"Theoretisches über neuere optische Beobachtungen zur Relativitätstheorie," Phys. Zeit. **21** (1920), 659-662.

Theoretical remarks on some recent optical observations about the theory of relativity

M. v. Laue (Berlin)

Translated by D. H. Delphenich

1. The calculation of the bending of light by the Sun rests upon the theorem that the propagation of light is represented by null geodetic lines in the general theory of relativity. Up to now, that theorem has not been proved, but was only conjectured by analogy with the special theory of relativity; in that theory, it is well-known. Indeed, the invariant equation of the "light cone":

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = 0$$

represents the forward and backward cones of the null point in the world, and thus, the totality of the geodetic lines (which are straight here), along which any segment will have the magnitude zero, that emanate from it.

Naturally, just as is true in the special theory of relativity, this theorem must be provable from the electrodynamics of empty space in general relativity, as well. Furthermore, one can completely adapt the time-honored way of doing that.

The one quadruple of **Maxwell** equations, in the four-dimensional notation:

$$\left.\begin{aligned} &\frac{\partial \mathfrak{M}_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial \mathfrak{M}_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial \mathfrak{M}_{\gamma\alpha}}{\partial x^\beta} = 0, \\ \text{or} \qquad & \\ &\Delta\text{iv}\, \mathfrak{M}^* = 0, \end{aligned}\right\} \tag{1}$$

actually demands that the six-vector of the field $\mathfrak{M}$ must revert to the four-potential Φ by way of the equations:

$$\left.\begin{aligned} &\mathfrak{M}_{\alpha\beta} = \frac{\partial \Phi_\beta}{\partial x^\alpha} - \frac{\partial \Phi_\alpha}{\partial x^\beta}, \\ \text{or} \qquad & \\ &\boldsymbol{\mathfrak{M}} = \boldsymbol{\mathfrak{Rot}}\, \boldsymbol{\mathfrak{M}}, \end{aligned}\right\} \tag{2}$$

such that (1) emerges from (2) with no further assumptions. However, in order for the potential to be defined completely, one can, moreover, as was known in the older theory, add the further condition ([1]) that:

$$\left.\begin{aligned} &\frac{1}{\sqrt{-g}}\sum_k \frac{\partial}{\partial x^k}(\sqrt{-g}\,\Phi^k) = 0, \\ \text{or} \\ &\mathrm{Div}\,\Phi = 0. \end{aligned}\right\} \tag{3}$$

With the Ansatz (2), we now go into the other quadruple of **Maxwell's** equations, which reads:

$$\left.\begin{aligned} &\frac{1}{\sqrt{-g}}\sum_k \frac{\partial(\sqrt{-g}\,\mathfrak{M}^{hk})}{\partial x^k} = P^h, \\ \text{or} \\ &\Delta\mathrm{iv}\,\mathfrak{M} = P \qquad (P \equiv \text{four-potential}). \end{aligned}\right\} \tag{3a}$$

Instead of the contravariant components of $\mathfrak{M}$, here we introduce the covariant ones, and we then go from the contravariant components of the vectorial divergence that appear on the left-hand side to the covariant ones. We then find that:

$$\Delta\mathrm{iv}^h\,\mathfrak{M} = \frac{1}{\sqrt{-g}}\sum_k \frac{\partial(\sqrt{-g}\,\mathfrak{M}^{hk})}{\partial x^k} = \frac{1}{\sqrt{-g}}\sum_{j,l} \frac{\partial}{\partial x^k}(\sqrt{-g}\,g^{kj}g^{kl}\mathfrak{M}_{jl}),$$

$$\Delta\mathrm{iv}_i\,\mathfrak{M} = \sum_k g_{ik}\Delta\mathrm{iv}^k\mathfrak{M} = \sum_{h,j,k,l}\left[\frac{g_{ik}g_{lj}}{\sqrt{-g}}\frac{\partial}{\partial x^k}(\sqrt{-g}\,g^{kl}\mathfrak{M}_{jl}) + g_{ih}g^{kl}\mathfrak{M}_{jl}\frac{\partial\gamma_{hj}}{\partial x^k}\right].$$

We can perform the sum over h in the first half of the sum that appears on the right-hand side:

$$P_i = \Delta\mathrm{iv}_i\,\mathfrak{M} = \frac{1}{\sqrt{-g}}\sum_{k,l}\frac{\partial}{\partial x^k}(\sqrt{-g}\,g^{kl}\mathfrak{M}_{il}) + \sum_{h,j,k,l} g_{ih}g^{kl}M_{jl}\frac{\partial g^{hj}}{\partial x^k}.$$

We now introduce Φ, from (2), and subtract the equation that emerges from (3):

$$0 = \frac{\partial}{\partial x^i}(\mathrm{Div}\,\Phi) = \frac{1}{\sqrt{-g}}\sum_{k,l}\left[\frac{\partial^2}{\partial x^i\partial x^k}(\sqrt{-g}\,g^{kl}\Phi_l) + \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^i}\cdot\frac{\partial}{\partial x^k}(\sqrt{-g}\,g^{kl}\Phi_l)\right].$$

([1]) All substitutions are performed from 1 to 4, unless some is said to the contrary.

