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THE PRESENT STATE OF OUR KNOWLEDGE

ON

THE ELASTICITY OF CRYSTALS

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Introduction.

1. *Goal and plan of the present report.* – The organizing committee of the Congrès international de Physique, while requesting that the author give a report on the present state of our knowledge of *the elasticity of crystals*, expressed the desire that this account should contain, in addition to a documentation of the recent progress in the theory and experiment, some general views on the use of symmetry relations in the physics of crystals, and some proposal for a uniform nomenclature in the domain of elasticity.

It can be only an honor and a pleasure for the author to contrast the experimental study of the elasticity of crystals (¹) with the general theory of elasticity, as he has dedicated part of his lifework to the exploration of those domains, and that proposal comes from the country in which the general theory of elasticity was born, and in which the premier scientific association has designated it (with three reiterations) to be one of the most important questions.

The following presentation, in which the author has sought to respond to that desire, is composed of six parts, which are preceded by some general remarks on the scientific nomenclature.

Conforming to a suggestion that the committee made to the author expressly, the first part is occupied with some general properties of the particular *directed* quantities that play a significant role in elasticity, and that the author has described with the name of *tensors* to the vectors that are essentially their parents. In that presentation, we have tried to principally summarize their properties that will be important in the rest of this paper,

(†) Translated from the French translation of the German original by P. Weiss, Maître de Conférences on the science faculty at Lyon.

(¹) Comptes rendus, **71** (1870), 160; *ibid.* **75** (1872), 1391; *ibid.* **81** (1875), 1369.

without at the same time neglecting the suggestions for the use of tensors in other parts of physics.

Part Two presents the methods that have been followed in the development of the theory of elasticity (viz., action at a distance and immediate action), and are attached to the general properties of the components of stress in deformed bodies.

Part Three deduces some relations between the stresses and deformations in elastic bodies, while especially taking into account the rotations of molecules with respect to the volume elements. It introduces two systems of elastic parameters (constants and moduli) and shows how they behave under changes of coordinates.

Part Four discusses the general principles of the application of some elements of crystalline symmetry to the use and specialization of various scalar functions, and especially the elastic potential.

Part Five begins by specifying some questions that can be answered by determining the elastic parameters of the crystal and then giving the set of formulas that are necessary for the calculation of the observations.

Part Six contains a general overview of some observations regarding the elasticity of crystals and their application to the explanation for some experimental facts that have been discussed for some time now and that belong to the realm of the elasticity of isotropic bodies.

An Appendix contains the statements of some laws of thermo-elasticity.

The conclusion of each part is composed of a small number of directed propositions – or *theses* – that are intended to clearly summarize the most important questions that were raised or resolved in that part.

2. Scientific nomenclature. – The branches of science in which the deeper study of the special problems that precede the development of the general theory are generally devoid of any systematic nomenclature. Elasticity, in which the cases of tension, flexion, and compression, in any sense of the terms, have been observed since the earliest times, is a striking example of that. The older terms – in particular, the terms for the simplest parameters (e.g., coefficient of elasticity, modulus of elasticity) – are less than characteristic and are not employed uniformly by all authors. One can also point out the indeterminacy in some of the terms of *large elasticity* (*grande élasticité*), or of *very elastic bodies*, which can be employed for simple deformations, as well as in cases of great resistance to the deformation.

The necessity for such reforms have been sensed for some time now, and has provoked the creation of a certain number of new terms for the case of isotropic bodies ⁽¹⁾ that realize some appreciable progress, but for the most part, were conceived in too narrow a context to be extended to crystals. By contrast, for quite a few years now, the author has appealed to a nomenclature for some parameters that was created especially for the elasticity of crystals and that can be transported immediately to isotropic bodies, and whose use, it seems, has been generalized little-by-little in Germany.

⁽¹⁾ W. THOMSON, Trans. Roy. Soc. 24 April 1856, 18 May 1865. Article “Elasticity” in the 9th edition of the *Encyclopaedia Britannica*, 1878.

Scientific terms must be (at least, in the opinion of the author) established in old languages, or be sufficiently clear that they can be easily translated. Brevity of expression implies that in the former case.

Indeed, that rule has been observed very generally up to now, and the striking exception of the expression *curl*, which originated in England during the last decade, apparently confirms that. That term has only the advantage of brevity. It is based less in the nature of the object than the Latin word *vortex* (Fr. *tourbillion*), whose significance has been established for thirty years now, and whose abbreviated form *vort* can be employed as a symbol, as well as *cos* for cosine.

How short a term can be is inversely proportional to the generality of its usage, as the terms “electric permeability” and “magnetic permeability” that Maxwell introduced show in a convincing manner. Although they are attached to a very specific interpretation of certain phenomena and are very lengthy, those terms were quickly popularized, thanks to their intuitive character.

The author’s proposals regarding the nomenclature of the elasticity of crystals will be developed later on. Here, in the Introduction, we shall remark only that this terminology, when adapted to the problems in the elasticity of crystals, was conceived in a very general manner. It is then perfectly compatible with a simplification and a modification of some terms in the case of isotropic bodies. However, in the interests of unity, there was good reason to proceed according to some principles that do not establish a separation between isotropic bodies and crystals that is too sharp.

I. – Some tensors and triple-tensors ⁽¹⁾.

3. Tensors. – In elasticity, in addition to scalars and vectors, one encounters a third type of functions that play an important role that is almost always attributed to vectors, although they possess completely different properties. The only common character to the last two types of functions – namely, that they are represented by a number and a direction – has no fundamental significance, as a deeper examination will show. Indeed, that direction has an essentially different character for the two types of quantities: Vectors possess two sides with *different values*, while these new quantities possess two *equivalent* sides, which entails some essential differences in their analytical properties.

The author has proposed the name *tensors* for those functions, which, like that of *vector*, is derived from a simple example and picture (viz., the simple elongation of a volume). We shall appeal to it in what follows.

Since the number of elements that fix a tensor *T* or a vector *V* is the same – viz., three – one also seeks to express the magnitude and direction of a tensor symmetrically in terms of components along the coordinate axes. However, while the projections of a vector onto the axes:

$$(1) \quad F = V \cos (V, X), \quad G = V \cos (V, Y), \quad H = V \cos (V, Z)$$

⁽¹⁾ W. VOIGT, *Die fundamentalen Eigenschaften der Krystalle*, Leipzig, 1898, pp. 20, *et seq.*

suggest themselves naturally, they are not useful for a tensor, because those projection, which will not change when one changes both the sign of V and its sense, correspond precisely to the *unilateral* character of the vector that distinguishes the tensor.

By contrast, the bilateral character of a tensor is expressed equally well by two different types of functions that can be considered to be its components along the axes, because each of those functions exhibits one of the axes with respect to the other two.

We call the expressions:

$$(2) \quad A = T \cos^2 (T, X), \quad B = T \cos^2 (T, Y), \quad C = T \cos^2 (T, Z)$$

components of the first kind.

They completely determine the magnitude of T by the formula:

$$(3) \quad T = A + B + C,$$

and its direction, by contrast, is determined incompletely by:

$$(4) \quad \cos^2 (T, X) = \frac{A}{T}, \quad \cos^2 (T, Y) = \frac{B}{T}, \quad \cos^2 (T, Z) = \frac{C}{T},$$

since the trihedron in which one finds T remains indeterminate. We call the expressions:

$$(5) \quad \begin{cases} A' = T \cos(T, Y) \cos(T, Z), \\ B' = T \cos(T, Z) \cos(T, X), \\ C' = T \cos(T, X) \cos(T, Y) \end{cases}$$

components of the second kind.

They determine the magnitude of T entirely by the formula:

$$(6) \quad T = \frac{B'C'}{A'} + \frac{C'A'}{B'} + \frac{A'B'}{C'},$$

except when T coincides with one of the axes. The expression is then indeterminate. By contrast, the direction of T is then always determined unambiguously by the formulas:

$$(7) \quad \cos (T, X) : \cos (T, Y) : \cos (T, Z) = \frac{1}{A} : \frac{1}{B} : \frac{1}{C}.$$

Since three elements suffice to determine the tensor, it will result that the two types of components are mutually dependent; indeed, one has:

$$(8) \quad A'^2 = BC, \quad B'^2 = CA, \quad C'^2 = AB,$$

$$(9) \quad A = \frac{B'C'}{A'}, \quad B = \frac{C'A'}{B'}, \quad C = \frac{A'B'}{C'}.$$

For certain positions that are determined by a coordinate system, a component of the first kind can be identical to the tensor itself, but that will never happen for a component of the second kind. *The components of first kind then have the same nature as the tensor, while the components of the second kind have a different nature.*

The vectorial quantities can always be made positive by an appropriate choice of positive direction for them. Tensors are neither positive nor negative. The components A, B, C always have the sign of T itself, while A', B', C' can have the same sign as T or the opposite one.

If one sets $T = \pm V^2$, according to its sign, and if one attributes one of the directions for T to V then the components A, B, C will be represented by the squares of the components F, G, H of the vector V , while the A', B', C' will be represented by the products of those components. The components of the first kind of tensors will then transform like the squares, while those of the second kind will transform like the components of vectors.

4. Tensor triples. – In nature, tensors are encountered mainly in groups of three whose directions are mutually orthogonal, but whose magnitudes are independent. We shall call such a system a *tensor triple* ⁽¹⁾ and denote it by T_1, T_2, T_3 .

A tensor triple is determined entirely, in magnitude and position, by the six sums of parallel components of the same type of the terms that comprise them:

$$(10) \quad A = \sum A_h, \quad B = \sum B_h, \quad C = \sum C_h,$$

$$(11) \quad A' = \sum A'_h, \quad B' = \sum B'_h, \quad C' = \sum C'_h, \quad h = 1, 2, 3.$$

In order to prove that, consider the tensor:

$$(12) \quad \Theta = A \cos^2 (\Theta, X) + B \cos^2 (\Theta, Y) + C \cos^2 (\Theta, Z) \\ + 2A' \cos (\Theta, Y) \cos (\Theta, Z) + 2B' \cos (\Theta, Z) \cos (\Theta, X) \\ + 2C' \cos (\Theta, X) \cos (\Theta, Y),$$

which has the same type as A, B, C , since it coincides with those three values for certain particular directions of Θ . Replace Θ with $1/r^2$ on the left, where r represents a length that is carried by both directions of Θ . The equation:

$$(12) \quad \frac{1}{r^2} = A \cos^2 (r, X) + B \cos^2 (r, Y) + C \cos^2 (r, Z) \\ + 2A' \cos (r, Y) \cos (r, Z) + 2B' \cos (r, Z) \cos (r, X) \\ + 2C' \cos (r, X) \cos (r, Y)$$

⁽¹⁾ In German: *Tensortripel*.

represents a centered surface that will have degree two that is referred to its principal axes when A', B', C' are zero. However, that will be true only if all the A'_h, B'_h, C'_h are zero individually. In that case, from what was said on page 5, each of the A, B, C will be identical with one of the T_1, T_2, T_3 . The principal axes of the surface (13) will then give the tensor triple T_1, T_2, T_3 in both magnitude and direction.

Just as the components of vectors are deduced by the *parallelepiped construction*, a tensor triple is determined by the construction of the second-degree surface that was considered above, which call the *ellipsoid construction*, to abbreviate.

Here, we point out that a very important category of functions in rational mechanics has the same character as the components of tensors. The moments of inertia of a body with respect to the coordinate axes are the components of a tensor *of the first kind*; the moments of deviation are the components *of the second kind*. The tensor triple that corresponds to those components is that of the principal moments of inertia.

Like the components of a unique tensor, those of a tensor triple behave like the squares and products of the components of a vector under coordinate transformations. It then results that they have a character that allows one to know whether one can consider six functions that are pair-wise referred to the same coordinate axis to be the components of a tensor triple. When the properties of requisite transformations exist, one can deduce the tensor triple that corresponds to those six quantities by the method of the ellipsoid.

One knows that the transformations of the components F, G, H of a vector into another coordinate system can be performed by means of the *same* coefficients as the reciprocal transformation. Let $\alpha_h, \beta_h, \gamma_h$ be the direction cosines of one of the systems X, Y, Z with respect to the other X_0, Y_0, Z_0 ; Consequently, one can represent the relations between the components in the two systems by the following table:

$$(14) \quad \begin{array}{c|ccc} & F & G & H \\ \hline F_0 & \alpha_1 & \beta_1 & \gamma_1 \\ G_0 & \alpha_2 & \beta_2 & \gamma_2 \\ H_0 & \alpha_3 & \beta_3 & \gamma_3 \end{array}$$

The same (orthogonal) relations exist for the six expressions:

$$F^2, G^2, H^2, GH\sqrt{2}, HF\sqrt{2}, FG\sqrt{2},$$

and consequently, they will also exist for the expressions that are obtained by means of the components of the tensor:

$$A, B, C, A'\sqrt{2}, B'\sqrt{2}, C'\sqrt{2}.$$

The system of coefficients of the transformation is then, upon setting $\sqrt{2} = r$:

$$(15) \quad \left\{ \begin{array}{c|cccccc} & A & B & C & rA' & rB' & rC' \\ \hline A_0 & \alpha_1^2 & \beta_1^2 & \gamma_1^2 & r\beta_1\gamma_1 & r\gamma_1\alpha_1 & r\alpha_1\beta_1 \\ & \dots & \dots & \dots & \dots & \dots & \dots \\ rA'_0 & r\alpha_2\alpha_3 & r\beta_1\beta_3 & r\gamma_2\gamma_3 & (\beta_2\gamma_3 + \gamma_2\beta_3) & (\gamma_2\alpha_3 + \alpha_2\gamma_3) & (\alpha_2\beta_3 + \beta_2\alpha_3) \\ & \dots & \dots & \dots & \dots & \dots & \dots \end{array} \right.$$

We represent that table in the abbreviated form:

$$(16) \quad \begin{array}{c|cccccc} & A & B & C & rA' & rB' & rC' \\ \hline A_0 & \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} & \delta_{15} & \delta_{16} \\ & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

5. Relations between scalars, vectors, and tensors. – When a scalar magnitude S is a function of a vector V , or to express that symmetrically, of its components F, G, H , the derivatives:

$$(17) \quad F_s = \frac{\partial S}{\partial F}, \quad G_s = \frac{\partial S}{\partial G}, \quad H_s = \frac{\partial S}{\partial H}$$

are, as one knows, the components of a vector of a particular type. The most important case is the one in which S is a function of the coordinates x, y, z .

In an analogous manner:

$$(18) \quad \left\{ \begin{array}{l} A_s = \frac{\partial^2 S}{\partial F^2}, \quad B_s = \frac{\partial^2 S}{\partial G^2}, \quad C_s = \frac{\partial^2 S}{\partial H^2}, \\ A'_s = \frac{\partial^2 S}{\partial G \partial H}, \quad B'_s = \frac{\partial^2 S}{\partial H \partial F}, \quad C'_s = \frac{\partial^2 S}{\partial F \partial G} \end{array} \right.$$

are, conforming to their transformation properties, the components of a tensor of a particular kind.

When the scalar S is a function of a tensor triple – i.e., of the six components A, B, C, A', B', C' – the derivatives:

$$\begin{aligned} A_1 &= \frac{\partial S}{\partial A}, & B_1 &= \frac{\partial S}{\partial B}, & C_1 &= \frac{\partial S}{\partial C}, \\ A'_1 &= \frac{1}{2} \frac{\partial S}{\partial A'}, & B'_1 &= \frac{1}{2} \frac{\partial S}{\partial B'}, & C'_1 &= \frac{1}{2} \frac{\partial S}{\partial C'} \end{aligned}$$

will likewise be the components of a tensor.

Those properties are attached to the property of the two derived quantities:

$$(20) \quad S = F_1 F_2 + G_1 G_2 + H_1 H_2,$$

and

$$(21) \quad S = A_1 A_2 + B_1 B_2 + C_1 C_2 + 2(A'_1 A'_2 + B'_1 B'_2 + C'_1 C'_2),$$

one of which is deduced from two vectors, and the other of which is deduced from two tensor triples, that they must be scalar functions.

