"Sopra una estensione della teoria Jacobi-Hamilton del calcolo delle variazioni," Rend. Acc. Lincei (5) **6** (1890), 127-138; *Opere matematiche*, v. I, 464-476.

On an extension of the Jacobi-Hamilton theory in the calculus of variations

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The papers that HAMILTON published in the *Philosophical Transactions* of the Royal Society of London in the years 1834 and 1835 became the point of departure for a series of research works that represent some of the most beautiful studies of our century in the fields of analysis and mechanics. It was JACOBI that generalized and modified HAMILTON's result in such a manner as to make its importance and fecundity obvious. The fundamental theorem, which was initially limited to the questions of dynamics, was extended by JACOBI to the case of isoperimetric problems in which the derivatives of the unknown function that appeared under the integral went only up to first order ([1]). CLEBSCH ([2]) then proved that the procedure that JACOBI had used would be applicable to the general question of annulling the first variation of a simple integral with several unknown functions when some differential relations exist between those functions. Any problem in the calculus of variations that relates to simple integrals can then be reduced to that question.

As far as I know, no attempt has been made to extend the JACOBI-HAMILTON theory to the case in which one must annul the first variation of multiple integrals. As soon as one proposes such a generalization, one will immediately encounter a difficulty. Let us say a few words about what it consists of. The JACOBI-HAMILTON procedure is based upon the examination of a simple integral (whose variation must be annulled) when it is considered to a function of its limits and the values that are assigned arbitrarily to the unknown functions at those limits. Such a function (viz., the characteristic function) will satisfy the partial differential equations that HAMILTON discovered, and which produce the integrals of the problem by means of the operation of differentiation. If one passes from the simple integrals to the case of double integrals then instead of two limits to the integral, one will have one or more *lines* that form the contour of the domain of integration and along which one must give *arbitrary values* to the unknown functions. Therefore, it is not possible in that case to obtain an *ordinary function* that is analogous to HAMILTON's *characteristic function.*

Moreover, the difficulty that was just described can be overcome. In this note, which I had the honor of presenting to the Accademia, I will show how for some purposes it can be useful to

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([1]) JACOBI, "Zur Theorie der Variations-Rechnung und der Differential-Gleichungen," Crelle's Journal, Bd. 17.
([2]) "Ueber diejenigen Probleme der Variations-Rechnung, welche nur eine unabhängige Variable enthalten," Crelle's Journal, Bd. 55.

introduce functions that, along with depending upon the points of space like ordinary functions, also depend upon lines, and in general can be considered to be quantities that depend upon all of the values of one or more function in given intervals.

Now the question that was examined above spontaneously suggests the thought of constructing an element that is analogous to the characteristic function and resorts to the use of the new type of function that was just recalled. One will find that the JACOBI-HAMILTON theory is susceptible to being extended to multiple integrals in that way. That generalization has defined the subject of some of my research, a sample of which I might be permitted to give in the present note.

1. – However, in this note, other than not leaving the case of double integrals, I shall limit myself to the consideration of those problems in the calculus of variations in which one treats the annulment of the first variation of an integral:

$$I = \iint U \, du \, dv ,$$

in which U is a function of $x_1, x_2, \ldots, x_n$ of u and v, and the determinants:

$$\frac{d\,(x_i, x_s)}{d\,(u, v)},$$

in which $x_1, \ldots, x_n$ are unknown functions of u and v.

That class of problems that relate to double integrals is closely-related to that of the isoperimetric problems.

Let us see what form the differential equations of the problem can be put into. If we set:

$$\frac{d\,(x_i, x_s)}{d\,(u, v)} = \xi_{is}$$

then we will have:

$$\delta I = \iint \left(\sum \frac{\partial U}{\partial \xi_{is}} \delta \xi_{is} + \sum_{i=1}^{n} \frac{\partial U}{\partial x_i} \delta x_i \right) du \, dv = 0 ,$$

so, upon supposing that the variations δx_i are zero at the limits and integrating by parts, we will find that:

$$\text{(1)} \qquad \frac{\partial U}{\partial x_i} - \sum_{h=1}^{n} \frac{d\left(\dfrac{\partial U}{\partial \xi_{ih}}, x_h \right)}{d\,(u, v)} = 0 \qquad (i = 1, 2, \ldots, n) .$$

If we set:

$$\text{(2)} \qquad \frac{\partial U}{\partial \xi_{ih}} = p_{ih} , \qquad p_{i,i} = 0$$

then we will have:

$$\sum_{h=1}^{n} \frac{d\,(p_{ih}, x_h)}{d\,(u,v)} = \frac{\partial U}{\partial x_i}\,.$$

