"Anwendungen der Motorrechnung," Zeit. angew. Math. u. Mech., **4** (1924), 193-213.

Applications of motor algebra.

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The following paper is immediately connected with the thesis "Motorrechnung, ein neues Hilfsmittel der Mechanik" that was published in the previous issue. The two parts of that thesis will be consistently referred to by the notation I and II, resp., with the addition of the number of the section or the equation.

In the choice of examples and the limits of the domain in which they are being dealt with, the driving consideration was not just that the old results should be derived anew. Rather, it was my goal, as well as the actual purpose, to make the explanation and clarification of the new calculation procedures for dealing with force and inertia seem consistent with the methods that are arrived at by other paths. One must therefore avoid going too far into the details. In the first two sections, the equations of motion for a rigid body were developed in sufficient generality that they subsumed the case of a materially extended Foucault pendulum as an example, which itself involves the consideration of the non-uniform equation of translation for the Earth. The third section presents a more fundamental argument that might be of interest in the systematic construction of mechanics. In sections **4** to **6**, I give a sketch of the treatment of the general equilibrium problems of structural mechanics. For spatial systems that are composed of elastic rods in a completely arbitrary way, the equilibrium equations and the general theorems on work done by deformation, etc., will be established for all special cases of the frameworks that are comprised of articulated or stiff nodes with continuous beams for the frame supports in a unified manner. The seventh and eighth sections are concerned with two hydrodynamical problems that mostly come up quite short in the usual presentations, namely, the calculation of the so-called "action" and "reaction" in moving water and the equations of motion for a rigid body in an ideal fluid. The ninth section speaks briefly on the equations of motion of an aircraft, which, as one knows, decompose into two groups under the transition to the consideration of small oscillations in certain cases: viz., the longitudinal equations and the lateral equation. This then allows us to go briefly into the more geometrically oriented question of the three-dimensional subgroups of the general concept of motor in the last section, in which the questions of the statics of rigid bodies come to the foreground.

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[1]) The basic ideas of the present work were already in existence in the year 1912, and were distributed at the time in a provisional version to a small circle of specialists. In many talks and university lectures since then, I have also communicated the individual parts of the theory. The complete elaboration should be dedicated to E. Study on his 60th birthday on 23 March 1922, although the final form has been delayed on various other grounds. The essay might now be devoted to firmly establishing the great fruitfulness of the, unfortunately, much-to-little noticed "Geometrie der Dynamen," by Study.

1. Basic equations for rigid bodies. We shall start with the following basic facts from the mechanics of rigid bodies: When a rigid body moves, at each instant, the totality of all forces that act on it yield a *force motor*, which is also called a *dyname* or a *force screw* $\mathfrak{K}$. The instantaneous velocity state will be determined by a *velocity motor* or *motion screw* $\mathfrak{G}$, whose first vector component is the rotational velocity, while the second one is the translation velocity. Ultimately, the mass inertia will be represented by a special symmetric motor dyadic: the *inertia dyadic* T, whose 36-element schema was already characterized in II.**3**. These three quantities $\mathfrak{K}$, $\mathfrak{G}$, and T are now coupled with each other by a fundamental law that is completely analogous to the simple *lex secunda* of Newton for the "material" point, and which we now write down in the following form:

$$\mathsf{T} \cdot \mathfrak{G} = \mathfrak{J}, \qquad \frac{d\mathfrak{J}}{dt} = \mathfrak{J}. \tag{1}$$

In words:

The product of the inertia dyadic with the velocity motor is called the impulse motor. *The derivative of the impulse motor with respect to time is equal to the force motor.*

The following remark will serve to explain (1): From the explanation that was given in II.**3** for the inertia dyadic, one can also regard its product with $\mathfrak{G}$ by saying that if the rigid body decomposes into mass elements *dm* and each such element with the position vector $\mathfrak{x}$ is "attached" to its "quantity of motion" $\mathfrak{v}$ *dm* as a special motor (rod, rotor) then $\mathfrak{J}$ is the sum of all these elementary motors; i.e., the first vector component is $\mathfrak{J} = \int \mathfrak{v}\, dm$ and the second one is $\mathfrak{J}_o = \int (\mathfrak{r} \times \mathfrak{v})\, dm$. In the same way, the derivative of $\mathfrak{J}$ with respect to time can be interpreted as the motor sum of the mass-times-acceleration products $\mathfrak{w}\, dm$ that are "attached" to the individual mass particles. For the first vector component, this is immediately obvious, since when one differentiates $\int \mathfrak{v}\, dm$ one arrives at $\int \mathfrak{w}\, dm$, precisely. For the second one, one must observe that the derivative of the position vector $\mathfrak{x}$ with respect to time is $\mathfrak{v}$ and that the vector product $\mathfrak{v} \times \mathfrak{v}$ vanishes. One thus has:

$$\frac{d}{dt} \int (\mathfrak{r} \times \mathfrak{v})\, dm = \int (\mathfrak{r} \times \mathfrak{w})\, dm + \int \left(\frac{d\mathfrak{x}}{dt} \times \mathfrak{v} \right) dm = \int (\mathfrak{r} \times \mathfrak{w})\, dm.$$

Therefore, the left-hand side of the second equation in (1) has, in fact, the vector components $\int \mathfrak{w}$ *dm and* $\int (\mathfrak{r} \times \mathfrak{w})\, dm$.

For a material point, the force (velocity, resp.) motor reduces to the force (velocity, resp.) vector, and the inertia dyadic reduces to the mass scalar. Eq. (1) then says: The product of mass with velocity vector, when differentiated with respect to time, is equal to the force vector. In the general case of the finitely-extended rigid body, the Ansatz (1) – whereby one imagines that the $\mathfrak{J}$ that comes from the first equation has been substituted into the second one – is equivalent to two vector equations, one of which is the so-called *center–of-mass theorem* and the other one expresses the *areal theorem;* we shall come back to this later in **2**.

A general scalar relation shall be derived from (1) that will be called the *energy theorem* or the *vis viva equation*. To that end, we introduce the following notations that are connected with the usual terminology: One calls the scalar product of the force motor and the velocity motor the *power*, and one-half the scalar product of the velocity motor and the impulse motor, the *vis viva*, or kinetic energy:

$$\mathfrak{K} \cdot \mathfrak{G} = L, \qquad \tfrac{1}{2}\mathfrak{G} \cdot \mathfrak{J} = E. \tag{2}$$

When one scalar multiplies the second of eq. (1) by $\mathfrak{G}$ then L appears on the right-hand side and the scalar product $\mathfrak{G} \cdot d\mathfrak{J} / dt$ appears on the left-hand side, from which we will prove that it amounts to the derivative of E. Namely, we have:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}\mathfrak{G} \cdot \mathfrak{J}\right) = \tfrac{1}{2}\frac{d\mathfrak{J}}{dt} + \tfrac{1}{2}\mathfrak{J}\frac{d\mathfrak{G}}{dt}, \tag{3}$$

and it is easy to see that the two summands on the right are equal to each other. Then, from the calculation rule that was mentioned at the end of I.**7** and derived in II.**7**, one has the following equation for the motor $\mathfrak{M}$:

$$\frac{d\mathfrak{A}}{dt} = \frac{d'\mathfrak{A}}{dt} + (\mathfrak{G} \times \mathfrak{A}), \tag{4}$$

if d'/dt denotes the "apparent" derivative with respect to time – i.e., the change in time as seen from the system that moves with $\mathfrak{G}$. One thus has:

$$\mathfrak{G}\frac{d\mathfrak{J}}{dt} = \mathfrak{G}\frac{d'\mathfrak{J}}{dt} + \mathfrak{G}\,(\mathfrak{G} \times \mathfrak{A}) = \mathfrak{G}\left(\mathbf{T}\frac{d'\mathfrak{G}}{dt}\right) = \mathfrak{G}\left(\mathbf{T}\frac{d\mathfrak{G}}{dt}\right) = \frac{d\mathfrak{G}}{dt}\;\mathfrak{G}. \tag{5}$$

The first equal sign follows immediately from the application of (4) to $\mathfrak{J}$; the second one is explained when one imagines that $\mathbf{T}$ is constant when considered from the moving body, and, on the other hand, the ternary product vanishes as a result of the commutation rule I (10) and since $\mathfrak{G}\times\mathfrak{G} = 0$. The equality of $d\mathfrak{G} / dt$ and $d'\mathfrak{G} / dt$ is likewise a consequence of $\mathfrak{G}\times\mathfrak{G} = 0$ and the relation (4), while the last equal sign follows from the symmetry of $\mathbf{T}$, according to II (49). The first and last term in (5) are now, in fact, the two components of the derivative of E in (3), such that we have proved the identity:

$$\frac{dE}{dt} = \mathfrak{G}\frac{d\mathfrak{J}}{dt}. \tag{3$'$}$$

The scalar multiplication of the Newtonian equation (1) by $\mathfrak{G}$ then yields:

$$\frac{dE}{dt} = L. \tag{6}$$

In words: *The power exerted by the applied force on the body is, at any instant, equal to the increase in vis viva during a unit time interval.*

For many purposes, it is useful to have the component representations for E, L, and $\mathfrak{J}$. For the sake of clarity, we would thus like to alter the notations from the ones in the general investigations of I and II somewhat. The three components of the resultant force $\mathfrak{K}$ may be called X, Y, Z, while the moment $\mathfrak{K}_o = \mathfrak{M}$ for the reference point o might be called M_x , M_y , M_z ; analogously, let u, v, w be the velocity components of the point o (vector $\mathfrak{v}$) and let ω_x, ω_y, ω_z be those of the rotational velocity $\bar{\omega}$. We denote the inertia and deviation moments by T_x , T_y , T_z (D_x , D_y , D_z , resp.) and the center-of-mass coordinates by a, b, c (vector $\mathfrak{r}$). The expression for the power L then reads:

$$L = Xu + Yv + Zw + M_x\,\omega_x + M_y\,\omega_y + M_z\,\omega_z = \mathfrak{K}\,\mathfrak{v} + \mathfrak{M}\,\bar{\omega}. \tag{7}$$

The inertia dyadic has the schema:

$$\left\{\begin{matrix} m & 0 & 0 & 0 & mc & -mb \\ 0 & m & 0 & -mc & 0 & mx \\ 0 & -mc & mb & mb & -mx & 0 \\ 0 & -mc & mb & T_x & -D_x & -D_y \\ mc & 0 & -ma & -D_x & T_y & -D_z \\ -mb & ma & 0 & -D_y & -D_z & T_z \end{matrix}\right\} \tag{8}$$

The scalar components of the impulse are, from II (14):

$$\left.\begin{aligned} J_1 &= m(u + c\omega_y - b\omega_z), & J_4 &= T_x\omega_x - D_z\omega_y - T_y\omega_z + m(bw - cv), \\ J_2 &= m(v - c\omega_z + a\omega_x), & J_5 &= T_y\omega_y - D_x\omega_z - T_z\omega_x + m(cu - aw), \\ J_3 &= m(w - a\omega_y + b\omega_x), & J_6 &= T_z\omega_z - D_y\omega_x - T_x\omega_y + m(av - bu). \end{aligned}\right\} \tag{9}$$

It is only when the reference point o is the center of mass that the first three impulse components depend upon just the translation velocity and the last three, on just the rotational velocity. The former will be called mu, mv, mw, which are the components of the vector $m\mathfrak{v}$ that is often simply called the "quantity of motion." If one chooses the axis direction, moreover, in such a way that they are principal axes of inertia (free axes) of the body then one also obtains likewise simple expressions for the second group of impulse components, namely, $T_x\,\omega_x$, $T_y\,\omega_y$, $T_z\,\omega_z$.

