"Motorrechnung, ein neues Hilfsmittel der Mechanik," Zeit. angew. Math. u. Mech., **4** (1924), 155-181.

Motor algebra – a new tool for mechanics.

By R. v. MISES in Berlin [1])

Translated by D. H. Delphenich

Anyone who is concerned with problems in mechanics today that start from the simplest place will scarcely avoid the convenient tool of vector algebra. We only need to go back a few decades from recent times to see that the definition of concepts and, above all, the formulas of vector analysis were so foreign to "applied mathematicians" that textbooks on mechanics dared to make use of it only with great suspicion, in sparse amounts, and after long preparations. A similar tool shall now be developed in what follows that will not, however, I will say in advance, replace the vector algebra or presume to supersede it, but only extend a certain circle of problems. Thus, we will not, by any means, treat any of the various forms in which one can regard the essentials of vector algebra. Whether one should employ the symbolism of the present-day vector algebra, the Grassmann calculus of extensions, quaternions, or matrix algebra in working with directed quantities is a pointless dispute, to which no attention will be paid here. The nucleus of "motor algebra," as we would like to call the new techniques that are connected with a word that was introduced by E. Study, is found, moreover, in the fact that one can advance the manner of thinking that leads from the coordinates or component calculations to the vectors by another step in many problems of mechanics. Just as it is the essential result of operating with vectors that all calculations remain unaffected by the arbitrary choice of coordinate directions, so can one, in many cases, also freely make an arbitrary choice of the coordinate origin when one computes with "motors."

Since one arrives at problems in three-dimensional space in the study of equilibrium and motion, one knows that the composition of arbitrary forces or velocities leads to a quantity that is no longer a simple vector, or a "line-bound" vector like a force with a fixed line of action, but something much more general: a complex of a force and a force-pair, for which one employs different notations: Dyname, screw, line or rod sum, central axis, etc. In the second half of the 19th Century, above all, the Englishman Sir Robert Ball, undertook the elaboration of "screw theory" in the geometric direction and, in Germany, Felix Klein has, with great vigor, exhibited the intuitive appeal, the elementary character, and the great utility of Ball's development in the mechanics of rigid bodies. The ground-breaking, but sadly much-too-little known, work of E. Study on the geometry

[1]) The basic ideas of the present work were already in existence in the year 1912, and were distributed at the time in a provisional version to a small circle of specialists. In many talks and university lectures since then, I have also communicated the individual parts of the theory. The complete elaboration should be dedicated to E. Study on his 60th birthday on 23 March 1922, although the final form has been delayed on various other grounds. The essay might now be devoted to firmly establishing the great fruitfulness of the, unfortunately, much-too-little noticed "Geometrie der Dynamen," by Study.

of dynames represented a very meaningful advance beyond Ball [1]), which appeared in 1903, and, in part, summarized the previously published work of its author into a large, complete system of ideas, and encroached upon a realm of mechanical applications that is still very difficult and remote, to this day. For us, of the elementary results of Study, above all, the most important ones are the fact that, just as the vector is determined by a point-pair (initial and final point of the line segment), the dyname – or motor – can be realized by a line-pair, and the fact that one can define a "geometric addition" of motors on the basis of this geometric representation. I shall now go a step further from Study and introduce, in complete analogy with the two definitions of product in vector algebra, a scalar and a motor product of two motors, which, as will be shown, both possess an immediate and elementary meaning in mechanics. In this way, we obtain the most important hand tool for arriving at the aforementioned goal of complete independence of the coordinate system. The type of calculation that is used here was not included in the work of Study, in which it was replaced by the symbolism of complex "dual" numbers, which relate to our motor analysis in a manner that is similar to the way that quaternions relate to ordinary vector algebra.

So widely distributed today are the knowledge and application of the simplest vector formulas that it is almost customary for one to generally revert to an earlier stage in the utilization of vector analytic tools, namely, in the technical literature. It is wrong to apply the customary vector presentation to the mechanical notions of force, velocity, acceleration, etc., and then, however, to abandon the completely analogous advantage for the advanced notions, such as moment of inertia, stress state, deformation, and similar things, when it would contribute a logical extension of any presentation and definition of concepts. One might find a certain explanation for this state of affairs in the fact that amongst the theoreticians in this domain, a much greater state of disunity prevails than amongst the practitioners of vector algebra, such that the scarcely-substantial argument of whether one should speak of "tensors," "dyadics," or "matrices" will be practically omitted, so the simple and, elevated beyond all debate, basic ideas will crystallize and be appropriated for the practical use. With the new motor algebra, it now also seems that its full utility in mechanics first proves itself when one directs one's attention to the structures of second order, such as the motor dyadic (or motor tensor, motor matrix). With hindsight of the aforementioned state of affairs, in what follows I would not like to assume acquaintance with the ordinary vectorial dyadic algebra, but employ the – perhaps to many readers, welcome – opportunity of giving an entirely skimpy, but I believe, intuitive and easily understood introduction into this domain, before I develop the structures of second order in motor algebra in immediate analogy to them [2]). The motor dyadic and everything connected with it was not considered by Study.

In all of its constructions, the present work subdivides into three sections [3]), of which the first one, in connection with a brief hint of the known basic notions of vector analysis, develops the simplest definitions of addition and multiplication of motors, and thus

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[1]) E. Study, *Geometrie der Dynamen, die Zusammensetzung von Kraften und verwendte Gegenstände der Geometrie.* Leipzig, Teubner, 1903.

[2]) As is well-known, the tensor (dydadic, resp.) algebra was founded by W. Voigt and J. W. Gibbs. However, G. Jaumann first established the role of the dyadic concepts in all of physics in a decisive way. Cf., perhaps, Archiv d. Mathem. u. Phys., Bd. 25, 1916, pp. 33 to 42.

[3]) Due to space limitations, the third section first appears in the following volume.

makes only fleeting contact with the meaning of the definitions for the various problems of mechanics. The second section, as we said, then brings a brief presentation of the main points of the dyadic algebra – i.e., the explanation for the dyadic concepts in the context of the ordinary three-dimensional vector analysis, and then couples it with the analogous developments in the domain of motor algebra. Finally, in the last section, a series of examples of the possible applications of the new concept definitions and formulas will be suggested from general mechanics, the dynamics of rigid bodies, structural mechanics, and hydraulics. I shall not enter into a somewhat outlying analogy that exists between the motor, as a geometric structure that is determined by six scalar quantities, and the so-called "six-vectors" of relativity theory. The well-informed might be satisfied with the fact that the motor appears, in a certain sense, to be the six-vector of real four-dimensional line geometry.

I. First-order motor algebra.

1. Vector; rod and wedge. If one finds the explanation at the summit of the customary presentation of vector algebra that the vector is a "directed quantity" then any reasonable person must recognize that almost nothing is expressed by this. The other definition that one likewise frequently encounters, that the vector is a "triple of numbers" is obviously too broad, since the number of men, women, and children at a location is certainly not a vector. In fact, one can give an analytic definition for a vector, as well as a geometric one; we will be concerned with the analytic one in II, but here we shall first start with the geometric one.

If the positions of two points *a* and *b* are given, and, in addition, one has established which of them is the first one and which of them is second then we would like to say: An ordered point-pair has been given. In the notation, one best represents an ordered point-pair when one draws a line with an arrowhead from one point to the other. The extent or definition of the vector concept will then be achieved by means of the following two theorems:

a) Any ordered point-pair determines a vector.

b) All ordered point-pairs that go to each other under parallel displacement determine the same vector.

From these theorems, one can immediately deduce that all vectors will be representable by point-pairs with one and the same starting point *o* and will then be determined only by the position of the endpoint. The manifold of vectors is then, like the points in space, a three-fold, and in the plane, a two-fold. If one takes *o* to be the origin of a Cartesian coordinate system then one can consider the three coordinates of the end point as the determining data of the vector, which are called its "components," etc., However, one can also take the direction and magnitude of the connecting line that is common to all point pairs that represent the same vector as its determining data. How one further develops all of the theorems and formulas of vector algebra on the basis of the definition that was given here is not the purpose of our efforts here; we will come back to one point or the other.