Now let:

$$H = -\,U + \sum p_{ih}\,\xi_{ih}\;\;.$$

Suppose that (2) can be solved with respect to ξ_{ih} . We then find how those quantities can be expressed in terms of the x_1 , …, x_n , the p_{ih} , the u and the v. If we substitute those values in H then we will get:

$$H = H\,(x_1\,,\,\ldots,\,x_n\,,\,p_{ih}\,,\,\ldots,\,u,\,v)\;,$$

so varying that while supposing that u and v are constant will give:

$$\delta H = -\sum \frac{\partial U}{\partial \xi_{ih}}\,\delta\xi_{ih} - \sum_{i=1}^{n} \frac{\partial U}{\partial x_i}\,\delta x_i + \sum p_{ih}\,\delta\xi_{ih} + \sum \xi_{ih}\,\delta\,p_{ih}$$

$$= -\sum_{i=1}^{n} \frac{\partial U}{\partial x_i}\,\delta x_i + \sum \xi_{ih}\,\delta\,p_{ih}\;\;,$$

or

$$\frac{\partial H}{\partial x_i} = -\,\frac{\partial U}{\partial x_i}\,, \qquad\qquad \frac{\partial H}{\partial p_{ih}} = \xi_{ih}\,.$$

The system of equations (1) can then be replaced with this other one:

(I) $$\frac{d\,(x_i, x_h)}{d\,(u,v)} = \frac{\partial H}{\partial p_{ih}}\;\;, \qquad \sum_{h=1}^{n} \frac{d\,(p_{ih}, x_h)}{d\,(u,v)} = -\,\frac{\partial H}{\partial x_i}\,,$$

which has a form that is perfectly analogous to the canonical form that HAMILTON gave to the equations of dynamics.

Now consider the system (I) of differential equations, in which H is an arbitrary function of the p_{ih} , the x_1, x_2, …, x_n, and u and v. One can easily prove the converse theorem to the one that was just proved, viz., that equations (I) can always be associated with a problem in the calculus of variations. Indeed, consider:

$$J = \iint \left(\sum p_{ih}\,\frac{d\,(x_i, x_h)}{d\,(u,v)} - H \right) du\,dv\;\;.$$

In order to have $\delta J = 0$ when one supposes that the δx_i are zero at the limits, one must have equations (I).

2. – In the study that we will now make, we start from the system (I) while supposing that the variables x_i are *three* in number. We assume that the system (I) is such that the unknown functions are defined when we know the values of the x_1, x_2, x_3 on the contour of a region $\mathcal{S}$ in which we suppose that u and v are variable. The region $\mathcal{S}$ in the plane of u, v is bounded by m lines $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m$. The equations of each of those $\mathcal{L}_i$ will be considered to have the forms:

$$u = f_i(t_i), \qquad v = \varphi_i(t_i), \qquad T_i \geq t_i \geq 0,$$

and denote the values of the x_1, x_2, x_3 that are assigned along $\mathcal{L}_i$ the by $\psi_i(t_i)$, $\chi_i(t_i)$, $\theta_i(t_i)$, resp. Those functions, along with the f_i and φ_i , are supposed to be continuous, periodic with a period T_i, and generally differentiable. Assume that the given elements are *characteristic elements* of the unknown functions, at least as long as the lines $\mathcal{L}_i$ and the arbitrary values that are assigned to the x on the contour remain between certain limits.

Let us see how we can consider the integrals of the problem under those hypotheses.

Each of them:

1. Will be a function of the variables u, v .

2. Will depend upon the functions $f_i(t_i)$, $\varphi_i(t_i)$, $\psi_i(t_i)$, $\chi_i(t_i)$, $\theta_i(t_i)$ ([3]).

Consider a five-dimensional space whose points are referred to the Cartesian coordinates y_1, y_2, y_3, y_4, y_5, and in which the lines Λ_i have the equations:

$$y_1 = f_i(t_i), \qquad y_2 = \varphi_i(t_i), \qquad y_3 = \psi_i(t_i), \qquad y_4 = \chi_i(t_i), \qquad y_5 = \theta_i(t_i).$$

The integrals in (I) can be kept as quantities that depend upon the lines $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$, and the two parameters u and v, i.e., upon adopting the notations that have already been used on other occasions, one can write:

$$(4) \qquad x_i = x_i \,|\, [\Lambda_1, \Lambda_2, \ldots, \Lambda_m, u, v] \,|, \qquad (4') \qquad p_{ih} = p_{ih} \,|\, [\Lambda_1, \Lambda_2, \ldots, \Lambda_m, u, v] \,|.$$

In a three-dimensional space whose points have the coordinates x_1, x_2, x_3, consider the lines L_i that have the equations:

$$x_1 = \psi_i(t_i), \qquad x_2 = \chi_i(t_i), \qquad x_2 = \theta_i(t_i).$$

([3]) Rendiconti R. Acc. d. Lincei, vol. III, fasc. 4 [in this volume: XVII, pp. 294-314.].

It is easy to see that the integrals in (I) cannot be regarded as depending separately on the lines $\mathcal{L}_1$, $\mathcal{L}_2, \ldots, \mathcal{L}_m$, $L_1, L_2, \ldots, L_m$, and the parameters u and v (4).