If we let F_i denote an expression that includes Φ and its first derivatives *linearly*, in addition to the g_{ik} and its first derivatives, but nothing else contributes to its form, then we will get:

$$P_i = -\frac{1}{\sqrt{-g}} \sum_{k,l} \frac{\partial^2 \Phi_i}{\partial x^k \partial x^l} g^{kl} + F_i \,. \tag{4}$$

With that, we have converted of the wave equation; in what follows, we shall set $P = 0$, and thus restrict ourselves to charge-free space.

We now carry out the transition from the wave equation to geometrical optics by the same process that **Debye** ([1]) applied to classical electrodynamics. Namely, we set:

$$\Phi_i = A_i \; e^{\sqrt{-1}\,kE} \,, \tag{5}$$

and understand E to mean a scalar function of the four coordinates and k to mean a constant that we will eventually allow to become infinitely large, and we understand that A_i are the components of vector that varies slowly in comparison to the exponential function. In the passage to the limit, we will arrive at infinitely-short waves, and thus to geometrical optics. However, the substitution of the values:

$$\frac{\partial \Phi_i}{\partial x^l} = e^{\sqrt{-1}\,kE} \left\{ \frac{\partial A_i}{\partial x^l} + \sqrt{-1}\,k\,A_i \frac{\partial E}{\partial x^l} \right\},$$

$$\frac{\partial^2 \Phi_i}{\partial x^j \, \partial x^l} = e^{\sqrt{-1}\,kE} \left\{ \frac{\partial^2 A_i}{\partial x^j \, \partial x^l} + \sqrt{-1}\,k \left(\frac{\partial A_i}{\partial x^j} \frac{\partial E}{\partial x^l} + \frac{\partial A_i}{\partial x^l} \frac{\partial E}{\partial x^j} + A_i \frac{\partial^2 E}{\partial x^j \, \partial x^l} \right) - k^2 A_i \frac{\partial E}{\partial x^j} \frac{\partial E}{\partial x^l} \right\}$$

into (4) will show that the some of the terms that are combined into F_i will not contain k at all and some of them will contain only k itself. Only one term is multiplied by k^2, as one sees in the sum that is written down in (4). If we preserve it alone then we will come to the equation:

$$\sum_{i,l} g^{il} \frac{\partial E}{\partial x^i} \frac{\partial E}{\partial x^l} = 0, \tag{6}$$

which should be considered to be a replacement for the equation:

$$\left(\frac{\partial \varepsilon}{\partial x}\right)^2 + \left(\frac{\partial \varepsilon}{\partial y}\right)^2 + \left(\frac{\partial \varepsilon}{\partial z}\right)^2 = \frac{1}{n^2}$$

for the eikonal function ε. *It says that the gradient of E has a null direction.*

The divergence equation (3) adds only a statement about the vector A that it will take form:

([1]) **A. Sommerfeld and J. Runge**, Ann. d. Phys. **35** (1911), 277.

$$\sum_i A^i \frac{\partial E}{\partial x^i} = 0$$

upon passing to the limit of $k = \infty$.

We now differentiate equation (6) with respect to x^m:

$$\sum_{i,l} g^{il}\left(\frac{\partial^2 E}{\partial x^i\,\partial x^m}\frac{\partial E}{\partial x^l}+\frac{\partial E}{\partial x^i}\frac{\partial^2 E}{\partial x^l\,\partial x^m}\right)+\sum_{i.l}\frac{\partial g^{il}}{\partial x^m}\frac{\partial E}{\partial x^h}\frac{\partial E}{\partial x^l} = 0.$$

We then make use of the formula ([1]):

$$\frac{\partial g^{il}}{\partial x^m} = -\sum_j\left(\begin{Bmatrix} j\,m \\ i \end{Bmatrix} g^{jl} + \begin{Bmatrix} j\,m \\ l \end{Bmatrix} g^{ji}\right),$$

and find that:

$$\sum_{i,l} g^{il}\left(\frac{\partial^2 E}{\partial x^i\,\partial x^m}\frac{\partial E}{\partial x^l}+\frac{\partial E}{\partial x^i}\frac{\partial^2 E}{\partial x^l\,\partial x^m}\right) = \sum_j\left(\begin{Bmatrix} j\,m \\ i \end{Bmatrix} g^{jl} + \begin{Bmatrix} j\,m \\ l \end{Bmatrix} g^{ji}\right)\frac{\partial E}{\partial x^i}\frac{\partial E}{\partial x^l}.$$

In both sums, the second half will become the first one when we switch the summation symbols i and l, so:

$$\sum_{i,l} g^{il}\frac{\partial E}{\partial x^l}\frac{\partial^2 E}{\partial x^i\,\partial x^m} = \sum_j \begin{Bmatrix} j\,m \\ i \end{Bmatrix} g^{jl}\frac{\partial E}{\partial x^i}\frac{\partial E}{\partial x^l}\ . \tag{7}$$