In the mechanics of rigid bodies, one learns of the concept of force, which is determined by its direction, magnitude, and the position of its line of action. One cares to call the force a "bound" or "line-bound" vector, but we, along with Study, will more briefly say that a force is a "rod." The rod will be defined in a similar way to the vector, namely, by the theorems:

a) Any directed point-pair determines a rod.

b) All directed point-pairs that go to each other under displacement along their connecting lines determine the same rod.

One derives from this, with no further assumptions, that all rods, except for certain exceptions, can be represented by point-pairs whose starting points lie in a fixed, arbitrarily chosen plane (Fig. 1); only the rods that are parallel to this plane are then omitted. If one takes the plane to be the *xy*-plane of a Cartesian coordinate system then one can regard the two coordinates of the starting point and the three coordinates of the endpoint to be the determining data of the rod. The manifold of rods in space is then a five-fold (this numbering will not be impaired by the rods that are parallel to the plane); the manifold of all rods that lie in a plane proves to be a three-fold by the same way of looking at things. However, an essential difference between the rod and the vector lies in the fact that one cannot give five determining data here with the same applicability as the three components of vectors. The aforementioned five coordinates obviously do not give us that, and when one takes, say, the distance *ab* and then looks for four quantities that determine the position of a line (complete with a sense of direction) in space, one then has only shifted the complications into the choice of suitable line coordinates. The difficulty is based in the nature of things and also finds its expression in the fact that there is no sort of addition for rods, as there is for vectors; we will come back to this later.

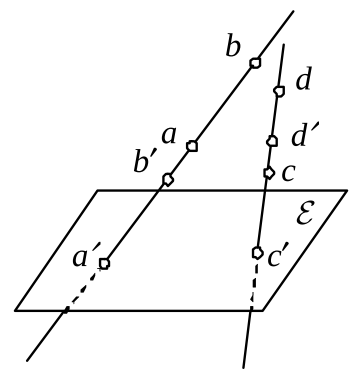

Figure 1

Study juxtaposed the concept of rod with one that was dual to it in the sense of geometry, that of "wedge," which is indeed of only theoretical interest, but due to its intuitive and elementary character, it will be mentioned here. Whenever two intersecting planes α, β (Fig. 2) are given and one of them is characterized as the "first" one, one can just as well speak of an ordered point-pair as one can of an "ordered plane-pair." One defines a wedge by the rules:

a) Any ordered plane-pair determines a wedge.

b) All ordered plane-pairs that go to each other by a rotation around their line of intersection determine the same wedge.

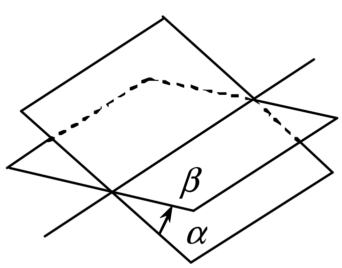

Figure 2

One easily recognizes that the manifold of wedges is likewise a five-fold. The wedge can perhaps be determined by the line that is its carrier and the "opening," namely, the angle between the planes (including the sense of rotation), as measured in some suitable manner. Study showed how one could associate the rods and wedges in space with each other in a one-to-one manner such that rods and wedges are interchangeable with each other. – In the mechanics of rigid bodies, along with force, the angular velocity also takes the form of a rod. The presentation of force may be so closely linked with the rod, namely, a point-pair of fixed distance bound to a fixed line that, on the other hand, it seems intuitive that angular velocity should be represented as a wedge: Its carrier gives the rotational axis, while the opening and sense of rotation from the first plane to the second one give the measure of the angular velocity. We must stop short of pursuing this behavior further here.

2. Introduction of the concept of motor. After these preparations, it is no longer difficult to define a "motor." Two lines *A*, *B* that are not parallel always possess one and only one common normal *N* that intersects both of them (Fig. 3), which shall be called the "axis" of the line pair *A*, *B*. If *A* and *B* lie in a plane then *N* is the altitude to the plane that is erected at the intersection point of *A*, *B*. If one displaces *A* and *B* parallel to itself through each line segments along the axis, namely, in such a way that the intersection of *A*, *B* slides along this axis, or if one rotates *A* and *B* through the same angle around the axis, then one always again obtains line pairs with the same axis. One can express this as follows:

If a line pair is screwed around its initial axis then it continually preserves that axis.

This theorem is the analogue of the following, entirely self-explanatory, one about a point-pair:

If a point-pair is parallel displaced along its original connecting line then that line continually remains the connecting line.

An entirely corresponding theorem on the rotation of a plane pair does need to be expressed explicitly.

We now add that a line-pair shall again be "ordered" when one of the two lines is characterized as the "first" one, so we can define a "motor" by the following two rules:

a) Any ordered line-pair determines a motor.

b) All ordered line-pairs that go to each other by screwing around their axis determine the same motor.

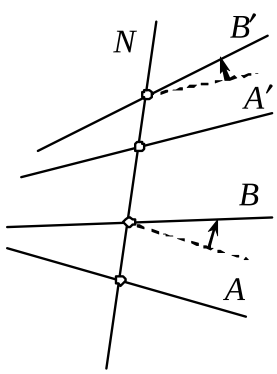

Figure 3

In Fig. 3, the line-pairs *A*, *B* and *A′*, *B′* represent the same motor if *N* is the common normal to *A*, *B*, as well as *A′*, *B′*, when the separation between *A* and *B*, as measured along *N*, is equal and equally-directed to that of *A′* and *B′*, and finally, when the angle and sense of rotation from *A* to *B* agrees with those of *A′* and *B′*. The motor will thus be determined by six pieces of data: the distance from the first line to the second one, their angle, and the four data that determine the axis. The manifold of motors in space is a six-fold. One knows from mechanics that the manifold of force systems and the state of angular velocity of a rigid body is likewise a six-fold, and we will see later that these two quantities, which are fundamental to the study of motion are "motors," just as the individual force is a rod and the velocity of a point is a vector. Study chose the term "motor" due to this interpretation and completely in accord with the word "vector." As will be explained in more detail, for a motor that represents a screw velocity the separation *AB* – we call it the "length" of the motor – is a measure of the translation and the tangent of the angle AB – we call it the "opening" of the motor – is a measure of the angular velocity, while naturally the "motor axis" – i.e., the axis of the line-pair A, B – likewise defines the screw axis.

In the defining rules, the case in which the two lines of the pair are parallel was not excluded, despite the fact that two parallel lines do not possess a uniquely defined axis. We must therefore extend our assignment and do it in the sense that any common normal to two parallel lines shall serve as their axis. According to theorem b), two parallel pairs A, B and A', B' then determine the same motor when the distance AB is equal and equally-directed to the distance $A'B'$. Then to such four lines (Fig. 4) there is one (possibly lying at infinity) common normal that intersects all four of them, and in order for this to be true, one must be able to screw AB into the position $A'B'$. We then refer to any of the ∞^2 lines that are parallel to the common normal as the *axis* of the motor represented by a parallel pair, while the motor that is represented by non-parallel lines possesses a uniquely defined axis. Where it is necessary to make a distinction in what follows, we will refer to a motor with many-to-one axes as "imaginary."

The manifold of line pairs that represent one and the same motor is a two-fold. Two simply counted mutually independent motions are then permissible: sliding along a fixed line and rotation around it. For an imaginary motor, this manifold, as one easily recognizes, is a three-fold. In any case, one has the important theorem:

Any line that intersects the axis of a motor perpendicularly can be chosen to be the starting line of a given line-pair that represents a motor.

It does not need to be expressly stated that when we speak of a screw here the exceptional cases of pure rotation or pure sliding shall be included and the translation or sliding motion takes the form of a rotation around an infinitely distant axis.

