69) $$\mu = -\frac{\varepsilon^2 \zeta R}{6\omega^2 \rho}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}) = -\frac{\varepsilon^2 \zeta R}{12\,\omega^2 \rho}(\mathrm{A} + \mathrm{B} + \Gamma),$$

which represents a simple generalization of the formula (62) for particles of spherical symmetry. The excitation by the *forming* magnetic field here is therefore also *diamagnetic*.

From (65) and (13), and due to the fact that $A = R\,\gamma_1$, $B = R\,\gamma_2$, $C = R\,\gamma_3$, the differential equations for the rotation in a *constant* magnetic field will read as follows:

(70) $$\begin{cases} \mathrm{A}\dfrac{df}{dt} + (\Gamma - \mathrm{B})gh = 2\mathrm{P}(g\gamma_3\mathfrak{C} - h\gamma_2\mathfrak{B}), \\ \mathrm{B}\dfrac{df}{dt} + (\mathrm{A} - \Gamma)hf = 2\mathrm{P}(h\gamma_1\mathfrak{A} - f\gamma_3\mathfrak{C}), \\ \Gamma\dfrac{df}{dt} + (\mathrm{B} - \Gamma)fg = 2\mathrm{P}(f\gamma_2\mathfrak{B} - g\gamma_1\mathfrak{A}). \end{cases}$$

The factors *f*, *g*, *h* directly imply a first integral:

(71) $$\mathrm{A}\,f^2 + \mathrm{B}\,y^2 + \Gamma\,h^2 = \Delta^2,$$

in which, one understands $\Delta^2$ to mean the integration constant; this is the equation for the conservation of *vis viva*.

A second integral will yield the factors $\gamma_1$, $\gamma_2$, $\gamma_3$ when one considers the relations:

(72) $$\frac{d\gamma_1}{dt} = \gamma_2 h - \gamma_3 g, \qquad \frac{d\gamma_2}{dt} = \gamma_3 f - \gamma_1 h, \qquad \frac{d\gamma_3}{dt} = \gamma_1 g - \gamma_2 f,$$

namely, the formula:

(73) $$\mathrm{A}\,\gamma_1 f + \mathrm{B}\,\gamma_2 g + \Gamma\,\gamma_3 h = \mathrm{K}' + \mathrm{P}(\mathfrak{A}\gamma_1^2 + \mathfrak{B}\gamma_2^2 + \mathfrak{C}\gamma_3^2),$$

in which the K′ denotes the second integration constant. By employing the relations $\frac{1}{2}$(B + Γ – A) = $\mathfrak{A}$, …, which correspond to formulas (4), and with the introduction of another constant K, this equation will assume the form:

$$\text{(74)} \qquad \mathrm{A}\,\gamma_1 f + \mathrm{B}\,\gamma_2 g + \Gamma\,\gamma_3 h = \mathrm{K} - \mathrm{P}\,(\mathfrak{A}\gamma_1^2 + \mathfrak{B}\gamma_2^2 + \mathfrak{C}\gamma_3^2)\,.$$

In this, the parentheses on the right-hand side is identical to instantaneous moment of inertia Z of the volume of the body around the *Z*-axis, while the expression on the left is identical to the surface moment around the same direction.

As with the analogous problem, one cannot apparently find a third integral, in general, but one will get one directly from one equations (70) when one considers a body with two equal principal moments of inertia.

For example, if A = B, and accordingly, $\mathfrak{A} = \mathfrak{B} = \frac{1}{2}\Gamma$, and $\mathfrak{C} = \mathrm{A} - \frac{1}{2}\Gamma$, then the three integrals will read:

$$\text{(75)} \qquad \mathrm{A}\,(f^2 + g^2) + \Gamma\,h^2 = \Delta^2,$$

$$\text{(76)} \qquad \mathrm{A}\,(\gamma_1 f + \gamma_2 g) + \Gamma\,\gamma_3 h = \mathrm{K} - \mathrm{P}\,[\mathrm{A} + (\Gamma - \mathrm{A})\,\gamma_3^2],$$

$$\text{(77)} \qquad h = \Lambda - \mathrm{P}\,\gamma_3\,,$$

in which $\Delta^2$, K, Λ represent the integration constants.

A further treatment of the general problem of rotation, which is very interesting in its own right, shall be omitted here. We shall restrict ourselves to the one that comes under consideration for our special problem.