When a vector V_1 is given as a function of another vector V , one can deduce the components of a vector or tensor by the following operations:

$$(22) \quad F_2 = \frac{\partial H_1}{\partial G} - \frac{\partial G_1}{\partial H}, \quad G_2 = \frac{\partial F_1}{\partial H} - \frac{\partial H_1}{\partial F}, \quad H_2 = \frac{\partial G_1}{\partial F} - \frac{\partial F_1}{\partial G},$$

$$(23) \quad A_2 = \frac{\partial F_1}{\partial F}, \quad B_2 = \frac{\partial G_1}{\partial G}, \quad C_2 = \frac{\partial H_1}{\partial H},$$

$$A'_2 = \frac{1}{2} \left(\frac{\partial H_1}{\partial G} + \frac{\partial G_1}{\partial H} \right), \quad B'_2 = \frac{1}{2} \left(\frac{\partial F_1}{\partial H} + \frac{\partial H_1}{\partial F} \right), \quad C'_2 = \frac{1}{2} \left(\frac{\partial G_1}{\partial F} + \frac{\partial F_1}{\partial G} \right).$$

If the functions F_1, G_1, H_1 have the same nature as the F_s, G_s, H_s in (17) then F_2, G_2, H_2 will be zero, and A_2, \dots, A'_2, \dots will become identical to the A_s, \dots, A'_s, \dots , resp., in (18).

Here again, the usual case is the one in which the components of the independent vector are the coordinates. The correlation that is represented in (22) is then the one that has been denoted by $V_2 = \text{curl } V_1$ or $V_2 = \text{vort } V_1$.

The general case gives rise to an important remark from the standpoint of principles. In several places in the physics of crystals, one encounters linear relations between vectors. For example, let:

$$(24) \quad \begin{cases} F_1 = \lambda_{11} F + \lambda_{12} G + \lambda_{13} H, \\ G_1 = \lambda_{21} F + \lambda_{22} G + \lambda_{23} H, \\ H_1 = \lambda_{31} F + \lambda_{32} G + \lambda_{33} H, \end{cases}$$

and apply equations (22) and (23) to those relations; they will become:

$$(25) \quad \lambda_{32} - \lambda_{23} = F_2, \quad \lambda_{13} - \lambda_{31} = G_2, \quad \lambda_{21} - \lambda_{12} = H_2,$$

$$(26) \quad \lambda_{11} = A_2, \quad \lambda_{22} = B_2, \quad \lambda_{33} = C_2, \\ \frac{1}{2}(\lambda_{32} + \lambda_{23}) = A'_2, \quad \frac{1}{2}(\lambda_{13} + \lambda_{31}) = B'_2, \quad \frac{1}{2}(\lambda_{21} + \lambda_{12}) = C'_2.$$

The nine constants in equations (24) then represent the three components of a vector and the six components of a tensor. In certain cases (for example, in the cases of magnetic or dielectric influence), one will have the relations $\lambda_{hk} = \lambda_{kh}$. In others (for example, in those of the conduction of heat and electricity), they will be valid only for certain crystalline systems. When they are true, the crystal will be characterized completely for the phenomena in question by just one tensor triple that belongs to it individually; in the contrary case, one must append a vector.

That remark, to which one can add some others, is important, since it shows that not just variable quantities, but also constant parameters, can have the nature of the components of vectors and tensors.

THESIS

Tensors have a certain kinship with vectors, insofar as they are directed quantities. However, the bilateral character of tensors will determine the essential differences between their properties.

Tensor triples, with their six independent components, which are generally encountered in tensorial phenomena, have no analogue in vectors.

Since the tensor triples that one encounters in the most varied topics in physics are variables, as well as constant parameters, in addition, one agrees to make tensors the object of an independent study.

II. – General properties of stresses in deformable bodies.

6. Theory of action-at-a-distance and immediate action. – Like many other branches of theoretical physics, elasticity was first explored with the aid of a particular conception of the mechanics of the phenomena that we call the *molecular hypothesis*. Under that hypothesis, the structure of a body is supposed to be constructed from distinct elementary masses – viz., molecules, in the broad sense of the term – which are maintained in their positions or displaced by mutual actions that have perceptible magnitudes only for imperceptible distances. More particularly, for an undeformed, homogeneous crystal, one assumes that its molecules are identical, oriented in the same manner, and regularly distributed in such a fashion that each molecule is surrounded by other ones in the same manner. The distances over which those mutual actions are exerted are considered to be very large in comparison to the distances between neighboring molecules, and one agrees that the deformations will vary in space slowly enough that, even in the deformed body, the distribution of the molecules can be considered to be regular, in the sense that was suggested above, in the extent of the sphere of molecular activity.

The molecular theory – or action-at-a-distance – that was founded upon these particular concepts by Navier ⁽¹⁾, Cauchy ⁽²⁾, and Poisson ⁽³⁾ soon fell into disfavor. Indeed, it made the elastic properties of an isotropic body depend upon just one parameter, while numerous observations seemed to contradict that result.

That is why a new theory was generally adopted after some time that established the elementary laws of elasticity by assuming that matter was continuous and that the mutual actions between the portions of neighboring matter were localized to their separation surface by a process that had served as an example for other branches of physics, and deduce some fundamental formulas of elasticity, such as the center of gravity theorem,

⁽¹⁾ NAVIER, Mém. de l'Acad. **7** (1827), 375.

⁽²⁾ A. CAUCHY, Exerc. de Math. **3** (1828), 188 and 214.

⁽³⁾ S.-D. POISSON, Mém. de l'Acad. **8** (1829), 357; Journ. de l'École Polyt., Cah. 20, **8** (1831).

the area theorem, and the energy equation, as it applied to conveniently-chosen volume elements ⁽¹⁾.

Contrary to the preceding theory, the theory that we shall call the *theory of immediate action* gives two constants that are characteristic of isotropic media, and all of its results found to be in accord with observations. Much later, after the molecular theory of elasticity had encountered some difficulties in the study of isotropic bodies, it also experienced one defeat after another in the context of crystals. Indeed, the laws that it provided for those bodies did not agree with observations, while the ones that were derived from the theory of immediate action were confirmed. We shall develop that point in a more detailed manner later on.

If one sets no other goal than that of obtaining the elementary laws of elasticity in a form that answers to reality then one can declare that they should be satisfied by the results of the theory of immediate action without demanding to know *why* the molecular hypothesis leads to inexact results. However, if one tends towards a general theory that is a consequence of matter and in which one will not lose sight of the remarkable results that were obtained in other domains with the aid of analogous hypotheses then one will find it informative to look for the reasons that would it seem to fail in the domain of elasticity. That search shows that the old molecular theory of elasticity started from a pointlessly-specialized fundamental concept, namely, the hypothesis of central molecular actions that depended upon only the distance, and that alone is why it failed.

Moreover, a very simple reflection on the mechanism of the growth of crystals would convince one of the inadmissibility of that concept, because the regular formation of a crystal in a solution or a melt is comprehensible only if a *directing moment* acts upon the particle that one associates with the crystal by giving it an orientation that is parallel to the ones that already compose the structure. However, from the principle of the conservation of energy, mutual actions that are directed along the line of centers and depend upon only the distance are incompatible with such moments. That is because the existence of moments demands a potential for the mutual action that depends upon the orientation, and that will lead to forces that vary with the orientation of the molecule and do not coincide with the line of centers, in general.

In his last, unfinished paper, Poisson ⁽²⁾ had already constructed a molecular theory that sought to take these circumstances into account. However, he made some further restrictive hypotheses, and consequently, his results were more specialized than those of the theory of immediate actions, and were partially incompatible with experiments. A more general presentation that has been envisioned by the author will be sketched out in what follows ⁽³⁾.

7. The components of stress in deformable bodies. – Ever since the fundamental work of Cauchy and Poisson, the general tendency has been to found the molecular theory of elasticity, not upon the consideration of a single molecule, but upon that of a

⁽¹⁾ S.-D. POISSON, Journ. de l'École Polyt. Cah. 20 (1831), 82; A. CAUCHY, Exerc. de Math. 4 (1829), 293; G. GREEN, Camb. Phil. Soc. 7 (1839), 121.

⁽²⁾ S.-D. POISSON, Mém. de l'Acad. 18 (1842), 3.

⁽³⁾ W. VOIGT, "Theoretische Studien über die Elasticitätsverhältnisse der Krystalle," Abh. d. Ges. d. Wiss. z. Göttingen 34 (1887), 3.

volume element that is large with respect to the sphere of molecular action, and consequently contains an extremely large number of molecules. The molecules around the volume element then act upon only the molecules that are inside the separation surface in its immediate neighborhood and that will combine to give the stress components.

As one knows, the latter were defined by Cauchy and Poisson ⁽¹⁾ to be the sums of the parallel components of the forces that all molecules (i) inside of a right cylinder that is raised over a surface element q experience as a result of all the molecules (e) that are situated on the other side of q , and divided by q ; i.e., when referred to a unit surface:

$$\frac{|i|}{q}.$$

Since the dimensions of the surface element are large in comparison to the distances over which the molecules act, in a homogeneous body, those components of the stress can be considered to depend upon only the orientation of the surface element. In a body whose state varies in a continuous manner, they will be functions of the coordinates, in addition.

It is customary to fix the orientation of the surface element by the direction of the normal n , when counted positively on the side (i) where one finds the attracting masses; i.e., in the case where one is dealing with a surface element that surrounds the volume considered, it will be *positive from the exterior to the interior*.

As far as the notations for the components of stress are concerned, there can hardly be any doubt that only a symbol with two letters, one of which denotes the direction of the components and the other of which denotes that of the normal, will achieve the proposed goal. One then arrives almost necessarily at the proposal that was undoubtedly made for the first time by Fr. Neumann ⁽²⁾ of letting S_n denote the component in the direction S of the pressure on a surface element whose (interior) normal is n . From the definition that was given above, one will then have:

$$(27) \quad q \cdot S_n = \sum_i \sum_e S_{ie},$$

in which S_{ie} is the component in the direction S of the force that is exerted by a molecule (e) on a molecule (i), and in which \sum_e is extended over all the molecules on the negative side of q , while \sum_i is extended over all molecules in the cylinder that is constructed on the positive side.

One will then have the system of notations:

$$X_x, X_y, X_z, \quad Y_x, Y_y, Y_z, \quad Z_x, Z_y, Z_z$$

⁽¹⁾ A. CAUCHY, Exerc. de Math. **3** (1828), 213; S.-D. POISSON, Mém. de l'Acad. **8** (1829), 373; *ibid.* **18** (1842), 47.

⁽²⁾ FR. NEUMANN, *Vorlesungen über Elasticitätstheorie*, written in 1830, but published in 1885. G. KIRCHHOFF, Crelle's Journal **56** (1856), 285.

for the components that are parallel to the coordinate axes of the pressure against the surface elements that are parallel to the coordinate planes, so when one abbreviates this by X_x, \dots, Z_z , one will always suppose that they are written in that order.

X_x, Y_y, Z_z can be called *normal* stresses, while $Y_z, Z_y, Z_x, X_z, X_y, Y_x$ are called *tangential* stresses; similarly, more generally, one will have S_s, T_t , on the one hand, and S_t, T_s , on the other, when $t \perp s$.

Poisson⁽¹⁾ showed that, from the definition that was given above, the components of the pressure against the planes that are parallel to the coordinate planes is calculated in the following manner: One has:

$$(28) \quad X_x = -\frac{1}{2}\nu \mathbf{S}_x X, \quad Y_x = -\frac{1}{2}\nu \mathbf{S}_x Y, \quad Z_x = -\frac{1}{2}\nu \mathbf{S}_x Z, \text{ etc.}$$

in which the sum \mathbf{S} must be taken over all forces X, Y, Z that a molecule feels on the part of all of the other ones whose relative coordinates are x, y, z , and in which ν represents the number of molecules per unit volume. From these values for the components of the stress, one will get the relations:

$$(29) \quad X_n = X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z), \dots, \text{ etc.}$$

One knows that the theory of immediate action likewise leads to these fundamental equations.

As long as one is dealing with pressures against the same surface element, the magnitudes X_n, Y_n, Z_n will be simple components of a vector and are consequently related to the resultant vector of the total pressure by the relations:

$$(30) \quad P_n^2 = X_n^2 + Y_n^2 + Z_n^2,$$

$$(31) \quad \cos(P, X) = \frac{X_n}{P_n}, \quad \cos(P, Y) = \frac{Y_n}{P_n}, \quad \cos(P, Z) = \frac{Z_n}{P_n}.$$

However, if one lets the direction n of the normal be arbitrary, while keeping the point of the body where one finds the surface element the same, then more complicated relations will come about.

Indeed, from the properties of the transformation of X_x, \dots, Z_z that result directly from their definitions that are contained in (28), and also from equations (29) and (31) in a more complicated manner:

$$X_x, Y_y, Z_z,$$

and

$$\frac{1}{2}(Y_z + Z_y), \quad \frac{1}{2}(Z_x + X_z), \quad \frac{1}{2}(X_y + Y_x)$$

will be the components of the first kind; by contrast:

$$(Y_z - Z_y), \quad (Z_x - X_z), \quad (X_y - Y_x)$$

⁽¹⁾ S.-D. POISSON, Mém. de l'Acad. **8** (1829), 374; *ibid.* **18** (1842), 51.

will be the components of a vector.

From page 5, the tensors that constitute a tensor triple have the same nature as the components of the first kind, and consequently, like them, they will be *normal pressures*. They are the *principal stresses* that we denote by P_1, P_2, P_3 . As one can see directly, the vector is the *moment per unit volume that the body experiences as a result of internal stresses*.

Up to now, we have considered the stresses that are due to *forces* that act between molecules, but it is certain that the moments – or *couples* – that act against them can be treated in the same manner.

They also intervene in the volume element considered only in the form of sums of the form of (27) that one can refer to by the expression *moment stresses*. With respect to an arbitrary axis D , one can write:

$$(32) \quad q D_n = \sum_i \sum_e D_{ie},$$

in which the sums must be performed in the way that was discussed on page 11.

In the particular case in which one takes the moments with respect to the coordinate axes and the stresses to act upon planes that are parallel to the coordinate planes, one will have the nine particular stress moments:

$$L_x, L_y, L_z, \quad M_x, M_y, M_z, \quad N_x, N_y, N_z,$$

which correspond to X_x, \dots, Z_z .

The energy equation establishes a well-defined relation between the molecular moments and the molecular forces. If one again employs the notations X_{ie}, \dots and L_{ie}, \dots for the components and moments that a molecule (i) experiences as a result of a molecule (e), and if one sets:

$$x_i - x_e = x_{ie}, \quad y_i - y_e = y_{ie}, \quad z_i - z_e = z_{ie}$$

for the relative coordinates then one will have ⁽¹⁾:

$$(33) \quad L_{ie} + L_{ei} + Z_{ie} y_{ie} - Y_{ie} z_{ie} = 0, \text{ etc.}$$

In particular, if the molecules are oriented in the same manner then due to the symmetry $L_{ie} = L_{ei}$, one will have:

$$(34) \quad L_{ie} = \frac{1}{2}(Y_{ie} z_{ie} - Z_{ie} y_{ie}), \quad \text{etc.}$$

Since, by hypothesis, the molecules are reasonably parallel in the interior of the sphere of activity, even in the deformed body, the latter formula can be employed in the sums that give L_x, \dots, N_z . However, since, from what was said above, the radius of activity must be considered to be an infinitely-small quantity (even a second order one), these sums will have the same infinitely-small nature with respect to the ones that give

⁽¹⁾ W. VOIGT, "Theoretische Studien über die Elasticitätsverhältnisse der Krystalle," Abh. d. Ges. d. Wiss. z. Göttingen **34** (1887), 71.

the components of the stress X_x, \dots, Z_z in a manner that is analogous to what happens in the theory of capillarity, in which the Laplace constant H is imperceptible with respect to the constant K . If we skip over the particular cases that have no reason to be addressed here then we can abstract the introduction of moment stresses from the foregoing.

8. The general equations of motion. – The problems that concern a function inside of a volume demand that one state some conditions that must be fulfilled at any instant for any point of the interior, along with some conditions that refer only to the points of the surface, and some other ones that refer only to a well-defined instant. There are no general terms for these various types of equations. A terminology that conforms to the nature of the object is to call the first type *fundamental equations*, the second type *surface conditions*, and the third type, *initial conditions*.

The forces that can be exerted on deformable bodies decompose into forces that act upon the interior points and forces that act upon the points of the surface. The former can be conveniently called *spatial forces*, or even *volume forces* or *mass forces*, according to whether they are referred to a unit volume of mass, respectively. The latter are called *exterior stresses*.