3. Geometric addition of motors. The known behavior of vector addition can be formulated in the following way (can be divided into two steps, resp.). Let two vectors be given by the associated point-pairs $a_1\, a_2$ and $b_1\, b_2$. One now represents:

1. The vectors by two pairs with the common starting point o; let them be oa and ob. One then looks for:

2. The fourth point c that is the opposite corner to the point o that is one of three points o, a, b of a parallelogram; oc then represents the vector sum.

In complete analogy with this, we define, as Study did [1]):

Let two motors be given by the ordered line pairs $A_1\, A_2$ and $B_1\, B_2$. One represents them by:

1. *Two line-pairs OA, OB with a common starting line O.*

One then looks for:

2. *The fourth point c in an arbitrary plane perpendicular to O that is the opposite corner to o in a parallelogram that includes the three piercing points o, a, and b of O, A, and B.*

[1]) *Geometrie der Dynamen*, pp. 54.

The locus of the point c is a line C, and the motor that is represented by OC is called the sum *of the given motors* (Fig. 5).

The line O cannot be chosen arbitrarily (this is different from the case of vector addition), since otherwise, from the theorem at the conclusion of the previous section, the axes of the given motors would intersect perpendicularly (for ideal motors, an axis of each motor, resp.) Such a choice is always possible, because there is at least one common normal to two arbitrary lines. It now remains to be proved that the locus of the point c, which is to be sought, according to the prescription of the definition, actually defines a line. In order to see this without much bother, we imagine that O is chosen to be the z-axis of an ordinary coordinate system and that the lines A and B are given by linear equations that are solved for x (y, resp.), so perhaps A is given by $x = \alpha_1 + \alpha_2 x$, $y = \alpha_3 + \alpha_4 z$ and B by $x = \beta_1 + \beta_2 x$, $y = \beta_3 + \beta_4 z$. If one substitutes a fixed value for z then this gives equations for the x and y coordinates of the points a and b in each case, while the point c, by construction and by its known properties, possesses a vector sum whose coordinates are:

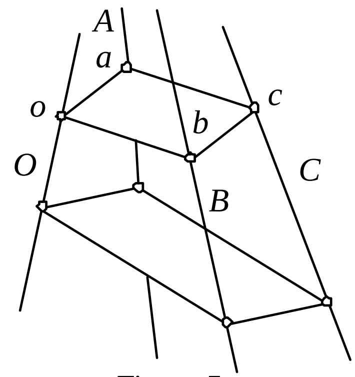

Figure 5.

$$x = \alpha_1 + \beta_1 + (\alpha_2 + \beta_2)\, x, \qquad y = \alpha_3 + \beta_3 + (\alpha_4 + \beta_4)\, z.$$

Since these are again linear equations for varying z the locus of c is, in fact, a line C.

The commutability of the summands is given immediately from our definition. Other simple theorems, line the arbitrary combinability of the summands in a sum of several motors, the relationships to multiplication by a whole number, the degeneracies in the special case of ideal motors, intersecting line-pairs, etc, may be deduced without much effort by further pursuing the construction of Fig. 5. We shall not go into this, since all of these results will become obvious in a later context. Whoever would work in the world of geometric constructions will, however, do well to work through each of these lines of reasoning.

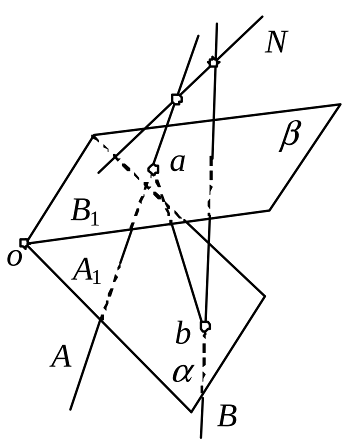

Figure 6.

4. Moment of a motor. In the final analysis, the geometric theory of the composition of forces in the plane rests upon a simple planimetric theorem that goes back to Varignon, the founder of planar statics, and a connection that is established between the lengths of the adjoining sides of a parallelogram, its diagonals, and the three distances of these lines from an arbitrary point. A similar role is played in general statics by the following theorem, which is fundamental to the geometry of motors: If one lays the altitude planes α and β to two lines A and B through a point o (Fig. 6) and intersects A with β at a and B with α at b then the line segment ab remains unchanged in magnitude and direction when one holds o fixed and screws the common line-pair AB

around its axis. From the definitions of vector and motor, this theorem reads, more briefly: The vector that is represented by the intersection points a, b of two lines A, B with the altitude planes β, α that fall upon a point o depend upon the motor AB, along with the point o. This vector will temporarily be called the *moment* of the motor for the point o. When we seek to prove the theorem, we obtain information from this about the behavior of this moment vector.

It is next clear that a pure sliding motion of the line-pair AB parallel to the common normal N leaves the vector ab unchanged; the altitude planes α and β then remain preserved and the intersection points a and b move uniformly with A and B in the direction of the axis, which is also the direction of the intersection of α and β. In order to appraise the influence of the rotation, we project the entire space figure onto a perpendicular to the normal N, thus, onto a plane parallel to A and B. In the projection (Fig. 7), the two lines whose projections are denoted by A', B' appear, the traces of A_1, B_1 of the altitude planes α, β, which are perpendicular to A', B' and include the projection o' of the point o, define the real angle. The vector ab consists of its projection $a'b'$ and a component that is parallel to N whose magnitude equals the distance to the line AB, so it is, in any event, unchanging. Therefore, all that must be proved is the planimetric theorem that in Fig. 7 the magnitude and direction of $a'b'$ remain unchanged when one rotates the line-pairs A', B' and A_1, B_1 around n (o', resp.) through equal angles in the same sense. Now, $a'b'$ is, in fact, perpendicular to the fixed line no', so in the triangle $n\ a\ b$, A_1 and B_1 are two altitudes and therefore the third altitude – i.e., the foot of n on $a'b'$ – must also go through their intersection point o'. From the similarity $a'b'b_1 \sim no'b_1$, it follows that $\overline{a'b'} : \overline{no'} = \overline{a'b_1} : \overline{nb_1}$, and since $\overline{a'b_1} = \overline{nb_1}\ \tan(A'\ B')$, one has $\overline{a'b'} = \overline{no'}\ \tan(A'\ B')$ – i.e., the magnitude of $a'b'$ is the product of the distance from the point o to the axis times the tangent to the angle between the lines A, B.

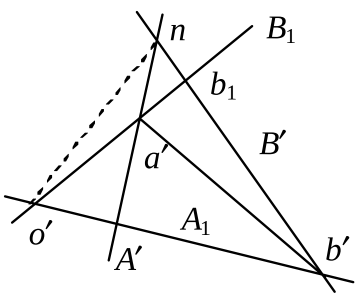

Figure 7.

One can express these results more briefly when one uses the notions and formulas of vector algebra. Let $\mathfrak{r}$ be the vector no', and let $\mathfrak{M}$ be a vector of magnitude tan(AB) that is known from any side of the normal N, from which, the rotation from A to B is seen to be positive (thus, under the image surface in Fig. 7), so the vector $a'b'$ is the vectorial product $\mathfrak{M}\times\mathfrak{r}$. This formula also remains true when $\mathfrak{r}$ means the vector that is drawn from an arbitrary point n of the axis to o, since the addition of a component parallel to $\mathfrak{M}$ to the vector product changes nothing. If we then call $\mathfrak{M}_o$ the moment of the motors relative to a point o, and call $\mathfrak{M}_n$, the one for the point n, then we have:

$$\mathfrak{M}_o = \mathfrak{M}_n + (\mathfrak{M}\times\mathfrak{r}), \tag{1}$$

where $\mathfrak{M}_n$ can be nothing but the vector that is parallel to N and points from A to B, and whose length is the distance AB. We summarize the result by saying:

Any point o of space of a vector $\mathfrak{M}_o$ *will be associated with a motor.*

The totality of these vectors is, from eq. (1), *given by two vectors* $\mathfrak{M}$ *and* $\mathfrak{M}_o$, *of which the latter is the value of* $\mathfrak{M}_o$ *for point on the motor axis.*

We call the vector $\mathfrak{M}$ that was defined above the *resultant vector* of the motor.