3. – Given that, substitute (4) and (4′) in place of the x_i and p_{ih} in:

$$\text{(II)} \qquad V = \iint_S \left[\sum \frac{\partial H}{\partial p_{ih}} p_{ih} - H\right] du\, dv \,.$$

Let W denote V after performing that substitution. One will obviously have:

$$W = W \mid [\Lambda_1, \Lambda_2, \ldots, \Lambda_m] \mid .$$

It follows from that when one give infinitesimal displacements to the lines $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$, or one varies the functions $f_s(t_i)$, $\varphi_s(t_i)$, $\psi_s(t_i)$, $\chi_s(t_i)$, $\theta_s(t_i)$ by infinitely little, one will find that the variation of W will be expressed by:

$$\text{(5)} \qquad \delta W = \sum_{s=1}^{m} \int_{\Lambda_s} \{(W'_{y_1})_s\, \delta f_s + (W'_{y_2})_s\, \delta\varphi_s + (W'_{y_3})_s\, \delta\psi_s + (W'_{y_4})_s\, \delta\chi_s + (W'_{y_5})_s\, \delta\theta_s\}\, dt_s \,,$$

in which W'_{y_i} is independent of the $\delta f_s, \ldots, \delta\theta_s$, and is what one calls *the derivative of W with respect to y_i relative to* Λ_i (5).

Since W must not change when one displaces Λ_s along itself, one must then have:

$$\text{(6)} \qquad (W'_{y_1})_s \frac{df_s}{dt_s} + (W'_{y_2})_s \frac{d\varphi_s}{dt_s} + (W'_{y_3})_s \frac{d\psi_s}{dt_s} + (W'_{y_4})_s \frac{d\chi_s}{dt_s} + (W'_{y_5})_s \frac{d\theta_s}{dt_s} = 0 \,.$$

One now supposes that one varies the functions $\psi_i(t_i)$, $\chi_i(t_i)$, $\theta_i(t_i)$ by infinitely little, while leaving the $f_i(t_i)$, $\varphi_i(t_i)$ unaltered, i.e., keeping the $\mathcal{L}_i$ unchanged.

Under that hypothesis, one will get:

$$\delta W = \iint_S \left\{\sum \frac{\partial H}{\partial p_{ih}} \delta p_{ih} + \sum \delta \frac{\partial H}{\partial p_{ih}} p_{ih} - \sum \frac{\partial H}{\partial p_{ih}} \delta p_{ih} - \sum \frac{\partial H}{\partial x_i} \delta x_i\right\} du\, dv$$

(4) That is why the x_i and p_{ih} must remain unaltered while displacing an arbitrary L_i along itself, while keeping the corresponding $\mathcal{L}_i$ unchanged.

(5) Rend. R. Acc. Lincei, vol. V, pp. 160 [in this volume: XXIII, pp. 405].

$$= \iint_S \left\{ \sum \delta \frac{\partial H}{\partial p_{ih}} p_{ih} - \sum \frac{\partial H}{\partial x_i} \delta x_i \right\} du\, dv ,$$

or:

$$\delta W = \iint_S \left\{ \sum \delta \frac{d\,(x_i, x_h)}{d\,(u,v)} p_{ih} + \sum_i \sum_h \frac{d\,(p_{ih}, x_h)}{d\,(u,v)} \delta x_i \right\} du\, dv ,$$

because of (I), from which it will follow from a calculation that presents no difficulty that:

$$\delta W = \iint_S \left\{ \frac{\partial}{\partial u} \sum p_{ih} \begin{vmatrix} \delta x_i & \delta x_h \\ \dfrac{\partial x_i}{\partial v} & \dfrac{\partial x_h}{\partial v} \end{vmatrix} - \frac{\partial}{\partial v} \sum p_{ih} \begin{vmatrix} \delta x_i & \delta x_h \\ \dfrac{\partial x_i}{\partial u} & \dfrac{\partial x_h}{\partial u} \end{vmatrix} \right\} du\, dv$$

$$= \sum_{s=1}^{m} \int_{\mathcal{L}_s} \left\{ \sum p_{ih} \begin{vmatrix} \delta x_i & \delta x_h \\ \dfrac{\partial x_i}{\partial v} & \dfrac{\partial x_h}{\partial v} \end{vmatrix} \frac{\partial v}{\partial t_s} + \sum p_{ih} \begin{vmatrix} \delta x_i & \delta x_h \\ \dfrac{\partial x_i}{\partial u} & \dfrac{\partial x_h}{\partial u} \end{vmatrix} \frac{\partial u}{\partial t_s} \right\} dt_s ,$$

so one finally has:

(7) $$\delta W = \sum_{s=1}^{m} \int_{\mathcal{L}_s} \sum p_{ih} \begin{vmatrix} \delta x_i & \delta x_h \\ \dfrac{\partial x_i}{\partial t_s} & \dfrac{\partial x_h}{\partial t_s} \end{vmatrix} dt_s .$$

Let us look for the significance of the determinants:

$$\begin{vmatrix} \delta x_i & \delta x_h \\ \dfrac{\partial x_i}{\partial t_s} & \dfrac{\partial x_h}{\partial t_s} \end{vmatrix} dt_s = \begin{vmatrix} \delta x_i & \delta x_h \\ dx_i & dx_h \end{vmatrix}_s = \Delta_{ih}^{(s)} .$$

To that end, observe that for the infinitesimal displacement that is given to each point of the curve L_s , any element δL_s of the arc of that curve will describe an infinitesimal area $d\sigma_s$. Let n_s denote the normal to that area. The projections of $d\sigma_s$ onto the coordinate planes $x_2\, x_3$, $x_3\, x_1$, $x_1\, x_2$, will be:

$$d\sigma_s \cos (n_s, x_1) , \qquad d\sigma_s \cos (n_s, x_2) , \qquad d\sigma_s \cos (n_s, x_3) ,$$

respectively.