From (8), the expression in the *left*-hand side of formula (76), which might be further denoted by Ω, is characteristic of the magnetic action || *Z* that emanates from the rotating body. If one finds a large number of such particles in a volume element that are initially in disordered motion then the spatial and temporal mean value $(\bar{\Omega})$ of Ω *before* the excitation of the external field – i.e., for P = 0 – must vanish. Now, if the field arises with all directions having equal probability to the distinguished *C*-axis then, from formula (76):

$$\text{(78)} \qquad 0 = \mathrm{K} - \mathrm{P}\,[\mathrm{A} + \tfrac{1}{3}(\Gamma - \mathrm{A})],$$

so the mean value of $\gamma_3^2$ will be 1/3 with the assumptions that were made. The temporal and spatial mean value $(\bar{\Omega})$ of Ω *with the continued existence of the field* is them given by:

$$(\bar{\Omega}) = \mathrm{K} - \mathrm{P}\,[\mathrm{A} + \tfrac{1}{3}(\Gamma - \mathrm{A})(\bar{\gamma}_3^2)]\,;$$

i.e., by:

$$\text{(79)} \qquad (\bar{\Omega}) = \mathrm{P}\,(\Gamma - \mathrm{A})[\tfrac{1}{3} - (\bar{\gamma}_3^2)]\,,$$

in which $(\bar{\gamma}_3^2)$ denotes the spatial and temporal mean value of $\gamma_3^2$.

In order to calculate this latter mean value, the integration of the problem must be continued one step further.

We introduce the value of $h$ from equation (77) into both sides of the foregoing one and write down the result:

(80) $$\mathrm{A}^2 (f^2 + g^2) = \mathrm{A}\,[\Delta^2 - \Gamma\,(\Lambda - \mathrm{P}\,\gamma_3)^2],$$

(81) $$\mathrm{A}\,(f\,\gamma_1 + g\,\gamma_2) = (\mathrm{K} - \Gamma\,\Lambda\,\gamma_3) - \mathrm{PA}\,(1 - \gamma_3^2);$$

we couple these with the third formula in (72) (the relation:

(82) $$\mathrm{A}\,(g\,\gamma_1 - f\,\gamma_2) = \mathrm{A}\,\frac{d\gamma_3}{dt},$$

respectively), and when we subtract the sum of the squares of the last two formulas from equation (80), multiplied by $(1 - \gamma_3^2)$, we will get:

(83) $$\mathrm{A}\,[\Delta^3 - \Gamma\,(\Lambda - \mathrm{P}\gamma_3)^2]\,(1 - \gamma_3^2) - [(\mathrm{K} - \Gamma\gamma_3) - \mathrm{PA}\,(1 - \gamma_3^2)]^2 = \mathrm{A}^2 \left(\frac{d\gamma_3}{dt}\right)^2,$$

which is a relation that couples $t$ with $\gamma_3$ by an elliptic integral.

From now on, we shall restrict ourselves to an approximation that assumes that the effect of the magnetic field on the motion is very slight, and that the magnetization is accordingly found to be proportional to the field (and thus, to P).

With that assumption, the last equation will reduce to:

(84) $$\mathrm{A}\,[\Delta^2 - \Gamma\Lambda^2 + 2\mathrm{PK}]\,(1 - \gamma_3^2) - [(\mathrm{K} - \Gamma\Lambda\gamma_3)^2 = \mathrm{A}^2 \left(\frac{d\gamma_3}{dt}\right)^2,$$

which we abbreviate to:

(85) $$U + 2V\,\gamma_3 - W\gamma_3^2 = \left(\frac{d\gamma_3}{dt}\right)^2,$$

such that one will have:

(86) $$\begin{cases} \mathrm{A}(\Delta^2 - \Gamma\Lambda^2 + 2\mathrm{PK}) - \mathrm{K}^2 = U\mathrm{A}^2, \qquad \Gamma\mathrm{K}\Lambda = V\mathrm{A}^2, \\ \mathrm{A}(\Delta^2 - \Gamma\Lambda^2 + 2\mathrm{PK}) - \Gamma^2\Lambda^2 = W\mathrm{A}^2. \end{cases}$$

If no magnetic field is present – i.e., P = 0 – then we will write (85) as:

(87) $$U_0 + 2V_0\,(\gamma_3)_0 - W_0\,(\gamma_3)_0^2 = \left(\frac{d(\gamma_3)_0}{dt}\right)^2,$$

in which $U_0$, $V_0$, $W_0$ emerge from $U$, $V$, $W$, resp., by dropping P. As is known, the motion in this case makes the distinguished principal moment axis circle around a fixed direction in space (viz., the normal to the so-called "invariable plane") at a constant angle and with a constant velocity. Here, $\gamma_3$ must then be periodic, from which, we can conclude that $U_0$ and $W_0$ will always have positive values, with no further discussion. We will then have:

(88) $$2W_0\,(\gamma_3)_0 - V_0 = \sqrt{4U_0W_0 + V_0^2}\,\sin(t - t_0)\sqrt{W_0}\,,$$

in which $t$ represents the integration constant.