5. Vectorial and scalar components. Addition and multiplication by scalars. If one develops the vector algebra in a purely geometric way then one must show at some later point that whenever three coordinate directions are chosen, everything turns into scalar computations. Correspondingly, we would now like to see that by the choice of a reference point everything in motor algebra (to the extent that we know up to now, and as we will further develop in the sequel) can be solved. To that end, we prove the theorem:

A motor is uniquely determined by its resultant vector $\mathfrak{M}$ *and its moment vector* $\mathfrak{M}_o$ *for any reference point.*

In fact, by definition, $\mathfrak{M}$ gives the direction of the axis and the angle of the line-pair *AB*, including the sense of rotation, and, from (1), the components of $\mathfrak{M}_o$ in the direction of $\mathfrak{M}$ give the distance *AB*. In order to then find the position of the axis *N*, from eq. (1) (by the construction that led to (1), resp.), we must extend a line segment *on* from *o* along the perpendicular to the plane that is defined by $\mathfrak{M}$ and $\mathfrak{M}_n$, which yields the component of $\mathfrak{M}_o$ that is perpendicular to $\mathfrak{M}$, multiplied by the length of $\mathfrak{M}$; the sense of direction on *on* is therefore determined by the sign convention for vector products.

We call $\mathfrak{M}$ the *first vector component* and $\mathfrak{M}_o$, the *second vector component* of the motor for the reference point *o*; the first vector component is the same for any reference point. We call the two times three scalar components of $\mathfrak{M}$ and $\mathfrak{M}_o$, when referred to any three perpendicular directions, the six *scalar components of the motor.* For everything that follows, we now introduce a unified notation: Motors shall be denoted by bold Gothic symbols. The same symbol, but not bold, shall denote the first vector component, and we refer to the second vector component relative to *o* by the symbol *o*. The corresponding Latin symbols refer to the magnitude of the first vector component, the symbols 1, 2, 3 refer to its scalar components, and the symbols 4, 5, 6 refer to the scalar components of the second vector component. The motor $\mathfrak{M}$ is then, after the addition of a reference point *o*, representable by the two vectors $\mathfrak{M}$ and $\mathfrak{M}_o$, after a further addition of three directions, by the six numbers M_1, M_2, M_3, M_4, M_5, M_6 .

The meaning of these definitions is immediately illuminated by the theorem:

In the geometric addition of motors, one adds their vector and scalar components.

One can also say that the motor equation $\boldsymbol{\mathfrak{A}} + \boldsymbol{\mathfrak{B}} = \boldsymbol{\mathfrak{C}}$ is equivalent to the two vector equations $\mathfrak{A} + \mathfrak{B} = \mathfrak{C}$ and $\mathfrak{A}_o + \mathfrak{B}_o = \mathfrak{C}_o$. One next recognizes that the theorem must be true for any reference point, as long as it is true for one of them. The difference between the moment vectors of a motor $\boldsymbol{\mathfrak{M}}$ for any two points o, o' is, from (1):

$$\mathfrak{M}_o - \mathfrak{M}_{o'} = \mathfrak{M} \times (\mathfrak{r} - \mathfrak{r}'). \tag{2}$$

One can then compute the position vectors $\mathfrak{r}$ and $\mathfrak{r}'$ for the two points o and o from an arbitrary starting point. If the $\mathfrak{M}$ and $\mathfrak{M}_o$ behave additively then this must also be the case for $\mathfrak{M}_o$, when it is increased by $\mathfrak{M} \times (\mathfrak{r} - \mathfrak{r}')$ – i.e., for $\mathfrak{M}_{o'}$. In Fig. 5, in which the geometric addition was introduced, we now choose a reference point o on the same starting line O that represents the motors $\boldsymbol{\mathfrak{A}}$, $\boldsymbol{\mathfrak{B}}$, $\boldsymbol{\mathfrak{C}}$. If one constructs the moment vector for this o by the prescription that was given in sect. 4 then one sees that it will be represented by the point-pairs oa, ob, oc, which lie in the normal plane to O. The existence of $\mathfrak{A}_o + \mathfrak{B}_o = \mathfrak{C}_o$ is therefore obvious here from the rule by which the search for c was given. However, this relation is also true for a second point o' of O and one therefore has $(\mathfrak{A} - \mathfrak{A}_{o'}) + (\mathfrak{B} - \mathfrak{B}_{o'}) = \mathfrak{C} - \mathfrak{C}_{o'}$. From (2), one can also write $(\mathfrak{A} + \mathfrak{B}) \times (\mathfrak{r} - \mathfrak{r}') = \mathfrak{C} \times (\mathfrak{r} - \mathfrak{r}')$ for this, where $\mathfrak{r} - \mathfrak{r}'$ denotes the vector $o'o$, and since it is certainly perpendicular to $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, it then follows that $\mathfrak{A} + \mathfrak{B} = \mathfrak{C}$.

By this process, motor addition reduces to the well-known vector addition, and therefore, to ordinary scalar addition, as well, so all of the essential properties of addition – e.g., the commutative and associative laws, etc – are established. We can also connect the explanation for the multiplication of a motor by a number, with no further ado, by saying:

The product of the motor $\mathfrak{M}$ with the number λ, which is written $\lambda\mathfrak{M}$, shall mean the motor whose components (vectorial and scalar) are λ times the components of $\mathfrak{M}$.

This multiplication is distributive with the previously-described addition, so, in particular, $\mathfrak{A} + \mathfrak{A} = 2\mathfrak{A}$ and $\mathfrak{A} - \mathfrak{B} = \mathfrak{A} + (-\mathfrak{B})$, where $-\mathfrak{B} = (-1)\,\mathfrak{B}$. Speaking briefly: One may work with the "linear" operations that were described up to now as if the objects in them were ordinary numbers.

An especially important application of the rules for computation is to the differentiation of motors. If a motor $\boldsymbol{\mathfrak{M}}$ is given as a function of the scalar quantity t then we understand the derivative $d\boldsymbol{\mathfrak{M}} / dt$ to mean the motor whose components are the derivatives of $\boldsymbol{\mathfrak{M}}$.

6. Example of an application. Before we go further, the applicability of the simple concepts that were introduced up to now shall be clarified by a beautiful example from statics. O. Mohr has given a proof of the theorem that goes back to Culmann on the relationship between the funicular polygons that belong to equal force systems, which actually works with the fundamental ideas of motor algebra. When we present Mohr's argument with the help of the foregoing results, we likewise obtain an essentially generalized theorem for spatial funicular polygons. It reads:

If two spatial n-gons are equilibrium funicular polygons for the same n individual forces then the common normal to any two corresponding sides of the n-gon will be intersected at right angles by a line. (Fig. 8).

If the forces and n-gons lie in a plane then the common normals are the perpendiculars to the plane that are erected at the points of intersection: From our theorem, if the perpendiculars themselves have a common normal that cuts everything then the intersection points lie on a line, which is the Culmann line.