However, if δx_1, δx_2, δx_3 are the components of the displacement along those coordinate axes and dx_1, dx_2, dx_3 are the components of dL_s then one will have that the projections of $d\sigma_s$ onto the coordinate planes will also be given by $\Delta_{23}^{(s)}$, $\Delta_{31}^{(s)}$, $\Delta_{12}^{(s)}$. One will then have:

$$\begin{vmatrix} \delta x_2 & \delta x_3 \\ dx_2 & dx_3 \end{vmatrix}_s = d\sigma_s \cos (n_s, x_1) , \quad \begin{vmatrix} \delta x_3 & \delta x_1 \\ dx_2 & dx_1 \end{vmatrix}_s = d\sigma_s \cos (n_s, x_2) , \quad \begin{vmatrix} \delta x_3 & \delta x_1 \\ dx_1 & dx_1 \end{vmatrix}_s = d\sigma_s \cos (n_s, x_3) .$$

Formula (7) can then be written:

(III) $$\delta W = \sum_{s=1}^{m} \int_{\mathcal{L}_s} (p_{23} \cos n_s\, x_1 + p_{31} \cos n_s\, x_2 + p_{12} \cos n_s\, x_3)\, d\sigma_s \,.$$

5. – Recall formula (7). It can be written:

$$\delta W = \sum_{s=1}^{m} \int_{\mathcal{L}_s} \left\{ \delta x_1 \begin{vmatrix} p_{12} & p_{31} \\ \dfrac{\partial x_3}{\partial t_s} & \dfrac{\partial x_2}{\partial t_s} \end{vmatrix} + \delta x_2 \begin{vmatrix} p_{23} & p_{12} \\ \dfrac{\partial x_1}{\partial t_s} & \dfrac{\partial x_2}{\partial t_s} \end{vmatrix} + \delta x_3 \begin{vmatrix} p_{31} & p_{23} \\ \dfrac{\partial x_3}{\partial t_s} & \dfrac{\partial x_1}{\partial t_s} \end{vmatrix} \right\} dt_s$$

$$= \sum_{s=1}^{m} \int_{\mathcal{L}_s} \left\{ \delta \psi_s \begin{vmatrix} p_{12} & p_{31} \\ \dfrac{d\theta_s}{dt_s} & \dfrac{d\chi_s}{dt_s} \end{vmatrix} + \delta \chi_s \begin{vmatrix} p_{23} & p_{12} \\ \dfrac{d\psi_s}{dt_s} & \dfrac{d\theta_s}{dt_s} \end{vmatrix} + \delta \theta_s \begin{vmatrix} p_{31} & p_{23} \\ \dfrac{d\chi_s}{dt_s} & \dfrac{d\psi_s}{dt_s} \end{vmatrix} \right\} dt_s \,.$$

When one compares that formula with (5), one will find that:

(8) $$(W'_{y_3})_s = \begin{vmatrix} p_{12} & p_{31} \\ \dfrac{d\theta_s}{dt_s} & \dfrac{d\chi_s}{dt_s} \end{vmatrix} , \qquad (W'_{y_4})_s = \begin{vmatrix} p_{23} & p_{12} \\ \dfrac{d\psi_s}{dt_s} & \dfrac{d\theta_s}{dt_s} \end{vmatrix} , \qquad (W'_{y_5})_s = \begin{vmatrix} p_{31} & p_{23} \\ \dfrac{d\chi_s}{dt_s} & \dfrac{d\psi_s}{dt_s} \end{vmatrix} ,$$

so

(9) $$(W'_{y_3})_s = \frac{d\psi_s}{dt_s} + (W'_{y_4})_s \frac{\delta\chi_s}{dt_s} + (W'_{y_5})_s \frac{\delta\theta_s}{dt_s} = 0 \,,$$

and consequently, due to (6), one will have:

(9′) $$(W'_{y_1})_s \frac{\delta f_s}{dt_s} + (W'_{y_2})_s \frac{\delta\varphi_s}{dt_s} = 0 \,.$$

The preceding two equations prove that W will not change when one displaces the lines L_s and $\mathcal{L}_s$ along themselves and independently of each other. That result leads one to state the following theorem:

W is a quantity that depends separately on the lines $L_1, L_2, \dots, L_m, \mathcal{L}_1, \mathcal{L}_2, \dots \mathcal{L}_3$ *. One can then write:*

$$W = W \,|\, [L_1, L_2, \dots, L_m, \mathcal{L}_1, \mathcal{L}_2, \dots \mathcal{L}_3] \,| \,,$$

while adopting the known symbols.

W is not a function of degree one in the lines L_s , but if one takes formula (III) then one will see immediately that when one extends a notation that was used before on another occasion ([6]), one can write:

(16) $$(p_{23})_s = \left(\frac{dW}{d(x_2, x_3)}\right)_s, \qquad (p_{31})_s = \left(\frac{dW}{d(x_3, x_1)}\right)_s, \qquad (p_{12})_s = \left(\frac{dW}{d(x_1, x_2)}\right)_s,$$

in which $(p_{23})_s$, $(p_{31})_s$, $(p_{12})_s$ denote the values of p_{23} , p_{31} , p_{12} for the values of *u* and *v* along the line $\mathcal{L}_s$.

