"Sulle equazioni differenziali che provengono da questioni di calcolo delle variazioni," Rend. Acc. dei Lincei (4) **6** (1890), 43-54; *Opere matematiche*, v. I, 454-463.

On the differential equations that are deduced from questions in the calculus of variations

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In a study that I hope that I can communicate to this academy, it was necessary for me to consider the integrals of a system of differential equations that is defined by annulling the first variation of a multiple integral as things that depend upon certain elements that are given at the limits of that integral. Therefore, permit me to present in this note some of my studies of the differential equations that are defined by the calculus of variations, including a contribution to the criteria for recognizing whether being given certain elements at the limits is sufficient to define the unknown functions of the problem. I shall not enter into the question of whether those elements that are given at the limits are *characteristic*. Those questions present great difficulties and can be solved only in a few cases, among which, one has the ones that were treated very remarkably by SCHWARZ and recently generalized by PICARD in an interesting paper that was published in Acta Mathematica.

JACOBI has observed that the differential equations that one finds by annulling the first variation of a simple integral can be transformed and reduced to a form that is equivalent to the one that HAMILTON provided for the equations of dynamics. Analogously, one can reduce the differential equations that are obtained by annulling the first variation of multiple integrals to that form. That form includes JACOBI's canonical form as a particular case. In the present note, I shall appeal to the differential equations when they are posed in that form.

I.

1. – Let $y_1, y_2, \ldots, y_p$ be functions of the n variables $x_1, x_2, \ldots, x_n$, and let F be a function of $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_p$, and their partial derivatives.

One can either suppose that the $y_1, y_2, \ldots, y_p$ are mutually independent or that they are coupled by certain relations:

$$F_1 = 0 , \qquad F_2 = 0 , \qquad \ldots, \qquad F_r = 0 . \tag{1}$$

As auxiliary variables, introduce the partial derivatives of the $y_1, y_2, \ldots, y_p$ of order *less than* the one that appears with the maximum index of derivation in F and in the given condition equations, that is to say, consider all of the:

$$z_h = \frac{\partial^{k_1+\cdots+k_n} y_i}{\partial x_1^{k_1}\, \partial x_2^{k_2} \cdots \partial x_n^{k_n}} = Y^{(i)}_{k_1 k_2 \ldots k_n}$$

that enter into F and the $Y^{(i)}_{h_1 h_2 \ldots h_n}$ in (1) for which one has:

$$h_1 \geq k_1 , \qquad h_2 \geq k_2 , \qquad \ldots, \qquad h_n \geq k_n ,$$

$$h_1 + h_2 + \ldots + h_n \geq k_1 + k_2 + \ldots + k_n .$$

One can consider F to be a function of $z_1, z_2, \ldots, z_p$ and their first derivatives:

$$z_h^{(i)} = \frac{\partial z_i}{\partial x_h},$$

while certain relations:

$$F_1 = 0 , \qquad F_2 = 0 , \qquad \ldots, \qquad F_r = 0 , \qquad F_{r+1} = 0 , \qquad \ldots, \qquad F_z = 0$$

exist between the z_i and the $z_h^{(i)}$.

The first r of them can be supposed to be the given relations (1), while the others will obviously be the linear relations between the z_i and the $z_h^{(i)}$.

2. – Now consider the problem of annulling the first variation of:

$$I = \int F\, dx_1 \cdots dx_n \ .$$

Set:

(2) $$\Phi = F + \sum_{t=1}^{s} \lambda_t F_t \ .$$

One will then get the equations:

(3) $$\begin{cases} \displaystyle\sum_{h=1}^{n} \frac{\partial}{\partial x_h} \frac{\partial \Phi}{\partial z_h^{(i)}} - \frac{\partial \Phi}{\partial z_i} = 0, \\ F_1 = 0, \quad F_2 = 0, \quad \ldots, \quad F_s = 0. \end{cases}$$

Suppose that only $z_{i_1}^{(i)}$, $z_{i_2}^{(i)}$, …, $z_{i_t}^{(i)}$ from among the $z_1^{(i)}$, $z_2^{(i)}$, …, $z_n^{(i)}$ enter into Φ. One can then write the preceding equations in the form:

$$\begin{cases} \dfrac{\partial}{\partial x_{i_1}}\dfrac{\partial\Phi}{\partial z_{i_1}^{(i)}}+\dfrac{\partial}{\partial x_{i_2}}\dfrac{\partial\Phi}{\partial z_{i_2}^{(i)}}+\cdots+\dfrac{\partial}{\partial x_{i_t}}\dfrac{\partial\Phi}{\partial z_{i_t}^{(i)}}=\dfrac{\partial\Phi}{\partial z_i}, \\ F_1=0,\quad F_2=0,\quad \ldots,\quad F_s=0. \end{cases}$$