For reasons that were given above, we have no means of exerting exterior moment stresses; by contrast, we must assume the possibility of exerting the *spatial moments* on the exterior. For example, that is possible when the elementary masses that we have called *molecules* possess electric or magnetic polarity, and the body is placed in an electric or magnetic field.

We represent the spatial components and the spatial moments with respect to the coordinate axes by X', Y', Z' , and L', M', N' , resp., call the density ρ , and suppose that the infinitely-small displacements s have the components u, v, w , and neglect the products of the angular accelerations with the infinitely-small moments of inertia of the molecules. The fundamental equations of elasticity then take the form:

$$(35) \quad \rho \frac{\partial^2 u}{\partial t^2} - X' + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0, \quad \text{etc.},$$

$$(36) \quad Z_y - Y_z - L' = 0, \quad \text{etc.}$$

We write down the surface conditions by introducing the exterior stresses by way of the components $(X), (Y), (Z)$. They will be:

$$(37) \quad \overline{X}_n = (X), \quad \overline{Y}_n = (Y), \quad \overline{Z}_n = (Z),$$

in which n denotes the *interior* normal to the surface element, and \overline{G} signifies that the function G must be taken on the surface.

In the particular case where one does not exert spatial moments on the body, formulas (36) will become:

$$(38) \quad Y_z = Z_y, \quad Z_x = X_z, \quad X_y = Y_x.$$

The only six independent stress components:

$$X_x, Y_y, Z_z, \quad Y_z = Z_y, \quad Z_x = X_z, \quad X_y = Y_x,$$

which we shall always arrange in that order, are then themselves the *components of the first and second kind of the tensor triple of principal stresses* P_1, P_2, P_3 .

Here, we remark that *the components* X_x, \dots, X_y *are not only the elements that determine the tensor triple, they are equivalent to it.*

THESIS

The notation S_n for a component of the pressure that is parallel to S in a surface element whose (interior) normal is n is more expressive and more practical than any other and must be preferred.

It is convenient to classify the condition equations for spatial problems into *fundamental equations, surface equations, and initial conditions.*

The external forces that act upon the interior points can be called *mass forces* when they are referred to a unit mass and *volume forces* when they are referred to a unit volume. In general, the latter notation is preferable.

III. – Particular laws for stresses in elastic bodies.

9. General relations. – From the molecular theory, the nine components of stress are defined in a completely general manner by formulas (28). Upon making the hypotheses that were stated in page 11 and assuming that the body is deformed by starting with a state in which all of the stresses are zero and applying continuous, infinitely-small displacements whose components are u, v, w and infinitely-small, continuous molecular rotations whose components are l, m, n , one can develop the sums that appear in those formulas ⁽¹⁾.

The result of the calculation will be a system of nine equations of the form:

$$(39) \quad \left\{ \begin{array}{l} -X_x = C_{11} \frac{\partial u}{\partial x} + C_{12} \left(\frac{\partial u}{\partial y} + n \right) + C_{13} \left(\frac{\partial u}{\partial z} - m \right) \\ \quad + C_{14} \left(\frac{\partial v}{\partial x} - n \right) + C_{15} \frac{\partial v}{\partial y} + C_{16} \left(\frac{\partial v}{\partial z} + l \right) \\ \quad + C_{17} \left(\frac{\partial w}{\partial x} + m \right) + C_{18} \left(\frac{\partial w}{\partial y} - l \right) + C_{16} \frac{\partial w}{\partial z}, \end{array} \right.$$

in which the C_{hk} are characteristic parameters of the body (in the undeformed state), and the orientation of the coordinate axes, which are determined, on the one hand, by the law of elementary actions, and on the other hand, by the distribution of molecules in space

⁽¹⁾ W. VOIGT, *loc. cit.*, pp. 21.

and between which, the relations $C_{hk} = C_{kh}$ are valid. As a result, the number of independent constants C_{hk} will be 45.

If there are no volume moments L' , M' , N' that act upon the body – so if one consequently has, from (36), that:

$$Y_z = Z_y, \quad Z_x = X_z, \quad X_y = Y_x,$$

then one can eliminate the components of the rotation l , m , n in the formulas for the stress components by means of those relations.

The result of that operation consists of six expressions for the stress components:

$$X_x, Y_y, Z_z, \quad Y_z = Z_y, \quad Z_x = X_z, \quad X_y = Y_x,$$

which are linear and homogeneous with respect to the six arguments:

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

which, as one knows, play a fundamental role in the entire theory of elasticity, and which we shall discuss later on. Here, we confine ourselves to simply introducing an abbreviated notation for those quantities.

Like the six components of stress, each of the preceding differential expressions refers to one or two coordinate axes. Consequently, it is entirely justified to appeal to a notation that is analogous to that of the stress components.

With Kirchhoff (¹), we set:

$$(40) \quad \left\{ \begin{array}{l} \frac{\partial u}{\partial x} = x_x, \quad \frac{\partial v}{\partial y} = y_y, \quad \frac{\partial w}{\partial z} = z_z, \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = y_z = z_y, \quad \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = z_x = x_z, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x_y = y_x. \end{array} \right.$$

However, we will show the inconvenience in that notation later on, despite its uncontested advantages.

In what follows, those six arguments x_x, \dots, x_y will be arranged in that order, which corresponds to the order in the stress components.

The result of the aforementioned elimination will then be six equations of the form:

$$(41) \quad -X_x = c_{11} x_x + c_{12} y_y + c_{13} z_z + c_{14} y_z + c_{15} z_x + c_{16} x_y,$$

in which c_{hk} are characteristic parameters of the substance that the body is composed of and the orientation of the coordinate system. One has the following relations between them:

(¹) G. KIRCHHOFF, *loc. cit.*, we should probably also point out the influence of Fr. Neumann; see Pogg. Ann. **31** (1834), 180.

$$(42) \quad c_{hk} = c_{kh} ;$$

the number of independent parameters c_{hk} is then 21.

In addition to equations (41), one will obtain three further linear relations between the components of the rotation l, m, n , and the nine derivatives $\partial u / \partial x, \dots, \partial w / \partial y$, which can be arranged in a remarkable manner.

Indeed, if one introduces the components:

$$(43) \quad \lambda = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right), \quad \mu = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right), \quad \nu = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

of the rotation of the volume element with respect to the coordinate axes then those equations will become linear and homogeneous in the nine arguments:

$$l - \lambda, \quad m - \mu, \quad n - \nu, \quad x_x, y_y, z_z, y_z, z_x, x_y .$$

The relative rotations of the molecules with respect to the volume elements are then represented by linear functions in x_x, \dots, x_y .

We shall now explain how the method that is based upon the theory of immediate actions leads to some analogous results. In order to make the symmetry in these developments complete, we shall first assume the existence of volume moments L', M', N' that act upon the elementary masses, as before. If we combine the three equations (35) with the factors du, dv, dw and the three equations (36) with the factors dl, dm, dn into a single equation in which the first of those factors represent the increments in the components of the displacement with respect to dt , while the last of them represent the components of the increments in the molecular rotations, and when one integrates the result of that combination over the entire extent of the body, or any part of it, one will obtain the *vis viva* equation for the volume considered.

The work that is done, which is equal to the increase in the *vis viva*, decomposed into a work that is done by volume forces, volume moments, exterior stresses, and finally, something that we represent by the expression $(d\alpha)$, which must be considered to be the work that is done by internal stresses in a unit volume:

$$(44) \quad \left\{ \begin{array}{l} d\alpha = X_x dx + Y_y dy + Z_z dz \\ \quad + \frac{1}{2}(Y_z + Z_y) dy_z + \frac{1}{2}(Z_x + X_z) dz_x + \frac{1}{2}(X_y + Y_x) dx_y \\ \quad + (Y_z - Z_y) d(l - \lambda) + (Z_x - X_z) d(m - \mu) + (X_y - Y_x) d(n - \nu). \end{array} \right.$$

Now, from the principles of thermodynamics, it is necessary that $d\alpha$ should be the (negative) differential of a function that depends upon only the current state of the body at the point considered, namely, the *general elastic potential* Φ per unit volume. Hence, it will result that Φ is a function of:

$$x_x, \dots, x_y, \quad l - \lambda, \quad m - \mu, \quad n - \nu,$$

and one must have:

$$(45) \quad \left\{ \begin{array}{lll} X_x = -\frac{\partial\Phi}{\partial x_x}, & Y_y = -\frac{\partial\Phi}{\partial y_y}, & Z_z = -\frac{\partial\Phi}{\partial z_z}, \\ \frac{1}{2}(Y_z + Z_y) = -\frac{\partial\Phi}{\partial y_z}, & \frac{1}{2}(Z_x + X_z) = -\frac{\partial\Phi}{\partial z_x}, & \frac{1}{2}(X_y + Y_x) = -\frac{\partial\Phi}{\partial x_y}, \\ Y_z - Z_y = -\frac{\partial\Phi}{\partial(l-\lambda)}, & Z_x - X_z = -\frac{\partial\Phi}{\partial(m-\mu)}, & X_y - Y_x = -\frac{\partial\Phi}{\partial(n-\nu)}. \end{array} \right.$$

The function Φ is assumed to be developed in powers of the nine arguments. The constant term in the result has no significance and can be omitted. The linear terms will lead to stresses in the undeformed body and must then disappear, from the hypotheses that were made above. One will then obtain the lowest-degree term in the development (which must be *the only one* that is conserved, as observation would permit) in the form of an expression that is *homogeneous of the second degree* in:

$$x_x, y_y, z_z, y_z, z_x, x_y, \quad l - \lambda, \quad m - \mu, \quad n - \nu.$$

We set:

$$(46) \quad \Phi = \varphi + \psi + \chi,$$

whose significance is:

$$(47) \quad \left\{ \begin{array}{l} 2\varphi = \alpha_{11}x_x^2 + 2\alpha_{12}x_xy_y + 2\alpha_{13}x_xz_z + \dots + 2\alpha_{16}x_xy_y \\ \quad + \alpha_{22}y_y^2 \quad + 2\alpha_{23}y_yz_z + \dots + 2\alpha_{26}y_yx_x \\ \quad \dots\dots\dots \end{array} \right.$$

$$(48) \quad \left\{ \begin{array}{l} \psi = (l - \lambda) (\beta_{11}x_x + \beta_{12}y_y + \dots + \beta_{16}x_y) \\ \quad + (m - \mu)(\beta_{21}x_x + \beta_{22}y_y + \dots + \beta_{26}x_y) \\ \quad + (n - \nu) (\beta_{31}x_x + \beta_{32}y_y + \dots + \beta_{36}x_y) \end{array} \right.$$

$$(49) \quad \left\{ \begin{array}{lll} 2\chi = \gamma_{11}(l - \lambda)^2 & + 2\gamma_{12}(l - \lambda)(m - \mu) & + 2\gamma_{13}(l - \lambda)(n - \nu) \\ & + \gamma_{22}(m - \mu)^2 & + 2\gamma_{23}(m - \mu)(n - \nu) \\ & & + \gamma_{33}(n - \nu)^2, \end{array} \right.$$

in which $\alpha_{hk} = \alpha_{kh}$, $\gamma_{hk} = \gamma_{kh}$, but not necessarily $\beta_{hk} = \beta_{kh}$.

The α_{hk} , β_{hk} , γ_{hk} represent characteristic parameters of the substance and the coordinate system. With those notations, one will have:

$$(50) \quad X_x = -\frac{\partial(\varphi + \psi)}{\partial x_x}, \text{ etc.,} \quad \frac{1}{2}(Y_z + Z_y) = -\frac{\partial(\varphi + \psi)}{\partial y_z}, \text{ etc.,}$$

$$(51) \quad -L' = Y_z - Z_y = -\frac{\partial(\psi + \chi)}{\partial(l - \lambda)}, \text{ etc.}$$

One obtains some relations that are even more specialized when one, like Lord Kelvin, makes some well-defined hypotheses on the distribution of molecules (¹).

As we already said above, observations are, in general, in contradiction with the *Poisson-Cauchy* relations (55), and in turn, with the concepts that they are based upon. One can then cease to make the mutual actions depend upon the relative orientation of the molecules. However, it seems possible that some specializations of a different kind will lead to relations that are confirmed by experiments, at least, in certain cases.

10. The components of the dilatation. – If one compares the definitions (40) of the six functions x_x, \dots, x_y with formulas (23), page 8, for the components of the tensor then one will see that it is not $x_x, y_y, z_z, y_z, z_x, x_y$, but:

$$x_x, y_y, z_z, \frac{1}{2}y_z, \frac{1}{2}z_x, \frac{1}{2}x_y,$$

that are the components of a tensor. One can doubt that it would be better to let y_z, z_x, x_y denote the expressions:

$$\frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\right), \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

as Christiansen (²) did. However, formulas (41) and (54), although symmetric, will undergo an unpleasant modification by the replacement of Y_z, Z_x, X_y with $2Y_z, 2Z_x, 2X_y$, that will make them give, instead of the components of a tensor of the second kind, twice those components. The factor 2, when suppressed from the y_z, z_x, x_y , will then be recovered in the Y_z, Z_x, X_y ; the asymmetry will be only displaced then.

One will obtain formulas that are entirely symmetric only if one introduces the orthogonal system A, B, C, rA', rB', rC' , in which $r = \sqrt{2}$, in place of the components A, B, C, A', B', C' of the tensor that was used up to now. In the case that we are presently occupied with, we will then have the two systems of components:

$$X_x, Y_y, Z_z, rY_z, rZ_x, rX_y$$

and

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{1}{r}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \frac{1}{r}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \frac{1}{r}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

However, the factor or denominator $\sqrt{2}$ that appears in the various formulas, as well as the less intuitive significance of those components of the second kind, prevents one from making the convention that would be most advantageous from the standpoint of form alone.

(¹) LORD KELVIN, Proc. Roy. Soc. of Edinburgh **16** (1890), 693; Proc. Roy. Soc. **14** (1893), 59. B. ÉLIE, Journ. de Phys. (2) **5** (1886), 204.

(²) C. CHRISTIANSEN, *Elemente der Theor. Phys.*, Leipzig, 1896; pp. 98. Compare also TODHUNTER and PEARSON, *A History of Elasticity*, v. I, Cambridge, 1886; pp. 881.

We then preserve the convention that we made, which has the advantage of being in accord with a very considerable number of authors.

As far as the geometric significance of the functions x_x, \dots, x_y is concerned, one knows that at each point of the body, x_x, y_y, z_z represent the linear dilatations parallel to the coordinate axes, and y_z, z_x, x_y represent the decreases in the angles between the elements of parallel lines and the coordinate axes to which the symbols refer before the deformation. The tensor triple whose components are the quantities $x_x, \dots, \frac{1}{2}x_y$ is then represented by three linear dilatations $\delta_1, \delta_2, \delta_3$ in the three mutually-perpendicular directions whose angles do not change under the deformation. Ordinarily, one calls those dilatations *principal dilatations*. One will then refer to the tensor triple $\delta_1, \delta_2, \delta_3$ as that of the principal dilatations, or more briefly, the *tensor triple of dilatations*. The term *components of the dilatation* is then completely justified for the $x_x, \dots, \frac{1}{2}x_y$. One can also apply it to the system x_x, \dots, x_y , with a slight imprecision.

For the sake of reference, we further give some deformations as functions of the components of the dilatation that are important for observations.

The *volume dilatation* \mathcal{D} is:

$$(56) \quad \mathcal{D} = x_x + y_y + z_z.$$

The *surface dilatation* normal to the Z-axis is:

$$(57) \quad \Theta_z = x_x + y_y.$$

The *linear dilatation* δ in a direction whose angles with the coordinate axes originally have α, β, γ for their cosines is:

$$(58) \quad \delta = x_x \alpha^2 + y_y \beta^2 + z_z \gamma^2 + y_z \beta\gamma + z_x \gamma\alpha + x_y \alpha\beta.$$

The increase η in the angle between the normals to two planes that originally have the direction cosines $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ is given by:

$$(59) \quad \left\{ \begin{array}{l} \eta \sin \varphi = 2(x_x \alpha_1 \alpha_2 + y_y \beta_1 \beta_2 + z_z \gamma_1 \gamma_2) \\ \quad + y_z (\beta_1 \gamma_2 + \gamma_1 \beta_2) + z_x (\gamma_1 \alpha_2 + \alpha_1 \gamma_2) + x_y (\alpha_1 \beta_2 + \beta_1 \alpha_2) \\ \quad - (\delta_1 + \delta_2) \cos \varphi, \end{array} \right.$$

in which δ_1 and δ_2 represent linear dilatations in the original direction of the two normals, and φ represents the angle between those two directions.