If one sets $\delta\psi_s = \delta\chi_s = \delta\theta_s = 0$ then one will have:

(11) $$\delta W = \sum_{s=1}^{m} \int_{\Lambda_s} \{(W'_{y_1})_s\, \delta f_s + (W'_{y_2})_s\, \delta\varphi_s +\}\, dt_s \ .$$

It follows from (6) that:

(12) $$\frac{(W'_{y_1})_s}{\left(\frac{d\varphi_s}{dt_s}\right)} = -\frac{(W'_{y_2})_s}{\left(\frac{df_s}{dt_s}\right)}.$$

If one calls that ratio M_s then one will have that (11) can be written:

$$\delta W = \sum_{s=1}^{m} \int_{\mathcal{L}_s} M_s \begin{vmatrix} \delta f_s & \delta\varphi_s \\ \frac{df_s}{dt_s} & \frac{df_s}{dt_s} \end{vmatrix} dt_s = \sum_{s=1}^{m} \int_{\mathcal{L}_s} M_s \ d\tau_s \ ,$$

in which $d\tau_s$ denotes the element of area that is described by the element $d\mathcal{L}_s$ under the infinitesimal displacement of the curve $\mathcal{L}_s$. One can write:

(13) $$M_s = \left(\frac{dW}{d(u, v)}\right)_s \ ,$$

while adopting a notation that is analogous to the one that was employed previously.

6. – If one supposes that equations (I) have been integrated then one will get expressions for x_1, x_2, x_3 as functions of *u* and *v* for all values of those variables in the region $\mathcal{S}$. Those functions:

$$x_1 = x_1(u, v) , \qquad x_2 = x_2(u, v) , \qquad x_3 = x_3(u, v)$$

([6]) See Acta Mathematica, vol. XII, pp. 247 [in this volume: XXII, pp. 373]; Rend. R. Acc. Lincei, vol. V, pp. 161 [in this volume: XXIII, pp. 406].

define a surface that is contained in the space (x_1, x_2, x_3) and might be called S, in such a way that any point in a piece of the plane $\mathcal{S}$ will correspond to a point in the surface $\mathcal{S}$, and therefore an arbitrary line $\mathcal{G}$ that is contained in $\mathcal{S}$ will correspond to a line G that is contained S. Call G a line that *corresponds* to $\mathcal{G}$. In particular, the contour lines $\mathcal{L}_1$, $\mathcal{L}_2$, …, $\mathcal{L}_m$ in $\mathcal{S}$ will correspond to the lines L_1, L_2, …, L_m that form a contour in S.

One now varies the lines $\mathcal{L}_1$, $\mathcal{L}_2$, …, $\mathcal{L}_m$ and, at the same time, the L_1, L_2, …, L_m , in such a way that the surface S will change only in size, but not change position in space, which is to say, one varies the $\mathcal{L}_1$, $\mathcal{L}_2$, …, $\mathcal{L}_m$ and chooses the L_1, L_2, …, L_m to be lines on S that correspond to those varied lines. To that end, if one gives the variations δf_s and $\delta\varphi_s$ to f_s and φ_s then one will need to give the variations:

$$\delta\psi_s = \left(\frac{\partial x_1}{\partial u}\right)_s \delta f_s + \left(\frac{\partial x_1}{\partial v}\right)_s \delta\varphi_s , \qquad \delta\chi_s = \left(\frac{\partial x_2}{\partial u}\right)_s \delta f_s + \left(\frac{\partial x_2}{\partial v}\right)_s \delta\varphi_s ,$$

$$\delta\theta_s = \left(\frac{\partial x_3}{\partial u}\right)_s \delta f_s + \left(\frac{\partial x_3}{\partial v}\right)_s \delta\varphi_s$$

to ψ_s , χ_s , θ_s , in which the partial derivatives of x_1, x_2, x_3 with respect to u and v are obtained form (4) and their values are taken at the points of the contour $\mathcal{L}_s$.

Upon applying (5) and (8), it will follow from those relations that the variation the W is subjected to will be:

$$\delta' W = \sum_{s=1}^{m} \int_{\mathcal{L}_s} \left\{ \left[\left(W'_{y_1}\right)_s - \begin{vmatrix} p_{23} & p_{31} & p_{12} \\ \dfrac{\partial\psi_s}{\partial t_s} & \dfrac{\partial\chi_s}{\partial t_s} & \dfrac{\partial\theta_s}{\partial t_s} \\ \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \end{vmatrix} \right] \delta f_s + \left[\left(W'_{y_2}\right)_s - \begin{vmatrix} p_{23} & p_{31} & p_{12} \\ \dfrac{\partial\psi_s}{\partial t_s} & \dfrac{\partial\chi_s}{\partial t_s} & \dfrac{\partial\theta_s}{\partial t_s} \\ \dfrac{\partial x_1}{\partial v} & \dfrac{\partial x_2}{\partial v} & \dfrac{\partial x_3}{\partial v} \end{vmatrix} \right] \delta f_s \right\} dt_s .$$