Set:

(4) $$\frac{\partial\Phi}{\partial z_{i_1}^{(i)}}=p_{i_1}^{(i)},\qquad \frac{\partial\Phi}{\partial z_{i_2}^{(i)}}=p_{i_2}^{(i)},\qquad \ldots,\qquad \frac{\partial\Phi}{\partial z_{i_t}^{(i)}}=p_{i_t}^{(i)}$$

and assume that the system of equations, together with:

(1) $$F_1=0\,,\qquad F_2=0\,,\qquad \ldots,\qquad F_r=0\,,$$

can be solved for the $z_{i_l}^{(i)}$ and the λ_h, that is to say, the Jacobian of the $\partial\Phi/\partial z_{i_l}^{(i)}$ and F_h with the respect to the $z_{i_l}^{(i)}$ and the λ_h is non-zero. Substitute the values that one obtains by means of that solution in:

(5) $$H=\Phi-\sum_{i=1}^{m}\left[z_{i_1}^{(i)}p_{i_1}^{(i)}+z_{i_2}^{(i)}p_{i_2}^{(i)}+\cdots+z_{i_t}^{(i)}p_{i_t}^{(i)}\right]\,.$$

Varying both sides of that equation will give:

$$\sum_{i=1}^{m}\left[\frac{\partial H}{\partial z_i}\delta z_i+\frac{\partial H}{\partial p_{i_1}^{(i)}}\delta p_{i_1}^{(i)}+\cdots+\frac{\partial H}{\partial p_{i_t}^{(i)}}\delta p_{i_t}^{(i)}\right]$$

$$=\sum_{i=1}^{m}\left[\frac{\partial\Phi}{\partial z_i}\delta z_i+\frac{\partial\Phi}{\partial p_{i_1}^{(i)}}\delta p_{i_1}^{(i)}+\cdots+\frac{\partial\Phi}{\partial p_{i_t}^{(i)}}\delta p_{i_t}^{(i)}\right]$$

$$+\sum_{i=1}^{m}\left[\frac{\partial\Phi}{\partial z_i}\delta z_i-z_{i_1}^{(i)}\delta p_{i_1}^{(i)}-\cdots-z_{i_t}^{(i)}\delta p_{i_t}^{(i)}\right]$$

$$+\sum_{i=1}^{m}\left[\left(\frac{\partial\Phi}{\partial z_i}-p_{i_1}^{(i)}\right)\delta z_{i_1}^{(i)}+\cdots+\left(\frac{\partial\Phi}{\partial z_t}-p_{i_t}^{(i)}\right)\delta z_{i_t}^{(i)}\right]$$

$$+\sum_{h=1}^{s}F_h\,\delta\lambda_h\,.$$

If one then takes into account (4) and (1) then one will have:

$$\sum_{i=1}^{m}\left[\frac{\partial H}{\partial z_i}\delta z_i+\frac{\partial H}{\partial p_{i_1}^{(i)}}\delta p_{i_1}^{(i)}+\cdots+\frac{\partial H}{\partial p_{i_t}^{(i)}}\delta p_{i_t}^{(i)}\right]$$

$$=\sum_{i=1}^{m}\left[\frac{\partial \Phi}{\partial z_i}\delta z_i-z_{i_1}^{(i)}\delta p_{i_1}^{(i)}-\cdots-z_{i_t}^{(i)}\delta p_{i_t}^{(i)}\right],$$

so

$$\frac{\partial H}{\partial z_i}=\frac{\partial \Phi}{\partial z_i},\qquad \frac{\partial H}{\partial p_{i_l}^{(i)}}=-z_{i_l}^{(i)}.$$

Equations (3) can then be replaced with the system of equations:

$$(6)\qquad \begin{cases} \dfrac{\partial p_{i_1}^{(i)}}{\partial x_{i_1}}+\dfrac{\partial p_{i_2}^{(i)}}{\partial x_{i_2}}+\cdots+\dfrac{\partial p_{i_t}^{(i)}}{\partial x_{i_t}}=\dfrac{\partial H}{\partial z_i}, \\ \dfrac{\partial z_i}{\partial x_{i_1}}=-\dfrac{\partial H}{\partial p_{i_1}^{(i)}}, \\ \dfrac{\partial z_i}{\partial x_{i_2}}=-\dfrac{\partial H}{\partial p_{i_2}^{(i)}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial z_i}{\partial x_{i_t}}=-\dfrac{\partial H}{\partial p_{i_t}^{(i)}}, \end{cases}\qquad (i = 1, 2, \ldots, m).$$

That system of equations has a form that is analogous to that of the canonical equations, except that any function z_i will be conjugate to many functions, in general, viz., the $p_{i_1}^{(i)}$, …, $p_{i_t}^{(i)}$, as opposed to just one function in the case of the ordinary canonical equations.