For the proof, we must assume as given that the forces that we are concerned with in statics are (special) motors and that the equilibrium condition consists of the statement that the motor sum must be zero. Equilibrium exists between the external nodal force and the two side forces at each corner point of the funicular polygon. If we call the external forces $\mathfrak{K}_1, \mathfrak{K}_2, \ldots, \mathfrak{K}_n$ and the side forces $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_n$, in such a way that, e.g., $\mathfrak{S}_2$ denotes the force that the side that goes from the second to the third node exerts on the last one, then the conditions of equilibrium read:

$$\mathfrak{K}_1 = \mathfrak{S}_1 - \mathfrak{S}_n, \qquad \mathfrak{K}_2 = \mathfrak{S}_2 - \mathfrak{S}_1, \qquad \mathfrak{K}_3 = \mathfrak{S}_3 - \mathfrak{S}_2, \ldots \qquad \mathfrak{K}_n = \mathfrak{S}_n - \mathfrak{S}_{n-1} .$$

The same equations are valid for the second funicular polygon, except that $\mathfrak{S}'$ appears in place of $\mathfrak{S}$. If one subtracts the one equation from the other one then one obtains:

$$\mathfrak{S}_1 - \mathfrak{S}'_1 = \mathfrak{S}_2 - \mathfrak{S}'_2 = \mathfrak{S}_3 - \mathfrak{S}'_3 = \ldots, \mathfrak{S}_n - \mathfrak{S}'_n .$$

The axis of a force-motor is the line of action of the force itself. From the definition in **3**, one can represent the motor $\mathfrak{S} - \mathfrak{S}'$ by a line-pair whose initial line O is the common normal to the axes of $\mathfrak{S}$ and $\mathfrak{S}'$. On the other hand, the axis of a motor cuts the lines represented perpendicularly. Therefore, the n common normals to the associated sides of the funicular polygon are cut perpendicularly by the axis of the motor $\mathfrak{S}_1 - \mathfrak{S}'_1 = \mathfrak{S}_2 - \mathfrak{S}'_2 = \ldots$

The representation that Mohr gave for the basic outline of statics and kinematics [1]), works with the essence of motor addition (except for the fact that the interpretation of a motor as a line-pair is missing). However, Mohr did not introduce a special notation for the motor-like quantities, but he changed the operation signs, using $\mp$ for addition and $\equiv$ for equality. For the advanced studies then, just as in vector algebra, the connection

[1]) O. Mohr, Abhandlungen a. d. Gebeit d. technischen Mechanik, Berlin 1906, Abh. I and II, esp., pp. 15, et seq.

between the motor quantities and their components then finds its expression in our system of notation, and with deep significance.

7. Scalar and motor product of two motors. We now go on to the explanation of the two types of multiplication, by whose introduction, the first-order motor algebra finds its completion. We first define: *The scalar product of two motors* $\mathfrak{A}$ *and* $\mathfrak{B}$, *which is written* $\mathfrak{A} \cdot \mathfrak{B}$, *or also* $\mathfrak{A}\,\mathfrak{B}$, *is the number:*

$$\mathfrak{A} \cdot \mathfrak{B} = \mathfrak{A}\mathfrak{B}_o + \mathfrak{A}_o\mathfrak{B} = A_1B_4 + A_2B_5 + A_3B_6 + A_4B_1 + A_5B_2 + A_6B_3 \,. \tag{3}$$

The second definition (which follows from the first one by the rules of vector algebra) expresses the scalar product in a way that is completely independent of the coordinate system, while the first one seems to depend only upon the choice of origin. If we denote the vector from o to o' by $\mathfrak{r}$, corresponding to the previous $\mathfrak{x}' - \mathfrak{x}$, then, from (2), we have:

$$\mathfrak{A}\mathfrak{B}_{o'} + \mathfrak{A}_{o'}\mathfrak{B} = \mathfrak{A}(\mathfrak{B}_o+(\mathfrak{B}\times\mathfrak{r}))+\mathfrak{B}(\mathfrak{A}_o+(\mathfrak{A}\times\mathfrak{r})) = \mathfrak{A}\mathfrak{B}_o + \mathfrak{A}_o\mathfrak{B} + \mathfrak{A}(\mathfrak{B}\times\mathfrak{r}) + \mathfrak{B}(\mathfrak{A}\times\mathfrak{r}). \tag{3$'$}$$

The two three-fold products on the right drop out as a result of the well-known commutation rule of vector algebra that says $\mathfrak{A}(\mathfrak{B} \times \mathfrak{r}) = - \mathfrak{B}(\mathfrak{A} \times \mathfrak{r})$. The invariance of the product defined in (3) is proved by this.

The meaning of the scalar product in mechanics corresponds to the scalar product of two vectors. If $\mathfrak{K}$ denotes a force motor and $\mathfrak{G}$ is a velocity motor then the scalar product $\mathfrak{K} \cdot \mathfrak{G}$ is the *power delivered in unit time.* If $\mathfrak{G}$ denotes an infinitely small displacement then $\mathfrak{K} \cdot \mathfrak{G}$ is the work that is done by this displacement. In contrast to the situation in vector algebra, here, one is spared the separation of the work done by translation and the work done by rotation, a separation that will be affected by the arbitrary choice of origin.

We define the *motor product* or *motor-like product* of two motors $\mathfrak{A}$ and $\mathfrak{B}$ to be a motor $\mathfrak{P}$ with the following vector components:

$$\mathfrak{P} = \mathfrak{A} \times \mathfrak{B}, \qquad \mathfrak{P}_o = (\mathfrak{A} \times \mathfrak{B}) + (\mathfrak{A}_o \times \mathfrak{B}_o). \tag{4}$$

The determination of the resultant component $\mathfrak{P}$ is immediately free of the concerns over the reference point. For the second component, on the basis of (2), we obtain, again with $\mathfrak{r} = \mathfrak{x}' - \mathfrak{x}$:

$$\begin{aligned}\mathfrak{P}_{o'} = (\mathfrak{A} \times \mathfrak{B}_{o'}) + (\mathfrak{A}_{o'} \times \mathfrak{B}) &= (\mathfrak{A} \times \mathfrak{B}_o) + \mathfrak{A} \times (\mathfrak{B} \times \mathfrak{r}) + (\mathfrak{A}_o \times \mathfrak{B}) + (\mathfrak{A} \times \mathfrak{r}) \times \mathfrak{B} \\ &= \mathfrak{P}_o + (\mathfrak{A} \times \mathfrak{B}) \times \mathfrak{r} = \mathfrak{P}_o + (\mathfrak{B} \times \mathfrak{r}).\end{aligned} \tag{4$'$}$$

In this, use has been made of the well-known theorem of vector algebra that for arbitrary vectors, one always has:

$$[\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{r})] + [\mathfrak{B} \times (\mathfrak{A} \times \mathfrak{r})] + [\mathfrak{r} \times (\mathfrak{A} \times \mathfrak{B})] = 0. \tag{5}$$

Eq. (4′) shows that under the transition to a new reference point o' the new moment vector, as defined by (4), is computed from the one for o in the same way that for any motor $\mathfrak{M}$ the moment $\mathfrak{M}_{o'}$ is computed from $\mathfrak{M}_o$ and $\mathfrak{M}$. With this, we have proved that under our definition (4) any pair of motors $\mathfrak{A}$, $\mathfrak{B}$ will actually be associated with a new motor $\mathfrak{P}$. The scalar components of $\mathfrak{P}$ are:

$$P_1 = A_2\, B_3 - A_3\, B_2\,,\ \ldots,\ P_4 = A_2\, B_6 - A_3\, B_5 + A_5\, B_3 - A_6\, B_2\,. \tag{4″}$$

An immediate invariant representation of this and the scalar product will follow later (**8**).

The mechanical interpretation for the motor product is the following one: Let $\mathfrak{M}$ be a motor that is fixed in space (i.e., a motor whose 6 components are constants relative to a fixed coordinate system) and let $\mathfrak{G}$ be the velocity motor of a moving rigid body. $\mathfrak{M}\times\mathfrak{G}$ is then the virtual increment that the motor $\mathfrak{M}$ experiences in a unit time when it is regarded from the moving body. One can also say that when $\mathfrak{M}$ has fixed components in a reference system that is moving with the velocity $\mathfrak{G}$, the components of $\mathfrak{G}\times\mathfrak{M}$ furnish the changes that the components of $\mathfrak{M}$ experience per unit time in rest space. This expression plays a decisive role in the presentation of the equations of motion for rigid bodies.