One obtains the explicit expression for $2E$ from the definition (2) in the form:

$$\left.\begin{aligned}2E = m(u^2+v^2+w^2)+T_x\omega_x^2+T_y\omega_y^2+T_z\omega_z^2 \\ +2m[a(v\omega_z-w\omega_y)+b(w\omega_x-u\omega_z)+c(u\omega_y-v\omega_x)] \\ -2[D_x\omega_y\omega_z+D_y\omega_z\omega_x+D_z\omega_x\omega_y].\end{aligned}\right\} \quad (10)$$

The second group of terms with the common factor of $2m$ drops away when the center of mass is the reference point, while the last group drops out when the coordinate directions are the principal axes. The middle group admits different rearrangements, since it represents the ternary product $2m\,\mathfrak{r}(\mathfrak{v}\times\bar{\omega}) = 2m\,\mathfrak{v}(\bar{\omega}\times\mathfrak{r}) = 2m\,\bar{\omega}(\mathfrak{r}\times\mathfrak{v})$.

2. General forms for the equations of motion. The second equation in (1), in which we imagine that $\mathfrak{J}$ has been substituted into the first one, gives, as we have said, the equation of motion for a rigid body in motor form (so it is equivalent to six scalar equations). In order to obtain the component decomposition for any fixed or moving reference system, one needs only to apply the rules of calculation of motor analysis in a purely schematic way. We next chose an axis cross that is rigidly linked with a moving body, but otherwise arbitrary, and imagine that the component notations that were introduced in **1** refer to this. When we make use of the differentiation rule (4), we obtain from (1):

$$\mathbf{T}\,\frac{d'\mathfrak{G}}{dt} + \mathfrak{G}\times(\mathbf{T}\,\mathfrak{G}) = \mathfrak{K}. \quad (11)$$

The components of the motor $\mathfrak{J}$ that appears in the brackets are already summarized in (9), in such a way that the components of the second summands on the left in (11) follow from I (4) immediately; those of the first one are found immediately from the components of **I**, when one replaces the u, v, w, ω_x, ω_y, ω_z with the derivatives u, v, etc., in each of the 6 expressions (9). It will suffice to write down the first and fourth component equation here, since the other ones emerge by cyclic permutation with no further assumptions:

$$m(u + c\omega_y - b\omega_z) + m\omega_y(w - b\omega_x + a\omega_y) - m\omega_z(v - c\omega_x + a\omega_z) = X, \quad (12a)$$

$$\begin{aligned}T_x\omega_x - D_z\omega_y - D_y\omega_z + m(bw - cv) + (T_z - T_y)\,\omega_y\,\omega_z + D_x(\omega_z^2 - \omega_y^2) \\ \omega_x(D_z\omega_z - D_y\omega_y) + m\omega_x(cw + bv) - mu(c\omega_z + c\omega_y) = M_x\,.\end{aligned} \quad (12b)$$

The former of these two equations makes the genesis suggested above immediately obvious, while in the latter, some contractions in the terms that arise from $\mathfrak{G}\times\mathfrak{J}_o + \mathfrak{G}_o\times\mathfrak{J}$ were carried out. If one chooses the center of mass to be the origin of the coordinates then the left-hand side of (12a) goes to the components of the vector:

$$m\left(\frac{d'\mathfrak{v}}{dt} - \mathfrak{v}\times\bar{\omega}\right) = m\frac{d\mathfrak{v}}{dt},$$

such that (12a) defines the well-known center of mass equation $m\, d\mathfrak{v} / dt = \mathfrak{K}$. From (12b), one achieves the familiar form of the Eulerian equation $T_x \omega_x + (T_z - T_y)\, \omega_y\, \omega_z = M_x$, when one also lets the axis directions coincide with the principal axes, moreover. The general form (12a), (12b) of the equations of motion (their left-hand sides, resp.) was given by K. Heun on the basis of vectorial derivatives, although his result is marred by some errors in calculation [1]).

One can derive (12a) and (12b) in a somewhat different way when one carries out the differentiation in (1) immediately, while bypassing (4):

$$\mathbf{T}\,\frac{d\mathfrak{G}}{dt} + \frac{d\mathbf{T}}{dt}\,\mathfrak{G} = \mathfrak{K}. \tag{13}$$

The first summand gives the actual acceleration terms in the same way as before (since, as was already mentioned in **1**, $d\mathfrak{G} / dt$ is identical with $d'\mathfrak{G} / dt$). One obtains the components of the second one as the product of the dyadic whose elements were written down in II (39) with the velocity motor. One now sees, since (11) must coincide with (13), that the following rule of computation must exist for the dyadic product of the motor and dyadic that was introduced in II.**7**:

$$(\mathfrak{G} \times \mathbf{\Pi})\, \mathfrak{G} = \mathfrak{G} \times (\mathbf{\Pi}\, \mathfrak{G}). \tag{14}$$

Eqs. (12) still do not represent the greatest generality that is either attainable or requisite. In many cases of the motion of rigid bodies – e.g., the Foucault pendulum, the vehicular gyroscope, the gyrocompass – one would like to employ a reference system, such that one of the bodies under scrutiny exhibits independent motion, like the rotating Earth, the moving gyroscope, the rocking ship, etc. We would thus like to assume that the motor $\mathfrak{R}$ determines the relative velocity of the body when compared to an axis cross whose motion is given by the motor $\mathfrak{F}$ of the guiding velocity, such that the absolute velocity is:

$$\mathfrak{G} = \mathfrak{F} + \mathfrak{R}. \tag{15}$$

If we substitute this in (1) and employ (4) then it becomes:

$$\mathbf{T}\frac{d'\mathfrak{F}}{dt} + \mathbf{T}\frac{d'\mathfrak{R}}{dt} + (\mathfrak{F} + \mathfrak{R}) \times (\mathbf{T}\mathfrak{F} + \mathbf{T}\mathfrak{R}) = \mathfrak{K}. \tag{16}$$

However, as a rule, one will assess the change in the guiding velocity, not from the moving body under scrutiny, but from rest space – or, what amounts to the same thing,

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[1]) K. Heun, *Lehrbuch der Mechanik I*, Leipzig, 1906, pp. 271. The signs in the terms $D_x \omega_y^2$ and $D_z\, \omega_x\, \omega_z$ have been inverted there when compared to (12b), and furthermore, some indices in the last terms have been switched. In the Encykl. d. math. Wissensch. IV, Art. 11 (K. Heun), pp. 398, the individual signs of the first group of terms do not coincide with the ones in (12a).

from the moving reference system; i.e., we would like to employ the derivative $d\mathfrak{F} / dt$, in place of $d'\mathfrak{F} / dt$. This requires that one has introduced the product $\mathbf{T}\,(\mathfrak{R} \times \mathfrak{F})$ in (16), according to (4). We then arrange that, at first, only the terms that depend upon relative motion, and then the ones that depend upon the guiding velocity, and finally, the mixed terms ("Coriolis" acceleration) appear then we obtain:

$$\left[\mathbf{T}\frac{d'\mathfrak{R}}{dt} + \mathfrak{R}\times(\mathbf{T}\mathfrak{R})\right] + \left[\mathbf{T}\frac{d\mathfrak{F}}{dt} + \mathfrak{F}\times(\mathbf{T}\mathfrak{F})\right] + [\mathfrak{F}\times(\mathbf{T}\,\mathfrak{R}) + \mathfrak{R}\times(\mathbf{T}\,\mathfrak{F}) - \mathbf{T}\,(\mathfrak{R}\times\mathfrak{F})] = \mathfrak{K}. \tag{17}$$

The writing down of the component equations results in a completely schematic way. For the sake of example, we assume that the axis cross that is fixed in the body coincides with the principal axes of inertia and let the guiding velocity be unchanging (say, the rotational velocity of the Earth). Furthermore, let the components of the relative motion be denoted by u, v, w, ω_x, ω_y, ω_z, where the same symbols with primes refer to the guiding motion. (17) then yields the equations:

$$m[u + wy\, w - wz\, v + \omega'_y w' - \omega'_z v' + 2(\omega'_y w - \omega'_z v)] = X. \tag{17a}$$

$$T_x\,(\omega_x - \omega_y\omega'_z + \omega_z\omega'_y) + (T_x - T_y)(\omega_y + \omega'_y)(\omega_z + \omega'_z) = M_x\,. \tag{17b}$$

If one would like to treat the physical Foucault pendulum (i.e., the oscillation of rigid body that is coupled to the moving Earth at a point) then one would do better to choose the reference point to be the point of suspension, for which $u = v = w = 0$. If the center of mass in the coordinate system, which is still assumed to be referred to the principal axis cross, has the coordinates 0, 0, c, then the term $mc\,(w'\omega'_x - u'\omega'_z)$ is added to the first moment equation, and the term $mc\,(w'\omega'_y - v'\omega'_z)$ to the second one, while the third one remains unchanged. The general Ansatz (17) is entirely suited to the case in which, besides the Earth rotation, the orbital motion of the Earth in the ecliptic is also considered.

3. Foundations of continuum mechanics. For the presentation of the equations of motion for continuously deformable bodies, one basically means an analogous resetting of Newton's axioms: Mass times acceleration equals the force that can be found acting on the volume element. In place of the mass of the "material point" that was considered by Newton, one finds the specific mass, and in place of the resultant force, one finds the specific force that acts on the unit volume, e.g., the specific weight as the length of a vector that is directed vertically downwards. First, Boltzmann, and after him, Hamel [1]) have clearly explained that one does not come to a single axiom in this way, but an assumption that is completely independent of it and must be introduced in some form as a "moment theorem" or "surface theorem." We are now in a position to give this

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[1]) Cf., G. Hamel, Mathem. Annalen, 66, 1908, pp. 350 and L. Boltzmann, Populär-wissenschaft. Schriften, Leipzig, 1905, pp. 298.

derivation in a somewhat simpler and more unified way using the tool of motor analysis, in order to also materially extend the foundations of the mechanics of continua in a direction that can possibly be once more meaningful in the applications.