When these normals coincide with two coordinate axes (for example, Y and Z), η will be equal to the corresponding components of the dilatation (so y_z , for example).

11. *The elastic parameters of crystals* ⁽¹⁾. – As was said on page 17, the general expressions (41) for the stress components as functions of the components of the dilatation contain twenty-one characteristic parameters of the substance and the system of coordinates that is employed. One calls them the *elastic constants* (and more rarely, the *elastic coefficients*) for the substance. One can reproach those terms for their total lack of color. It would be more expressive, and entirely harmonious with the nomenclature that is used in heat and electricity, to call then the *coefficients of elastic resistance*.

At present, it is undoubtedly impossible to get away from the neutral term *elastic constants* in the fundamental system (41), but it would be nonetheless recommended, in all of the laws that express an elastic modification by means of a force on a length, an angle, a surface, or a volume, to call the combination of the c_{hk} that appears in the formula *as a divisor of the force*, the *elastic resistance of that modification*, and to always give it the dimension of an elastic constant. Therefore, for example, in the well-known formula for the elongation of a prism by a weight P :

$$(60) \quad \delta L = \frac{P \cdot L}{E \cdot Q},$$

the denominator E (which is generally known by the name of *elastic coefficient* or *Young's modulus*) must be called the (specific) *resistance to elongation* of that prism.

The solution of equations (41) for the components of dilatation is provided by expressions that we will write in the form:

$$(61) \quad -x_x = s_{11} X_x + s_{12} Y_y + s_{13} Z_z + s_{14} Y_z + s_{14} Z_x + s_{16} X_y.$$

Here, the s_{hk} are certain ratios of the determinants of the c_{hk} , between which, as a result of the relations:

$$c_{hk} = c_{kh},$$

there will exist the relations:

$$(62) \quad s_{hk} = s_{kh},$$

which will reduce their number to 21.

Formulas (61), which are reciprocal to formulas (41), are the point of departure for the theory of deformations that is most important in practice. Consequently, they also define the basis for the theory of all the important observation methods. Measurements will always first lead to the parameters s_{hk} , which one can infer from the elastic constants, properly speaking, only by some calculations that are often very complicated.

Since the probable error in the determination by these calculations increases extremely, it would seem rational to characterize elastic media by their s_{hk} , rather than their c_{hk} .

⁽¹⁾ W. VOIGT, Abh. der Kgl. Ges. d. Wiss. zu Göttingen **36** (1890), 40; Wied. Ann. **41** (1890), 715. *Die fundamentalen Eigenschaften*, etc., pp. 137, et seq.

The great importance of these new parameters, which is derived from what we just said, justifies a special terminology. Since the s_{hk} , as proportionality factors in the expressions for the components of the dilatation, also measure them, the author has proposed the name of *elastic moduli* for them, which presently has a certain measure of usage in Germany.

In a general manner, it is recommended that when an observable elastic modification is expressed by a force, one should say *modulus of that modification* to mean the combination of the c_{hk} or s_{hk} that appear *as a factor of that force* and which will always have the dimension of an elastic modulus. For example, if we write equation (60) for the elongation of a prism in the form:

$$(63) \quad \delta L = \frac{D \cdot P \cdot L}{Q}$$

then D will be the modulus of elongation (for longitudinal traction) in the prism considered.

The elastic moduli and the elastic resistances are then reciprocal in the same way that resistivities and conductivities are reciprocal for heat and electricity.

In (53), the expression for the elastic potential is given as a function of the components of the dilatation; its parameters are the constants of electricity. With the aid of equations (61), one can first define the symmetric quadratic form in the two types of tensor components:

$$(64) \quad -2f = X_x x_x + Y_y y_y + Z_z z_z + Y_z y_z + Z_x z_x + X_y x_y .$$

If one recalls the expression (21) (pp. 8) then one will infer immediately that:

$$(65) \quad \left\{ \begin{array}{lllll} 2f = s_{11} X_x^2 & + 2s_{12} X_x Y_y & + 2s_{13} X_x Z_z & + \cdots & + 2s_{16} X_x X_y \\ & + s_{22} Y_y^2 & + 2s_{23} Y_y Z_z & + \cdots & + 2s_{26} Y_y X_y \\ & & \cdots & \cdots & \cdots \end{array} \right.$$

The number of moduli and elastic constants that are exhibited by the preceding is *twenty-one*. Meanwhile, we remark that this enumeration is performed according to another principle that will give *three* for the number of resistance or conductivity constants for heat or electricity. The first evaluation supposes an *arbitrary* coordinate system, while the second one supposes a *completely special* system, namely, the principal axes. For arbitrary coordinate systems, the number of thermal or electric constants is six, while principal axis systems require three of those six constants for their definition.

One can also refer equations (41) or (61) to a special system of axis that is chosen line the principal axis system and is defined three relations between the parameters. The number of them will then be reduced from twenty-one to *eighteen*. It is obvious that once the system of principal axis is introduced by three relation between the elastic constants, one must introduce only equations between the moduli that *are deduced from them*, and no others, and conversely.

We denote the principal axes of elasticity by X_0, Y_0, Z_0 , and the parameters that refer to them by c_{hk}^0 and s_{hk}^0 . Later on, we shall speak of the principals that will lead one to choose a system of elastic principal axes.

The introduction of the principal axes presents a noticeable advantage, and above all in certain general studies. In order to treat some particular problems – for example, the deformation of truncated prisms with an arbitrary orientation in a crystal – it is, by contrast, generally more advantageous to choose the coordinate system X, Y, Z in such a manner that the surface conditions will take on a particularly simple form in that problem. In that case, one will be dealing with the expression of the parameters s_{hk} and c_{hk} in terms of the parameters of the principal system X_0, Y_0, Z_0 , when the former parameters refer to the arbitrary system X, Y, Z and can be called *secondary parameters* where clarity would demand it.

That problem is solved in the simplest manner by means of an elastic potential f that is expressed in (53) in terms of the components of the dilatation, and in (65) by the components of the stress, and that one begins to define by referring to the principal axes X_0, Y_0, Z_0 . If one then transforms the components of the stress or dilatation in the arbitrary system x, y, z by means of known properties of the transformation of tensor components then one will obtain a form that is similar to (53) or (65), in which one will find some linear functions of all the c_{hk}^0 or the s_{hk}^0 in place of the c_{hk} or s_{hk} , resp.

The relations that one then obtains between the two types of parameters will generally be very complicated. Here, it will suffice to give the formulas for the moduli, which are the only ones that are important in the applications.

If one lets d_{mn} denote what the coefficients δ_{mn} in (15) and (16) (pp. 7) will become when one replaces r with 2 in the first three columns, and by 1 in the last three, then the secondary moduli, expressed in terms of the principal moduli, will be (¹):

$$(66) \quad s_{hk} = \sum_m \sum_n s_{mn}^0 d_{mh} d_{nk} \quad \left. \begin{array}{l} h, k \\ m, n \end{array} \right\} = 1, 2, 3, \dots, 6.$$

One has equations of the same form for the constants, with a meaning for the factor d that has changed somewhat.

THESIS

A definition of the arguments that determine the deformation that would be convenient in every regard seems impossible. The notation that Kirchhoff proposed seems preferable, due to its intuitive and practical character and its analogy with Neumann's symbols for the components of stress, as well as for the symmetry that it gives to the formulas.

There is no reason to favor the elastic constants to the detriment of the moduli, as one generally does. There is no advantage to giving names to the elastic constants that would express anything but their place in the expression for the potential.

(¹) W. VOIGT, Wied. Ann. **16**, pp. 404; "Theoretische Studien über die Elasticitätsverhältnisse der Krystalle; Abh. d. Ges. d. Wiss. z. Göttingen, **34** (1887).

In the particular laws that are intended to be compared to experimental results, it is desirable to characterize unambiguously the factors that the elastic parameters depend upon as resistances or moduli according to their places in the formulas.

IV. – Introduction of symmetry elements into crystals.

12. Independent symmetry elements for the 32 crystal groups. – The basis for the specialization of the preceding general formulas that conforms to the properties of the various crystal groups is the rule that is derived from experiments and was undoubtedly stated explicitly for the first time by F. Neumann ⁽¹⁾, which says that *any symmetry element that manifests itself in the phenomena of the growth and dissolution of a crystal also enters into all other physical phenomena that are based in it.*

Since other symmetry elements occasionally result from the particular laws of those phenomena that are superimposed with the laws of their form, one can express that hypothetical rule by saying that *of all physical phenomena that are based upon a crystal, those of growth and dissolution possess the lowest degree of symmetry.*

From that proposition, it is impossible to deduce the complete symmetry of the form of a crystal from some other physical property, as is sometimes attempted.

The first applications of that rule of crystal elasticity were made by Fr. Neumann himself, and were repeated, in a simplified form, by Kirchhoff ⁽²⁾. In that era, there was no way of classifying the most important examples, since observations were also completely lacking. Their number was ultimately augmented by the author ⁽³⁾ and by Aron ⁽⁴⁾. That question was treated systematically for the first time by Minnigerode ⁽⁵⁾. However, the procedures that are inferred from the theory of groups that address it are somewhat disproportionate to the simplicity of the problem. In what follows, we shall show how the formulas of elasticity (and similarly, those of some other topics in crystal physics) can be established for all crystal groups by some entirely elementary procedures. Lack of space prohibits us from giving anything but a simple outline, but I hope that it will suffice to give an overview of the question.

As one knows, symmetry of form consists of the property of a crystal (once it has been suitably reduced to a well-defined normal form) that it can be superimposed over itself by certain geometric operations (e.g., rotation, reflection in a mirror, inversion with respect to a center) in such a way that the directions in the first position will all coincide with the directions in the second position that they are equivalent to from the standpoint of the phenomena of growth and dissolution.

Now, the rule that was developed at the beginning amounts to saying that the positions that are equivalent from the standpoint of *form* are also equivalent as *physical properties*, in such a way that two primary determining actions (for example, two deformations) that are superimposed by those transformations will always have two

⁽¹⁾ FR. NEUMANN, *Vorlesungen...* some suggestions pertaining to that question can be found in Pogg. Ann. **31** (1834), 177 *et seq.*

⁽²⁾ G. KIRCHHOFF, *Mechanik*, Leipzig, 1879; pp. 390.

⁽³⁾ W. VOIGT, Wied. Ann. **16** (1882), 275.

⁽⁴⁾ H. ARON, Wied. Ann. **20** (1883), 272.

⁽⁵⁾ B. MINNIGERODE, *Nachr. d. Ges. d. Wiss. z. Göttingen* (1884), pp. 195, 374, 484.

secondary phenomena as a consequence (for example, two systems of stresses) that are similarly superimposed.

In order to conveniently apply that rule to the various topics in crystal physics, it is advantageous to define, once and for all, a table of the *independent* symmetry elements that characterize the various groups (¹). *Only those elements* will enter into the questions of physics. The union of *all* the symmetry elements of a group is useful only in crystallography, and rather tends to confuse the clarity in the problems that we shall treat here.

For the characters of the various symmetry elements, we refer to any treatise on crystallography. We represent the existence of a *symmetry center* by the symbol C , that of a *symmetry axis of order n* by A^n , that of a *mirage axis* (²) of order n by S^n , and that of a *symmetry plane* by P .

We suppose that we have a system of principal axes that coincides, as much as possible, with the symmetry axes or are perpendicular to the planes of symmetry. A coordinate axis, which then agrees with a symmetry element, will then be indicated by a lower index on the symbol for that element. For example, A_z^3 signifies that the z -axis is a ternary symmetry axis, and P_x signifies that the x -axis is perpendicular to a symmetry plane. When the crystal group possesses an obvious symmetry axis, one can always make it coincide with the z -axis; alternatively, one can prefer the x -axis.

Developing what was said on page 23, we remark that for all crystal groups in which the principal coordinate system is fixed completely by the principles that were just pointed out, while the three relations between the elastic parameters that were mentioned in page 23 and reduce their number from 21 to 18 are introduced by that itself. One sees that there are only two groups in which the principal axis system will remain completely arbitrary when one has accepted those conventions, and that for three other ones, only one of the axis will be fixed. The purely crystallographic process for fixing the system of the principal axes of elasticity will then suffice in the greatest number of cases.

As for the classification of the 32 crystal groups, as before, we shall adopt a proposal that was made by Schoenflies (³) that seems to be very suitable.

In the holohedral groups of some systems, one finds two different ways of indicating the independent symmetry elements. They are *equivalent*, since, as one easily recognizes, a symmetry center, a symmetry plane, and a binary symmetry axis that is perpendicular to that plane will have relationships between them in such a way that the presence of two of those elements will have that of the third one as a consequence. Both of those forms have been listed in the following table in order to make the relationships between the holohedral groups and the ones that are not more clearly recognizable.

(¹) W. VOIGT, *Komp. d. theor. Physik*, Bd. I, Leipzig, 1895; pp. 133. “Die fundamentalen phys. Eigenschaften, etc.,” pp. 191.

(²) *I.e., mirror axis*. That symmetry element corresponds to the following definition: A rotation of $\pi/2$ and a reflection with respect to a plane that is perpendicular to the axis that reproduces the solid. One can just as well say *mirage plane of order n* . Example: The line that joins the middles of two opposite edges of a regular tetrahedron is a mirage axis of order 2 by the rule that one vertex successively reproduces the other three. (French translator’s note.)

(³) SCHOENFLIES, *Krystallsysteme und Krystallstruktur*, 1891; pp. 146, 147.

INDEPENDENT GENERAL SYMMETRY ELEMENTS

Anorthic system.

1. Holohedry C .
2. Hemihedry $--$.

Clinorhombic system.

3. Holohedry CA_z^2 of CP_z .
4. Hemihedry P_z .
5. Hemimorphism A_z^2 .

Orthorhombic system

6. Holohedry $CA_z^2A_x^2$ or $CA_z^2P_x$.
7. Hemihedry $A_z^2A_x^2$.
8. Hemimorphism $A_z^2P_x$.

Rhombohedral system

9. Holohedry $CA_z^3A_x^2$ or $CA_z^3P_x$.
10. Hemihedral enantiomorphism $A_z^3A_x^2$.
11. Hemihedral hemimorphism $A_z^3P_x$.
12. Hemihedral paramorphism CA_z^3 .
13. Tetartohedry A_z^3 .

Quadratic system

14. Holohedry $CA_z^4A_x^2$ or $CA_z^4P_x$.
15. Hemihedral enantiomorphism $A_z^4A_x^2$.
16. Hemihedral hemimorphism $A_z^4P_x$.
17. Hemihedral paramorphism CA_z^4 .
18. Tetartohedry A_z^4 .
19. Hemihedry with a mirage axis $S_z^2A_x^2$.
20. Tetartohedry with a mirage axis S_z^2 .

Hexagonal system

21. Holohedry $CA_z^6A_x^2$ or $CA_z^6P_x$.

22. Hemihedral enantiomorphism $A_z^6 A_x^2$.
23. Hemihedral hemimorphism $A_z^6 P_x$.
24. Hemihedral paramorphism CA_z^6 .
18. Tetartohedry A_z^6 .
19. Hemihedry with a ternary axis $A_z^3 P_z A_x^2$.
20. Tetartohedry with a ternary axis $A_z^3 P_x$.

Cubic system

28. Holohedry $CA_z^4 A_x^4$.
29. Hemihedral enantiomorphism $A_z^4 A_y^4$.
30. Hemihedral hemimorphism $S_x^2 S_y^2$.
31. Hemihedral paramorphism $CA_z^2 \approx A_y^2 \approx A_x^2$.
32. Tetartohedry $A_x^2 \approx A_y^2 \approx A_z^2$.

In the last two groups, the symbol \approx represents the equivalence of the three mutually-normal symmetry axes.

The preceding table shows the extreme variety and the nonetheless simple and orderly character of the symmetry relations that are at the basis for the physical of crystals.

The application of the principle that was presented to begin with for the specialization of any general formula from the particular groups will become particularly simple when that formula can be reduced to one such scalar function. How such a function can be introduced in certain cases, even when one is dealing with vectors or tensors, in principle, will result from what was said on page 7.