3. – We have seen how the differential equations that are produced by annulling the first variation of an integral can be put into the form (6). Conversely, we can prove that any system of equations of the form (6), in which H is a function of the z_i and the $p_{i_l}^{(i)}$, can be made to depend upon a problem in the calculus of variations. Indeed, consider:

$$\int\left[\sum_{i=1}^{m}z_i\left(\frac{\partial p_{i_1}^{(i)}}{\partial x_{i_1}}+\frac{\partial p_{i_2}^{(i)}}{\partial x_{i_2}}+\cdots+\frac{\partial p_{i_t}^{(i)}}{\partial x_{i_t}}\right)-H\right]dx_1\cdots dx_n\ .$$

The condition for the first variation of that integral to vanish, when one supposes that the z_i and the $p_{i_l}^{(i)}$ are mutually independent, is precisely (6).

II.

4. – Some noteworthy relations exist between the function H and the functions Φ, F, F_1, …, F_r. If Φ contains $z_{h_l}^{(h)}$ then it will follow from (6) that:

$$\frac{\partial H}{\partial p_{h_l}^{(h)}} = - \; z_{h_l}^{(h)} .$$

Consequently, one can write:

$$\sum_{i=1}^{m} \left[\frac{\partial H}{\partial z_i} z_i - \frac{\partial H}{\partial p_{i_1}^{(i)}} p_{i_1}^{(i)} - \cdots - \frac{\partial H}{\partial p_{i_t}^{(i)}} p_{i_t}^{(i)} \right]$$

$$= \sum_{i=1}^{m} \left[\frac{\partial \Phi}{\partial z_i} z_i + \frac{\partial \Phi}{\partial z_{i_1}^{(i)}} z_{i_1}^{(i)} + \cdots + \frac{\partial \Phi}{\partial z_{i_t}^{(i)}} z_{i_t}^{(i)} \right]$$

$$= \sum_{i=1}^{m} \left[\frac{\partial F}{\partial z_i} z_i + \frac{\partial F}{\partial z_{i_1}^{(i)}} z_{i_1}^{(i)} + \cdots + \frac{\partial F}{\partial z_{i_t}^{(i)}} z_{i_t}^{(i)} \right]$$

$$+ \sum_{h=1}^{s} \lambda_h \sum_{i=1}^{m} \left[\frac{\partial F_h}{\partial z_i} z_i + \frac{\partial F_h}{\partial z_{i_1}^{(i)}} z_{i_1}^{(i)} + \cdots + \frac{\partial F_h}{\partial z_{i_t}^{(i)}} z_{i_t}^{(i)} \right] .$$

However, F_{r+1} , …, F_s are homogeneous functions of degree one, so:

(7)
$$\sum_{i=1}^{m} \left[\frac{\partial H}{\partial z_i} z_i - \frac{\partial H}{\partial p_{i_1}^{(i)}} p_{i_1}^{(i)} - \cdots - \frac{\partial H}{\partial p_{i_t}^{(i)}} p_{i_t}^{(i)} \right]$$

$$= \sum_{i=1}^{m} \left[\frac{\partial F}{\partial z_i} z_i + \frac{\partial F}{\partial z_{i_1}^{(i)}} z_{i_1}^{(i)} + \cdots + \frac{\partial F}{\partial z_{i_t}^{(i)}} z_{i_t}^{(i)} \right]$$

$$+ \sum_{h=1}^{r} \lambda_h \sum_{i=1}^{m} \left[\frac{\partial F_h}{\partial z_i} z_i + \frac{\partial F_h}{\partial z_{i_1}^{(i)}} z_{i_1}^{(i)} + \cdots + \frac{\partial F_h}{\partial z_{i_t}^{(i)}} z_{i_t}^{(i)} \right] .$$

If all of the F_h were homogeneous functions then the last term on the right-hand side of the preceding equation would disappear.

5. – Consider two systems of solutions of (6), i.e., z_i , $p_{i_l}^{(i)}$, and u_i , $q_{i_l}^{(i)}$, and set $u_i - z_i = \zeta_i$, $q_{i_l}^{(i)} - p_{i_l}^{(i)} = \tilde{\omega}_{i_l}^{(i)}$; one will have:

$$u_{i_l}^{(i)} - z_{i_l}^{(i)} = \zeta_{i_l}^{(i)} .$$

It follows from the relations:

$$\frac{\partial \Phi}{\partial z_{i_l}^{(i)}} = p_{i_l}^{(i)}, \quad z_{i_l}^{(i)} = -\frac{\partial H}{\partial p_{i_l}^{(i)}}, \quad \frac{\partial \Phi}{\partial z_i} = \frac{\partial F}{\partial z_i},$$

upon applying TAYLOR's theorem, that:

$$\sum_{h=1}^{m}\left[\frac{\partial^2\Phi}{\partial z_{i_l}^{(i)}\,\partial z_h}\zeta_h + \frac{\partial^2\Phi}{\partial z_{i_l}^{(i)}\,\partial z_{h_1}}\zeta_{h_1}^{(h)} + \cdots + \frac{\partial^2\Phi}{\partial z_{i_l}^{(i)}\,\partial z_{h_t}}\zeta_{h_t}^{(h)}\right] + \varphi_{i_l}^{(i)} = \tilde{\omega}_{i_l}^{(i)}, \tag{8}$$

$$\zeta_{i_l}^{(i)} = -\sum_{h=1}^{m}\left[\frac{\partial^2 H}{\partial p_{i_l}^{(i)}\,\partial z_h}\zeta_h + \frac{\partial^2 H}{\partial p_{i_l}^{(i)}\,\partial p_{h_l}^{(h)}}\tilde{\omega}_{h_1}^{(h)} + \cdots + \frac{\partial^2 H}{\partial p_{i_l}^{(i)}\,\partial p_{h_l}^{(h)}}\tilde{\omega}_{h_t}^{(h)}\right] + \psi_{i_l}^{(i)}, \tag{8'}$$

$$\begin{aligned}\sum_{h=1}^{m}&\left[\frac{\partial^2\Phi}{\partial z_i\,\partial z_h}\zeta_h + \frac{\partial^2\Phi}{\partial z_i\,\partial z_{h_1}^{(h)}}\zeta_{h_1}^{(h)} + \cdots + \frac{\partial^2\Phi}{\partial z_i\,\partial z_{h_t}^{(h)}}\zeta_{h_t}^{(h)}\right] + \varphi_i \\ &= \sum_{h=1}^{m}\left[\frac{\partial^2 H}{\partial z_h\,\partial z_h}\zeta_h + \frac{\partial^2 H}{\partial z_i\,\partial p_{h_l}^{(h)}}\tilde{\omega}_{h_1}^{(h)} + \cdots + \frac{\partial^2 H}{\partial z_i\,\partial p_{h_l}^{(h)}}\tilde{\omega}_{h_t}^{(h)}\right] + \psi_i ,\end{aligned} \tag{8''}$$

in which $\varphi_{i_l}^{(i)}$, φ_i are homogeneous functions of degree two in the ζ_h and $\zeta_{h_\lambda}^{(h)}$ whose coefficients are the third derivatives of Φ taken at intermediate values of the variables from among the values of z_h, $z_{h_\lambda}^{(h)}$ and u_h , $u_{h_\lambda}^{(h)}$, while the $\psi_{i_l}^{(i)}$, ψ_i are homogeneous functions of degree two in the ζ_h, $p_{h_\lambda}^{(h)}$ whose coefficients are the third derivatives of H taken at intermediate values of the variables from among the values z_h , $p_{h_\lambda}^{(h)}$ and u_h , $q_{h_\lambda}^{(h)}$.

If one multiplies corresponding sides of (8) and (8′) and multiplies (8″) by ξ_i and sums then one will find that:

$$\sum\sum\frac{\partial^2\Phi}{\partial z_i\,\partial z_h}\zeta_i\,\zeta_h + \sum\sum\frac{\partial^2\Phi}{\partial z_{i_l}^{(i)}\,\partial z_{h_\lambda}^{(h)}}\zeta_{i_l}^{(i)}\,\zeta_{h_\lambda}^{(h)} + \varphi \tag{9}$$

$$= \sum\sum \frac{\partial^2 H}{\partial z_i\, \partial z_h} \zeta_i\, \zeta_h - \sum\sum \frac{\partial^2 H}{\partial p_{i_l}^{(i)}\, \partial p_{h_\lambda}^{(h)}} \tilde{\omega}_{i_l}^{(i)}\, \tilde{\omega}_{h_\lambda}^{(h)} + \psi\ ,$$

in which φ is a homogeneous function of degree three in the ζ_i , $\zeta_{i_l}^{(i)}$, and ψ is homogeneous of degree three in just the ζ_i , $\tilde{\omega}_{i_l}^{(i)}$. Denote the z_i and $z_{i_l}^{(i)}$ that are contained in Φ by $v_1, v_2, \ldots, v_g$, and the corresponding u_i and $u_{i_l}^{(i)}$ by $w_1, w_2, \ldots, w_g$. Set:

$$w_i - v_i = \nu_i$$

and recall that F_{r+1} , …, F_s are linear functions. The preceding equations can then be written:

(10)
$$\sum\sum \frac{\partial^2 \Phi}{\partial v_i\, \partial v_h} v_i\, v_h + \varphi = \sum\sum \frac{\partial^2 H}{\partial z_i\, \partial z_h} \zeta_i\, \zeta_h - \sum\sum \frac{\partial^2 H}{\partial p_{i_l}^{(i)}\, \partial p_{h_\lambda}^{(h)}} \tilde{\omega}_{i_l}^{(i)}\, \tilde{\omega}_{h_\lambda}^{(h)} + \psi$$
$$= \sum\sum \frac{\partial^2 F}{\partial v_i\, \partial v_h} v_i\, v_h + \sum_{g=1}^{r} \lambda_g \sum\sum \frac{\partial^2 F_g}{\partial v_i\, \partial v_h} v_i\, v_h + \varphi\ .$$

III.