The concept of velocity motor, and therefore, that of inertia dyadic, loses all meaning for a body whose individual parts are not rigidly coupled to each other. For each point of the body, there is only one velocity vector $\mathfrak{v}$ and one specific mass μ, which equals the limiting value $dm : dV$ of the quotient of mass over volume. Nonetheless, one can define the impulse motor $\mathfrak{I}$ of such a body when one uses the second explanation for $\mathfrak{I}$ that was given in **1**: $\mathfrak{I}$ represents the "motor sum" of the rods (specialized motors) $\mathfrak{v}\ dm$ that are "attached to" the individual mass particles, and accordingly, $d\mathfrak{I}/dt$ (cf., the passage quoted above on this) is the analogous sum taken over $\mathfrak{w}\ dm$, when $\mathfrak{w}$ denotes the acceleration vector of the mass element dm. For the mechanics of continua, this then leads to the following theorem, which can call the "extended" Newton Law:

For every part of an arbitrary body, the derivative of the impulse motor with respect to time equals the force motor,

where this is defined by summing over all external forces and moments that act on the volume elements, as well as the surface elements of the parts of the body. The first of eq. (1) for the rigid body emerges immediately, and the second one can be written as:

$$\frac{d\mathfrak{I}}{dt} = \mathfrak{K}^F + \mathfrak{K}^V, \tag{18}$$

from which, the decomposition into volume and surface forces is likewise proved. The fact that the Ansatz (18) or the one that was given in words above is not attained when one considers vectors instead of motors is the essential content of the converse argument. The most general equation of motion of a deformable body comes about when one applies eq. (18) to a volume element.

We construct the following picture of the volume and surface forces that appear in (18) that is somewhat extended when compared to the usual one: First, let a be an arbitrary point in the interior of the body and let V be a region of space that contains a (Fig. 1). All of the volume forces that act on the points of V then yield the motor $\mathfrak{K}^V$, which might perhaps be reduced to a as a reference point. If V is reduced more and more, while a always remains an interior point, then we assume that there exists a limiting value $\mathfrak{K}^V : V$; it is the "specific force motor" for the point a whose vector components relative to a will be denoted by $\mathfrak{k}$ and $\mathfrak{m}$. As a rule, one is usually only concerned with forces for which the second vector component (referred to a) of this motor vanishes, but there is nothing to prevent the presence of – say – magnetic effects in the form of "specific force moments." In a similar way, the following is true for a point b on the outer surface of the part of the body that was considered in (18): If $\mathfrak{K}^F$ is the resultant force motor for all forces that act on a surface patch F that surrounds b then the limiting value $\mathfrak{K}^F : F$ exists when one concentrates F at the point b. We call this limiting value the *stress motor* for the surface element at *b* that is more rigorously determined by its normal direction. The generalization of the Ansatz that we used is again the same as before. We denote the vector components of the stress motor referred to *b* by $\mathfrak{p}_\nu$ and $\mathfrak{q}_\nu$, where ν refers to the direction of the surface element.

The coexistence of volume and surface forces is possible only when the boundary surface forces that act on a closed boundary surface yield a sum over all pieces that has the same magnitude as the sum of the volume forces that act inside of it. If one pursues this line of reasoning for an oriented tetrahedron (Fig. 2) of coordinate directions then, as is known, one obtains a relation that allows $\mathfrak{p}_\nu$ to be computed from $\mathfrak{p}_x$, $\mathfrak{p}_y$, $\mathfrak{p}_z$, and which reads the same way for the $\mathfrak{q}$ in our case:

$$\left.\begin{aligned} \mathfrak{p}_\nu &= \mathfrak{p}_x \cos(\nu, x) + \mathfrak{p}_y \cos(\nu, y) + \mathfrak{p}_z \cos(\nu, z), \\ \mathfrak{q}_\nu &= \mathfrak{q}_x \cos(\nu, x) + \mathfrak{q}_y \cos(\nu, y) + \mathfrak{q}_z \cos(\nu, z). \end{aligned}\right\} \tag{19}$$

From II (2), this means that $\mathfrak{p}$, as well as $\mathfrak{q}$, defines a vectorial dyadic at each point.

If one now considers a parallelepiped volume element *dx dy dz* (Fig. 3) then the first vector component of (18) delivers the well-known differential equation:

$$\mu\,\mathfrak{w} = \mathfrak{k} + \frac{\partial \mathfrak{p}_x}{\partial x} + \frac{\partial \mathfrak{p}_y}{\partial y} + \frac{\partial \mathfrak{p}_z}{\partial z}. \tag{19a}$$

In the second component equation, the left-hand side drops out because $\int (\mathfrak{r} \times \mathfrak{w})\, dm$ goes to zero with the reduction in volume, using its linear measure. On the right-hand side, along with the terms that are analogous to (19a) that are determined by $\mathfrak{m}$ and $\mathfrak{q}$, there are ones that originate in the equal and opposite (up to terms of higher order) tangential components of $\mathfrak{p}$ on the opposite surface such that one obtains:

$$0 = \mathfrak{m} + \frac{\partial \mathfrak{q}_x}{\partial x} + \frac{\partial \mathfrak{q}_y}{\partial y} + \frac{\partial \mathfrak{q}_z}{\partial z} + (p_{yz} - p_{yz})\,\mathfrak{i}_1 + (p_{zx} - p_{zz})\,\mathfrak{i}_2 + (p_{xy} - p_{yx})\,\mathfrak{i}_3\,, \tag{19b}$$

where $\mathfrak{i}_1$, $\mathfrak{i}_2$, $\mathfrak{i}_3$ refer to the unit vectors in the coordinate directions. If, as usual, one sets the moment quantities $\mathfrak{m}$ and $\mathfrak{q}$ equal to zero from now on then (19b) says that $p_{yz} = p_{yz}$, $p_{zx} = p_{zz}$, $p_{xy} = p_{yx}$, such that the stress dyadic is therefore symmetric. This somewhat unexpected requirement follows casually from the Newton Ansatz when it is extended in the sense of motor algebra. The case in which $\mathfrak{m}$ and $\mathfrak{q}$ do not both vanish, so the stress dyadic loses its symmetry, has occasionally been treated in relation to the quasi-elastic ether theory of A. Brill [1]).

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[1]) A. Brill, *Vorlesungen zur Einführung in die Mechanik raumerfüllender Massen*, Leipzig and Berlin 1909.

4. The elastic rod. One finds a particularly convenient validation for the concept definitions and formulas of motor algebra in the treatment of the equilibrium problem of general elastic framework (ideal frames and frame structures are special cases of these). We begin with an investigation of the individual elastic rod, which we regard as a very slender straight elastic prism, according to the usual assumptions. The loads consist of isolated forces and moments, and are no larger then the corresponding elastic limit of the material. We also assume that for the relation between forces and deformations only linear laws apply and the effects of the individual load components simply superpose; a reaction to the load during the deformation (kinking) shall remain beyond the scope of consideration.

A piece of the rod of length l that is free from external applied forces (Fig. 4) will be in equilibrium by means of the forces and moments that are applied to the two end cross-sections 1 and 2. If we combine the forces X, Y, Z and moments M_x, M_y, M_z that act on 2 into a motor $\mathfrak{S}$ – briefly called a "rod force" in the sequel – then the motor that is applied to 1 must be – $\mathfrak{S}$. We can imagine a rigid body, or, what will also suffice, any axis cross as being rigidly fixed in the rod-ends, whose change in position, rotation and displacement under the transition from the unloaded to the loaded state will likewise be represented by motors $\mathfrak{U}^1$ and $\mathfrak{U}^2$. One may then consider all displacements as being infinitely small, as is also done in the equilibrium problems of elasticity theory, and such changes in position are, just like the velocities of rigid bodies, representable by motors. The first three scalar components of $\mathfrak{U}^1$ then mean the components of the rotation of the left-hand rod end for the chosen axis cross and the other three are components of the displacement, taken at the reference point, which is thought of as rigidly fixed on the rod end. Our first problem is to ascertain the relations between the rod force $\mathfrak{S}$ and the relative displacement of the two rod ends $\mathfrak{U} = \mathfrak{U}^2 - \mathfrak{U}^1$ on the grounds of the well-known equations of elasticity.

We establish the coordinate system in the following way (Fig. 4): We let the longitudinal axis of the rod that links the centroids of the cross-sections be the z-axis, and let the origin be the midpoint of this axis, so the distance to either end is $l / 2$. The x and y-axis, which run through the central cross-section, shall be the principal bending axes of the cross-section. The positive direction of the z-axis runs from 1 to 2, so the sense of direction of the x and y axes shall fulfill the condition that the three axes define a right-hand system. The first three scalar components of $\mathfrak{S}$ are now equal to the three components of the end force applied to 2: $S_1 = X$, $S_2 = Y$, $S_3 = Z$. By comparison, the next three components do not coincide completely with the components M_x, M_y, M_z of the stress moment on 2, but one has $S_4 = M_x - Yl / 2$, $S_5 = M_y + Xl / 2$, and $S_6 = M_z$, since these quantities define the moment of the force system that is defined by X, Y, Z, M_x, M_y, M_z for the chosen reference point at the center of the rod. Now, let u, v, w be the components of the relative displacement that endpoint 2 of the z-axis experiences compared to the endpoint 1 as a result of the load through $\mathfrak{S}$ (i.e., the difference that equalsthe displacement of 2 minus the displacement of 1) and let ϑ_x, ϑ_y, ϑ_z be the analogous components of the relative rotation of an axis cross that is thought of as fixed in the rod-end 2 when compared to the rod-end 1. One then has, analogously, the following relations for the six components of the motor $\mathfrak{U}$ that was introduced above: $U_1 = \vartheta_x$, $U_2 = \vartheta_y$, $U_3 = \vartheta_z$; $U_4 = u - \vartheta_y\, l / 2$, $U_5 = v + \vartheta_x\, l / 2$, $U_6 = w$. We seek the connection between the $S_1, \ldots, S_6$, on the one hand, and the $U_1, \ldots, U_6$, on the other.

The simplest components to account for are the z-components. If E is the elastic modulus and F is the cross-sectional area then $w = Zl / EF$, and thus under this displacement the effect of the longitudinal component Z is exhausted. In a completely analogous way, the moment M_z (torsion moment) acts merely along the longitudinal axis as a rotation that is directly proportional to the moment M_z and length l, inversely proportional to the shear modulus G, and is to be set equal to a cross-sectional magnitude J (which is $\pi d^4/32$, for a circle). One thus has:

$$U_6 = S_3 \frac{l}{EF}, \qquad U_3 = S_6 \frac{l}{GJ}. \tag{20}$$

We now consider, at the same time, the effect of the force Y and the moment M_x, which both provoke a bend of the rod along the x-axis (Fig. 5). The bending moment at a distance z from the left-hand end is $M_x - (l - z)\, Y$, and if J_x refers to the moment of inertia then the bending equations read, with the conditions $y = y' = 0$ for $z = 0$ imposed on its integrals:

$$EJ_x\, y'' = - M_x + (l - z)Y, \quad EJ_x\, y' = - zM_x + z\left(l - \frac{z}{2}\right)Y, \quad EJ_x y = -\frac{z^2}{2} M_x + \frac{z^2}{2}\left(l - \frac{z}{3}\right)Y.$$

If one inserts $z = l$ on the right then y goes to v and y' goes to $-\vartheta_x$, such that:

$$\vartheta_x = \frac{l}{EJ_x}\left(M_x - \frac{l}{2}Y\right), \qquad v = -\frac{l^2}{2EJ_x}M_x + \frac{l^3}{3EJ_x}Y, \qquad v + \frac{l}{2}\vartheta_x = \frac{l^3}{12EJ_x}Y.$$