In the case of elasticity, we have already encountered some scalar functions that are endowed with a physical significance that permits that operation. The elastic potential Φ , in the broad sense of the word, was introduced on page 17. On the same page, it was decomposed into three parts φ , χ , ψ of differing characters that are, like Φ , naturally scalars, and one ultimately deduces the elastic potential f , in the narrow sense of the word, from it. φ and f have the same form and are distinguished by only the values of the parameters.

None of these functions change sign when one reverses the directions of all axes.

For φ , χ , f , this is obvious from their definitions in (47), (49), (53), resp. However, it is also true for ψ , which, from (48), has a form that is bilinear in x_x, \dots, x_y , and $l - \lambda, m - \mu, n - \nu$, because the two types of arguments that enter into it individually possess the property of keeping their sign when one reverses the senses of all coordinate axes.

From what was said on page 25, *one always superimposes a center of symmetry with the crystallographic symmetry elements for the visible phenomena of elasticity, as well as for the invisible molecular rotations that accompany them.* Consequently, the table above simplifies noticeably. A great number of groups will become equivalent for elastic

phenomena and are subsumed by *higher groups*, because, on the one hand, the symmetry planes are now equivalent to the binary axes that they are normal to, and on the other hand, the mirage axes are equivalent to the symmetry axes of order twice as high. Upon suppressing the symbol *C* that is common to all of the groups and has no significance, we will obtain the following table ⁽¹⁾:

INDEPENDENT SYMMETRY ELEMENTS OF CENTRAL PHENOMENA

<i>Anorthic system</i>	
1, 2	No symmetry element
<i>Clinorhombic system</i>	
3, 4, 5	A_z^2
<i>Orthorhombic system</i>	
6, 7, 8	$A_z^2 A_x^2$
<i>Rhombohedral system</i>	
9, 10, 11	$A_z^3 A_x^2$
12, 13	A_z^3
<i>Quadratic system</i>	
14, 15, 16, 19	$A_z^4 A_x^2$
17, 18, 20	A_z^4
<i>Hexagonal system</i>	
21, 22, 23, 26	$A_z^6 A_x^2$
24, 25, 27	A_z^6
<i>Cubic system</i>	
28, 29, 30	$A_z^4 A_y^2$
31, 32	$A_x^2 \approx A_y^2 \approx A_z^2$.

⁽¹⁾ W. VOIGT, *Kompendium*, pp. 134; “Die fundamentalen phys. Eigenschaften, etc.,” pp. 193.

The 32 crystal groups then reduce in the central phenomena to 11 higher groups that are all characterized by only their symmetry axes.

13. Specialization of the elastic potential for the various crystal groups. – We attach the following considerations to the function f that is expressed in terms of elastic constants in (53) and in terms of moduli in (65).

The best procedure for specializing that potential for the various higher crystal groups consists of first looking for relations between the parameters c_{hk} , or s_{hk} in the simplest case in which one axis (the z -axis, for example) is a symmetry axis of order n . One does that by transforming the potential into a new coordinate system that has turned, upon starting from the first one, through an angle of $2\pi / n$ around the symmetry axis and equating corresponding terms of the expression that is obtained from the original expression in that way. That simply amounts to applying the general formulas to that rotation of the coordinate system and equating each c_{hk} or s_{hk} thus-obtained to the c_{hk}^0 or s_{hk}^0 , resp., that is referred to the original system. (66) represents the necessary transformation formula for the moduli; as one will easily see, it is valid for the constants that are distinguished, in part, by the values of the coefficients d_{mn} .

Once one has defined a table of relations between the parameters for $n = 2, 3, 4, 6$, one can pass from the z -axis to the x -axis by a cyclic permutation through one unit of the indices 1, 2, 3, on the one hand, and 4, 5, 6, on the other. One likewise obtains (for $n = 2$) the values that are valid when the y -axis is the symmetry axis.

With the aid of those relations one can, with no calculation, define the systems of parameters, principal constants c_{hk}^0 , and principal moduli s_{hk}^0 that are referred to the principal axes X_0, Y_0, Z_0 by simply combining the formulas that are valid for the various symmetry elements. In the last higher group (31, 32), one must, in addition, take into account the equivalence of the three coordinate axes by equating the parameters c_{hk}^0 or s_{hk}^0 , which transform into each other by a cyclic permutation of the indices 1, 2, 3, or 4, 5, 6, resp.

Lack of space prohibits us from characterizing all of the higher groups in Table II by the c_{hk}^0 or s_{hk}^0 .

Here, we shall give those constants and moduli in view of some ultimate applications in just the four cases in which the z -axis is a symmetry axis of order 2, 3, 4, or 6.

3. *Systems of elastic constants and moduli in the cases of different types of symmetry axes.*

$$A_z^2: \begin{array}{cccccccccccc} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} & s_{11} & s_{12} & s_{13} & 0 & 0 & s_{16} \\ & c_{22} & c_{23} & 0 & 0 & c_{26} & & s_{22} & s_{23} & 0 & 0 & s_{26} \\ & & c_{33} & 0 & 0 & c_{36} & & & s_{33} & 0 & 0 & s_{36} \\ & & & c_{44} & c_{45} & 0 & & & & s_{44} & s_{45} & 0 \\ & & & & c_{55} & 0 & & & & & s_{55} & 0 \\ & & & & & c_{66} & & & & & & s_{66} \end{array}$$

$$\begin{array}{l}
 A_z^3: \quad c_{11} \quad c_{12} \quad c_{13} \quad c_{14} \quad -c_{15} \quad 0 \quad s_{11} \quad s_{12} \quad s_{13} \quad s_{14} \quad -s_{25} \quad 0 \\
 \quad \quad c_{11} \quad c_{13} \quad -c_{14} \quad c_{25} \quad 0 \quad s_{11} \quad s_{13} \quad -s_{14} \quad s_{25} \quad 0 \\
 \quad \quad \quad c_{33} \quad 0 \quad 0 \quad 0 \quad s_{33} \quad 0 \quad 0 \quad 0 \\
 \quad \quad \quad \quad c_{44} \quad 0 \quad c_{25} \quad s_{44} \quad 0 \quad 2s_{25} \\
 \quad \quad \quad \quad \quad c_{44} \quad c_{14} \quad s_{44} \quad 2s_{14} \\
 \quad \quad \quad \quad \quad \quad \frac{1}{2}(c_{11} - c_{12}) \quad 2(s_{11} - s_{12}) \\
 \\
 A_z^4: \quad c_{11} \quad c_{12} \quad c_{13} \quad 0 \quad 0 \quad c_{16} \quad s_{11} \quad s_{12} \quad s_{13} \quad 0 \quad 0 \quad s_{16} \\
 \quad \quad c_{11} \quad c_{13} \quad 0 \quad 0 \quad -c_{16} \quad s_{11} \quad s_{13} \quad 0 \quad 0 \quad -s_{16} \\
 \quad \quad \quad c_{33} \quad 0 \quad 0 \quad 0 \quad s_{33} \quad 0 \quad 0 \quad 0 \\
 \quad \quad \quad \quad c_{44} \quad 0 \quad 0 \quad s_{44} \quad 0 \quad 0 \\
 \quad \quad \quad \quad \quad c_{44} \quad 0 \quad s_{44} \quad 0 \\
 \quad \quad \quad \quad \quad \quad c_{66} \quad c_{66} \\
 \\
 A_z^6: \quad c_{11} \quad c_{12} \quad c_{13} \quad 0 \quad 0 \quad 0 \quad s_{11} \quad s_{12} \quad s_{13} \quad 0 \quad 0 \quad 0 \\
 \quad \quad c_{11} \quad c_{13} \quad 0 \quad 0 \quad 0 \quad s_{11} \quad s_{13} \quad 0 \quad 0 \quad 0 \\
 \quad \quad \quad c_{33} \quad 0 \quad 0 \quad 0 \quad s_{33} \quad 0 \quad 0 \quad 0 \\
 \quad \quad \quad \quad c_{44} \quad 0 \quad 0 \quad s_{44} \quad 0 \quad 0 \\
 \quad \quad \quad \quad \quad c_{44} \quad 0 \quad s_{44} \quad 0 \\
 \quad \quad \quad \quad \quad \quad \frac{1}{2}(c_{11} - c_{12}) \quad 2(s_{11} - s_{12})
 \end{array}$$

The following table then results, for example, for the higher group (9, 10, 11), which is characterized by the simultaneous existence of A_z^3 and A_z^2 :

$$\begin{array}{l}
 A_z^3 A_x^2: \quad c_{11}^0 \quad c_{12}^0 \quad c_{13}^0 \quad c_{14}^0 \quad 0 \quad 0 \quad s_{11}^0 \quad s_{12}^0 \quad s_{13}^0 \quad s_{14}^0 \quad 0 \quad 0 \\
 \quad \quad c_{11}^0 \quad c_{12}^0 \quad -c_{14}^0 \quad 0 \quad 0 \quad s_{11}^0 \quad s_{13}^0 \quad -s_{14}^0 \quad 0 \quad 0 \\
 \quad \quad \quad c_{33}^0 \quad 0 \quad 0 \quad 0 \quad s_{33}^0 \quad 0 \quad 0 \quad 0 \\
 \quad \quad \quad \quad c_{44}^0 \quad 0 \quad 0 \quad s_{44}^0 \quad 0 \quad 0 \\
 \quad \quad \quad \quad \quad c_{44}^0 \quad c_{14}^0 \quad s_{44}^0 \quad 2s_{14}^0 \\
 \quad \quad \quad \quad \quad \quad \frac{1}{2}(c_{11}^0 - c_{12}^0) \quad 2(s_{11}^0 - s_{12}^0)
 \end{array}$$

When the analogous operation is performed on all of the eleven higher groups in the table on page 29, that will show that two of them (with two repetitions) behave in the same manner, and that consequently only nine distinct higher groups will exist for the elastic potential f . Their numbers of constants will then be as follows:

Anorthic system.....	(1, 2)	21 constants
Clinorhombic “	(3, 4, 5)	13 “
Orthorhombic “	(6, 7, 8)	9 “
Rhombohedral “	(9, 10, 11)	6 “
“ “	(12, 13)	7 “
Quadratic “ ...	(14, 15, 16, 19)	6 “

	“ ...	(17, 18, 19)	7	“
Hexagonal	“ ...	(21 to 27)	5	“
Cubic	“ ...	(28 to 32)	3	“

The preceding tables also permit one to easily write down the particular forms for the expressions for the components of stress and dilatation (41) and (61) for each of the groups. For example, for the important higher group (9, 10, 11), the system (61) will take the form:

$$(67) \quad \left\{ \begin{array}{l} -x_x^0 = s_{11}^0 X_x^0 + s_{12}^0 Y_y^0 + s_{13}^0 Z_z^0 + s_{14}^0 Y_z^0, \\ -y_y^0 = s_{12}^0 X_x^0 + s_{11}^0 Y_y^0 + s_{13}^0 Z_z^0 - s_{14}^0 Y_z^0, \\ -z_z^0 = s_{13}^0 X_x^0 + s_{13}^0 Y_y^0 + s_{33}^0 Z_z^0, \\ -y_z^0 = s_{14}^0 X_x^0 - s_{14}^0 Y_y^0 + s_{44}^0 Y_z^0, \\ -z_x^0 = s_{44}^0 Z_x^0 + 2s_{14}^0 X_y^0, \\ -x_y^0 = 2s_{14}^0 Z_x^0 + 2(s_{11}^0 - s_{12}^0)X_y^0. \end{array} \right.$$

One remarks that, from the table on pages 30 and 31, the tables of elastic constants differ from those of the moduli. That (small) inconvenience is a direct consequence of the usual definitions of the components of stress and dilatation. One can avoid it by appealing to the orthogonal components $X_x, Y_y, Z_z, r Y_z, r Z_x, r X_y$, and $x_x, y_y, z_z, \frac{y_z}{r}, \frac{z_x}{r}, \frac{x_y}{r}$, which we mentioned on page 20, and whose use can be recommended in certain general studies ⁽¹⁾. However, for some particular problems in physics, one can hardly abandon the usual notation, which conforms to the remark on page 20.

Here, we point out that property of the method that was outlined above by applying it to the elastic potential f , as well as the functions φ, ψ, χ that were introduced on page 18. The last two are especially interesting because, from equations (52), they determine the rotations of the molecules relative to the volume elements for the ordinary elastic deformations.

For the higher group (9, 10 11), for example, which is characterized by $A_z^3 A_x^2$, one will have the following table for the constants β_{hk} and γ_{hk} :

β_{11}	$-\beta_{11}$	0	β_{14}	0	0	γ_{11}	0	0
0	0	0	0	$-\beta_{14}$	$-\beta_{11}$		γ_{11}	0
0	0	0	0	0	0			γ_{33}

The molecules then turn around the ternary axis along with the volume element. In addition, the phenomenon depends upon the two constants β_{11} / γ_{11} and β_{14} / γ_{11} , which cannot be deduced from observations on elasticity.

Up to the present, no phenomena that one could use to determine that value are known. However, those of the formation of slip surfaces are truly consistent with those molecular rotations. Certainly, they will also play a role on the phenomena of

⁽¹⁾ W. VOIGT, *Kompendium*, pp. 139.

piezoelectricity and electrostriction. In addition, the theory shows that if one does not suppose that the moments of inertia of the elementary masses that one calls *molecules*, and that one accounts for in formulas (36), on the contrary, are infinitely small then certain vibrations in crystals will have a speed of propagation that depends upon the period. Will that truly feeble dependency be detected experimentally someday? That question presently remains open.

THESES

It suffices to consider the objects of crystal physics to be only the mutually-independent symmetry elements.

As a result of the conventions that were made on the components of stress and dilatation, the tables of elastic constants will take different forms from the tables of moduli. One can make them agree by introducing the orthogonal components that were defined on page 20.

However, that way of doing things presents some inconveniences that prevent one from recommending it.

V. – Summary of the theory of methods of observation.

14. General problems in experimentation for the elasticity of crystals. – The first experimental problem in crystal elasticity that *can* be posed (and which in fact *has been* posed) is to establish *whether crystals can truly be distinguished from isotropic bodies in a detectable manner from the standpoint of elasticity.*

As one knows, that question was answered in the affirmative by the observations of Savart⁽¹⁾ on the nodal lines of vibrating quartz plates. By showing that a circular plate that is parallel to the optical axis will give different nodal figures when one proceeds from the same excitation at various points of the boundary, he established the elastic anisotropy of matter without any measurements. Savart could not infer more advanced theoretical conclusions from his experiments for the simple reason that in that era the fundamental equations of elasticity for a body, such as a quartz one, had not yet been established.

Once the elastic anisotropy of crystals was established, the need for numerical determinations was felt, all the more so because in the interim the theories that had been constructed upon the two bases that were developed in Part Two had arrived at different results. The quantitative observations obviously first have to answer the following question:

a. Does the most general theory that rests upon twenty-one constants explain all observations, or do they demand an even more general basis for their representation?

Then, if that theory is always in accord with experiments:

⁽¹⁾ F. SAVART, Ann. d. Chimie, **40** (1821), 5; Mém. de l'Acad. **9** (1830), 405.

b. Are all of the crystal groups that can be distinct from the viewpoint of the general theory of elasticity distinct in reality?

That question is one part of another question that is posed in all branches of crystal physics, namely, whether all of the phenomena that are possible as a result of symmetry actually exist. Recall that the answer that has been given up to now by experiment has not been in the affirmative. In the conduction of heat and electricity, certain crystal groups, by their symmetries, separate from the set that belongs to a system and admit some *rotational* properties that the other ones do not ⁽¹⁾. However, up to now, it was not possible to exhibit those rotational effects in an arbitrary crystal ⁽²⁾. The same thing is true for piezomagnetic effects, which, from symmetry, are possible in a large number of crystal groups, but have not been observed for even one of them ⁽³⁾.

In those two cases, some possible theoretical differences between the different groups do not seem to exist in reality, and that result makes the question that was formulated in *b* particularly interesting. It is attached to another question that has a close kinship with it, but that we shall formulate separately for more clarity:

c. For the same substance, are all of the constants that the general theory assumes to be independent actually independent?

In regard to that question, one must observe especially that, whether or not the *general* molecular theory leads to the same results as the theory of immediate actions, from page 19, its specialization will be given in the following six relations between the twenty-one independent constants of the general theory:

$$(68) \quad \left\{ \begin{array}{l} c_{23} = c_{44}, \quad c_{31} = c_{55}, \quad c_{12} = c_{66}, \\ c_{14} = c_{56}, \quad c_{25} = c_{64}, \quad c_{36} = c_{45}. \end{array} \right.$$

The proof of all these relations, or of some of them, permits one simplify, and in turn, to correct the concepts of the most general molecular theory, which would represent a result of great theoretical importance.