6. – Recall the fundamental equations (6). As before, let z_i , $p_{i_l}^{(i)}$ denote one system of integrals and let u_i , $q_{i_l}^{(i)}$ denote another system of integrals.

Let S_n be an n-dimensional region that is bounded by the contour S_{n-1}, between which the preceding two systems of integrals are functions that are finite and continuous, along with all of their derivatives.

Multiply the two sides of (6) by u_i, $-\ q_{i_l}^{(i)}$, in succession, and then sum and integrate over all of S_n . One will get:

$$\int_{S_n} \sum_{i=1}^{m} \left[u_i \frac{\partial H}{\partial z_i} + q_{i_1}^{(i)} \frac{\partial H}{\partial p_{i_1}^{(i)}} + \cdots + q_{i_l}^{(i)} \frac{\partial H}{\partial p_{i_l}^{(i)}} \right] dS_n$$

$$= \int_{S_n} \left[\sum_{i=1}^{m} u_i \left(\frac{\partial p_{i_1}^{(i)}}{\partial x_{i_1}} + \cdots + \frac{\partial p_{i_t}^{(i)}}{\partial x_{i_t}} \right) - \sum_{i=1}^{m} \left(q_{i_1}^{(i)} \frac{\partial z_i}{\partial x_{i_1}} + \cdots + q_{i_t}^{(i)} \frac{\partial z_i}{\partial x_{i_t}} \right) \right] dS_n$$

$$= - \int_{S_{n-1}} \sum_{i=1}^{m} u_i\, (p_{i_1}^{(i)} \cos \nu\, x_{i_1} + \cdots + p_{i_t}^{(i)} \cos \nu\, x_{i_t})\, dS_{n-1}$$

$$+ \int_{S_{n-1}} \sum_{i=1}^{m} z_i \, (q_{i_1}^{(i)} \cos \nu \, x_{i_1} + \cdots + q_{i_t}^{(i)} \cos \nu \, x_{i_t}) \, dS_{n-1}$$

$$- \int_{S_n} \left[\sum_{i=1}^{m} \left(p_{i_1}^{(i)} \frac{\partial u_i}{\partial x_{i_1}} + \cdots + p_{i_t}^{(i)} \frac{\partial u_i}{\partial x_{i_t}} \right) - z_i \left(\frac{\partial q_{i_1}^{(i)}}{\partial x_{i_1}} + \cdots + \frac{\partial q_{i_t}^{(i)}}{\partial x_{i_t}} \right) \right] dS_n \ ,$$

in which ν is the normal to S_{n-1} that points inward to S_n . One will then have:

(11)
$$\int_{S_{n-1}} \sum_{i=1}^{m} z_i \, (q_{i_1}^{(i)} \cos \nu \, x_{i_1} + \cdots + q_{i_t}^{(i)} \cos \nu \, x_{i_t}) \, dS_{n-1}$$

$$- \int_{S_{n-1}} \sum_{i=1}^{m} u_i \, (p_{i_1}^{(i)} \cos \nu \, x_{i_1} + \cdots + p_{i_t}^{(i)} \cos \nu \, x_{i_t}) \, dS_{n-1}$$

$$= \int_{S_n} \sum_{i=1}^{m} \left(u_i \frac{\partial H}{\partial z_i} + q_{i_1}^{(i)} \frac{\partial H}{\partial p_{i_1}^{(i)}} + \cdots + q_{i_l}^{(i)} \frac{\partial H}{\partial p_{i_t}^{(i)}} \right) dS_n$$

$$- \int_{S_n} \sum_{i=1}^{m} \left(z_i \frac{\partial H}{\partial u_i} + p_{i_1}^{(i)} \frac{\partial H}{\partial q_{i_1}^{(i)}} + \cdots + p_{i_l}^{(i)} \frac{\partial H}{\partial q_{i_t}^{(i)}} \right) dS_n \ .$$

That relation is analogous to that of GREEN.

In the case where H is an entire rational function that is homogeneous of degree two, the right-hand side will vanish. One can deduce prof. BETTI's fundamental theorem of elasticity from that fact.