However:

$$\frac{\partial\psi_s}{\partial t_s} = \frac{\partial x_1}{\partial u}\frac{\partial u}{\partial t_s} + \frac{\partial x_1}{\partial v}\frac{\partial v}{\partial t_s} = \frac{\partial x_1}{\partial u}\frac{df_s}{dt_s} + \frac{\partial x_1}{\partial v}\frac{d\varphi_s}{dt_s} ,$$

$$\frac{\partial\chi_s}{\partial t_s} = \frac{\partial x_2}{\partial u}\frac{\partial u}{\partial t_s} + \frac{\partial x_2}{\partial v}\frac{\partial v}{\partial t_s} = \frac{\partial x_2}{\partial u}\frac{df_s}{dt_s} + \frac{\partial x_2}{\partial v}\frac{d\varphi_s}{dt_s} ,$$

$$\frac{\partial\theta_s}{\partial t_s} = \frac{\partial x_3}{\partial u}\frac{\partial u}{\partial t_s} + \frac{\partial x_3}{\partial v}\frac{\partial v}{\partial t_s} = \frac{\partial x_3}{\partial u}\frac{df_s}{dt_s} + \frac{\partial x_3}{\partial v}\frac{d\varphi_s}{dt_s} ,$$

so

$$\delta' W = \sum_{s=1}^{m} \int_{\mathcal{L}_s} \left\{ M_s + \begin{vmatrix} p_{23} & p_{31} & p_{12} \\ \dfrac{\partial \psi_s}{\partial t_s} & \dfrac{\partial \chi_s}{\partial t_s} & \dfrac{\partial \theta_s}{\partial t_s} \\ \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \end{vmatrix} \right\} \left(\delta f_s \frac{d\varphi_s}{dt_s} - \delta\varphi_s \frac{df_s}{dt_s} \right) dt_s$$

$$= \sum_{s=1}^{m} \int_{\mathcal{L}_s} \left\{ M_s + \begin{vmatrix} p_{23} & p_{31} & p_{12} \\ \dfrac{\partial \psi_s}{\partial t_s} & \dfrac{\partial \chi_s}{\partial t_s} & \dfrac{\partial \theta_s}{\partial t_s} \\ \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \end{vmatrix} \right\} d\tau_s ,$$

in which M_s represents the ratio (13), and $d\tau_s$ is the infinitesimal area that is described by the element $d\mathcal{L}_s$ during the infinitesimal displacement of the curve $\mathcal{L}_s$.

However, one immediately gets from (II) that:

$$\delta' W = \sum_{s=1}^{m} \int_{\mathcal{L}_s} \left[\sum \frac{\partial H}{\partial p_{ih}} p_{ih} - H \right] d\tau_s ,$$

so, since the deformations of the curve $\mathcal{L}_s$ is arbitrary, one will have:

$$M_s + \begin{vmatrix} p_{23} & p_{31} & p_{12} \\ \dfrac{\partial \psi_s}{\partial t_s} & \dfrac{\partial \chi_s}{\partial t_s} & \dfrac{\partial \theta_s}{\partial t_s} \\ \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \end{vmatrix} = \sum \frac{\partial H}{\partial p_{ih}} p_{ih} - H ,$$

or, when one takes into account the first of the relations (I):

$$M_s + \sum \frac{\partial H}{\partial p_{ih}} p_{ih} = \sum \frac{\partial H}{\partial p_{ih}} p_{ih} - H ,$$

so one will finally have:

$$M_s + H = 0 \qquad (s = 1, 2, \ldots, m) .$$

7. – Observe that because of (10) and (13), the preceding equations can be written:

$$\text{(IV)} \qquad 0 = \left(\frac{dW}{d(u,v)}\right)_s + H\left(\left(\frac{dW}{d(x_2,x_3)}\right)_s, \left(\frac{dW}{d(x_3,x_1)}\right)_s, \left(\frac{dW}{d(x_1,x_2)}\right)_s, x_1, x_2, x_3, u, v\right),$$

in which one has substituted the values of p_{23}, p_{31}, p_{12} that are given in (10) in H, and s has the values 1, 2, …, m.

One then has that W, when considered to be dependent upon the lines $\mathcal{L}_1$, $\mathcal{L}_2$, …, $\mathcal{L}_m$, L_1, L_2, …, L_m must satisfy the preceding m differential equations, which are perfectly analogous to the partial differential equations to which one arrives in the theory of ordinary differential equations when they are posed in canonical form ([7]).

8. – The function W of the lines that is obtained in that way is not generally a function of degree one. One now supposes that H is independent of u and v; one can prove the theorem:

If one knows the integrals of the differential equations:

$$\text{(I}'\text{)} \qquad \frac{d(x_i, x_s)}{d(u,v)} = \frac{\partial H}{\partial p_{is}}, \qquad \sum \frac{d(p_{ih}, x_h)}{d(u,v)} = -\frac{\partial H}{\partial x_i} \qquad (i = 1, 2, 3)$$

then one can determine a function of degree one W that satisfies the relation:

$$\text{(IV}'\text{)} \qquad H\left(\frac{\partial W}{\partial(x_2,x_3)}, \frac{\partial W}{\partial(x_3,x_1)}, \frac{\partial W}{\partial(x_1,x_2)}, x_1, x_2, x_3\right) + h = 0,$$

in which h is a constant, and the derivatives of the function W are substituted for the p in H.