The two definitions (3) and (4), which reduce the product of two motors to the analogous product of vectors, show us that all of the rules of calculation emerge from vector algebra (scalar algebra, resp.). In particular, one has the commutative law for the motor product with a sign change:

$$\mathfrak{A} \cdot \mathfrak{B} = \mathfrak{B} \cdot \mathfrak{A}, \qquad \mathfrak{A} \times \mathfrak{B} = -\,\mathfrak{B} \times \mathfrak{A},$$

the distributive law:

$$\mathfrak{A} \cdot (\mathfrak{B} + \mathfrak{C}) = \mathfrak{A} \cdot \mathfrak{B} + \mathfrak{A} \cdot \mathfrak{C}, \qquad \mathfrak{A} \times (\mathfrak{B} + \mathfrak{C}) = \mathfrak{A} \times \mathfrak{B} + \mathfrak{A} \times \mathfrak{C},$$

the associative law:

$$\lambda\, \mathfrak{A} \cdot \mathfrak{B} = \lambda(\mathfrak{A} \cdot \mathfrak{B}), \qquad \lambda\, \mathfrak{A} \times \mathfrak{B} = \lambda(\mathfrak{A} \times \mathfrak{B}).$$

In a word: *One may calculate with the scalar and motor products of two motors just like the scalar and vectorial product of two vectors.*

8. Invariant representation and construction of the product. The invariant – i.e., free from the choice of arbitrary reference point – definitions of the two products (corresponding to the ones in vector algebra for length, projection, area) seem to be less simple here, but they still exhibit a remarkable symmetry. In addition, a noteworthy phenomenon manifests itself, namely, that the products are constructible without assuming a unit, as long as one performs them with quantities of the same dimension as the factors, while the products in the vector algebra have higher dimensions.

We will first assume that the resultant vectors of $\mathfrak{A}$ and $\mathfrak{B}$ are non-zero, so one is dealing with an actual motor. If one chooses an arbitrary reference point *o* on the axis of $\mathfrak{A}$ then the moment vector $\mathfrak{A}_0$ has the direction of $\mathfrak{A}$ – namely, it is equal to the vector from the initial line of the motor to the final line that is drawn along the axis – and can perhaps be denoted by $a\mathfrak{A}$. We would like to call the number *a* the *pitch* of the motor (corresponding to the pitch of a screw), in order that the magnitude *A* of the resultant vector and the position of the axis should together determine the motor $\mathfrak{A}$ in an invariant way. In the same way, let the motor $\mathfrak{A}$ be given by its axis, along with *b* and $\mathfrak{B}$, although one must observe that $b\mathfrak{B}$ is the moment vector relative to a point on the axis of $\mathfrak{B}$, such that from (2):

$$\mathfrak{A}_o = a\mathfrak{A}, \qquad \mathfrak{A}_o = b\mathfrak{B} - (\mathfrak{B}\times\mathfrak{d}). \tag{7}$$

Here, $\mathfrak{d}$ can mean any vector that points from the first axis to a point on the second one, although we will assume that $\mathfrak{d}$ shall denote the shortest vector, so it must be drawn along the common normal to the two axes. If the two motors were represented with a common initial line *O*, as in **3**, then $\mathfrak{d}$ would be parallel to *O* and our reference point *o* would lie on *O*.

From (7), the expression (3) for the scalar product assumes the form:

$$\mathfrak{A}\cdot\mathfrak{B} = b\,\mathfrak{A}\cdot\mathfrak{B} - \mathfrak{A}(\mathfrak{B}\times\mathfrak{d})\times a\,\mathfrak{A}\cdot\mathfrak{B} = (a+b)\,\mathfrak{A}\mathfrak{B} - \mathfrak{d}(\mathfrak{A}\times\mathfrak{B}), \tag{8}$$

while, from (4), the components of the motor product become:

$$\mathfrak{P} = \mathfrak{A}\times\mathfrak{B},\ \mathfrak{P}_o = b(\mathfrak{A}\times\mathfrak{B}) - \mathfrak{A}\times(\mathfrak{B}\times\mathfrak{d}) + b(\mathfrak{A}\times\mathfrak{B}) = (a+b)\,(\mathfrak{A}\times\mathfrak{B}) + \mathfrak{d}(\mathfrak{A}\mathfrak{B})\,. \tag{9}$$

The scalar product of the motors $\mathfrak{A}$ and $\mathfrak{B}$ is, from (8), equal to the sum of the pitches times the scalar product of the resultant vectors minus the scalar triple product of the two resultant vectors and the shortest distance between the axes. This consequence of (8) is even more important and intuitive: The scalar product of two motors vanishes for arbitrary values of *a*, *b*, *A*, *B* when and only when the two axes cut each other at right angles – since then $\mathfrak{A}\mathfrak{B} = 0$, $\mathfrak{d} = 0$. Except for this case and that of the vanishing of one factor, the scalar product also vanishes when $a + b = d\tan(\mathfrak{A}, \mathfrak{B})$, where $(\mathfrak{A}, \mathfrak{B})$ denotes the angle between the motor axes.

Eq. (9) next shows that the axis of the product motor $\mathfrak{P}$ is the common normal *O* to the axes of $\mathfrak{A}$ and $\mathfrak{B}$. Then, $\mathfrak{P}$, as well as $\mathfrak{P}_o$, has the direction of *O*. If we assume that $\mathfrak{P}$ does not have length 0 then we obtain the pitch *p* of the product motor as the quotient $\mathfrak{P}_o : \mathfrak{P} = a + b + d\cot(\mathfrak{A}, \mathfrak{B})$. One can then easily put the definition of $\mathfrak{P}$ into words.

One also sees that the product motor vanishes for arbitrary *a, b, A* when and only when the axes of $\mathfrak{A}$ and $\mathfrak{B}$ coincide – because the $\mathfrak{A}\times\mathfrak{B} = 0$, $\mathfrak{d} = 0$.

If a motor is represented by a line-pair then its resultant vector appears as the unknown number (tangent of the angle) and the moment vector as the length. Equations (3) and (4) or (8) and (9) show us that the scalar product of two motors is a length and the motor product is a motor with the same dimensions as each factor. From this, it follows that one can construct the product, as was mentioned already, without a special unit of length, or as the case may be, having to choose one. Here, we next give the characteristic construction for the moment vector $\mathfrak{P}_o$ of the motor product.

In Fig. 9, let the motors $\mathfrak{A}$ and $\mathfrak{B}$ be represented by the line-pairs *OA* and *OB*, such that *O* is the common normal to their axes, and likewise, to the axis of the motor product $\mathfrak{P}$. When we define the moments $\mathfrak{A}_o$ and $\mathfrak{A}$ for any two points *o*, *b* that determine the vector $\mathfrak{r}$ as its starting point and end point then from (2), $\mathfrak{A}_b - \mathfrak{A}_o = \mathfrak{A} \times \mathfrak{r}$. One can then construct the product $\mathfrak{A} \times \mathfrak{B}_o$ that appears in (4), when one chooses *o* and *b* such that *ob* represents the moment vector $\mathfrak{B}_o$. We then choose, similar to what we did in sec. 5, the points *a* and *b* on *A* and *B* in a perpendicular plane to *O*, drop the perpendicular plane onto *A* that meets *O* at a_1 from *b*, and the perpendicular plane onto *B* that meets *O* at b_1 from *a*, so a_1b_1 represents the desired vector $\mathfrak{P}_o$. One then has, if *o* denotes the intersection of *O* with the perpendicular plane to *O* through *a* and *b:*

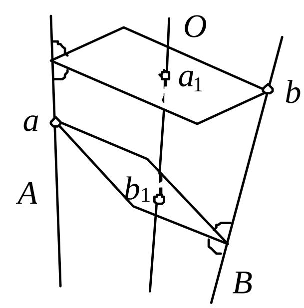

Figure 9

$$\overline{oa} = \mathfrak{A}_o, \qquad \overline{ob} = \mathfrak{B}_o, \qquad \overline{a_1a} = \mathfrak{A}_b, \qquad \overline{b_1b} = \mathfrak{B}_a,$$
$$\overline{oa_1} = \mathfrak{A}_o - \mathfrak{A}_b = -\mathfrak{A} \times \overline{ob} = \mathfrak{B}_o \times \mathfrak{A}, \qquad \overline{ob_1} = \mathfrak{B}_o - \mathfrak{B}_a = -\mathfrak{A} \times \overline{oa} = \mathfrak{A}_o \times \mathfrak{B},$$

and therefore one actually has $\mathfrak{P}_o = \overline{ob_1} - \overline{oa_1} = \overline{a_1b_1}$. The fact that $\mathfrak{P}_o$ is independent of the choice of point *o* on *O* delivers a simple stereometric theorem [1]).