Aside from these general questions, one can naturally pose the problem of the numerical determination of the elastic parameters in view of applications. Among the branches of science for which the knowledge of such parameters would be indispensable, we cite thermoelasticity ⁽⁴⁾, piezoelectricity ⁽⁵⁾, electrostriction ⁽⁶⁾, and the optics of deformed bodies ⁽⁷⁾, or ones that are subject to the action of electric fields ⁽¹⁾.

⁽¹⁾ G.-G. STOKES, Camb. and Dubl. Math. Jour. **6** (1851), 233.

⁽²⁾ CH. SORET, Arch. Sciences phys. et nat. **29** (1893), 355; *ibid.* **32** (1894), 631.

⁽³⁾ The formulas that refer to them will be published next in the Annales de Drude.

⁽⁴⁾ General theory by W. THOMSON, Quart. Jour. of Math. **1** (1857), 57; application to crystals by employing the elastic parameters W. VOIGT, Wied. Ann. **36** (1889), 743.

⁽⁵⁾ W. VOIGT, "Allgemeine theorie, etc.," Abh. d. Ges. d. Wiss. z. Göttingen **36** (1890); E. RIECKE and W. VOIGT, Wied. Ann. **45** (1892), 523.

⁽⁶⁾ FR. POCKELS, Neues Jahrb. f. Mineralogie, Supplement **7** (1890), 253; W. VOIGT, Wied. Ann. **55** (1895), 701.

⁽⁷⁾ FR. POCKELS, Wied. Ann. **38** (1889), 144, 269, 373; *ibid.* **39** (1890), 440.

However, s_{14} , s_{24} , s_{34} are, at the same time, the moduli of linear dilatations, s_{44} , s_{54} , s_{64} are those of the changes of angle by the system of tangential stresses $Y_z = Z_y$, and the moduli s_{13} , ..., s_{63} , s_{16} , ..., s_{66} play analogous roles for the stresses $Z_x = X_z$, $X_y = Y_x$.

Only the case of normal pressures has been realized in practice, and then only when either all of the faces of the prism support the same pressure or when two parallel faces support an arbitrary pressure when the other ones support a negligible pressure.

We shall develop the first case here. One can construct a body of arbitrary form from elements that are parallelepipeds that are compressed uniformly on all sides and keep it in that state by exerting the same normal pressure p on its surface elements. Indeed, one can satisfy the fundamental equations and the surface conditions with:

$$(70) \quad X_x = Y_y = Z_z = p, \quad Y_z = 0, \quad Z_x = 0, \quad X_y = 0.$$

Since, from (29), these formulas are valid for an arbitrary coordinate system, it will be convenient to refer then to the principal axes:

$$X_0, Y_0, Z_0,$$

and one consequently writes them:

$$(71) \quad X_x^0 = Y_y^0 = Z_z^0 = p, \quad Y_z^0 = 0, \quad Z_x^0 = 0, \quad X_y^0 = 0.$$

It then simply results that:

$$(72) \quad -x_x^0 = p (s_{11}^0 + s_{12}^0 + s_{13}^0), \quad \text{etc.}, \quad -y_z^0 = p (s_{41}^0 + s_{42}^0 + s_{43}^0), \quad \text{etc.}$$

We add a general remark in regard to these formulas. On page 26, we observed that the principles that are deduced from the symmetry of the form do not suffice in some cases to fix the system of principal axes. The deformation by a stress that acts uniformly in every direction that we just treated will provide a basis for fixing it in the cases where it remains undetermined. It is, in fact, natural to take the principal axes of elasticity X_0 , Y_0 , Z_0 to be the three mutually-perpendicular directions whose angles do not vary under a compression that is uniform in every direction; i.e., the directions of tensor triple of dilatation in the case of pressure that is uniform in every direction.

That manner of proceeding will lead to the three conditions:

$$(73) \quad s_{41}^0 + s_{42}^0 + s_{43}^0 = 0, \quad s_{51}^0 + s_{52}^0 + s_{53}^0 = 0, \quad s_{61}^0 + s_{62}^0 + s_{63}^0 = 0,$$

by which the number of moduli, and in turn, the elastic constants, is also reduced from 21 to 18.

We remark that in all of the higher groups in the table on page 27, except for the first two, those conditions will be fulfilled identically with the axes that are chosen by way of symmetry considerations.

The new determination of the axes then agrees with the old one, and generalizes it only in such a fashion as to make it applicable to all groups.

The laws of deformation of volumes, surfaces, lengths, and angles by a pressure that is uniform in all directions will result from the general formulas (56) to (59) by substituting the values (72) of the components of dilatation ⁽¹⁾.

To save space, we shall not give them. We only point out the modulus s of cubic compression that relates to that case, due to its particular importance:

$$(74) \quad s = s_{11}^0 + s_{22}^0 + s_{33}^0 + 2(s_{23}^0 + s_{31}^0 + s_{12}^0).$$

16. Prismatic beams of constant deformation along the axis. – The most important methods of observation by far are attached to the deformations of prismatic beams whose longitudinal dimensions are large in comparison to their transversal dimensions as a result of forces that act upon the bases of the prism.

As one knows, de Saint-Venant ⁽²⁾ attacked the theory of deformations of cylindrical bodies by looking for the circumstances under which an elementary fiber that is parallel to the axis of the cylinder would experience only longitudinal stresses as a result of the neighboring fibers. That method, which is extremely fruitful for isotropic bodies, is less so for crystals, since the hypothesis on the pressures already ceases to be verified in the simplest case of torsion. As a consequence, de Saint-Venant could treat only some of the most particular cases of the deformation of prismatic crystals with his method.

For our problem, it seems more advantageous to give a somewhat different objective to the investigations by directing them towards *the general properties of deformations that either do not at all depend upon the coordinate that is parallel to the axis of the beam or depend upon it by way of a term of first degree, second degree, etc.* Like de Saint-Venant, we suppose that the beam is sufficiently long with respect to its transversal dimensions that in the part that is dominated by its length, the deformation depends upon only the components of the total external force and moment that is exercised upon the terminal sections and not on their distribution over those sections.

The most important case is then that of a cylinder that is deformed uniformly along its axis ⁽³⁾. Analysis shows that this state can be obtained by means of actions on the bases that have a resultant that is parallel to the axis of the cylinder and which provides moments with respect to the longitudinal axis as well as with respect to the transverse axes that are situated in the terminal sections.

We make the Z -axis of the X, Y, Z coordinate system coincide with the line that joins the centers of gravity of all the sections, while the X and Y axes coincide with the principal axes of inertia of the terminal section $z = 0$, which must be considered to be fixed, in general. We then suppose that the forces that are exerted on the other (free) base $z = l$ yield a resultant Γ that is parallel to Z and the moments A, M, N with respect to the coordinate axes. In order to fix the section $z = 0$, one must subject it to equal and

⁽¹⁾ Particular case by FR. NEUMANN, Pogg. Ann. **31** (1834), pp. 177 *et seq.*; *Vorlesungen, etc.* pp. 179 *et seq.*; the general problem by W. Voigt, Wied. Ann. **16** (1882), 419.

⁽²⁾ DE SAINT-VENANT, Mém. des Sav. étr. **14** (1857), 233; Jour. d. Liouville (2) **1** (1856), 89.

⁽³⁾ W. VOIGT, Wied. Ann. **16** (1882), pp. 280, *et seq.*; “Theoretischen Studien über die Elasticitätsverhältnisse der Krystalle,” Abh. d. Ges. d. Wiss. z. Göttingen **34** (1887), pp. 53, *et seq.*; particular case by C. CHREE, Proc. London **44** (1888), pp. 214; G.-C. MICHAELIS, Arch. Néerl. **21** (1886), pp. 387.

opposite actions. Conforming to the three types of efforts that are exerted, we must consider three types of elastic problem.

a. Effect of a longitudinal traction. – The deformation of the cylinder is homogeneous, so the components of the dilatation can be denoted once and for all and independently of the form of the section. They will result directly from (69) when one sets $X_x = 0, Y_y = 0, Y_z = 0, Z_x = 0, X_y = 0$, in it, but $Z_z = -\Gamma / q$, in which q represents the section of the cylinder. As a result, one has:

$$(75) \quad \begin{cases} qx_x = s_{13}\Gamma, & qy_y = s_{23}\Gamma, & qz_z = s_{33}\Gamma, \\ qy_z = s_{43}\Gamma, & qz_x = s_{53}\Gamma, & qx_y = s_{63}\Gamma. \end{cases}$$

The moduli of the dilatations that are parallel to the axes are s_{13}, s_{23}, s_{33} ; from (57), the modulus of dilatation of the section is $(s_{13} + s_{23})$. The moduli of the variations of the angles between the axes or coordinate planes are s_{43}, s_{53}, s_{63} . From the table (pp. 30), all of the three angular variations will be zero when the Z -axis is a symmetry axis of order 3, 4, or 6, and the first two y_z and z_x will be zero only when it is a binary symmetry axis (¹).

b. Effect of moments with respect to the transverse axes. – The components of the dilatation and the pressure component Z_z are linear functions of x and y for all forms of the section; X_x, Y_y, Y_z, Z_x, Y_y disappear.

The fiber that is situated along the axis $x = 0, y = 0$ of the cylinder is flexed into a curve whose equations are:

$$(76) \quad \xi = \frac{M s_{33}}{2q \kappa_y^2} \zeta(l - \zeta), \quad \eta = \frac{\Lambda s_{33}}{2q \kappa_x^2} \zeta(l - \zeta),$$

in which κ_x and κ_y denote the radii of gyration of the cylindrical section with respect to its principal axes of inertia, which coincide with X and Y , and in which one assumes that the two extremities $z = 0$ and $z = l$ of the axial fiber are situated along the Z -axis.

From (76), the projections of the axial curve have constant radii of curvature:

$$(77) \quad R_1 = \frac{q \kappa_y^2}{M s_{33}}, \quad R_2 = \frac{q \kappa_x^2}{\Lambda s_{33}},$$

and are, as a result, circumferences of the circle.

If just one of the two moments Λ and M is non-zero then one of the two radii of curvature will be infinite. The axial curve will then be situated in the plane that is normal to the principal axis of inertia with respect to which the moment will be zero.

Due to the constancy of R_1 and R_2 , one calls the flexure by M and Λ *uniform*. *The modulus of uniform flexure s_{33} is identical with the modulus of axial elongation by longitudinal traction.*

¹) Particular case by FR. NEUMANN, *loc. cit.*; the general problem by W. VOIGT, Wied. Ann. **16** (1882), 421; "Theoretischen Studien über die Elasticitätsverhältnisse der Krystalle," Abh. d. Ges. d. Wiss. z. Göttingen **34** (1887), pp. 65.

In the general case, the uniform flexure of a cylinder that is composed of a crystalline substance is accompanied by a uniform torsion. The rotation ω of a section that is situated at a distance $z = \zeta$ of the fixed section $z = 0$ is given by:

$$(78) \quad 2q \omega \kappa_x^2 \kappa_y^2 = (M s_{34} \kappa_x^2 - \Lambda s_{34} \kappa_y^2) \zeta.$$

From (75), that remarkable torsion possesses the same moduli s_{34} and s_{35} as the angular variations y_z and z_x for a longitudinal traction.

(From pp. 38), it will disappear when the z -axis is a symmetry axis of arbitrary order ⁽¹⁾.

c. Effect of a moment with respect to the longitudinal axis. – In this problem, which is generally referred to as the problem of *torsion*, the difficulties are incomparably larger than in the preceding two. One can obtain *just one* result that is common to *all* forms of section, namely, the magnitude of the uniform flexure ⁽²⁾ that accompanies the torsion. With the notations of (77), one finds that this flexure is:

$$(79) \quad R_1 = \frac{2q \kappa_y^2}{N s_{34}}, \quad R_2 = -\frac{2q \kappa_x^2}{N s_{33}}.$$

The secondary flexure then possesses the same moduli s_{34} and s_{33} as the second torsion that was in question above and will disappear with it.

Apart from that, each form for the section will require another solution. The simplest one is obtained for the elliptic section ⁽³⁾. In that case, one can satisfy all of the conditions by functions that are linear in x and y for all of the components of the dilatation, as well as for the stress components Y_z and Z_x , and the zero values of X_x , Y_y , X_y , Z_z . Under those conditions, one will find the value:

$$(80) \quad 4q \omega \kappa_x^2 \kappa_y^2 = N \zeta (s_{44} \kappa_x^2 + s_{55} \kappa_y^2)$$

for the rotation ω of the section $z = \zeta$.

The magnitude ω then has two moduli s_{44} and s_{55} that reduce to one only in the cases of a very elongated section and a circular section.

The case of a rectangular section, which is exceptionally important for observations, leads to some extremely complicated differential equations that seem resistant to a rigorous solution. In general, one can establish the formula ⁽⁴⁾:

⁽¹⁾ W. VOIGT, Wied. Ann. **16** (1882), pp. 282; “Theoretische Studien, etc.,” (1887), pp. 68.
⁽²⁾ W. VOIGT, Wied. Ann. **16** (1882), pp. 300; “Theoretische Studien, etc.,” (1887), pp. 72.
⁽³⁾ W. VOIGT, Wied. Ann. **16** (1882), pp. 300; “Theoretische Studien, etc.,” (1887), pp. 73. A remark that was attached to it by G. KIRCHHOFF, *Mechanik*, pp. 415; CHREE, *loc. cit.*
⁽⁴⁾ W. VOIGT, Wied. Ann. **16** (1882), pp. 609.

$$(81) \quad \omega = \frac{3N\zeta \left[s_{35} + \frac{s_{34}^2 b^2}{s_{33} a^2} + \left(\frac{s_{34}^2}{a^2} + \frac{s_{35}^2}{b^2} \right) \frac{b^2 F}{a s_{33}} \right]}{16ab^2 \left(1 + \frac{b}{a} F \right)},$$

in which a and b represent one-half the edges of the rectangular section that are parallel to the X and Y axes, resp. By contrast, F signifies a function of the ratio a/b , which, by the manner in which it was presented, so it can be considered to be *constant* when the section is somewhat elongated (for example, $a/b > 3$).

That formula is not a complete solution to the problem of torsion then, but a procedure for deducing numerical values for the elastic moduli from observations, in particular s_{55} , in which one considers F to be an unknown constant (after a prior examination of the legitimacy of that hypothesis) that one eliminates by appropriate combinations of the measurements.

The general formula will simplify when one makes the approximation that consists of neglecting $\frac{s_{34}^2 b^2}{s_{33} s_{55} a^2}$ and the square of $\left(\frac{s_{34}^2}{a^2} + \frac{s_{35}^2}{b^2} \right) \frac{b^2 F}{s_{33} s_{55}}$, in comparison to unity. One will then have:

$$(82) \quad \omega = \frac{3N\zeta s_{55}}{16ab^3 \left(1 + \frac{b}{a} F_1 \right)},$$

in which F_1 denotes another constant. That approximation is a very good one, even when a/b is relatively small, because, in reality, s_{34} and s_{35} are usually small in comparison to s_{33} and s_{55} .

In the particular case in which s_{34} and s_{35} are zero (which will be true, as we have said, when the Z -axis is an arbitrary elastic symmetry axis), one will have:

$$(83) \quad \omega = \frac{3N\zeta s_{55}}{16ab^3 \left(1 + \frac{b}{a} F \right)},$$

rigorously.

Here, the function F can be determined completely in a more or less simple manner ⁽¹⁾, in such a way that it will not be necessary to eliminate it by a combination of observations. It is only because F contains moduli of elasticity of unknown numerical values that one can employ a process of approximation to calculate them.

17. Prismatic beams with deformations that varies linearly along the axis. – The case that follows from the one that we just treated in order of increasing complexity, namely, the one in which the components of the dilatation and pressure are linear functions of z ,

⁽¹⁾ DE SAINT-VENANT, Sav. etr. **14** (1857), pp. 263, *et seq.*; W. VOIGT, Wied. Ann. **24** (1886), pp. 612.

will once more lead to some relatively simple results ⁽¹⁾. One can show that these hypotheses are compatible with the pressures on the free base that yield the results that are perpendicular to the beam axis, as well as the forces on the volume that are external and of constant magnitude and parallel to the beam axis. We shall give the results for those two cases separately.