If one compares this result with the connection that was presented above between U_1 and ϑ_x , U_5 and v, S_2 and Y, S_4 and M_x then one sees that:

$$U_1 = \frac{l}{EJ_x} S_4, \qquad U_5 = \frac{l^3}{12EJ_x} S_2 \tag{21}$$

appear, in addition to (20), as further conditions between forces and displacements. In precisely the same way, one ultimately finds by examining the bending around the y-axis:

$$U_2 = \frac{l}{EJ_y} S_5, \qquad U_4 = \frac{l^3}{12EJ_y} S_1. \tag{22}$$

Eq. (20), (21), (22) deliver the complete connection between the force motor $\mathfrak{S}$ and the motor $\mathfrak{U}$ of the relative displacement of the rod ends. If we introduce a motor dyadic **K** with the component schema:

$$\left\{\begin{matrix} \frac{l}{EJ_x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{l}{EJ_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{l}{GJ} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{l^3}{12EJ_y} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{l^3}{12EJ_x} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{l}{EF} \end{matrix}\right\} \tag{23}$$

then we can write:

$$\mathfrak{U} = \mathfrak{U}^3 - \mathfrak{U}^1 = \mathbf{K} \cdot \mathfrak{S}, \tag{24}$$

and say:

The relative displacement of the ends of the elastic rod is the product of a motor dyadic that is determined by the rod constants whose schema is represented in the chosen reference system by (23) *with the motor of the rod force.*

One sees that the "dyadic of rod elasticity," like the inertia dyadic that was treated in II.**10**, is of completely symmetric type. The reference system that we chose, which recommends itself as the only uniquely-distinguished one from the outset, stands out as the principal cross for the dyadic. From the rules that were given II.**6** and **7**, one can, starting from one's knowledge of the six quantities that appear in (23) as the component schema, also present them for an arbitrary reference system, and thus give a representation of the connection between arbitrary displacements and the load components.

5. The elastic rod-work, equilibrium conditions. We now consider a general "rod-work" (generalization of "framework"), which we summarize as follows: A number of straight elastic rods of the type that was treated in section **4**, which are stressed by isolated forces and moments at arbitrary points, are coupled to each other at their ends into some arrangement by rigid bodies, which call "nodal bodies" and which are likewise subject to external forces and moments (Fig. 6). For the sake of unity, we would like to count any force-free piece of a rod as a special rod and think of the point of application of a force as a nodal body (which is, in reality, non-existent) that accepts the external force and is rigidly coupled with the two partial rod ends that meet there; i.e., it takes part in the displacement and twisting of the continuously-connected ends. By this enumeration, the rod-work consists of s rods and k-nodal bodies, and we let the external forces, which now merely act at the nodes, be given by k motors $\mathfrak{P}^1, \mathfrak{P}^2, \ldots, \mathfrak{P}^k$. We denote the displacements that the individual nodal bodies experience as a result of the various applied forces by $\mathfrak{U}^1, \mathfrak{U}^2, \ldots, \mathfrak{U}^k$. We would now like to not enumerate the individual rods, but characterize them by the two indices that go with the nodes at their ends. (Rods that either flow into an actual node at one end or carry a load can be excluded from consideration.) Let the rod-force that is conceived of as acting on the rod that runs between the nodes ι and κ from the node ι to the other be $\mathfrak{S}^{\iota\kappa}$, and accordingly $\mathfrak{S}^{\iota\kappa} = -\mathfrak{S}^{\kappa\iota}$. Finally, we call the displacement that the end of this rod at the node ι experiences $\mathfrak{B}^{\iota\kappa}$, so the displacement of the second end of the same rod must be $\mathfrak{B}^{\kappa\iota}$. Between $\mathfrak{S}^{\iota\kappa}$ and the difference $\mathfrak{B}^{\iota\kappa} - \mathfrak{B}^{\kappa\iota}$, there exists a relation like the one that was written down in **4**:

$$\mathfrak{B}^{\iota\kappa} - \mathfrak{B}^{\kappa\iota} = \mathbf{K}^{\iota\kappa} \cdot \mathfrak{S}^{\iota\kappa}. \tag{25}$$

In this, $\mathbf{K}^{\iota\kappa}$ means a motor dyadic of the form (23), where the constants l, E, J_x, … are thought of as being set equal to the values that correspond to the rod $\iota\kappa$. One naturally has $\mathbf{K}^{\iota\kappa} = \mathbf{K}^{\kappa\iota}$, and the exchange of the two indices in (25) only reiterates the fact that the rod force motor changes sign under this exchange.

We need a precise explanation for the way that the rod-ends are connected to their nodal bodies. If we generally assume that the connection is a rigid one then the possibility of relative displacements (or twists) of the rods that meet at a node (as they would be for a framework) is eliminated. In order to encompass the most general case, we assume that each rod-end is elastically bound with the nodal bodies; i.e., a linear relation exists between the rod-force $\mathfrak{S}^{\iota\kappa}$ and the relative displacement $\mathfrak{B}^{\iota\kappa} - \mathfrak{U}^{\iota}$, where $\mathfrak{U}^{\iota}$ denotes the displacement motor of the ι^{th} nodal body. For example, it might be the case that for a certain coordinate cross whose starting point lies on a rod end the first three (rotational) components of $\mathfrak{B}^{\iota\kappa} - \mathfrak{U}^{\iota}$ are proportional to the three moment components of $\mathfrak{S}^{\iota\kappa}$, and the last three are the resultant components. If all of the proportionality coefficients are null then we have the case of a rigid coupling; conversely, one obtains loose coupling by increasing the individual coefficients by passing to the limit for the components in question. The ideal framework is characterized by the assumption that the coupling is rigid with respect to displacements and completely loose with respect to rotations. In general, we must assume a linear relation in the form:

$$\mathfrak{B}^{\iota\kappa} - \mathfrak{U}^{\iota} = \mathbf{\Lambda}^{\iota\kappa}\, \mathfrak{S}^{\iota\kappa}, \tag{26}$$

in which $\mathbf{\Lambda}^{\iota\kappa}$ denotes a motor dyadic. The aforementioned assumption of a simple proportionality would correspond to the following schema for $\mathbf{\Lambda}^{\iota\kappa}$:

$$\left\{\begin{matrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_3 \end{matrix}\right\} \tag{27}$$

When the α are all zero and the β take on excessively high values, we have the limiting case of nodes in an ideal framework; if all of the α and β are zero then one is dealing with the case of the stiff coupling, as one has for a supporting structure. In what follows, we will only assume that the schema of $\mathbf{\Lambda}^{\iota\kappa}$ is symmetric, which is certainly true for the case (27) in question.

It should be expressly remarked here that the actual nodal plates or joints must be regarded as "nodal bodies;" however, that is also the case when such things are absent and the "nodal bodies: can be replaced by a purely idealized, geometrically defined axis cross. If we assume that "nodal body" is present in the middle of an elastic rod, simply because a notion of force exists there, this does not mean that there is a coordinate cross by which the two partial rods are rigidly coupled. If such a node has the index ι, while the two partial rods extend to the nodes ι and κ then the following two equations replace (26):

$$\mathfrak{B}^{\iota\kappa} - \mathfrak{U}^{\iota} = 0, \qquad \mathfrak{B}^{\lambda\kappa} - \mathfrak{U}^{\lambda} = 0. \tag{26′}$$

We now go on to specifying the conditions for equilibrium in our general elastic rod-work. The Ansatz is no more tedious or protracted than it is for an ideal framework in the ordinary notation.

First, one must have equilibrium at every node; i.e., the sum of the rod-form motors that start from a nodal body must be equal to the motor of external forces that act upon this node:

$$\sum_{\kappa} \mathfrak{S}^{\iota\kappa} = \mathfrak{P}^{\iota}. \tag{28}$$

The summation of naturally extended over all κ whose incidence with ι corresponds to an actual rod of the framework being present. If one thinks of eq. (28) as being written down for all nodes and added then every rod force appears on the left-hand side twice, and in fact, with opposite signs, such that one obtains the condition:

$$\sum_{\iota=1}^{k} \mathfrak{P}^{\iota} = 0, \tag{29}$$

which shows that the $\mathfrak{P}^{\iota}$ might not perhaps be written down independently of each other. Eq. (28), which gives the equilibrium conditions in the restricted sense, succeeds in determining the rod forces only in very rare cases (viz., so-called static determinacy). In general, one must add the elastic relations that follow from (25) and (26). If one subtracts the analogously defined equation for the opposite index combination from (26) then, since $\mathfrak{S}^{\iota\kappa} = -\,\mathfrak{S}^{\kappa\iota}$, what comes about is:

$$\mathfrak{B}^{\iota\kappa} - \mathfrak{B}^{\kappa\iota} = \mathfrak{U}^{\iota} - \mathfrak{U}^{\kappa} + (\mathbf{\Lambda}^{\iota\kappa} + \mathbf{\Lambda}^{\kappa\iota})\,\mathfrak{S}^{\iota\kappa},$$

and when one combines this with (25):

$$\mathfrak{U}^{\iota} - \mathfrak{U}^{\kappa} = (\mathbf{K}^{\iota\kappa} - \mathbf{\Lambda}^{\iota\kappa} - \mathbf{\Lambda}^{\kappa\iota}) = \mathfrak{S}^{\iota\kappa} = \mathbf{M}^{\iota\kappa}\,\mathfrak{S}^{\iota\kappa}. \tag{30}$$

Here, $\mathbf{M}^{\iota\kappa}$ has been used as an abbreviation for the sum of three motor dyadics that is inside the parentheses.

There are s such equations (30) in all, one for each rod, just as there are k equilibrium conditions (28), namely, one for each nodal body. Both groups of equations collectively solve the problem when perhaps all $\mathfrak{P}^{\iota}$ up to $\mathfrak{P}^{\iota}$ and $\mathfrak{U}^{\iota}$ (e.g., $\mathfrak{U}^{1} = 0$), as well: the $k + s$ motor equations then serve to determine the s unknown rod force motors $\mathfrak{S}^{\iota\kappa}$, the $k - 1$ unknown displacement motors $\mathfrak{U}^{2}$, $\mathfrak{U}^{3}$, …, $\mathfrak{U}^{k}$, and the support reaction $\mathfrak{P}^{\iota}$. However, in order to also include the case of arbitrary support conditions, as well as elastically flexible supports, we add the following equation for each node to eq. (28) and (30):

$$\mathbf{A}^{\iota}\,\mathfrak{U}^{\iota} + \mathbf{B}^{\iota}\,\mathfrak{P}^{\iota} = \mathfrak{G}, \tag{31}$$

in which $\mathbf{A}^{\iota}$, $\mathbf{B}^{\iota}$, and $\mathfrak{G}$ shall denote the given coefficient system (any two dyadics and a motor). $\mathbf{A}$ vanishes for every node at which the external load $\mathfrak{P}$ is given, and $\mathbf{B}$ can perhaps be the identity dyadic, such that eq. (31) reads simply $\mathfrak{P}^{\iota} = \mathfrak{G}$. If $\mathfrak{U}$ is prescribed, in stead of $\mathfrak{P}$, then this means that $\mathbf{B}$ vanishes and $\mathbf{A}$ is set to the identity dyadic. Finally, if one has an elastically flexible support then $\mathfrak{G} = 0$, and from (31), one will have a linear, homogeneous relation between $\mathfrak{U}$ and $\mathfrak{P}$. Naturally, cases in which the individual components of $\mathfrak{U}^{\iota}$ and $\mathfrak{P}^{\iota}$ are given (unknown, resp.) are included in (31) immediately.