From the product motor $\mathfrak{P}$, one now knows the position of the axis and the moment vector for its points. The magnitude of the resultant vector depends only upon the directions *O*, *A*, *B*. If one directs three parallels to them through a fixed point, and they intersect through a perpendicular plane to *O* at a distance *c* from the origin then twice the area of the intersection point triangle *oab* gives c^2 times the desired quantity, which is therefore easy to construct as the tangent of the angle (triangle of equal area with *c* as the leg).

One can construct the magnitudes of the scalar products of $\mathfrak{A}$ and $\mathfrak{B}$ as lengths on *o* when one alters the construction of $\mathfrak{P}_o$ somewhat: Since one needs the products $AB_o \cos(\mathfrak{A}, \mathfrak{B}_o)$ and $A_oB \cos(\mathfrak{A}_o, \mathfrak{B})$ in place of the analogues with the sine of the angle, then one must perform only corresponding rotations through 90°.

[1]) Cf., the problem that I posed in the Jahresber. d. dtsch. Mathematiker-Verein 31, 1922, pp. 65.

9. Multiple products, rules of calculation. The commutation rule for the combination of a scalar and a motor product is completely analogous to the one in vector algebra. Let $\boldsymbol{\mathfrak{A}}$, $\boldsymbol{\mathfrak{B}}$, $\boldsymbol{\mathfrak{C}}$ be three arbitrary motors, so one has:

$$\boldsymbol{\mathfrak{A}} \cdot (\boldsymbol{\mathfrak{B}} \times \boldsymbol{\mathfrak{C}}) = \boldsymbol{\mathfrak{B}} \cdot (\boldsymbol{\mathfrak{C}} \times \boldsymbol{\mathfrak{A}}) = \boldsymbol{\mathfrak{C}} \cdot (\boldsymbol{\mathfrak{A}} \times \boldsymbol{\mathfrak{B}}). \tag{10}$$

The proof follows from the definitions (3) and (4), as long as one makes use of the known commutation theorem of vector algebra. From (3) and (4), the first product in (10) is equal to:

$$\mathfrak{A}\,(\mathfrak{B} \times \mathfrak{C}_o) + \mathfrak{A}\,(\mathfrak{B}_o \times \mathfrak{C}) + \mathfrak{A}_o\,(\mathfrak{B} \times \mathfrak{C}),$$

and the sequence of factors in each summand here may be cyclically permuted, from which (10) is proved.

The commutability also emerges immediately from the representation in scalar components. By definition, one has:

$$\boldsymbol{\mathfrak{A}} \cdot (\boldsymbol{\mathfrak{B}} \times \boldsymbol{\mathfrak{C}}) = \begin{vmatrix} A_4 & A_5 & A_6 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} + \begin{vmatrix} A_1 & A_2 & A_3 \\ B_4 & B_5 & B_6 \\ C_1 & C_2 & C_3 \end{vmatrix} + \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_4 & C_5 & C_6 \end{vmatrix}. \tag{10$'$}$$

The three determinants arise when one replaces one row in the matrix of nine components of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ with the components of $\mathfrak{A}_o$, $\mathfrak{B}_o$, $\mathfrak{C}_o$ in succession.

The decomposition of the triple product that arises from repeated motor multiplication into two summands is not as simple. From (3) and (4), one has:

$$\boldsymbol{\mathfrak{P}} = \boldsymbol{\mathfrak{A}}\times(\boldsymbol{\mathfrak{B}}\times\boldsymbol{\mathfrak{C}}), \quad \mathfrak{P} = \mathfrak{A}\times(\mathfrak{B}\times\mathfrak{C}), \quad \mathfrak{P}_o = \mathfrak{A}\times(\mathfrak{B}\times\mathfrak{C}_o)\times \mathfrak{A}\times(\mathfrak{B}_o\times\mathfrak{C}) + \mathfrak{A}_o\times(\mathfrak{B}\times\mathfrak{C}). \tag{11}$$

If one applies the development theorem for the triple product of vector algebra here, namely:

$$\mathfrak{P} = \mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C}) = \mathfrak{B}\,(\mathfrak{A} \cdot \mathfrak{C}) - \mathfrak{C}\,(\mathfrak{A} \cdot \mathfrak{B}),$$

then one sees that $\boldsymbol{\mathfrak{P}}$ may be, in fact, represented as the difference $\boldsymbol{\mathfrak{P}} = \boldsymbol{\mathfrak{K}} - \boldsymbol{\mathfrak{L}}$ of two motors:

$$\left.\begin{aligned} \mathfrak{K} &= \mathfrak{B}(\mathfrak{A}\cdot\mathfrak{C}), \;\; \mathfrak{K}_o = \mathfrak{B}_o(\mathfrak{A}\cdot\mathfrak{C}) + \mathfrak{B}(\boldsymbol{\mathfrak{A}}\cdot\boldsymbol{\mathfrak{C}}), \\ \mathfrak{L} &= \mathfrak{C}(\mathfrak{A}\cdot\mathfrak{B}), \;\; \mathfrak{L}_o = \mathfrak{C}_o(\mathfrak{A}\cdot\mathfrak{B}) + \mathfrak{C}(\boldsymbol{\mathfrak{A}}\cdot\boldsymbol{\mathfrak{B}}), \end{aligned}\right\} \tag{12}$$

however, the components $\boldsymbol{\mathfrak{K}}$ and $\boldsymbol{\mathfrak{L}}$ are not immediately expressible as products of the motors $\boldsymbol{\mathfrak{A}}$, $\boldsymbol{\mathfrak{B}}$, $\boldsymbol{\mathfrak{C}}$.

This minor deviation makes things no longer valid, until one considers the ternary product that emerges from (11) by cyclic permutation of the factors. Analogous to the vector formula $\mathfrak{a} \times (\mathfrak{b} \times \mathfrak{c}) + \mathfrak{b} \times (\mathfrak{c} \times \mathfrak{a}) + \mathfrak{c} \times (\mathfrak{a} \times \mathfrak{b}) = 0$, one has:

$$\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C}) + \mathfrak{B} \times (\mathfrak{C} \times \mathfrak{A}) + \mathfrak{C} \times (\mathfrak{A} \times \mathfrak{B}) = 0. \tag{13}$$

One obtains (13) when one develops each of the three summands according to (12).

The four-fold product displays a remarkable symmetry that arises from the scalar multiplication of two motor products. If one applies the commutation formula (10) to:

$$Q = (\mathfrak{A} \times \mathfrak{B}) \cdot (\mathfrak{C} \times \mathfrak{D}), \tag{14}$$

when one focuses on $\mathfrak{C}$ (heraushebt), then the second factor that remains is $\mathfrak{D} \times (\mathfrak{A} \times \mathfrak{B})$, and the development formula (12) may be then applied to this. One then obtains:

$$Q = (\mathfrak{B}\ \mathfrak{D})\ (\mathfrak{A}\ \mathfrak{B}) + (\mathfrak{A}\ \mathfrak{C})\ (\mathfrak{B}\ \mathfrak{D}) - (\mathfrak{A}\ \mathfrak{D})\ (\mathfrak{B}\ \mathfrak{C}) - (\mathfrak{B}\ \mathfrak{C})\ (\mathfrak{A}\ \mathfrak{D}). \tag{15}$$

This equation appears in place of the vector formula:

$$(\mathfrak{a} \times \mathfrak{b})\ (\mathfrak{c} \times \mathfrak{d}) = (\mathfrak{a}\ \mathfrak{c})(\mathfrak{b}\ \mathfrak{d}) - (\mathfrak{a}\ \mathfrak{b})(\mathfrak{b}\ \mathfrak{c}),$$

which is derived in an analogous way, as is known.