If one consider the expressions (86) for $U$, $V$, $W$, and observes that P is treated as a first-order quantity then one can write:

(89) $$\gamma_3^2 = (\gamma_3)_0^2 + \left(\frac{\partial(\gamma_3)_0^2}{\partial U_0} + \frac{\partial(\gamma_3)_0^2}{\partial W_0}\right)\frac{2\mathrm{PK}}{\mathrm{A}}\,.$$

The spatial-temporal mean value of the second term vanishes. Namely, due to the order of magnitude of P, and according to (76), one can consider K to be the initial value of the surface moment around the *Z*-axis in this term, which is positive, as well as negative, just as often for the assumed disorder in the initial state. However, as one easily recognizes, the expression in parentheses does *not* change its value when one switches K with – K. Accordingly, the spatial-temporal mean value of $\gamma_3^2$ is identical to its value for P = 0 – i.e., to $\frac{1}{3}$ – and formulas (79) yields:

(90) $$(\bar{\Omega}) = 0.$$

The result that was expressed on pp. 17 can be generalized in the following way here:

*If very many rotating, electrically-charged particles with two equal principal moments of inertia are present in a volume element, and their rotational axes and velocities are randomly-distributed in such a way that the volume element does not possess a total magnetic moment then such a volume element will be initially excited diamagnetically by the formation of an external magnetic field. However, that excitation will not persist for an external field that is kept constant when the motion of the particles is kept disordered by a steady series of new random collisions.*

An extension of these considerations to bodies that have three different principal moments of inertia and are charged inhomogeneously would undoubtedly lead to analogous results, and for that reason, it shall be omitted.

**7.** In the foregoing, we have overlooked the direct effect of a force of resistance that would counteract the rotation. In order to conclude our investigation, we must still discuss the question of *the extent to which such a resistance might modify the magnetic force that emanates from a charged, rotating body by altering the motion.* Thus, the consideration shall, in turn, be restricted to homogeneous bodies with two equal principal moments of inertia for its volume, so, e.g., to bodies of revolution, and restrict ourselves moreover, to a resistance that is proportional to the angular velocity and has an analogous symmetry, which we introduce in connection with the dispersion equations without making any hypotheses about its origin, initially.

For that case, we write the equations of motion (70):

$$(91)\qquad \begin{cases} \mathrm{A}\dfrac{df}{dt}+(\Gamma-\mathrm{A})gh = 2\mathrm{P}(g\gamma_3\mathfrak{C}-h\gamma_2\mathfrak{A})-\mathfrak{a}\, f, \\ \mathrm{B}\dfrac{dg}{dt}-(\Gamma-\mathrm{A})hf = 2\mathrm{P}(h\gamma_1\mathfrak{A}-f\gamma_3\mathfrak{C})-\mathfrak{a}\, g, \\ \Gamma\dfrac{dh}{dt} \qquad\qquad\quad = 2\mathrm{P}\,\mathfrak{A}(f\gamma_2-h\gamma_1)-\mathfrak{c}\, h, \end{cases}$$

in which we have introduced two resistance parameters $\mathfrak{a}$ and $\mathfrak{c}$ for the coordinate system *A*, *B*, *C* that is fixed in the body. When they are combined with the factors $\gamma_1$, $\gamma_2$, $\gamma_3$, and with consideration given to the conditions (72), they will give:

$$(92)\qquad \frac{d}{dt}[\mathrm{A}\,(f\,\gamma_1+g\,\gamma_2)+\Gamma\,h\,\gamma_3] = -\,\mathrm{P}\,(\Gamma-\mathrm{A})\frac{d\gamma_3^2}{dt}-[\mathfrak{a}\,(f\,\gamma_1+g\,\gamma_2)+\mathfrak{c}\,h\,\gamma_3].$$

in the manner that was applied on pp. 19.