7. – Multiply both sides of (6) by z_i, $p_{i_1}^{(i)}$, …, $p_{i_t}^{(i)}$, in succession and integrate over S_n . One will find that:

$$\int_{S_n} \sum_{i=1}^{m} \left(z_i \frac{\partial H}{\partial z_i} - p_{i_1}^{(i)} \frac{\partial H}{\partial p_{i_1}^{(i)}} - \cdots - p_{i_l}^{(i)} \frac{\partial H}{\partial p_{i_t}^{(i)}} \right) dS_n$$

$$= \int_{S_n} \sum_{i=1}^{m} \left(\frac{\partial (z_i \, p_{i_1}^{(i)})}{\partial x_{i_1}} + \cdots + \frac{\partial (z_t \, p_{i_t}^{(i)})}{\partial x_{i_t}} \right) dS_n$$

$$= - \int_{S_{n-1}} \sum_{i=1}^{m} z_i \, (p_{i_1}^{(i)} \cos \nu \, x_{i_1} + \cdots + p_{i_t}^{(i)} \cos \nu \, x_{i_t}) \, dS_{n-1} \ .$$

If one takes (7) into account and supposes that F, F_1, …, F_r are homogeneous, while F has degree k, then one will have:

$$k_1\, I = k \int_{S_n} F\, dS_n = -\int_{S_{n-1}} \sum_{i=1}^{m} z_i\, (p_{i_1}^{(i)} \cos \nu\, x_{i_1} + \cdots + p_{i_t}^{(i)} \cos \nu\, x_{i_t})\, dS_{n-1}\ .$$

8. – It follows from the formulas that were found in § **5** that:

$$(12)\qquad \begin{cases} \dfrac{\partial \tilde{\omega}_{i_1}^{(i)}}{\partial x_{i_1}} + \cdots + \dfrac{\partial \tilde{\omega}_{i_t}^{(i)}}{\partial x_{i_t}} = \displaystyle\sum_{h=1}^{m} \left(\frac{\partial^2 H}{\partial z_i\, \partial z_h} \zeta_h + \frac{\partial^2 H}{\partial z_i\, \partial p_{h_1}^{(h)}} \tilde{\omega}_{h_1}^{(h)} + \cdots + \frac{\partial^2 H}{\partial p_{i_1}^{(i)}\, \partial p_{h_t}^{(h)}} \tilde{\omega}_{h_t}^{(h)} \right) + \psi_i\ , \\ \dfrac{\partial \zeta_i}{\partial x_{i_1}} = -\displaystyle\sum_{h=1}^{m} \left(\frac{\partial^2 H}{\partial p_{i_1}^{(i)}\, \partial z_h} \zeta_h + \frac{\partial^2 H}{\partial p_{i_1}^{(i)}\, \partial p_{h_1}^{(h)}} \tilde{\omega}_{h_1}^{(h)} + \cdots + \frac{\partial^2 H}{\partial p_{i_1}^{(i)}\, \partial p_{h_t}^{(h)}} \tilde{\omega}_{h_t}^{(h)} \right) + \psi_{i_1}^{(i)}\ , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial \zeta_i}{\partial x_{i_t}} = -\displaystyle\sum_{h=1}^{m} \left(\frac{\partial^2 H}{\partial p_{i_t}^{(i)}\, \partial z_h} \zeta_h + \frac{\partial^2 H}{\partial p_{i_t}^{(i)}\, \partial p_{h_1}^{(h)}} \tilde{\omega}_{h_1}^{(h)} + \cdots + \frac{\partial^2 H}{\partial p_{i_t}^{(i)}\, \partial p_{h_t}^{(h)}} \tilde{\omega}_{h_t}^{(h)} \right) + \psi_{i_t}^{(i)}\ , \end{cases}$$

so

$$\int_{S_n} \left(\sum_i \sum_h \frac{\partial^2 H}{\partial z_i\, \partial z_h} \zeta_i\, \zeta_h - \sum \sum \frac{\partial^2 H}{\partial p_{i_l}^{(i)}\, \partial p_{h_\lambda}^{(h)}} \tilde{\omega}_{i_l}^{(i)}\, \tilde{\omega}_{h_\lambda}^{(h)} + \psi \right) dS_n$$

$$= \int_{S_n} \sum_{i=1}^{m} \left(\frac{\partial(\zeta_i\, \tilde{\omega}_{i_1}^{(i)})}{\partial x_{i_1}} + \frac{\partial(\zeta_i\, \tilde{\omega}_{i_2}^{(i)})}{\partial x_{i_2}} + \cdots + \frac{\partial(\zeta_i\, \tilde{\omega}_{i_t}^{(i)})}{\partial x_{i_t}} \right) dS_n\ ,$$

which ultimately gives the formula:

$$(13)\qquad -\int_{S_{n-1}} \sum_{i=1}^{m} \zeta_i\, (\tilde{\omega}_{i_1}^{(i)} \cos \nu\, x_{i_1} + \tilde{\omega}_{i_2}^{(i)} \cos \nu\, x_{i_2} + \cdots + \tilde{\omega}_{i_t}^{(i)} \cos \nu\, x_{i_t})\, dS_{n-1}$$

$$= \int_{S_n} \left(\sum_i \sum_h \frac{\partial^2 H}{\partial z_i\, \partial z_h} \zeta_i\, \zeta_h - \sum \sum \frac{\partial^2 H}{\partial p_{i_l}^{(i)}\, \partial p_{h_\lambda}^{(h)}} \tilde{\omega}_{i_l}^{(i)}\, \tilde{\omega}_{h_\lambda}^{(h)} + \psi \right) dS_n$$

$$= \int_{S_n} \left(\sum \sum \frac{\partial^2 \left(F + \sum_{g=1}^{r} \lambda_g\, F_g \right)}{\partial v_i\, \partial v_h} v_i\, v_h + \varphi \right) dS_n\ .$$

IV.

9. – The last formula in the preceding section yields a fundamental theorem that relates to equations (6):

If $\mathbf{z}_i$, $\mathbf{p}_{i_l}^{(i)}$ *define a system of integrals of* (6) *such that the two quadratic forms:*

$$\sum\sum \frac{\partial^2 H}{\partial \mathbf{z}_i\, \partial \mathbf{z}_h}\, \alpha_i\, \alpha_h\,, \quad \sum\sum \frac{\partial^2 H}{\partial \mathbf{p}_{i_l}^{(i)}\, \partial \mathbf{p}_{h_\lambda}^{(h)}}\, \beta_{i_l}^{(i)}\, \beta_{h_\lambda}^{(h)}$$

are definite and of opposite sign then all of the integrals z_i , $p_{i_l}^{(i)}$ *that belong to a region* S_n *and differ from* $\mathbf{z}_i$, $\mathbf{p}_{i_l}^{(i)}$ *by less than a certain value will be determined when one knows the values of the* z_i *or those of the sum:*

$$p_{i_1}^{(i)} \cos \nu\, x_{i_1} + p_{i_2}^{(i)} \cos \nu\, x_{i_2} + \cdots + p_{i_t}^{(i)} \cos \nu\, x_{i_t} = P_i$$

on the contour S_{n-1} *or, more generally, they will be determined when one knows the values of* z_1, z_2, …, z_k *and the remaining ones* P_{k+1} , …, P_m *at any point on the contour.*

Indeed, let z_i , $p_{i_l}^{(i)}$ and u_i , $q_{i_l}^{(i)}$ be two systems of integrals of (6) such that one will have:

$$z_1 = u_1\,,\ z_2 = u_2\,, \ldots, z_k = u_k\,, \quad P_{k+1} = Q_{k+1}\,, \ldots, P_m = Q_m$$

at any point of S_{n-1} , in which:

$$Q_i = q_{i_1}^{(i)} \cos \nu\, x_{i_1} + \cdots + q_{i_t}^{(i)} \cos \nu\, x_{i_t}\,.$$

When one applies that to (13), one will find that:

$$\int_{S_n} \left(\sum\sum \frac{\partial^2 H}{\partial z_i\, \partial z_h}\, \zeta_i\, \zeta_h - \sum\sum \frac{\partial^2 H}{\partial p_{i_l}^{(i)}\, \partial p_{h_\lambda}^{(h)}}\, \tilde{\omega}_{i_l}^{(i)}\, \tilde{\omega}_{h_\lambda}^{(h)} + \psi \right) dS_n = 0\,. \tag{14}$$

Now, if the z_i , u_i , $p_{i_l}^{(i)}$, $q_{i_l}^{(i)}$ differ from $\mathbf{z}_i$, $\mathbf{p}_{i_l}^{(i)}$ by less than a certain value M then one will have:

$$|\, \zeta_i \,| < 2M\,, \qquad \left|\, \tilde{\omega}_{i_l}^{(i)} \,\right| < 2M\,.$$

One can then determine a value μ of M such that:

1. The form:

$$f = \sum\sum \frac{\partial^2 H}{\partial z_i\, \partial z_h} \zeta_i\, \zeta_h - \sum\sum \frac{\partial^2 H}{\partial p_{i_l}^{(i)}\, \partial p_{h_\lambda}^{(h)}} \tilde{\omega}_{i_l}^{(i)}\, \tilde{\omega}_{h_\lambda}^{(h)}$$

is definite.

2. The homogeneous form ψ of degree three in the ζ_i , $\tilde{\omega}_{i_l}^{(i)}$ always assumes absolute values that are less than $|f|$ (except for $\zeta_i = \tilde{\omega}_{i_l}^{(i)} = 0$).