In summary, we can then say that the $(2k + s)$ motors of the rod forces $\mathfrak{S}^{\iota\kappa}$, the displacements $\mathfrak{U}^{\iota}$, and the external loads $\mathfrak{P}^{\iota}$ are coupled together by the following $(2k + s)$ linear motor equations:

(a)	k	equilibrium conditions	$\sum_{\kappa}\mathfrak{G}^{\iota\kappa} = \mathfrak{P}^{\iota}$	for each node,
(b)	s	elasticity equations	$\mathfrak{U}^{\iota} - \mathfrak{U}^{\iota} = \mathbf{M}^{\iota\kappa}\,\mathfrak{S}^{\iota\kappa}$	for each rod,
(c)	k	external data	$\mathbf{A}^{\iota}\,\mathfrak{U}^{\iota} + \mathbf{B}^{\iota}\,\mathfrak{P}^{\iota} = \mathfrak{G}$	for each node.

This Ansatz subsumes every imaginable rod-work that can be constructed from straight rods, the continuous beams with arbitrary supports and isolated loads, the ideal framework in space and in the plane with arbitrary static indeterminacy, frameworks with stiff or elastically flexible joints, frame supporting structures of any sort, as well as arbitrary spatial supports when one considers the torsion of rods.

6. Deformation work. reciprocity of displacements. From the Ansatz (*a*), (*b*), (*c*), one can effortlessly derive a series of general statements, through which, the known theorems of the theory of frames first take on their true place and illumination. Next, eq. (*a*) can be written in the form of the "principle of virtual displacements." Namely, if one denotes any motors by $\mathfrak{B}^1$, $\mathfrak{B}^2$, …, $\mathfrak{B}^k$, which we would like to regard as arbitrary displacements of the nodal bodies, then scalar multiplication of the *k* equations (*a*) by the $\mathfrak{B}^\iota$ in question and the subsequent addition of the right-hand sides delivers that work done by this displacement of the rod-work by the external forces, namely, the sum of all products $\mathfrak{P}^\iota\ \mathfrak{B}^\iota$. On the left-hand side, there is a double sum of products $\mathfrak{S}^{\iota\kappa}\ \mathfrak{B}^\iota$, which is taken over all of the number combinations ι, κ that are realized by rods. Any rod-force appears twice in it, and indeed with vanishing sign, once multiplied by the $\mathfrak{B}$ of the first endpoint and then with the $\mathfrak{B}$ of the second endpoint. Thus, one can write for (*a*):

$$\sum_{(\iota,\kappa)} \mathfrak{S}^{\iota\kappa}(\mathfrak{B}^\iota - \mathfrak{B}^\kappa) = \sum_\iota \mathfrak{P}^\iota \mathfrak{B}^\iota \tag{32}$$

in which the summation symbols in parentheses imply that every value combination ι, κ is to be included just once (thus, e.g., only with $\iota < \kappa$). Eq. (32) says:

For any infinitely small displacement of the rod-work the work done by the applied forces equals the elastic work (deformation work) done by the rod forces.

The fact that $\mathfrak{B}$ might also coincide with the actual displacements is self-explanatory; for that reason, the smallness of the **B** must be assumed, since otherwise the left-hand side of (32) could not be spoken of as the deformation work.

An entirely simple step now leads us to the most general form of Maxwell's theorem on the "reciprocity of displacements." One lets $\mathfrak{B}^\iota$ denote those nodal displacements that correspond to an equilibrium state with the external loads $\mathfrak{Q}^\iota$ and the rod forces $\mathfrak{T}^{\iota\kappa}$. The connection between these quantities will then originate with eq. (*a*) and (*b*), of which, we write down the first one in the form that is precisely analogous to (32), with the use of the symbol $\mathfrak{U}$ for the arbitrary displacements:

$$\sum_{(\iota,\kappa)} \mathfrak{T}^{\iota\kappa}(\mathfrak{U}^\iota - \mathfrak{U}^\kappa) = \sum_\iota \mathfrak{Q}^\iota \mathfrak{U}^\iota, \qquad \mathfrak{B}^\iota - \mathfrak{B}^\kappa = \mathbf{M}^{\iota\kappa}\mathfrak{T}^{\iota\kappa}. \tag{33}$$

If one inserts the second of these equations in (32) then one obtains:

$$\sum_{(\iota,\kappa)} \mathfrak{S}^{\iota\kappa}(\mathbf{M}^{\iota\kappa}\mathfrak{T}^{\iota\kappa}) = \sum_\iota \mathfrak{P}^\iota \mathfrak{B}^\iota,$$

and when one introduces the value from (*b*) into the first of eq. (33):

$$\sum_{(\iota,\kappa)} \mathfrak{T}^{\iota\kappa}(\mathbf{M}^{\iota\kappa}\mathfrak{S}^{\iota\kappa}) = \sum_{\iota} \mathfrak{Q}^{\iota}\mathfrak{U}^{\iota} .$$

From theorem II (49), the expressions on the left-hand side are equal when the dyadic **M** is assumed to be symmetric. In this case, one then has:

$$\sum_{\iota} \mathfrak{P}^{\iota}\mathfrak{V}^{\iota} = \sum_{\iota} \mathfrak{Q}^{\iota}\mathfrak{U}^{\iota} ; \tag{34}$$

in words:

If $\mathfrak{P}$ *and* $\mathfrak{Q}$ *are two equilibrium systems of external loads and* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are the displacements of the nodes that they provoke then the work that* $\mathfrak{P}$ *does under the displacement* $\mathfrak{V}$ *equals the work done by* $\mathfrak{Q}$ *under the displacement* $\mathfrak{U}$*.*

This is the generalized Maxwell theorem that one can employ in a well-known way for the calculation of the displacements $\mathfrak{U}$ by a special choice of $\mathfrak{Q}$. As one sees, its true source is in the symmetry of the dyadic **M** that links the rod forces with the relative displacements of the nodes. It is remarkable that eq. (*c*) was not employed in the derivation of (34), so the support conditions have no influence on the validity of (34); naturally, the support reactions are included in $\mathfrak{P}$ and $\mathfrak{Q}$, which then drop out only when they exert no work.

Finally, in order arrive at the analogue of Castigliano's theorem, we would like to specialize the support conditions that enter into (*c*) somewhat. We assume that the 6*k* scalar equations into which (*c*) resolves might be arranged such that for a part of them the coefficients of the displacements vanish, while in the remaining one the coefficients of the force components and the right-hand sides are zero, such that perhaps the former equations determine 6*k*–*m* force components, while the latter ones make the displacement equal to zero for the remaining *m* components. In other words, at the ι^{th} node, let either all six force components be given or let one of them be unknown, but the let the relevant displacement components be zero. We then define the "reduced system of equations (*a*)," when we omit those *m* of the 6*k* component equations that correspond to the unknown forces and vanishing displacements. We obtain an expression for the deformation work from (32) by the substitution of (*b*):

$$A = \tfrac{1}{2}\sum_{\iota} \mathfrak{P}^{\iota}\mathfrak{V}^{\iota} = \tfrac{1}{2}\sum_{(\iota,\kappa)} \mathfrak{S}^{\iota}\mathbf{M}^{\iota\kappa}\mathfrak{S}^{\iota\kappa} . \tag{35}$$

From what we accomplished in II.**10**, the right-hand side is a quadratic form in the 6*s* components of the rod force motors. We have now proved the theorem:

The rod forces $\mathfrak{S}$ that are provoked by the external loads $\mathfrak{P}$ are determined by the fact that they make the deformation work A a minimum with the reduced system of eq. (*a*) *for auxiliary conditions.*

In order to see this, we must only convince ourselves that eq. (*b*) can be regarded as the conditions for an extremum of *A* in the given sense.

How is one to solve the extremum problem that was suggested? One must add to the expression *A*, the 6*k*–*m* components of $\left(\sum_\kappa \mathfrak{S}^{\iota\kappa} - \mathfrak{P}^\iota\right)$ that appear in the reduced system (*a*), when multiplied by undetermined factors $\lambda_1, \lambda_2, \ldots, \lambda_{6k-m}$, and then set the derivatives of each of the 6*s* components of the $\mathfrak{S}^{\iota\kappa}$ equal to zero. The 6*s* equations thus defined, together with the reduced system (*a*), then determine the *S* and λ. For the multipliers λ, one now chooses a better notation with double indices, say, $\lambda_1^\iota, \lambda_2^\iota, \ldots, \lambda_6^\iota$ where each upper or lower index might coincide with the corresponding one in the component of $\mathfrak{P}^\iota$. One now sees immediately that the differentiation of the additional term, for example, with respect to the component $S_3^{1,\,2}$ delivers precisely the expression $\lambda_3^1 - \lambda_3^2$; $S_3^{1,\,2}$ then enters into the third component equation of $\sum \mathfrak{S}^{1\kappa} = \mathfrak{P}^1$ with a positive sign and in the third component equation of $\sum \mathfrak{S}^{2\kappa} = \mathfrak{P}^2$ with a negative sign. The derivative of *A* in (35) with respect to $S_3^{1,\,2}$ is, however, from the remark in II.**10**, equal to the third component of the product $\mathbf{M}^{1,2} \cdot \mathfrak{S}^{1,2}$. Correspondingly, setting the derivative with respect to $S_3^{1,\,2}$ equal to zero leads to the equation:

$$\lambda_3^1 - \lambda_3^2 + (\mathbf{M}^{1,2} \cdot \mathfrak{S}^{1,2}) = 0, \tag{36}$$

and completely corresponding equations arises by differentiation with respect to the remaining components. However, one sees that these 6*s* equations are nothing but the component decomposition of (*b*) when each U_ρ^1 is replaced with $-\lambda_\rho^\iota$. If one therefore eliminates the 6*k*–*m* multipliers from the 6*s* eq. (36) and the reduced system (*a*) then what remain to be determined for the 6*s* components *S* are precisely the same equations that arise from eliminating the displacements from (*a*), (*b*), (*c*). With that, the proof of the general theorem on the minimum deformation work is complete.

Here, we shall not go further into the issue of how one arrives at the analogues of the other forms of Castigliano's theorem and further generalizations of it.

7. Action and reaction in flowing fluids. We consider a rigid body (*A*, in Fig. 7) whose instantaneous velocity is given by the motor $\mathfrak{G}$, and which is in contact with a flowing – i.e., moving – fluid in whatever way. Nothing will be assumed about the mechanical nature of the fluid – e.g., its viscosity, etc. We seek an expression for the force $\mathfrak{K}$ that the fluid exerts on the rigid body or a particular piece *O* of its outer surface. This leads us to the application of Newton's equation to the fluid – while observing the reaction principle for the internal stresses – and a generalized Gauss integral conversion.