If one again introduces the notation $\Omega$ for the expression in brackets on the left, which, from formula (8), is characteristic of the magnetic effect of the body that is parallel to the lines of force of the external field, then one can write that result as:

$$(93)\qquad \frac{d\Omega}{dx}+\frac{\mathfrak{a}}{\mathrm{A}}\Omega = -\,\mathrm{P}\,(\Gamma-\mathrm{A})\frac{d\gamma_3^2}{dt}-\frac{\mathfrak{c}\mathrm{A}-\mathfrak{a}\Gamma}{\mathrm{A}}h\,\gamma_3,$$

or also:

$$(94)\qquad \frac{d\Omega}{dx}(\Omega\, e^{\mathfrak{a}t/\mathrm{A}}) = -\,\mathrm{P}\,(\Gamma-\mathrm{A})\,\mathrm{e}^{\mathfrak{a}t/\mathrm{A}}\,\frac{d\gamma_3^2}{dt}-\frac{\mathfrak{c}\mathrm{A}-\mathfrak{a}\Gamma}{\mathrm{A}}h\,\gamma_3\, e^{\mathfrak{a}t/\mathrm{A}},$$

after multiplying by $e^{\mathfrak{a}t/\mathrm{A}}$.

One then gets from this, *for the special case of three equal principal moments of inertia and equal resistance moments around all three principal moment axes* (so, e.g., for the case of a *ball*):

$$(95)\qquad \frac{d\Omega}{dx}(\Omega\, e^{\mathfrak{a}t/\mathrm{A}}) = 0, \qquad \text{i.e.,} \qquad \Omega = C\, e^{\mathfrak{a}t/\mathrm{A}},$$

in which $\mathrm{A} = \Gamma$, $\mathfrak{a} = \mathfrak{c}$, and *C* means the integration constant. For such a body, the function $\Omega$ will then die off in a manner that is entirely independent of the sense of rotation. Therefore:

*If the mean value* $\bar{\Omega}$ *of* $\Omega$ *is zero for a large number of rotating, charged balls at any time then it will also keep that value – viz., no magnetic effect will arise as a result of the damping.*

In other cases, one will then arrive at simple results when one restricts oneself to terms of first order in the effect of the external magnetic field, and thus in the quantity P, as would correspond to the proportionality of the magnetic excitation with the external field that is always observed in dielectrics. One can, e.g., employ the properties of $\gamma_3$ that would exist for *no* magnetic field, e.g., in the term in formula (93) that is multiplied by P. If one takes, e.g., the spatial mean value of all terms in equation (93) for a system of very many initially-disordered moving bodies then one can set the mean value of $d\gamma_3^2 / dt$ equal to zero in each term. Upon neglecting the magnetic effect, positive and negative directions of rotation will then be equally probable for the same position of the body.

If one then denotes the spatial mean value of a function $\varphi$ by $\bar{\varphi}$ then (93) will imply:

$$\text{A}\frac{d\bar{\Omega}}{dt} + \mathfrak{a}\,\bar{\Omega} = (\mathfrak{a}\,\Gamma - \mathfrak{c}\,\text{A})\,\overline{h\gamma_3}\,. \tag{96}$$

If one further constructs the *vis viva* equation from the system (91) as on pp. 22 then it will read:

$$\frac{d}{dt}\tfrac{1}{2}[\text{A}\,(f^2 + g^2) + \Gamma\,h^2] = -\,[\mathfrak{a}\,(f^2 + g^2) + \mathfrak{c}\,h^2]. \tag{97}$$

This shows that the *vis viva* does not necessarily vanish steadily in the cases where one of the two resistance constants $\mathfrak{a}$ or $\mathfrak{c}$ is vanishingly small. In those special cases, the motion of the rotating body will approach a *stationary* state in which only a *resistance-free* rotation will still persist, and in which $\overline{d\gamma_3^2} / dt$ will vanish strictly, so formula (96) will not just be valid approximately.

If, e.g., $\mathfrak{a} = 0$ then the body will enter the stationary state with vanishing $h$ for a non-zero $f^2 + g^2$; if $\mathfrak{c} = 0$ then the opposite will be true.