We will prefix that with the following lemma:

The integrals of equations (I′) *satisfy the condition that:*

$$H = \text{const.}$$

Indeed, one has:

$$\frac{\partial H}{\partial u} = \sum \frac{\partial H}{\partial p_{ih}} \frac{\partial p_{ih}}{\partial u} + \sum_i \frac{\partial H}{\partial x_i} \frac{\partial x_i}{\partial u} = \sum \frac{d(x_i, x_h)}{d(u,v)} \frac{\partial p_{is}}{\partial u} - \sum_i \sum_h \frac{\partial(p_{ih}, x_h)}{\partial(u,v)} \frac{\partial x_i}{\partial u} = 0.$$

One will find analogously that $\partial H / \partial v = 0$, which proves the lemma.

([7]) See JACOBI's lectures on dynamics, Lecture 19.

Suppose that one has found the integrals:

(14) $x_i = x_i\,(u, v, C, C_1)$ (14′) $p_{is} = p_{is}\,(u, v, C, C_1)$

for equations (I′). When one substitutes them in H, it will reduce to something that is equal to a constant h, so one will have:

$$H\,(p_{23}, p_{31}, p_{12}, x_1, x_2, x_3) = \varphi\,(C, C_1) = h\,.$$

Solve the preceding equation for C_1 and substitute the value that is obtained in (14) and (14′). One will have:

(15) $x_i = x_i\,(u, v, C, h)\,,$ (15′) $p_{is} = p_{is}\,(u, v, C, h)\,.$

Suppose that:

(16) $$\frac{d\,(x_1, x_2, x_3)}{d\,(C, u, v)} \neq 0\,.$$

If one solves (15) for u, v, C and substitutes the values that one obtained in (15′) then one will get:

(17) $$p_{is} = p_{is}\,(x_1, x_2, x_3, h)\,.$$

Substitute those values in H and denote the function that results from the substitution by H'. One will have $H' = H'\,(x_1, x_2, x_3, C, h)$.

If one puts the values (15) in place of x_1, x_2, x_3 then H' will reduce to h identically, so:

$$\sum \frac{\partial H'}{\partial x_i}\frac{\partial x_i}{\partial C} = 0\,, \qquad \sum \frac{\partial H'}{\partial x_i}\frac{\partial x_i}{\partial u} = 0\,, \qquad \sum \frac{\partial H'}{\partial x_i}\frac{\partial x_i}{\partial v} = 0\,,$$

from which it will follow from (16) that $\partial H' / \partial x_i = 0$, so one must have:

(18) $$H' = h$$

identically.

If one substitutes (17) in (I′) then one will find that:

$$-\frac{\partial H}{\partial x_i} = \sum_h \frac{d\,(p_{ih}, x_h)}{d\,(u, v)} = \sum_h \sum_s \frac{\partial p_{ih}}{\partial x_s}\frac{d\,(x_s, x_h)}{d\,(u, v)} = \sum_h \sum_s \frac{\partial p_{ih}}{\partial x_s}\frac{\partial H}{\partial p_{sh}}\,,$$

so

$$\frac{\partial H}{\partial p_{23}}\left(\frac{\partial p_{13}}{\partial x_2}-\frac{\partial p_{12}}{\partial x_3}\right)+\frac{\partial H}{\partial p_{12}}\frac{\partial p_{12}}{\partial x_1}+\frac{\partial H}{\partial p_{13}}\frac{\partial p_{13}}{\partial x_1}+\frac{\partial H}{\partial x_1}=0\,.$$

However, one has $H' = h$, so one will have:

$$\frac{\partial H}{\partial p_{23}}\frac{\partial p_{23}}{\partial x_1}+\frac{\partial H}{\partial p_{31}}\frac{\partial p_{31}}{\partial x_1}+\frac{\partial H}{\partial p_{12}}\frac{\partial p_{12}}{\partial x_1}+\frac{\partial H}{\partial x_1}=0\,,$$

from which it will follow that:

$$\frac{\partial H}{\partial p_{23}}\left(\frac{\partial p_{23}}{\partial x_1}+\frac{\partial p_{31}}{\partial x_2}+\frac{\partial p_{12}}{\partial x_3}\right)=0\,.$$

Analogously, one will have:

$$\frac{\partial H}{\partial p_{31}}\left(\frac{\partial p_{23}}{\partial x_1}+\frac{\partial p_{31}}{\partial x_2}+\frac{\partial p_{12}}{\partial x_3}\right)=0\,,$$

$$\frac{\partial H}{\partial p_{12}}\left(\frac{\partial p_{23}}{\partial x_1}+\frac{\partial p_{31}}{\partial x_2}+\frac{\partial p_{12}}{\partial x_3}\right)=0\,,$$

and therefore:

$$\frac{\partial p_{23}}{\partial x_1}+\frac{\partial p_{31}}{\partial x_2}+\frac{\partial p_{12}}{\partial x_3}=0\,.$$

There will then exist a line function W of degree one whose derivatives are p_{23}, p_{31}, p_{12}, and that will satisfy the condition (18), due to (18).