Under those hypotheses, the relation (14) will not be satisfied by anything but:

$$\zeta_i = \tilde{\omega}_{i_l}^{(i)} = 0$$

in all of the region S_n, which proves the theorem.

10. – Suppose that *H is independent* $z_1, z_2, \ldots, z_m$. Under those hypotheses, the preceding theorem will be modified in the following manner:

If $\mathbf{z}_i$, $\mathbf{p}_{i_l}^{(i)}$ *form a system of integrals of* (6) *and the* $\mathbf{p}_{i_l}^{(i)}$ *are such that the quadratic form:*

$$\sum\sum \frac{\partial^2 H}{\partial \mathbf{p}_{i_l}^{(i)}\, \partial \mathbf{p}_{h_\lambda}^{(h)}} \beta_{i_l}^{(i)}\, \beta_{h_\lambda}^{(h)}$$

is definite then all of the integrals z_i , $p_{i_l}^{(i)}$ *such that the* $p_{i_l}^{(i)}$ *that belong to a region* S_n *and differ from* $\mathbf{p}_{i_l}^{(i)}$ *, respectively, by less than a certain value will be determined when one knows the values of the* z_i *on the contour* S_{n-1} .

Indeed, if z_i and u_i are such that one has $z_i = u_i$ on the contour then one will have:

$$\int_{S_n} \left(\sum\sum \frac{\partial^2 H}{\partial p_{i_l}^{(i)}\, \partial p_{h_\lambda}^{(h)}} \tilde{\omega}_{i_l}^{(i)}\, \tilde{\omega}_{h_\lambda}^{(h)} + \psi \right) dS_n = 0 .$$

If one then repeats the argument that was presented in the preceding section then one will find that if $p_{i_l}^{(i)}$ and $q_{i_l}^{(i)}$ do not wander from $\mathbf{p}_{i_l}^{(i)}$ by more than a certain value then one must have that:

$$\tilde{\omega}_{i_l}^{(i)} = 0$$

in order for the preceding equality to be satisfied. However, it follows from (12) that:

$$\frac{\partial \zeta_i}{\partial x_{i_l}} = 0 ,$$

so ζ_i must be independent of x_{i_l}. Now any parallel to the x_{i_l}-axis will meet S_{n-1} where ζ_i is zero, so the ζ_i will be zero, and consequently, one will have:

$$p_{i_l}^{(i)} = q_{i_l}^{(i)} , \qquad z_i = u_i$$

inside S_n, which proves the theorem.

11. – The theorem in § **10** can also be interpreted in a different way.

If $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m, \lambda_1, \lambda_2, \ldots, \lambda_r$ *are a system of integrals of equations* (3) *such that the quadratic form* ([1]):

$$\sum\sum \frac{\partial^2 \left(F + \sum_{\gamma=1}^{r} \lambda_\gamma F_\gamma \right)}{\partial \mathbf{v}_i \, \partial \mathbf{v}_h} \alpha_i \, \alpha_h$$

is definite then if all of the integrals $z_1, \ldots, z_m$ *differ from* $\mathbf{z}_1, \ldots, \mathbf{z}_m$, *respectively, by less than a certain value within a region* S_n, *they will be determined when one knows their values on the contour* S_{n-1}.

Indeed, if $z_1, \ldots, z_m$ and $u_1, \ldots, u_m$ are two systems of integrals of (3) such that one has $z_i = u_i$ on the contour then one can deduce from formula (13) that:

$$\int_{S_n} \left(\sum\sum \frac{\partial^2 \left(F + \sum_{\gamma=1}^{r} \lambda_\gamma F_\gamma \right)}{\partial v_i \, \partial v_h} v_i \, v_h + \varphi \right) dS_n = 0 .$$

When one repeats an argument that was made before, one will then arrive at the conclusion that if the z_i and u_i differ from the $\mathbf{z}_i$ by less than a certain value then in order for the preceding equation to be satisfied, one must have:

$$v_i = 0 ,$$

so (see § **5**) the z_i and $z_h^{(i)}$ that appear in $F, F_1, \ldots, F_g$ will prove to be equal to the corresponding u_i, $u_h^{(i)}$ within S_n. However, if:

([1]) The $\mathbf{v}_1, \ldots, \mathbf{v}_1$ denote the $\mathbf{z}_i$ and $\mathbf{z}_{i_l}^{(i)}$ that are contained in Φ (§ **5**).

$$z_h^{(i)} - u_h^{(i)} = \frac{\partial (z_i - u_i)}{\partial x_h} = 0$$

then it will result that:

$$z_i - u_i = \text{const.}$$

along all of the parallels to the x_h-axis, since they will meet the contour S_{n-1} where $z_i = u_i$, so one must have $z_i = u_i$ at all points in S_n, which proves the theorem.