The *first* extreme case (viz., $\mathfrak{a} = 0$) is hardly simple to realize, but it still serves as a simple limiting case that might clarify a problem that is generally complicated. Here, $h$ = 0 for the stationary state, so, from (96), one will also have:

$$\bar{\Omega} = 0,$$

which is valid for $d\bar{\Omega}/dt$. *Therefore, the resistance does not modify the effect of the field on the system here.*

The *second* extreme case (viz., $\mathfrak{c} = 0$) would arise approximately for, e.g., a body of the form of a thin circular disc that moves in a frictionless fluid. Equation (77) is true for it, so $h$ will be constant in time, along with $\gamma_3$, and it will follow here immediately from equation (96) that *for the stationary state* ($d\bar{\Omega}/dt = 0$):

$$\bar{\Omega} = \Gamma\overline{h\gamma_3}\,. \tag{98}$$

Since $f^2 + g^2$, and therefore $f$ and $g$, vanish individually in this case, from the first two formulas (91) – at least, when $h$ does not vanish – $\gamma_1$ and $\gamma_2$ will both be equal to zero, $\gamma_3$ will be equal ± unity. The distinguished $C$-axis will then point in the direction of the lines of force of the external field, and an argument concerning stability behavior will imply that $h\ \gamma_3 > 0$, so the *C*-axis must be directed *parallel* to the field strength *R* for *positive h.*

If $\gamma_3$ and $h$ are true for stationary state, while $\gamma_3^0$ and $h^0$ are true for the initial state then it will follow from (77) that:

(99) $$h = h^0 - \text{P}\,(\gamma_3 - \gamma_3^0),$$

so the sign of $h$ can be *inverted* as a result of the field effect. A considerable complication of the situation is based upon this that makes the computational analysis of the processes more difficult. In any case, if one denotes the absolute value of a function $\varphi$ by $|\ \varphi\ |$ then one can write $\overline{h\gamma_3} = \left|\bar{h}\right|$, so:

(100) $$\bar{\Omega} = +\,\Gamma\left|\bar{h}\right|.$$

This implies that the magnetic force that emanates from the center from the body, which assumed to be rotating, and parallel to the lines of force of the external field will be, according to (8):

(101) $$(\bar{Z}_1) = +\,\frac{\varepsilon\,\Gamma\,|\,\bar{h}\,|}{\omega E_0^3}.$$

The unit volume of medium that contains $\zeta$ such bodies will then possess the magnetic moment:

(102) $$\mu = +\,\frac{\varepsilon\,\zeta\,\Gamma\,|\,\bar{h}\,|}{2\omega},$$

and *the excitation will always be paramagnetic.*

The situation will simplify when all of the initial velocities $h^0$ (or at least, the majority of them) possess a magnitude whose absolute value is greater than 2P. If one then aligns the positive *C*-axis such that $h^0 > 0$ then $h$ will be positive for all bodies, and $\gamma_3 = +\,1$, so for initially-disordered motion, one will have $\bar{\gamma}_3^0 = 0$, so $\left|\bar{h}\right| = \bar{h}^0 - \text{P}$, and since $\text{P} = \varepsilon R / 2\rho\,\omega$, one will have:

(103) $$\mu = \frac{\varepsilon\,\zeta\,\Gamma}{2\omega}\left(\bar{h}^0 - \frac{\varepsilon R}{2\rho\omega}\right).$$

The result that is included in the foregoing exhibits many surprising features. Since the ultimate value of $\gamma_3$ is equal to $\pm\,1$, every particle with its distinguished axis will ultimately be directed parallel to the lines of force by the magnetic field in the case considered. A mere deviation from the original position whose magnitude is determined by the external field, as one assumed in the older theory, does not take place. That is connected with the fact that the equivalent mean moment $\bar{\mu}$ contains a considerable part that is completely independent of the strength the magnetic field that acts upon it and will remain when the external field is arbitrarily small. If the external field is strictly equal to zero then $\mu$ will nevertheless be simultaneously zero, because in that case equations (91) will not lead to $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = \pm 1$, at all, so there will be no orientation of the particles either. Along with this part that is independent of *R*, which corresponds to a *para*magnetic excitation, formula (103) also includes a second part that is proportion to *R*, represents *dia*magnetism, and originates from the fact that the magnetic field will *reduce* the initial angular velocity during the alignment of the particles according to the formula:

$$h = h^0 - \mathrm{P}\,(1 - \gamma_0^3).$$

The apparent discontinuity that arises from the transition from infinitely-small to strictly-vanishing field strengths *R* will vanish by the argument that our formulas refer to the *stationary* state, which later arises completely in a noticeable way the smaller that *R* gets, while for vanishing *R* it will first arise after an infinitely-long time, and therefore never.