In general, the outer surface piece O will not bound any closed region of space, in its own right. We then extend it by a surface F that runs through all of the fluid, and assume that F, together with O, bounds a volume V that is completely filled with a fluid that has a continuous velocity distribution, and in whose interior one therefore also finds no other fixed bodies or free surfaces. All of the spatial integrals that appear in the sequel are taken over this volume V and all boundary surface integrals, over F. Now, let m denote the specific mass of the fluid – which is assumed constant – and let $\mathfrak{P}$ denote the total external forces (gravity) that act on all particles in V. Furthermore, let $\mathfrak{v}$ be the velocity vector at an arbitrary point of V and let $\mathfrak{p}$ be the stress vector at a point of the boundary F. When we "attach" $\mathfrak{v}$ and $\mathfrak{p}$ to the points that they belong to, we make rods of the vectors – and thus, special motors – and then write $\boldsymbol{\mathfrak{v}}$ ($\boldsymbol{\mathfrak{p}}$, resp.) for them. The motor $\boldsymbol{\mathfrak{v}}$ thus has the two vector components $\mathfrak{v}$ and $\mathfrak{r} \times \mathfrak{v}$, if $\mathfrak{r}$ denotes the vector from the reference point to the point with the velocity $\mathfrak{v}$. Newton's equation, when applied to all mass particles in V and integrated over V, immediately delivers:

$$\boldsymbol{\mathfrak{K}} = \boldsymbol{\mathfrak{P}} + \int \boldsymbol{\mathfrak{p}}\, dF - \int \mu \frac{d\boldsymbol{\mathfrak{v}}}{dt} dV \,. \tag{37}$$

The expressions – $\boldsymbol{\mathfrak{K}}$, $\boldsymbol{\mathfrak{P}}$, and $\int \boldsymbol{\mathfrak{p}}\, dF$ are then the forces that the accelerations of the fluid particles are attributed to. We shall now treat a conversion of the last expression on the right, by which one primarily arrives at an examination of the differential processes.

Any fluid point p possesses a velocity $\mathfrak{c}$ relative to the rigid body, which differs from $\mathfrak{v}$ by the guiding velocity vector $\mathfrak{G}_p$. The vector lines of $\mathfrak{c}$ define the "relative streamlines," of which it is certain that they do not go through the body A and the surface O. One lets ds refer to the element of length of such a streamline, and df, to the cross-section of a stream tube that they define then it results from the continuity condition that $c\ df$ is constant along the tube and equal to the flux dQ. Now, since the following differentiation rule is valid:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + c \frac{\partial}{\partial s}$$

when one regards a variable as a function of time and position relative to the body A, and on the other hand, one can write $dV = df \cdot ds$, one then obtains for the integral in (37):

$$\int \mu \frac{d\boldsymbol{\mathfrak{v}}}{dt} dV = \int \mu \frac{\partial \boldsymbol{\mathfrak{v}}}{\partial t} dV + \int \mu c \frac{\partial \boldsymbol{\mathfrak{v}}}{\partial t} ds = \int \mu \frac{\partial \boldsymbol{\mathfrak{v}}}{\partial t} dV + \mu \int_{(F)} \boldsymbol{\mathfrak{v}}\, dQ \,. \tag{37′}$$

The last integral is taken over the entire surface F, so dQ is taken to be positive wherever the fluid leaves it and negative wherever it enters. One can correspondingly think of F as divided into two parts F_1 and F_2, such that any relative streamline begins at a point of F_1 and ends at a point if F_2, and as long as one establishes that the absolute value of dQ is always taken, one can form the expression $\boldsymbol{\mathfrak{R}} – \boldsymbol{\mathfrak{A}}$ in question, where:

$$\mathfrak{R} = \mu\int_{(F_2)} \mathfrak{v}\, dQ\,, \qquad \mathfrak{A} = \mu\int_{(F_1)} \mathfrak{v}\, dQ\,. \tag{38}$$

These two quantities are the ones that one refers to, as a rule, as the "reaction" and "action" of the flowing water; they are motors whose resultant (moment, resp.) components one obtains when one sets $\mathfrak{v}$ in place of $\boldsymbol{\mathfrak{v}}$ in one case under the integral sign and $\mathfrak{x} \times \mathfrak{v}$, in the other. It is expressly emphasized that $\mathfrak{R}$ and $\mathfrak{A}$ are defined by the *absolute* velocity $\boldsymbol{\mathfrak{v}}$, although the *relative* flux $dQ = c\ df$ must then be taken.

The value (37) for $\mathfrak{K}$ now reduces to:

$$\mathfrak{K} = \mathfrak{P} + \int \mathfrak{p}\ dF + \mathfrak{A} - \mathfrak{R}, \tag{39}$$

or the dynamic effect of the flow will be given by the difference "action minus reaction" when the first integral in the right in (37′) vanishes, thus certainly when for each fixed point, the absolute velocity $\boldsymbol{\mathfrak{v}}$ relative to *A* is unchanging when it is evaluated from rest space. For example, this is the case for a rocket (Fig. 8), as long as one makes sure that the absolute exhaust velocity of the gas remains constant; as the enclosing surface *F*, what will serve the purpose most simply here is the plane across the exhaust opening, and *F* coincides with F_2 .

If the rocket does not move in an acceleration-free way then, as a rule, the assumption of constant absolute velocity will not be fulfilled, but rather, the one that the relative velocity $\mathfrak{c}$ does not change at a point that is fixed in *A*. For such cases, one must bring (37) into another form. If we denote the guiding velocity at that point by $\mathfrak{f}$, which is then a motor with the vector components $\mathfrak{G}_p$ and $\mathfrak{x} \times \mathfrak{G}_p$, then the first integral on the right in (37′) decomposes into two of them:

$$\int \mu \frac{\partial \mathfrak{f}}{\partial t} dV + \int \mu \frac{\partial \mathfrak{f}}{\partial t} c dV\,.$$

The first one gives the derivative with respect to time for the impulse that the fluid possesses when it moves with the rigid body. If we call this $\mathfrak{J}$ then we have, in place of (39):

$$\mathfrak{K} = \mathfrak{P} + \int \mathfrak{p}\ dF + \frac{d\mathfrak{J}}{dt} + \mathfrak{A} - \mathfrak{R}, \tag{40}$$

when the relative – but not the absolute – velocity is stationary in the moving body.

As we already mentioned, it is true for (39), as well as (40), that the required invariability of $\boldsymbol{\mathfrak{v}}$ ($\boldsymbol{\mathfrak{c}}$, resp.) is to be assessed in rest space. If the body *A* does not execute a pure translational motion and $\boldsymbol{\mathfrak{v}}$ or $\boldsymbol{\mathfrak{c}}$ remains stationary relative to *A* then the assumption of (39) ((40), resp.) is not fulfilled. Eq. (39) and (40) are then not true for a turbine or a propeller, in general, but only those of their scalar component equations, for which the components of $\mathfrak{v}$ or $\mathfrak{c}$ in question are also stationary when seen from rest space. If one sets using (4):

$$\frac{\partial\mathfrak{v}}{\partial t} = \frac{\partial'\mathfrak{v}}{\partial t} + (\mathfrak{G} \times \mathfrak{v}),$$

then one sees that when only the first part on the right vanishes, an expression:

$$-\,\mathfrak{G} \times \mu \int \mathfrak{v}\, dV$$

must appear in (39), of which, one only lets the resultant component in a boundary integral vary. However, since, for a motor product, one of whose factors is $\mathfrak{G}$, the resultant vector and the moment vector for a point on the axis of $\mathfrak{G}$ that is chosen to be the reference point are both perpendicular to this axis, it follows that:

For relatively stationary $\mathfrak{v}$ *(*$\mathfrak{c}$*, resp.), the first scalar resultant and first moment component is employed in eq. (39) ((40), resp.) when the 1-axis coincides with the instantaneous screw axis of A.*

In fact, for turbines and propellers, only the axial thrust and the moment around the rotational axis can be calculated from the theorem of action and reaction. It is not futile to observe that in the example of a propeller whose advance is not in the axis direction eq. (39) ((40), resp.) must also be corrected in this regard.

The influence of internal viscosity or external motion resistance is not specifically knowable by our Ansatz. However, it might be the case that – perhaps, for a given entry velocity – the end velocity will be different, and furthermore $\mathfrak{p}$ will not be perpendicular to dF, in general.

Theoretical hydromechanics does not care to make note of the foregoing formulas and the concepts of "action" and "reaction," so in the technical literature the derivation is mostly full of ambiguities and – not so seldom – flaws, as well. However, when the mechanical foundations are clarified completely, the separate derivation of the moment components of (39) and (40), in particular, also brings with it some complications that will be lessened by the use of the concept of motor that is appropriate to the problem [1]).

8. Inertia increase of a rigid body in an ideal fluid. We would now like to specialize the Ansatz (37) of the foregoing section for the case in which the fluid is an ideal one and the motion is vortex-free and as a result, without circulation. Moreover, the boundary surface F will be assumed to be a level surface of the potential φ, such that one can assume $\varphi = 0$ along F, with no further restrictions in generality (Sec. **7** and **9**). The potential φ must be determined at all points of A by this condition and the further one that

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[1]) I gave a complete, but somewhat more general, basis for the theorem of action and reaction in § 9 of my Habilitationsschrift: "Theorie der Wasserräder," Leipzig, 1908, and also in Zeitschr. f. Math. u. Phys., 57, 1908, pp. 1 to 120. The special case of a guiding surface O at rest was treated briefly by U. Cisotti, Rendic. Lombard. Ist., ser. II, v. 50, pp. 502 to 515.

the normal derivative $\partial\varphi / \partial n$ must coincide with the normal component of the velocity $\mathfrak{G}_p$ of A on O. If we let $\mathfrak{N}$ denote the unit vector in the normal direction that is "attached" to the point p of O then this velocity component has the value of the scalar product $\mathfrak{G} \cdot \mathfrak{N}$; from I.**3**, since the moment component of $\mathfrak{N}$ relative to p vanishes, one has $\mathfrak{G} \cdot \mathfrak{N} = \mathfrak{G}_p \cdot \mathfrak{N}$.

We now make the Ansatz for the potential φ:

$$\varphi = \mathfrak{U} \cdot \mathfrak{G} = G_1U_4 + G_2U_5 + G_3U_6 + G_4U_1 + G_5U_2 + G_6U_3 , \tag{41}$$

and subject the six scalar functions U to the following conditions:

1. The U must individually satisfy the potential equation $\Delta U = 0$.
2. Each of the U vanish along F.
3. The derivative condition:

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\mathfrak{U}}{\partial n} \cdot \mathbf{G} = \mathfrak{N} \cdot \mathfrak{G}$$

must be satisfied along O, which is certainly the case when the normal derivatives of the six U coincide with the six scalar components of $\mathfrak{N}$ in sequence. We have thus introduced a new type of "motor potential" $\mathfrak{U}$ that will be determined by the following equations in what follows:

$$\Delta\mathfrak{U} = 0 \qquad \text{in } V; \qquad \mathfrak{U} = 0 \quad \text{along } F; \qquad \frac{\partial\mathfrak{U}}{\partial n} = \mathfrak{N}. \tag{42}$$

Naturally, $\mathfrak{U}$ still depends upon time, insofar as the position of O and possibly that F changes in time.