9. – Let us now move on and prove the converse proposition:

Let W be a line function of degree one in the space of x_1, x_2, x_3 *that satisfies the equation:*

$$\text{(IV}''\text{)}\qquad H\left(\frac{\partial W}{\partial(x_2,x_3)},\frac{\partial W}{\partial(x_3,x_1)},\frac{\partial W}{\partial(x_1,x_2)},x_1,x_2,x_3\right)=h\,,$$

in which h is a constant. Set:

$$\frac{\partial W}{\partial(x_2,x_3)}=p_{23}\,,\qquad \frac{\partial W}{\partial(x_3,x_1)}=p_{31}\,,\qquad \frac{\partial W}{\partial(x_1,x_2)}=p_{12}\,.$$

If one substitutes the given value in the equations:

$$(\mathrm{I}_1) \qquad \frac{d\,(x_i, x_s)}{d\,(u, v)} = \frac{\partial H}{\partial p_{is}},$$

which are compatible, then one will also satisfy the equations:

$$(\mathrm{I}_2) \qquad \sum_h \frac{\partial\,(p_{ih}, x_h)}{\partial\,(u, v)} = -\frac{\partial H}{\partial x_i}.$$

In addition to that, one can prove:

If W depends upon a constant parameter a and one sets $W' = \partial W / \partial a$, $W'' = \partial W / \partial h$ *then* W' *and* W'' *will be two functions of the line L in the space of* x_1, x_2, x_3 . *When one displaces the line L over any of the surfaces:*

$$(19) \qquad x_1 = x_1\,(u, v)\,, \qquad x_2 = x_2\,(u, v)\,, \qquad x_3 = x_3\,(u, v)$$

that are obtained by integrating (I3), *one will have:*

$$(20) \qquad W'\,|\,[L]\,| = \frac{\partial W}{\partial a} = a'\,, \qquad (20') \qquad W''\,|\,[L]\,| = \frac{\partial W}{\partial a} = \iint_\sigma du\,dv + h'\,,$$

in which σ *is the portion of the surface* (19) *that is enclosed by the line L, and* a' *and* h' *are two constants.*

Indeed, when one substitutes the integrals (19) in the p_{ih} in (I_1), one will have:

$$\sum_h \frac{\partial\,(p_{ih}, x_h)}{\partial\,(u, v)} = \sum_h \sum_r \frac{\partial p_{ih}}{\partial x_r} \frac{d\,(x_r, x_h)}{d\,(u, v)} = \sum \sum \frac{d\,(x_r, x_h)}{d\,(u, v)} \left(\frac{\partial p_{ih}}{\partial x_r} + \frac{\partial p_{ri}}{\partial x_h} \right) = \sum \frac{\partial H}{\partial p_{rh}} \frac{\partial p_{rh}}{\partial x_i}\,.$$

However, since one has H = const., one will have:

$$\sum \frac{\partial H}{\partial p_{rh}} \frac{\partial p_{rh}}{\partial x_i} = -\frac{\partial H}{\partial x_i}\,,$$

so

$$\sum_h \frac{\partial\,(p_{ih}, x_h)}{\partial\,(u, v)} = -\frac{\partial H}{\partial x_i}\,.$$

In order to prove (20) and (20), consider two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ that belong to the surface (19) and enclose a portion σ' of that surface between them. From a known formula ([8]), one will have:

$$W' \mid [\mathcal{L}_1] \mid - W' \mid [\mathcal{L}_2] \mid = \int_{\sigma'} \sum \frac{\partial W'}{\partial (x_i, x_s)} \frac{d\,(x_i, x_s)}{d\,(u,v)}\, du\, dv$$

$$= \int_{\sigma'} \sum \frac{\partial H}{\partial p_{is}} \frac{\partial p_{is}}{\partial a}\, du\, dv = \int_{\sigma'} \frac{\partial H}{\partial a}\, du\, dv = 0 \,,$$

$$W'' \mid [\mathcal{L}_1] \mid - W'' \mid [\mathcal{L}_2] \mid = \int_{\sigma'} \sum \frac{\partial W''}{\partial (x_i, x_s)} \frac{d\,(x_i, x_s)}{d\,(u,v)}\, du\, dv$$

$$= \int_{\sigma'} \sum \frac{\partial H}{\partial p_{is}} \frac{\partial p_{is}}{\partial h}\, du\, dv = \int_{\sigma'} \frac{\partial H}{\partial h}\, du\, dv = \int_{\sigma'} du\, dv \;.$$

10. – If $H = \frac{1}{2}(p_{23}^2 + p_{31}^2 + p_{12}^2)$ then equations (I) refer to the problem of the surfaces of minimal area. In that case, formula (III) will give rise to a well-known theorem of GAUSS. Let use interpret the theorems of § **8** and § **9** .

Suppose that we have a double infinitude of lines. All of the ones that start from the points of the contour that lie within an infinitesimal area constitute a small tube that can be called a *filament.*

The theorem in § **8** can be stated in the following way:

The orthogonal trajectories to a system of surfaces of minimal area form a system of filaments with constant section.

The theorem in § **9** gives rise to the proposition:

If a system of filaments with constant section admits orthogonal surfaces then those surfaces will have minimal area.

Those two theorems were given by Prof. PADOVA in his note “Sulla teoria delle coordinate curvilinee” ([9]).

([8]) See Rend. Acc. Lincei, vol. III, 2nd sem., pp. 277 [in this volume: XVIII, pp. 234].
([9]) Rend. Acc. Lincei, vol. IV, pp. 373.