We now follow one of the "lines of force" of the four-dimensional vector fields of the gradient of E along the segment $d\tau$ whose contravariant components might be the dx^i. The quotients $\partial x^i\ /\ d\tau$ are then proportional to the contravariant components of the gradients, which are sums:

$$\sum_l g^{il}\frac{\partial \xi}{\partial x^l}.$$

We can then replace these sums with those quotients in (7):

$$\sum_i \frac{\partial^2 E}{\partial x^m\,\partial x^i}\frac{dx^i}{d\tau} = \sum_{i,j}\begin{Bmatrix} j\,m \\ i \end{Bmatrix}\frac{\partial E}{\partial x^j}\frac{dx^i}{d\tau}$$

and then find immediately that:

$$\frac{d}{d\tau}\left(\frac{\partial E}{\partial x^m}\right) = \sum_{i,j}\begin{Bmatrix} j\,m \\ i \end{Bmatrix}\frac{\partial E}{\partial x^j}\frac{dx^i}{d\tau}\ .$$

([1]) **H. Weyl**, *Raum, Zeit, Materie*, 1st ed., pp. 105, eq. 41.

However, this is the known formula for the variation of the components of a covariant vector under parallel displacement ([1]). If we shift a gradient along the "lines of force" then its direction will then be preserved. However, its direction is that of the line of force itself, so the latter will preserve its direction under translation in any case; it is then a geodetic line, and from (6), a null geodetic line, as well.

If we finally argue that the gradient of E represents the propagation of the oscillation that is given by the Ansatz (5) then we will have proved the desired theorem.

2. The theory of redshift is ordinarily presented as follows: From the metric:

$$d\tau^2 = \sum_{i,k} g_{ik}\, dx^i dx^k \qquad (x^4 = t),$$

the ratio of the proper time $d\vartheta$ of a clock in the chosen rest system to the time t will be equal to $\sqrt{g_{44}}$. g_{44} is smaller for the Sun than it is for the Earth, so the proper time $d\vartheta_1$ for a clock on the Sun will be smaller than the proper time $d\vartheta_2$ for one on the Earth, *when one refers both $d\vartheta$ to the same dt.* However, the oscillation numbers will behave like the proper times when they are referred to t.

By contrast, one can object that from the complete arbitrariness in the choice of time variable, one cannot see what the physical meaning of this relationship to the same dt might be. Otherwise stated: Can the light wave on the Earth have the same oscillation number relative to t that it does on the Sum? If it changes its oscillation number under propagation then the argument above will not permit any conclusion whatsoever regarding any physically-verifiable effects.

Moreover, that does, in fact, seem to be the case, because it might be impossible for the law of conservation of oscillation number for a light wave to be true in any reference system. In fact, if there were a system in which it were true then we would only need to carry out the transformation $t' = \varphi(t)$, in which φ is not just a linear function with constant (i.e., independent of x^1, x^2, x^3) coefficients, and we would then have a system in which it was false.

However, the usual theory of redshift does not relate to an arbitrary system at all, but to a very specific system whose metric is:

$$d\tau^2 = V^2\,(dx^4)^2 - \sum_{i,k=1}^{3} \gamma_{ik}\, dx^i dx^k\ ,$$

in which all coefficients are independent of $x^4 = t$.

In order to arrive at sinusoidal oscillations, we now extend the Ansatz (5) by establishing that:

$$E = \alpha\, x^4 + f\,(x^1, x^2, x^3),$$

in which we first understand α to mean a function of x^1, x^2, x^3, so equation (6) will yield:

([1]) *Ibidem*, pp. 101, eq. 36.

$$V^2\,\alpha^2 - \sum_{i,k=1}^{3} \gamma_{ik} \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^k} - x^4 \sum_{i,k} \gamma_{ik} \frac{\partial \alpha}{\partial x^i}\frac{\partial \alpha}{\partial x^k} = 0.$$

This function, which is linear in t, can vanish identically only when the positive-definite quadratic form becomes zero when multiplied by t – i.e., when α, and likewise the oscillation number $2\pi k\alpha$ of the wave, is constant in *this* coordinate system.

Discussion

Hamel: Herr **v. Laue** has shown us that it follows from **Einstein's** theory that:

1. A complete analogy exists between it and geometrical optics for light-fast oscillations.
2. A redshift possibly exists on the Sun.

Please permit me the following question: Is there an exact proof that the redshift actually takes place? From the way that the theory looks at things, a clock on the Sun will be different from one on the Earth, and with no further analysis, one would assume that the light that is emitted by oscillating atoms would be governed by that clock, and with that, the redshift would come about. Is there any Ansatz for proving this rigorously, in the sense of Herrn **v. Laue**?

Einstein: It is a logical weakness in the theory of relativity, in its present state, that yardsticks and clocks must be introduced in a different manner from constructing them as solutions to differential equations. However, as far as the admissibility of the consequences in regard to their relationship to the empirical foundations of the theory is concerned, the consequences that relate to the behavior of rigid bodies and close are the ones that are the best confirmed. Since the emitting atoms are to be regarded as "clocks" in the eyes of the theory, the redshift belongs to the best-confirmed results of the theory.