With the help of the motor potential, one can now give the integral in (37) a definite characteristic form in a simple manner. It is known that the velocity vector $\mathfrak{v}$ at any point of V equals grad φ. If we denote the gradients as a motor **Grad** φ that is "attached" to its position then we have $\mathfrak{v}$ = **Grad** φ. One now has the extended Gaussian integral formula:

$$\int \mathbf{Grad}\ \varphi \cdot dV = \int \varphi\,\mathfrak{N}\, dO, \tag{43}$$

if the integral on the right-hand side is taken over the entire boundary of V. The first three scalar components of (43) are then immediately (scalar) Gaussian transformation formulas and one treats the three other ones according to the template:

$$\int\left(y\frac{\partial\varphi}{\partial z} - z\frac{\partial\varphi}{\partial y}\right)dV = \int\left(\frac{\partial(y\varphi)}{\partial z} - \frac{\partial(z\varphi)}{\partial y}\right)dV = \int \varphi[y\cos(\mathfrak{N}, z) - z\cos(\mathfrak{N}, y)]\, dO.$$

If we call $\mathfrak{J}'$ the *total impulse* of the fluid mass that is enclosed within *V*, whose derivative with respect to *t* appears in the right in (37), then we have, from (41) and (43):

$$\mathfrak{J}' = \mu \int \mathfrak{v} \cdot dV = \mu \int \textbf{Grad}\ \varphi \cdot dV = \mu \int (\mathfrak{U} \cdot \mathfrak{G})\ \mathfrak{N}\ dO,$$

in which the last integral can be taken over just *O*, since $\mathfrak{U}$ vanishes along *F* from the second of eq. (42). We now apply the dynamic conversion II (21) to the last integrands: $\mathfrak{N}\ (\mathfrak{U} \cdot \mathfrak{G}) = (\mathfrak{N};\ \mathfrak{U})\ \mathfrak{G}$. One can take the motor $\mathfrak{G}$ out of the integration, and one gets:

$$\mathfrak{J}' = \left[\int \mu(\mathfrak{N};\mathfrak{U})\, dO\right]\mathfrak{G} = \mathbf{T}\ \mathfrak{G} \quad \text{with} \quad \mathbf{T}' = \int \mu\, (\mathfrak{N};\ \mathfrak{U})\ dO = \int \mu\left(\frac{\partial \mathfrak{U}}{\partial n};\mathfrak{U}\right)dO, \tag{44}$$

while using the last of eq. (42). The impulse motor of the fluid proves to be the product of a motor dyadic $\mathbf{T}'$ with the velocity motor $\mathfrak{G}$ of the body *A*, which determines the motion of the fluid. The elements of the dyadic are found immediately from II (19′):

$$\mathbf{T}'_{\iota\kappa} = \int \mu U_\iota \frac{\partial U_\kappa}{\partial n} dO, \qquad \iota,\ \kappa = 1, 2, \ldots, 6. \tag{44′}$$

With regard to the first of eq. (42), the Green formula:

$$\int\left(U_\iota \frac{\partial U_\kappa}{\partial n} - U_\kappa \frac{\partial U_\iota}{\partial n}\right)dO = \int (U_\iota\, \Delta U_\kappa - U_\kappa\, \Delta U_\iota)\ dV = 0$$

yields the symmetry of the nine "inertia dyadics" **T**.

If the rigid body *A* possesses an inertia **T** and moves under the influence of a force $\mathfrak{K}$ that originates in the flow and other forces with the resultant $\mathfrak{P}'$ then one can write its equation of motion:

$$\frac{d}{dt}(\mathbf{T} \cdot \mathfrak{G}) = \mathfrak{P}' + \mathfrak{K},$$

when one brings the last term of $\mathfrak{K}$ into the left-hand side, into the form:

$$\frac{d}{dt}\ [(\mathbf{T} + \mathbf{T}')\ \mathfrak{G}] = \mathfrak{P} + \mathfrak{P}' + \int \mathfrak{p}\ dF. \tag{45}$$

The "dynamical influence" of the fluid flow on the rigid body thus asserts itself completely as an apparent "increase in inertia;" the individual supplementary terms that are added to the elements of the original inertia dyadic are given by (44′). Since **T** is indeed symmetric, but generally does not possess the much more specialized form of the inertia dyadic, one cannot maintain the notion of an "inertia increase" in full detail. For example, there is generally no number m' that can be added to m, etc. We shall not go further into the relations the come about in regard to the principal axes and similar things.

The behavior of motor algebra likewise gives a very simple representation of the kinetic energy E' of the fluid that is contained in V. From Gauss's law, one has:

$$E' = \tfrac{1}{2}\int \mu(\mathrm{grad}\,\varphi)^2\, dV = \tfrac{1}{2}\mu\int \varphi\frac{\partial\varphi}{\partial n}\,dO\,.$$

If one substitutes the value of φ from (41) then one obtains:

$$\left.\begin{aligned} 2E' &= \int \mu(\mathfrak{G}\cdot\mathfrak{U})\left(\mathfrak{G}\cdot\frac{\partial\mathfrak{U}}{\partial n}\right)dO = \int \mu\mathfrak{U}\left(\mathfrak{G}\cdot\frac{\partial\mathfrak{U}}{\partial n}\right)dO \\ &= \mathfrak{G}\cdot\int\mu\left(\mathfrak{U};\frac{\partial\mathfrak{U}}{\partial n}\right)\mathfrak{G}\cdot dO = \mathfrak{G}\cdot\mathfrak{J}', \end{aligned}\right\} \tag{46}$$

i.e., the *vis viva* of the fluid is expressed by its impulse $\mathfrak{J}$ and the velocity motor $\mathfrak{G}$, like that of a rigid body. In other words: One can also employ the concept of "inertia increase" for the kinetic energy of the system that is composed of the rigid body and fluid.

The total energy of the system is equal to that of a rigid body that moves with the velocity $\mathfrak{G}$ *and possesses the inertia dyadic* $\mathbf{T} + \mathbf{T}'$.

The eq. (45) is ordinarily applied to the motion of a rigid body in an infinitely extended fluid. The boundary surface $\varphi = 0$ then lies completely at infinity, and one infers from the general theorems of potential theory that $\mathfrak{p} = 0$ the integral on the right in (45) may thus be omitted. The conceptualization of the equations of motion, as well as the computational derivation, will be simplified essentially by the notion of motor [1]).

9. Motion of an aircraft. Kinematically, an aircraft can be regarded as a rigid body; if its individual parts are more or less elastically flexible then this flexibility affects the distribution of velocity and acceleration only very slightly. We refrain from considering the relative motion of the control units, such as the propeller and motor. In the equation of motion (1), which contains the velocity motor $\mathfrak{G}$ and the inertia dyadic $\mathbf{T}$ on the left-hand side, the following forces enter on the right: The weight $\mathfrak{S}$, the propeller thrust $\mathfrak{P}$, and the aerodynamic forces, which act on the different components (minus $\mathfrak{P}$), such as the airfoil, control surfaces, etc. Of the aerodynamic forces, we may assume that their magnitudes and relative positions depend upon only the velocity motor $\mathfrak{G}$, which is also assessed relative to the aircraft. In other words: The six components of the resultant

[1]) The presentation in the German edition of Lamb, *Hydromechanik*, Leipzig, 1907, pp. 117, *et seq.*, is completely confused. As long as one is dealing with an isolated rigid body, appealing to the Lagrangian equations is entirely unfounded.

aerodynamic force are functions of the six components of $\boldsymbol{\mathfrak{G}}$, when everything is referred to an axis system that is fixed in the aircraft. If we then write down the equation in the form (11) then we have an Ansatz in the form of:

$$\mathbf{T}\,\frac{d'\boldsymbol{\mathfrak{G}}}{dt} + \boldsymbol{\mathfrak{G}}\,(\mathbf{T}\,\boldsymbol{\mathfrak{G}}) = \boldsymbol{\mathfrak{S}} + \boldsymbol{\mathfrak{P}} + \boldsymbol{\mathfrak{K}}, \tag{47}$$

which – as far as it relates to the left-hand side and $\boldsymbol{\mathfrak{K}}$ – includes the six components u, v, w; ω_x, ω_y, ω_z of $\boldsymbol{\mathfrak{G}}$ and their derivatives with respect to time as the variable under a decomposition in a co-moving coordinate system only. We can regard the propeller thrust $\boldsymbol{\mathfrak{P}}$ as given by its relative position and magnitude; however, the weight $\boldsymbol{\mathfrak{S}}$ brings a complication along with it that is given immediately either relative to the aircraft or to rest space. $\boldsymbol{\mathfrak{S}}$ is constant in direction relative to rest space, but the line of action will follow the moving body. Now, since, in general, from (4) one has:

$$\frac{d\boldsymbol{\mathfrak{S}}}{dt} = \frac{d'\boldsymbol{\mathfrak{S}}}{dt} + (\boldsymbol{\mathfrak{G}} \times \boldsymbol{\mathfrak{S}}),$$

then one has the first vector component from this:

$$0 = \frac{d'\mathfrak{S}}{dt} + (\mathfrak{G} \times \mathfrak{S}). \tag{48}$$

If one imagines that the vector $\mathfrak{r}$ from the reference point to the center of mass has unchanging components in our reference system, such that the components 4 to 6 of $\boldsymbol{\mathfrak{S}}$ are given by $\mathfrak{S}_0 = \mathfrak{r} \times \mathfrak{S}$ when the first three are given then one sees that the motor equation (47), together with the vector equation (48), define a complete system of 9 differential equations of first order for the 6 components of $\boldsymbol{\mathfrak{G}}$ and the 3 components of $\boldsymbol{\mathfrak{S}}$. By scalar multiplication of (48) by $\mathfrak{S}$, one easily finds the relation $d'(\mathfrak{S}^2) / dt = 0$, which is naturally known, *a priori*, and which gives the integral $S_1^2 + S_2^2 + S_3^2$ = const., such that one recognizes the integration problem as being one of eighth order.

The so-called "gyroscopic effect" in the rotating parts of the aircraft may also be easily considered to a high degree of approximation. The force on the right in (47) actually acts on the entire system that is composed of the aircraft itself and the propeller. Therefore, the impulse increment for both rigid bodies must also appear on the left-hand side. As we may assume, the rotating parts have an approximately unchanging inertia dyadic $\mathbf{T}'$ relative to the aircraft (this is true precisely only for pure rotating bodies), and an unchanging relative velocity $\boldsymbol{\mathfrak{G}}'$. The first part of the impulse $\mathbf{T}'(\boldsymbol{\mathfrak{G}} + \boldsymbol{\mathfrak{G}}')$ has already been considered. When one calculates $\mathbf{T}$ accordingly, the second one gives an increment $\boldsymbol{\mathfrak{J}}'$ to the impulse, namely, the product of the rotational velocity and the moment of inertia of the rotating parts, as a moment vector in the direction of the propeller axis. One then adds $\mathfrak{G} \times \mathfrak{J}'$ on the left-hand side of (47), which only affects the moment components; the expression $\mathfrak{J}' \times \mathfrak{G}$ on the right-hand side, which comes with the impressed force, represents the "gyroscopic effect" in the usual terminology.