That is based upon the fact that a system of particles of the type considered can exhibit a total excitation that contains *no* part that is independent of *R* when the ongoing alignment of the particles will advance even more by randomly-distributed collisions. Since the alignment advances even more for equal mean time intervals T between two collisions by the effect of larger field strengths, the mean moment that is produced will increase with the field strengths (in a complicated way). Shortening the time duration T (perhaps by raising the temperature) will *lower* the excitations that correspond to the same fields *R*.

We can summarize the result that was obtained for the special case that was just treated as follows:

*A system of homogeneous and homogeneously-charged rotating bodies that move around the directions of the resistance moment that are normal to the distinguished axis, and not the latter axis, will be excited paramagnetically by a constant external field with the mean moment that is given by (102); conversely, if a resistance moment acts only around the distinguished axis then a magnetic excitation will not take place.*

In the general case where none of the resistance moments vanish, the motion of a rotating, charged body will approach no other stationary state than that of *rest* (which is analogous to the special case of the ball that was treated on pp. 22). A constant, finite mean value of the energy can be maintained here only by continually renewing the driving forces. If the latter are found to be completely random then that will raise the question in regard to the magnetic effect of a system of particles of that kind of whether the mean value $\bar{\Omega}$ starts from an initial value of zero, while the subsidence of the motion is positive or negative. Thus, the following (approximate) formula:

$$\frac{d\bar{\Omega}e^{\mathfrak{a}t/\mathrm{A}}}{dt} = \frac{\mathfrak{a}\Gamma - \mathfrak{c}\mathrm{A}}{\mathrm{A}}\,\overline{h\gamma_3}\,e^{\mathfrak{a}t/\mathrm{A}}$$

or

$$\bar{\Omega} = \frac{\mathfrak{a}\Gamma - \mathfrak{c}\mathrm{A}}{\mathrm{A}}\,e^{-\mathfrak{a}t/\mathrm{A}}\int_0 \overline{h\gamma_3}\,e^{-\mathfrak{a}t/\mathrm{A}}\,dt$$

will be true for $\bar{\Omega}$.

The general treatment of the problem raises some difficulties. In the case where the resistance moment around the distinguished *C*-axis is small compared to the moments around the directions that are normal to it, and thus $\mathfrak{c}$ is small compared to $\mathfrak{a}$, one can, from the foregoing, then deduce from the evolution of $\overline{h\gamma_3}$ with some likelihood that this quantity will initially increase and then approach zero. $\bar{\Omega}$ would then behave analogously, the excitation of the system would generally prove to be paramagnetic, and diamagnetism would come about only for an especially large value of Α / Γ.

In the foregoing, in connection with the equations of optics, we introduced resistance moments that acted upon the charged particles of ponderable matter (i.e., electrons in the general sense), which were linear functions of the angular velocities and could be interpreted by imagining that the motion took place in a resisting medium. Meanwhile, in the more recent ways of conceptualizing matters, at least a part of the damping of each electron motion will be due to the emission of energy by means of the electromagnetic waves that are produced.

It is remarkable that when one attributes *all* the damping to radiation by the rotating, charged particles in question, the second extreme case (viz., $\mathfrak{c} = 0$) that was treated will result approximately for rotating bodies. A charged rotating body that rotates with constant velocity around its axis will then emit *no* electromagnetic waves at all, and when the velocity is *slowly-varying*, only ones of extremely small energy will occur. Thus, the more recent picture will provide a close analogy to the resistance-free molecular currents of the older theory in this case, and formulas (100)-(103), which express a *para*magnetic excitation, would take on an essential meaning here.

Whether it will suffice for the quantitative derivation of observed phenomena of magnetization to include rotations of appreciable *vis viva*, along with the advancing motion of the negative electrons that are employed in recent theoretical optics or whether one must revert to positively-charged atoms of ponderable matter, has still not be resolved as of yet. Only the problem of its general validity was clarified here.

## Conclusion.

The result of the investigation can be summarized by saying that the electron hypothesis, in its present form, as long as one directs one's attention to only regular *orbital* motions of the electrons under the action of quasi-elastic forces, and thus excludes resistances, and accordingly, driving forces, along with any other sort of perturbation, and thus approaches the older hypotheses of molecular currents as closely as possible, the effect of a magnetic field will not produce any magnetic excitations at all. However, it will produce paramagnetic, as well as diamagnetic, effects when one assumes that random, recurring driving forces are present that would be necessary to compensate for the energy loss to resistance. Furthermore, the assumption of a charged mass that is subject to no resistance and *rotates* in a magnetic field will give only diamagnetism, but will admit paramagnetism, as well as diamagnetism, when resistance and driving forces act upon it.

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