18. Calculation of the principal moduli and principal constants. – If one decomposes the resultant that acts upon the free terminal section into two components A and B that are parallel to the principal axes of inertia X and Y then one will obtain:

$$(84) \quad \xi = \frac{A s_{33}}{2q \kappa_y^2} \zeta^2 \left(l - \frac{1}{3} \zeta \right), \quad \eta = \frac{B s_{33}}{2q \kappa_x^2} \zeta^2 \left(l - \frac{1}{3} \zeta \right)$$

as the equations of the curve of the axis, upon supposing that its first element is kept along the Z -axis.

The axial curve has degree three, and in turn, a non-uniform curve. The modulus of that non-uniform flexure is the same s_{33} as that of the uniform flexure.

The flexure is generally accompanied by a torsion, although there is no moment with respect to the longitudinal axis. The mean torsion of a section at a distance $z = \zeta$ from the fixed section is given by:

$$(85) \quad \omega = \frac{1}{2} \zeta^2 \left(\frac{A s_{34}}{\kappa_y^2} - \frac{B s_{35}}{\kappa_x^2} \right).$$

The moduli are the same as the ones that measure the torsion that accompanies the uniform flexure ⁽²⁾.

b. Volume force that is longitudinal and of constant magnitude. – This case is theoretically important only because of the smallness of the only force that can be applied, namely, gravity. We shall then only mention briefly that a cylinder that is erected vertically or suspended and is subject to the action of its weight and is composed of a crystalline substance will not only be elongated, but also flexed. Its axial curve will be represented by the equations:

$$(86) \quad \xi = -\frac{1}{2} Z' s_{35} \zeta^2, \quad \eta = -\frac{1}{2} Z' s_{35} \zeta^2,$$

in which Z' is the volume force that acts parallel to the Z -axis. The projections of the axial curve onto the coordinate planes will then be circles of radius:

$$(87) \quad R_1 = Z' s_{35}, \quad R_2 = Z' s_{34},$$

⁽¹⁾ W. VOIGT, "Theoret. Studien, etc." (1887), pp. 80.

⁽²⁾ W. VOIGT, *loc. cit.*, pp. 87.

resp., and a point at a distance z from the fixed extremity $\zeta = 0$ will be subject to a displacement:

$$(88) \quad w = \frac{1}{2} Z' s_{33} \zeta.$$

The moduli s_{33} , s_{34} , s_{35} that come into play here will then be the same ones that we first encountered together in the problem of homogeneous deformation (¹).

Somigliana (²) has undertaken a generalization of that problem to the case in which the components of pressure and dilatation are proportional to powers of z that are higher than the first.

18. Calculation of the principal moduli and principal constants. – One result of the preceding is a way to deduce certain moduli s_{hk} from observations when they are referred to a particular X, Y, Z coordinate system. However, these moduli are not individual parameters of the crystal considered; they vary with the direction of the X, Y, Z axes within the crystal. It is then necessary to treat the problem of the determination of the characteristic principal moduli s_{hk}^0 in terms of the observed secondary moduli s_{hk} (³).

On page 23, that was a question of finding relations between the two types of parameters. When s_{hk} are represented by formula (66) as linear functions of all the s_{hk}^0 , they will be coefficients that depend upon the orientation of the X, Y, Z coordinate system with respect to the system of principal elastic axes X_0, Y_0, Z_0 . One will then arrive at the principal moduli by observing, along with (n) conveniently-chosen second moduli that the crystal possesses, some independent principal moduli, upon expressing each of them in terms of the principal moduli and solving the equations thus-obtained for the principal moduli s_{hk}^0 .

Those secondary moduli must be chosen, as much as the material will permit, in such a manner that those equations will take as simple a form as possible in order to reduce the probable errors in the calculated principal moduli.

On first glance, one can suppose that one can determine all of the principal moduli of a crystal by observing the same deformations for n different ways of preparing the orientation; i.e., by determining just one secondary modulus s_{hk} with respect to n different X, Y, Z coordinate systems.

However, one sees very simply that this manner of proceeding will not lead to the goal. The trigonometric factors d_{mh} , d_{nk} in the expression (66) for the secondary moduli are not all independent of each other, and consequently, the principal moduli will not all be distinct in those formulas, but partially combined with each other. *The observation of just one type of deformation does not generally suffice then to determine all of the principal moduli of a crystal.*

The most convenient method for determining an elastic modulus is undoubtedly the flexure of a beam of rectangular section. From formulas (76) and (84), it will yield the

(¹) W. VOIGT, *loc. cit.*, pp. 95.

(²) C. SOMIGLIANA, *Giorn. di Min. Crist. e Petrogr.* (1) **4** (1893), pp. 1.

(³) W. VOIGT, *Wied. Ann.* **16** (1882), pp. 404.

modulus s_{33} . If one expresses the principal moduli s_{hk}^0 by equation (66) then one will obtain a relation that contains the s_{hk}^0 only in 15 combinations; for example, the following ones:

$$s_{11}^0, s_{22}^0, s_{33}^0, s_{24}^0, s_{34}^0, s_{35}^0, s_{15}^0, s_{16}^0, s_{26}^0 \\ (s_{44}^0 + 2s_{23}^0), (s_{55}^0 + 2s_{31}^0), (s_{66}^0 + 2s_{12}^0), (s_{14}^0 + 2s_{56}^0), (s_{25}^0 + 2s_{64}^0), (s_{36}^0 + 2s_{45}^0).$$

One is easily convinced that, even for some special crystal groups, the number of combinations that enter into s_{33} is always less than then the number of s_{hk}^0 themselves.

It then results that in order to experimentally determine all of the moduli one must always combine the observation of flexures with those of some other deformations. The practical fact that the same preparations that can serve for the flexure will also serve for torsion can give preference to the latter deformation over the other one.

However, from (81) and (83), the torsion of a beam of rectangular section will lead to the modulus s_{55} . In order for the combination of flexure and torsion to lead to the objective, the expression for s_{55} in terms of the principal moduli must contain the moduli:

$$s_{44}^0, s_{55}^0, s_{66}^0, s_{23}^0, s_{31}^0, s_{12}^0, s_{14}^0, s_{25}^0, s_{36}^0, s_{56}^0, s_{64}^0, s_{45}^0$$

in other combinations than s_{33} . Indeed, the calculation that is performed will show that s_{55} can be expressed in terms of:

$$s_{11}^0 + s_{23}^0 - s_{12}^0 - s_{31}^0, \quad s_{22}^0 + s_{31}^0 - s_{23}^0 - s_{12}^0, \quad s_{33}^0 + s_{12}^0 - s_{31}^0 - s_{23}^0, \\ s_{24}^0 - s_{14}^0, \quad s_{34}^0 - s_{14}^0, \quad s_{35}^0 - s_{25}^0, \quad s_{15}^0 - s_{25}^0, \quad s_{16}^0 - s_{36}^0, \quad s_{26}^0 - s_{36}^0, \\ s_{44}^0, s_{55}^0, s_{66}^0, s_{56}^0, s_{64}^0, s_{45}^0,$$

so one can infer six independent combinations of the preceding in several ways.

The combination of observations of flexure and torsion then permits one to determine all of the principal moduli for each crystal group. In addition, it yields some advanced resources for checking the theory. Indeed, any observation in excess of the number that is necessary to determine the principal moduli must be capable of being calculated in terms of the other observations.

Once the principal moduli has been found, the constants will be calculated by considering that formulas (61) must give equations (41) when they are solved for the stress components. The principal constants c_{hk}^0 will then be ratios of the determinants of the moduli, and conversely. It is easy to establish the necessary formulas in terms of some known rules. However, their calculation is much less convenient for the crystal groups with fewer constants, and the probable error in the principal constants is, in turn, incomparably larger than that of the principal moduli, as was said already on page 22.

THESIS

The combination of observations of flexure and torsion that are made on prismatic beams provides a method that best agrees with the determination of the elastic constants of crystals.

One determines the greatest possible number of parameters with the aid of flexures, and one adds the torsion only in order to complete them.

In practical applications, the moduli are less convenient than the constants for the characterization of a crystal.

It is particularly recommended that one should take the principal elastic axes to be three rectangular directions whose angles do not change under uniform compression in every direction.

VI. – Observations on the elastic parameters of crystals and their usage.

19. Determination of some incomplete systems of constants. – We first enumerate the observations that are directed towards certain special questions and do not lead to the determination of a complete system of principal moduli.

The first quantitative measurements of the elasticity of crystals, which in itself was epochal, were due to Baumgarten ⁽¹⁾. In our language, its objective was the determination of the modulus s_{33} by observations of flexure for prisms that were made of calcite in various orientations, and a comparison of it with the general expression that results for those bodies from formula (66). The observations were found to be in accord with the theoretical law and, in turn, contributed to give an answer to the general question *a* on page 33.

The determination of the numerical values for the modulus s_{33} for some cubic crystals was also the objective of the observations of Koch ⁽²⁾ and Beckenkamp ⁽³⁾. The former did research with rock salt, sylvite, and sodium chlorate, while the latter did research with potassium alum and chromium alum. The two series of observations did not suffice to give a comparison with the theory. The observations of Coromilas ⁽⁴⁾ on two clinorhombic crystals (gypsum and mica) were likewise concerned with the modulus s_{33} , but extended only in the directions that were contained in a plane, and were consequently incomplete.

The values of s_{33} in the directions of the principal crystallographic axes were determined by Niedmann ⁽⁵⁾ with barite. Some isolated numbers for various crystals are given by Mallock ⁽⁶⁾.

All of those observers appealed to prismatic beams, but they were not always careful to fulfill the fundamental condition that is assumed by the formula that is employed for

⁽¹⁾ G. BAUMGARTEN, Pogg. Ann. **152** (1874), 369.

⁽²⁾ K.-R. KOCH, Wied. Ann. **18** (1883), 325.

⁽³⁾ J. BECKENKAMP, Zeit. Kryst. **12** (1887), 419.

⁽⁴⁾ L.-A. COROMILAS, Inaugural dissertation, Tübingen, 1877.

⁽⁵⁾ H. NIEDMANN, Zeit. Kryst. **13** (1888), 362.

⁽⁶⁾ A. MALLOCK, Proc. Roy. Soc. **49** (1891), 380.

the calculation of the observations, namely, that the transverse dimensions must be small in comparison to the longitudinal dimensions.

The flexure of circular discs that are supported by two parallel knife edges ⁽¹⁾ does not lend itself to exact determinations, and similarly, the flattening of an isotropic sphere that is pressed against a plate of crystallized substance ⁽²⁾, because those phenomena cannot be treated theoretically. Meanwhile, the method of the circular disc can serve to show, with a small expenditure of matter, that in these particular cases, the elasticity has the symmetry of a solid of revolution around the axis that is normal to the plane of the disc. It was in that sense that the method that was indicated by Groth was employed by Vater ⁽³⁾ in order to confirm a consequence of the theory with plates of calcite and apatite that were cut perpendicularly to the axes of order 3 or 6 in those crystals.

From (74), the determination of the cubic compressibility of a crystal with a piezometer yielded only one combination of principal moduli, and in turn, attracted no great interest for the determination of a complete system of moduli, especially since exact measurements of that type presented the greatest difficulties. Nevertheless, that observation, when calculated in terms of moduli that were determined in some other way, can provide an independent verification of the theory ⁽⁴⁾.

20. Determination of some complete systems of constants. – Up to now, the determination of complete systems of moduli or constants has been done only by the author [by abstracting from a series of observations ⁽⁵⁾ that he had suggested and that has not decisive significance, due to the inferior quality of the materials]. The measurements that he made referred to the following crystals ⁽⁶⁾:

Cubic system

Group 28.....	Rock salt, fluorite
“ 29.....	Sylvite
“ 31.....	Pyrite
“ 32.....	Sodium chlorate

Hexagonal system

Group 21.....	Beryl
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⁽¹⁾ H. NIEDMANN, *loc. cit.*; H. VATER, *Zeit. Kryst.* **11** (1886), 549.

⁽²⁾ F. AUERBACH, *Wied. Ann.* **43** (1891), 61.

⁽³⁾ H. VATER, *loc. cit.*

⁽⁴⁾ W.-C. RÖNTGEN and J. SCHNEIDER, *Wied. Ann.* **31** (1887), 1000.

⁽⁵⁾ H. KLANG, *Wied. Ann.* **12** (1881), 321. (Observations with fluorite, calculated with the inexact torsion formulas of FR. NEUMANN.)

⁽⁶⁾ W. VOIGT, *Pogg. Ann. Ergänzungsband 7* (1875), pp. 1 and 177. (Observations on rock salt, likewise calculated with the inexact formulas); *Wied. Ann.* **35** (1888), 642. (Fluorite, pyrite, rock salt, sylvite); *Ibid.* **49** (1893), 710. (Sodium chlorate); *Ibid.* **31** (1887), 474. (Beryl and quartz); *Ibid.* **39** (1890), 412. (Calcite, while appealing to some observations of flexure by Baumgarten as an alternative); *Ibid.* **40** (1890), 642. (Dolomite); *Ibid.* **41** (1890), 712. (Brazilian tourmaline); *Ibid.* **34** (1888), 981. (Topaz and barite).

Rhombohedral system

Group 9.....	Calcite
“ 10.....	Quartz
“ 11.....	Tourmaline
“ 12.....	(Dolomite)

Orthorhombic system

Group 6.....	Topaz, Barite
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Strictly speaking, the observations on dolomite have no place in this table, since the material that was available was not sufficient to determine all of the parameters. However, if one takes one of the general questions that were formulated above into consideration then they will give an important contribution to the answer to that question when one compares their results with the ones that were obtained for the other crystals with rhombohedral systems. They are thus intimately linked with the other observations.

To return to those general questions in their own right, one finds no divergence between the most general theory (of action-at-a-distance or immediate action) and experiment (question *a*) in the calculations for the observations of rock salt, beryl, and calcite that are directed in such a manner that they would verify certain surprising consequences of the theory. One can then consider formulas with 21 constants to embrace the observations, especially since it is difficult to see in what way they can be further generalized while one preserves the proportionality between the stress components and those of dilatation that has been confirmed to a very great extent by experiments.

Up to now, the differences in the properties of the various crystal groups that are possible from symmetry have always been encountered in reality. In particular, cubic crystals, *although optically isotropic, are strongly elastically anisotropic*, and the crystals of the rhombohedral system, *while optically equivalent to those of the hexagonal system, are completely different from them in regard to their elastic properties*. The elasticity of dolomite has an entirely special interest in comparison to that of calcite, which resembles it crystallographically. The diversity of symmetry that is sometimes not manifested in the form at all and can be exhibited only by etchings (*figures de corrosion*) is expressed in an extremely striking way in the elastic properties.

Observation has not exhibited any well-defined numerical relations between the elastic constants of the same crystal (question *c*). In particular, the Poisson-Cauchy relations (69) have not been confirmed. For rock salt, it is true that the single relation that one deduces for the cubic system – namely, $c_{12} = c_{44}$ – is fulfilled approximately. By contrast, it is not true at all for the other crystals of that system; for pyrite and sodium chlorate, even the signs of c_{12} and c_{44} disagree. Similarly, in the other systems, along with isolated approximate disagreements, such as for beryl, one will also find large divergences.

The hypothesis of molecular forces that act along the line of centers and depend upon only the distance must be regarded as having been definitively refuted by those results. The fact that it is unlikely for other reasons was explained already on page 10.

As for the numerical values of the elastic moduli and constants, without a doubt, there is no reason to enumerate them here. We make the following remarks:

The absolute numerical values of the elastic moduli are extremely different for the various crystals that were studied. In general, the bodies with the greatest rigidity possess the smallest moduli and the largest elastic resistances.