If we choose the reference system to be the principal axes that goes through the center of mass of the aircraft then the component equations of (47) read:

$$\left.\begin{array}{ll} m(u + \omega_y w - \omega_z v) = S_1 + P_1 + K_1, & \cdots \\ T_x \omega_y + (T_z - T_y)\omega_y \omega_z + \omega_y J'_z - \omega_z J'_y = P_4 + K_4, & \cdots \end{array}\right\} \tag{49}$$

Naturally, in this P_4, K_4, … are the components of the propeller thrust and the aerodynamic forces. The three components of (48) are then added to the six equations (49):

$$0 = \dot{S}_1 + \omega_y S_3 - \omega_z S_2 = \dot{S}_2 + \omega_z S_1 - \omega_x S_3 = \dot{S}_3 + \omega_x S_2 - \omega_y S_1 . \tag{50}$$

A stationary motion under which all derivatives with respect to time vanish is possible as a result of (48) only when $\mathfrak{G}$ is vertical, so the motion consists of an arbitrary translation and a rotation around a vertical axis. If we assume that the vertical is a principal axis of inertia (and neglect the gyroscopic effect of the propeller) then (49) shows that the forces must possess a horizontal resultant that goes through the center of mass, is perpendicular to the center of mass velocity, and is equal to the product of this velocity with the mass and the rotational velocity. Under a pure translation, the forces naturally define an equilibrium system. Since $\mathfrak{K}$ depends upon the velocities, it is imperative to know if there are solutions for u, v, w, S_1, S_2, S_3 that satisfy the equations:

$$\mathfrak{S} + \mathfrak{P} + \mathfrak{K} = 0, \tag{51}$$

along with $\omega = 0$, and how they behave. The ratios of the S determine the position of the aircraft with respect to the vertical, while the u, v, w are the magnitudes and positions of the translation vector relative to the aircraft.

If we assume that $\mathfrak{G}^0$ is a possible stationary velocity state, $\mathfrak{K}^0$ is the associated value of the aerodynamics forces, etc., and the actual motion might be a small deviation from stationary, such that in $\mathfrak{G} = \mathfrak{G}^0 + \mathfrak{G}'$, the higher powers of $\mathfrak{G}'$ can be neglected. The system of linear differential equations for $\mathfrak{G}'$ and $\mathfrak{S}'$ is:

$$\left.\begin{array}{l} \mathbf{T}\dfrac{d'\mathfrak{G}}{dt} + \mathfrak{G}' \times (\mathbf{T}\mathfrak{G}^0) + \mathfrak{G}^0 \times (\mathbf{T}\mathfrak{G}') = \mathfrak{S}' + \mathfrak{P}' + \mathfrak{K}', \\ \dfrac{d'\mathfrak{S}}{dt} + (\mathfrak{G}^0 \times \mathfrak{S}') + (\mathfrak{G}' \times \mathfrak{S}^0) = 0. \end{array}\right\} \tag{52}$$

In this, $\mathfrak{K}'$ is a linear homogeneous function of $\mathfrak{G}'$, so it will be represented by a motor dyadic:

$$\mathfrak{K}' = \mathbf{M} \cdot \mathfrak{G}'. \tag{52′}$$

Eqs. (52) and (52′), in which the elements of **M** are considered to be given constants that possibly depend upon $\mathfrak{G}^0$, determine the small oscillations around the stationary state of motion; the component equations are read off immediately. We would like to write them down explicitly for the following special case: The aircraft has a symmetry plane that is vertical to the stationary motion; let this be a pure translation with the components u^0, w^0, and $v^0 = 0$ if the symmetry plane is the *xz*-plane. The system of equations:

$$\left.\begin{array}{lll} m(\dot{u}' + w^0\omega_y) = S_1' + K_1', & m(\dot{v}' + u^0\omega_z' - w^0\omega_x) = S_2' + K_2', & m(\dot{w}' - u^0\omega_y') = S_3' + K_3', \\ T_x\omega_x' = K_4', & T_y\omega_y' = K_5', & T_z\omega_z' = K_6', \\ \dot{S}_1' = -S_3\omega_y', & \dot{S}_2' = -S_3\omega_x' - S_1\omega_z', & \dot{S}_3' = S_1\omega_y' \end{array}\right\} \tag{53}$$

then follows from (52), or also from (49) and (50). The presence of a symmetry plane, in which the stationary velocity vector also falls, has, however, a peculiarity for $\mathfrak{K}'$ (**M**, resp.) in (52) as a consequence. Namely, if the additional motion consists of only a rotation around an axis that is perpendicular to the symmetry plane (which can also lie at infinity) then the additional aerodynamic force certainly has a resultant in the plane or defines a force-couple that lies in the plane. Conversely, if the additional motion is a rotation around an axis that lies in the symmetry plane then the additional force is perpendicular to the plane. Analytically expressed: K_1', K_3', K_5' depend upon only u', w', ω_y', and K_2', K_4', K_6' depend upon only v', ω_x', ω_z'. One sees that eqs. (53) divide into two mutually independent groups. The one, which subsumes the first, third, fifth, seventh, and ninth equation includes only the variables u', w', ω_y', S_1', S_3', while only the variables v', ω_x', ω_z', S_2' appear in the four remaining equations. Recalling the aforementioned general integral, each group of equations defines an integration problem of fourth order. As is known in stability theory, one distinguishes the two parts of the total problem as the theory of longitudinal and transverse oscillations [1]).

10. The special case of three dimensions. Application to statics. The consideration of the decomposition of a six-dimensional motion problem into two three-dimensional ones that was just presented is closely related to the examination of the special case that arises when one rigorously examines individual components of the six motor components and thus comes down to lower-dimensional structures in this way. We would like to make some brief remarks about them, and thus concern ourselves chiefly with the questions of the statics of rigid bodies.

It is first clear that one comes to ordinary vector equations by examining the fourth through the sixth scalar components of any motor. From the standpoint of statics, this means the problem of the equilibrium of a material point. In precisely this way, the

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[1]) The equations are treated without any use of the concept of motor in closer detail regarding the connection between aerodynamic forces and velocity in my reference "Dynamische Probleme der Maschinenlehre," Enzykl. d. math. Wiss., v. IV, article 10. Cf., v. IV, sub-volume **2**, pp. 343, *et seq.*

conservation of the fourth through sixth components alone yields the statics of bodies when one of the points is fixed. Here, however, there are two other three-dimensional special cases of greater interest, namely: the examination of a (scalar) resultant component and the two other moment components, and conversely, two resultant components and the third moment component. In the first case, when we preserve, say, A_1, A_2, and A_6 for any motor $\mathfrak{A}$, statically speaking, we have a plane force system before us, and indeed one with the *xy*-plane as its force plane. A_1, A_2 are the two components of the force and A_6 is the moment of the force referred to a point of the plane itself; in planar statics, one has nothing to do with anything else except for this moment. The second case, which is, in a certain sense, dual to the first one, is the preservation of A_3, A_4, A_5 . In statics, it refers to a system of parallel forces that one can also regard as a system of listed (kotierter) points in the plane or ones that are endowed with masses. A_3 is the magnitude of the force (in the *z*-direction) or the magnitude of the mass distribution of its piercing point with the *xy*-plane, A_4 and A_5 are the components of the moment of the force (static moment of the mass, resp.) when referred to a point of the plane.

It is now interesting to see that both special cases of motors admit an invertible single-value map to the vectors of three-dimensional space, in the sense, that the problems of planar statics and the parallel statics in such planes go to point statics. For the case of the planar force systems, I have the carried out the map on a previous occasion, with the objective of making the problems of statics for spatial force systems accessible to a constructive treatment in a drafting plane [1]). The connection between the "planar" force A_1, A_2, A_3 and a space vector A_1', A_2', A_3' will be mediated here by the equations:

$$A_1 = A_1', \qquad A_2 = A_2', \qquad A_3 = A_3', \tag{54}$$

in which c denotes a reduction line segment that has been chosen once and for all. The geometric relationship between the two structures is expressed in Fig. 10. If we start from the space vector oa then the magnitude and direction of the planar force that is mapped equals the projection oa' of the vector onto the plane, while the line of action A is displaced from the reference point o through the distance $c \tan \varphi$. The fact that the rules of addition carry over under this map emerges immediately from (54).

For the case of parallel systems of forces, a map onto vectors is already given in the basics of the so-called barycentric calculus. Analytically, the relation between the "parallel force" A_3, A_4, A_5, and the space vector A_1'', A_2'', A_3'' is represented by:

$$A_1'' = c\,A_3\,, \qquad A_2'' = A_5\,, \qquad A_3'' = -A_4\,, \tag{55}$$

and the geometric relationship is suggested by Fig. 11. If one again starts with the space vector oa then one obtains the position of the mass point or the line of action of the parallel force when one cuts oa with a plane that is perpendicular to the direction of the force and at a distance c from o, and chooses the magnitude of force or the mass to be the quotients of oa by the distance oa' to the piercing point. The transformation formulas (55) show immediately that the addition of the space vectors and that of parallel forces

[1]) "Graphische Statik räumlicher Kräftesysteme," Zeitschr. f. Math. u. Phys., 64, 1917, pp. 209 to 232.

correspond to each other. Since one can conveniently solve barycentric addition problems constructively when one is dealing with just a few mass points, Runge has sometimes employed the map that was defined by (55) in order to convert problems in spatial vector analysis to planar ones. However, since the constructive methods are, by no means, as well developed as in the case of planar force systems it is often useful to carry out the transition from parallel systems to planar systems, as is defined by the equations:

$$A_4 = c\,A_1, \qquad A_5 = c\,A_2, \qquad c\,A_3 = A_6\,. \tag{56}$$

Geometrically, when one interprets the parallel forces as point masses, this yields the connection by a purely planar construction, as is suggested in Fig. 12: If m is the position of the mass point (piercing point or parallel force) then one obtains the magnitude and direction oa of the mapped planar force when one assigns the angle whose tangent equals $A_3 : c$ to om and regards the line of action A as the antipolar of m relative to the circle around o with radius c, when one makes oa' equal to c and drops the perpendicular to ma' at a'. On the basis of this transition, any planar center-of-mass determination will be achieved by the construction of a force diagram and a funicular polygon.

These relations do not carry over to the problems of kinetics completely. Only the planar motion of a disc defines a self-contained three-dimensional special case of general kinetics (naturally, like the point motion and the rotation of a body around a fixed point). Here, the velocity motor is a rotor with an axis that is perpendicular to the plane, so it belongs to a parallel system, while impulse and force motors fall into the plane. Any problem of planar motion may be completely mapped to a three-dimensional point motion. However, things are different in the case of the motion of a body with a symmetry plane that is yet to be addressed here. If the velocity is a rotor that falls in this plane then this yields an impulse that is perpendicular to the plane; so far, the analogy still works. However, if we now also assume that the force is likewise perpendicular to the plane then impulse does not retain this property. The perturbation acts on the second expression on the left-hand side of the equation of motion: the motor product of two mutually perpendicular rotors lies in the common normal to the two axes, so it is coplanar to each of the two factors, but parallel to none of them. For the aircraft problem that was treated in **9**, the anomaly lies in the fact that one was dealing with small oscillations such that as a result of neglecting the terms of higher dimensions the product of velocity and impulse was not completely valid.