Upon taking the unit of force to be the gram-weight and the unit of length to be the millimeter, one will have, for sodium chlorate, for example:

$$s_{11}^0 = 24.1 \times 10^{-8}, \quad s_{12}^0 = 12.3 \times 10^{-8}, \quad s_{44}^0 = 82.1 \times 10^{-8},$$

for topaz:

$$\begin{aligned} s_{11}^0 &= 4.34 \times 10^{-8}, & s_{23}^0 &= -0.65 \times 10^{-8}, & s_{44}^0 &= 9.06 \times 10^{-8}, \\ s_{12}^0 &= 3.46 \times 10^{-8}, & s_{31}^0 &= -0.84 \times 10^{-8}, & s_{45}^0 &= 7.37 \times 10^{-8}, \\ s_{33}^0 &= 3.77 \times 10^{-8}, & s_{12}^0 &= -1.35 \times 10^{-8}, & s_{46}^0 &= 7.49 \times 10^{-8}. \end{aligned}$$

However, that rule is far from general: For example, for pyrite, one will have:

$$s_{11}^0 = 2.83 \times 10^{-8}, \quad s_{12}^0 = 0.43 \times 10^{-8}, \quad s_{44}^0 = 9.30 \times 10^{-8}.$$

From what was said on page 35, $s_{hk}^0 = s_{kh}^0$ for h and k equal to 1, 2, 3, but $h > k$ and $h < k$ measures the transverse dilatation in the direction of the coordinate axis that corresponds to h for tractions that are parallel to the coordinate axis that corresponds to k . For those moduli, as one would expect from the experiments that were done with isotropic bodies, observations would give negative values in most cases. The *longitudinal dilatation* is accompanied by a *transversal contraction*. Topaz is one example. However, that property is not general. The numbers above that relate to pyrite and sodium chlorate show that cylinders that are cut from those substances parallel to one of the principal axes and dilated longitudinally will also experience a transverse dilatation, which is a result that seems a bit surprising.

The general research into the existence of unique solutions in these elastic problems (i.e., on the possibility of several states of equilibrium for given external actions and infinitely-small displacements u , v , w) has led to the result that the existence of just one solution can be proved in a general manner only in the case where the elastic potential f is essentially positive.

The general conditions for that to be true were given by Jacobi ⁽¹⁾. They are satisfied or not according to the numerical values of the elastic parameters. It is then interesting to see whether the complete systems of parameters that have been determined satisfy the Jacobi conditions. Wesendonck ⁽²⁾ has done that research and showed that, in reality, the elastic potential f is an essentially-positive function in the crystals that were studied.

21. Application of the preceding results to isotropic bodies. – The results that were developed above give rise to some interesting applications to *isotropic bodies*.

⁽¹⁾ Jacobi, Crelle's Journal **53** (1857), 281; K. WESENDONCK, Wied. Ann. **35** (1888), 21.

⁽²⁾ K. WESENDONCK, Wied. Ann. **36** (1888), 725.

In Part Two, we have shown that the elastic formulas with one constant that yield the oldest molecular theory are not confirmed by experiments. Now, one can suppose that the introduction of the general laws of molecular action, with which, the most recent theory of action-at-a-distance operates, and by which the formulas of crystal elasticity arrive at an agreement with the theory of immediate action and with experiments, will lead in an analogous manner for isotropic bodies to formulas with two constants that the theory of immediate action yields and which are in agreement with observations.

However, that is not true immediately, because if one considers an isotropic body to be a collection of elementary masses called *molecules* that all possess relative orientations then the definition on page 12 of the stress components will show that the manner by which the molecular actions depend upon the directions in them will have no influence on the values of the sums in question. The latter are the same ones that one will get if one makes a constant form of a certain mean magnitude act in place of the force that varies with direction, and the same relation between the two elastic constants that gave the old molecular theory will result from it.

Direct examination suggests another way of imagining isotropic bodies that will lead to some new and more satisfying results ⁽¹⁾.

All metals and almost all compact rocks immediately present themselves as collections of crystals whose size varies with the circumstances, which are juxtaposed in all possible orientations. In other bodies – for example, in certain types of glass – the same structure is made visible by the etching of a polished surface. One can then assume that a structure of that type (which is called *quasi-isotropic*, by the author) is the rule in nature.

When the individual crystallites are large with respect to the sphere of activity of molecular force, but small with respect to the dimensions of the body, and when they fill up space completely without any intermediate layers that might have a loose and powdery consistency, one can calculate the mean values of the stress components, and in turn, the elastic constants of the quasi-isotropic bodies by starting with those of a homogeneous crystal.

From the fundamental hypothesis, any plane that is laid through a quasi-isotropic body will cut the individual crystallites in all possible orientations, and the portions of the plane that is contained in each of them will be large with respect to the molecular sphere of action. It then results that *the components of the pressure against such a plane can be equated to the arithmetic mean of the values that the analogous components take in the homogeneous crystal for all possible orientations of the surface element.*

Consequently, if one starts with the formulas:

$$(89) \quad \begin{cases} -X_x = c_{11}x_x + c_{12}y_y + \cdots + c_{16}x_y, \\ -Y_z = c_{41}x_x + c_{42}y_y + \cdots + c_{46}x_y \end{cases}$$

then if one forms these expressions for all possible orientations of the X, Y, Z system with respect to the principal system X_0, Y_0, Z_0 , when one takes their arithmetic mean, one will obtain the values of the normal and tangential components of the stress for the quasi-isotropic body. The resulting expressions take the form:

⁽¹⁾ W. VOIGT, "Theoretische Studien, etc." pp. 48; Wied. Ann. **38** (1889), pp. 573.

$$(90) \quad \begin{cases} -X_x = cx_x + c_1y_y + c_1z_z, \\ -Y_z = \frac{1}{2}c_2y_z, \end{cases}$$

in which c , c_1 , c_2 signify:

$$(91) \quad c = \frac{1}{5}(3A + 2B + 4\Gamma), \quad c_1 = \frac{1}{5}(A + 4B - 2\Gamma), \quad c_2 = \frac{1}{5}(2A - 2B + 6\Gamma),$$

in which one has set:

$$(92) \quad c_{11}^0 + c_{22}^0 + c_{33}^0 = 3A, \quad c_{23}^0 + c_{31}^0 + c_{12}^0 = 3B, \quad c_{44}^0 + c_{45}^0 + c_{46}^0 = 3\Gamma,$$

to abbreviate.

A , B , Γ are then certain simple means of $3 \times 3 = 9$ principal constants with similar meanings. The other 12 principal constants do not enter into the formulas for the constants of a quasi-elastic body at all.

From (91), one will always have the relation:

$$(93) \quad c_2 = (c - c_1)$$

between the constants c , c_1 , c_2 of equations (90) that the old molecular hypothesis provided, along with the theory of immediate actions. *However, there is generally no other relation between the constants.* Formulas (90) and (93) are then identical with the results of the theory of immediate action.

It is only in the case where the molecules of the individual crystals possess no polarity, and in which, from (59), the Poisson-Cauchy formulas:

$$c_{44}^0 = c_{23}^0, \quad c_{45}^0 = c_{31}^0, \quad c_{66}^0 = c_{12}^0$$

will consequently be valid, that one will have $B = \Gamma$, and in turn:

$$c_1 = \frac{1}{2}c_2, \quad c = 3c_1.$$

It is the Poisson relation for isotropic bodies that establishes the transition to the elastic formulas with just one constant and for whose verification one will carry out research with the most varied successes.

Starting from the viewpoint that was explained above, the diversity of the ratio c / c_1 in various bodies is perfectly comprehensible. For crystals whose elastic constant have been determined, the ratio c / c_1 is easy to calculate with formulas (91) and (92), and one will find the most extreme variety in the numbers, which starts with 13.7 for rock salt and descends down to negative values for pyrite and sodium chlorate.

As for the verification of that theory by experiments, one must take into account the difficulty that resides in the rarity of the quasi-isotropic bodies of the supposed composition, which exists simultaneously with the crystals of the same material that are susceptible to measurements. The compact (quasi-isotropic) varieties generally present individual crystals that are weakly agglomerated by powdery masses of extremely small elastic resistance. It is then natural that they should give resistances that are noticeably

weaker than the ones that were derived by means of formulas (91) and (92) from the measurements that were made by the author in collaboration with Drude (¹). Meanwhile, one can show that it is plausible that those perturbations have effects that are proportional to the different sums that enter into the formulas for the stresses, and in turn, despite the diminution of the absolute value of the constants c and c_1 , their ratio must have roughly the theoretical value.

If one compares the values of c / c_1 that are observed for compact varieties with the ones that are calculated by means of the elastic constants of a homogeneous crystal then one will find an approximate agreement. What is particularly striking is the manner by which the two varieties of amorphous silica compare to those of quartz. As was said above, the latter will give the exceptionally high number of 13.7, in place of Poisson's number 3. For flint, observation gave 11.7, and for opal, it was 15.6. In order to better judge the significance of those results, it is convenient to consider that, on the one hand, it is not certain that flint and opal actually contain silica in the same modification as quartz, and that, on the other hand, the directly-observable values of the resistances to elongation and torsion E and T combine in the expression for the ratio of the constants c / c_1 in such a fashion that the result will possess a relatively low degree of certainty.

Consequently, it is better to make the comparison of the theory and observation on the basis of the ratio $E : T$ itself. From the Poisson-Cauchy relations, that quantity is equal to 2.5 for all bodies, so the theory that was developed above will demand that one get 2.13₀ for quartz, while observations give 2.15₀ for flint and 2.12₀ for opal. That excellent agreement is a confirmation of the greatest value for the theory.

THESIS

The observations that have been made up to today agree with the equations of elasticity with 21 constants in all of their aspects.

Some crystal groups that can have different elastic properties from the general theory have always been found to be different, up to the present.

The *Poisson-Cauchy* relations between the elastic constants are verified approximately in certain cases, but in most of them, they are not even close.

The elastic properties of isotropic bodies can be explained by the molecular hypothesis only if one assumes that those bodies are composed of crystal fragments.

APPENDIX

Theory of thermoelasticity (²).

In the preceding exposition, one exclusively imagined mechanical forces that acted upon crystals (except for volume moments, which were discussed in passing, and which are barely realizable mechanically). However, one knows that elastic forces can be put into play by other phenomena, such as the action of heat, and electric and magnetic fields, in particular. Each of those actions will give rise to theoretical considerations that are

⁽¹⁾ W. VOIGT and P. DRUDE, Wied. Ann. **42** (1891), 537. – W. VOIGT, *ibid.* **44** (1891), 170.

⁽²⁾ W. THOMSON, Quart. Jour. Math. **1** (1857), 57. – N. SCHILLER, Jour. russ. phys. Ges. **11** (1879), 6. Especially for crystals: W. VOIGT, Wied. Ann. **36** (1889), 743.

important generalizations of the preceding ones. Space permits us to give only some brief suggestions about those questions.

One considers those phenomena to be reversible in the thermodynamic sense, and one consequently represents their laws most conveniently by means of the function that is called the *thermodynamic potential*, whose existence is precisely the expression for that reversibility, and whose derivatives, one knows, provide all of the characteristic quantities of the phenomena considered in the simplest manner.

Let Ω denote the thermodynamic potential per unit volume for a crystal. In the phenomena considered, it will depend upon not only the dilatation components x_x, \dots, x_y , but also the variation τ of the temperature and the components X, Y, Z and A, B, C of the electric and the magnetic field.

From the principles of thermodynamics, minus the first partial derivatives of Ω with respect to the components of the dilatation x_x, \dots, x_y will then provide the general components of the stress Ξ_x, \dots, Ξ_y , and the derivatives with respect to the components of the electric and magnetic field will give the electric moments ξ, η, ζ and magnetic moments α, β, γ per unit volume. Minus the first partial derivative with respect to temperature will give the increase in entropy per unit volume that is determined by the other arguments. If one is always dealing with dilatation components that are extremely small, as well as temperature variations that are likewise very small and weak field components, then one can develop Ω in powers of those variables and keep only the terms of lowest degree that appear (viz., the ones of second degree). The specialization of that expression for the various crystallographic systems can then be made from the principles on page 30, and the best way of doing that is to make a convenient decomposition of Ω into simpler terms, as was done on page 18.

For the phenomena of thermoelasticity, only the terms that depend upon x_x, \dots, x_y , and τ will occur. Here, Ω is then composed of a function that is homogeneous of second degree in the x_x, \dots, x_y – i.e., an ordinary elastic potential f – a term in τ^2 , and a term that is bilinear in τ and x_x, \dots, x_y . We can then set:

$$(94) \quad \Omega = f - \tau (q_1 x_x + q_2 y_y + \dots + q_6 x_y) - \frac{1}{2} r \tau^2,$$

in which the q_h and r represent constants.

One will then have the formulas:

$$(95) \quad \Xi_x = X_x + q_1 \tau, \dots, \Xi_y = X_y + q_6 \tau$$

for the stress components, in which X_x, \dots, X_y have the values that were indicated in (41). The terms $q_1 \tau$ are then the thermal components of stress, which are added to the purely elastic or isothermal components X_x, \dots, X_y when the temperature changes.

For example, if there are no external actions then Ξ_x, \dots, Ξ_y will be zero, and upon substituting the values (41) for X_x, \dots, X_y , some expressions for the components of the dilatation that corresponds to the variation in temperature τ will result from (95) that will have the form:

$$(96) \quad x_x = a_1 \tau, \quad y_y = a_2 \tau, \quad \dots, \quad x_y = a_6 \tau,$$

The *coefficients of thermal deformation* a_h , which are defined by these equations, will then be given by the formulas:

$$(97) \quad a_h = q_1 s_{h1} + q_2 s_{h2} + \dots + q_6 s_{h6},$$

which will also give:

$$(98) \quad q_h = a_1 c_{h1} + a_2 c_{h2} + \dots + a_6 c_{h6}$$

when it is solved for the q_h .

The latter formulas are particularly interesting, because they make it possible to calculate the constants q_h of the thermal stresses by starting with the constants of elastic and thermal deformation that are accessible to experiment. Moreover, their determination will permit one to perceive, on the one hand, the absolute magnitude of the stresses that are produced by the temperature variations, and on the other hand, they permit one to appreciate if and how much those stresses vary with direction in the crystal.

In regard to those questions, observation shows that the thermal stresses, like the elastic stresses, are functions of the orientation of the surface element on which they act, that they are positive, even when the crystal contracts in one direction by heating, and that the effect of an elevation in temperature can generally be annulled by a pressure that is uniform in every direction.

From the preceding, one infers the expression:

$$(99) \quad H = q_1 x_x + q_2 y_y + \dots + q_6 x_y + r \tau$$

from (94) for the increase in entropy H per unit volume that will correspond to the deformation x_x, \dots, x_y , and the variation of temperature τ .

If Θ_0 represents the initial absolute temperature by starting from which one will count the (small) increase τ then, from the principles of thermodynamics, $\Theta_0 H$ will be the quantity of heat per unit volume in the crystal, as measured in mechanical units, and Q / τ will be the *caloric capacity* Γ per unit volume. *Consequently, $r \Theta_0$ will be equal to that capacity in the case where the deformation is lacking.*

When the phenomenon is adiabatic – i.e., there is no exchange of heat – one will have $H = 0$, and that equation will give the value τ of the variation of temperature that is a consequence of only the deformation when one substitutes the expression given by (99) for H . If one substitutes that value in the expressions (95) for the general stress components Ξ_x, \dots, Ξ_y then one will obtain expressions that are homogeneous in x_x, \dots, x_y that play the same role for all adiabatic elastic phenomena that the fundamental formulas (41) play for isothermal phenomena. In place of the isothermal elastic constants c_{hk} , one will find adiabatic constants c'_{hk} that are related to the preceding ones by the formulas:

$$(100) \quad c'_{hk} = c_{hk} + \frac{q_h q_k}{r}.$$

One must similarly append adiabatic *moduli* to the isothermal elastic moduli, which were all that was employed above. The author has calculated the adiabatic parameters for the crystals when one knows the isothermal parameters. The differences between those two types of quantities are very noticeable for some bodies.

The isothermal and adiabatic phenomena are the extreme cases in which the real phenomena are approached closely enough to be identified with them in certain cases.

We consider an elastic phenomenon to be isothermal when it is produced in a medium at constant temperature slow enough that the variation of temperature that is produced by the deformation itself, by the preceding, will have enough time to disappear almost completely by thermal conduction. We consider it to be adiabatic when the modifications are produced fast enough that the thermal conduction can have only extremely small effects. The former case is that of the methods of static observation, while the latter one is the one in which all vibrations are rapid, which is that of acoustic phenomena. In the former case, one is consequently authorized to operate with the isothermal constants and moduli, while in the latter, one can work with the adiabatic constants and moduli.

Just as the considerations above provide, at the same time, the laws of thermal deformation and those of the heat of deformation, similarly, when one takes into account the electric field components, some analogous considerations with a generalized potential Ω will give the laws of piezoelectricity and pyroelectricity, as well as the reciprocal phenomena of electrostriction and the electric heating of crystals. However, we are obliged to settle for a simple mention of the existence of those relationships here.