10. Simple geometric applications. A motor $\mathfrak{A}$ that is represented by two intersecting lines is analytically characterized by the fact that its moment vector vanishes for the intersection point of the lines and then for all points of its axis. From (1), for an arbitrary reference point *o* the moment $\mathfrak{A}_o$ is therefore perpendicular to $\mathfrak{A}$; i.e., the well-known equation:

$$\mathfrak{A} \cdot \mathfrak{A}_o = A_1\, A_4 + A_2\, A_5 + A_3\, A_6 = 0 \tag{16}$$

exists between the 6 scalar components, which goes by the name of the *Plücker relation.* (The mean of the expression in this equation is, moreover, one-half the scalar product of the motor with itself, so it is an invariant of the motor $\mathfrak{A}$.) Study called such a special motor a "rotor;" in dynamics, it corresponds to an individual force, and in kinematics, to a pure rotation. The position of a line in space, namely, the *axis* of the motor $\mathfrak{A}$, will be determined by the ratios of the six components A_1 to A_6, in long as they satisfy (16) and A_1, A_2, A_3 do not vanish identically. The rotor components are thus nothing but the Plücker coordinates.

A motor that is represented by two parallel lines is analytically characterized by the fact that its resultant component vanishes, so $A_1 = A_2 = A_3 = 0$. From (1), the moment component is then the same for any reference point, and the motor is given completely by three scalar numbers A_4, A_5, A_6. Study called such a motor a "translator;" in dynamics, it corresponds to a force-pair, and in kinematics, to a pure translation. Naturally, the ratios of the A_4, A_5, A_6 determine only one direction. It is recommended that one introduce the notion of "unit motor," by analogy with that of unit vector. We would like to use the term *unit motor* to refer to:

1) All motors, for which $A_1^2 + A_2^2 + A_3^2 = 1$.

2) All motors, for which $A_1 = A_2 = A_3 = 0$ and $A_4^2 + A_5^2 + A_6^2 = 1$; these are the translators whose moment vector has length 1.

Later, we will make extensive use of this notion.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two rotors, such that one then has:

$$A_1 A_4 + A_2 A_5 + A_3 A_6 = 0, \qquad B_1 B_4 + B_2 B_5 + B_3 B_6 = 0. \tag{17}$$

The two-times-six numbers $A_1, \ldots, A_6$ and $B_1, \ldots, B_6$ – or really their ratios – determine two lines A, B, as was shown above. As long as they are not perpendicular to each other, they again determine the starting line and end-line of a motor $\mathfrak{M}$, and it must therefore be possible, to calculate this $\mathfrak{M}$ from $\mathfrak{A}$ and $\mathfrak{B}$. One can see that:

$$\mathfrak{M} = \frac{\mathfrak{A} \times \mathfrak{B}}{\mathfrak{A} \cdot \mathfrak{B}}. \tag{18}$$

From the definition in **7**, the motor that is defined by (18) has the common normal of A and B for its axis and the resultant vector $(\mathfrak{A} \times \mathfrak{B})$ $(\mathfrak{A} \cdot \mathfrak{B})$, whose length is therefore $\tan(A, B)$. From (9), the second vector component of $\mathfrak{M}$ for a point of the axis as reference point is equal to $\mathfrak{o}$ $(\mathfrak{A}\mathfrak{B}) : (\mathfrak{U}\mathfrak{B}) = \mathfrak{o}$, since $a = b = 0$ now for the rotors $\mathfrak{A}$ and $\mathfrak{B}$, from which, the assertion is proved. The six component equations of (18):

$$\left.\begin{aligned} &M_1 = A_2 B_3 - A_3 B_2, && M_2 = A_3 B_1 - A_1 B_3, && \cdots \\ &M_4 = A_5 B_3 - A_6 B_2 + A_2 B_6 - A_3 B_5, && M_5 = A_6 B_1 - A_4 B_3 + A_3 B_4 - A_1 B_6, && \cdots \end{aligned}\right\} \tag{19}$$

deliver the components of a motor, as expressed in terms of the Plücker coordinates of its representative line-pair.

If one gives fixed values to $M_1, \ldots, M_6$ then eq. (19), together with (17), determine all line pairs that go to each other by a screwing motion around their axis. In this way, one can represent the various line loci through motor equations and their component decompositions, if not all of motor analysis, as a geometry whose element is the line, similarly to the way that vector analysis employs the ordinary point-geometry. We shall not go into this further here.

II. Second-order motor algebra (motor-dyadics).

1. Vector and vector-dyadic. In the first section, we gave a geometric definition of a vector, and fixed out attention on an analytic one. If we now turn to it, then we must naturally start with the concept of vector components that are associated with the directions of space. However, as we already mentioned, it does not suffice to say that a vector is a triple of numbers that related to a well-defined axis cross. The two numbers of people strolling along the one line or the other in a right-angled street cross certainly does not determine any actual vector. The definitive and analytically essential property of a vector is the fact that every direction is associated with a certain number – the "value" of the vector for this direction – and that the totality of all these numbers satisfies a certain legitimacy condition. If a spatial direction ν is established by the three direction cosines $\cos(\nu, x)$, $\cos(\nu, y)$, $\cos(\nu, z)$ then one can call an expression of the form $v_\nu = a \cos(\nu, x) + b \cos(\nu, y) + c \cos(\nu, z)$, in which a, b, c are constants, a *linear direction function*. If one chooses a new reference system with the axes x', y', z' then, as one easily sees, the linear expression that we just considered again goes to another such expression – say, $v_\nu = a' \cos(\nu, x') + b' \cos(\nu, y') + c' \cos(\nu, z')$. Thus, a, b, c mean the values of the function in the directions of the x, y, and z axis, resp., and can also be reasonably denoted by v_x, v_y, v_z (since – e.g., for $\nu = x$ – the first cosine equals 1, the other ones will be zero, etc), while the a', b', c' means the values for the x', y', z' axis, resp. We arrive at the following analytical definition of a vector: *By the term* vector $\mathfrak{v}$, *we understand this to mean the totality of directions in space that are associated with a scalar number* v_ν *by a certain linear law:*

$$v_\nu = v_x \cos(\nu, x) + v_y \cos(\nu, y) + v_z \cos(\nu, z). \tag{1}$$

The vector is thus determined by the three numbers v_x, v_y, v_z – its "values" in the three coordinate directions. One easily calculates from (1) that v_ν possesses the maximum value $\sqrt{v_x^2 + v_y^2 + v_z^2}$, which it assumes for the direction ν that is given by $\cos(\nu, x) : \cos(\nu, y) : \cos(\nu, z) = v_x, v_y, v_z$, and which vanishes for all directions v_ν that are perpendicular to it. When one takes the magnitude of the maximum and its direction as the determining data (and thus, chooses only the corresponding reference system), one achieves the link to the geometric definition of vector.

The way that a vector appears in mechanics or physics thus corresponds to our definition. When we say that the velocity of a point is a vector, this rests upon the fact that a moving point exhibits a certain advance per unit time in every direction of space, and that all of these magnitudes are connected by (1). This becomes clearer when we consider the statement: The first derivative of a scalar space function $f(x, y, z)$ is a vector (gradient). In fact, by the rules of differential calculus, any direction will be associated with a number – the "derivative in this direction" – and the totality of all these numbers will be determined by three of them using (1).

Moreover, it is an effortless task to introduce vector structures of second and higher order. In mechanics and physics, there are also cases in which any spatial direction is not associated with a number, but a vector. For example, at each point in the interior of a body, any direction of intersection n corresponds to a stress vector φ_ν, namely, the stress on the surface element that is perpendicular to ν. From well-known equilibrium theorems, one easily deduces that the infinitude of stress directions at a point are by no means independent of each other, but have a relationship that one must further refer to as "linear." Namely, one expresses $\mathfrak{p}_\nu$ in the form $\mathfrak{a}\cos(\nu, x) + \mathfrak{b}\cos(\nu, y) + \mathfrak{c}\cos(\nu, z)$, where $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are constant vectors, which naturally equal the values of $\mathfrak{p}_\nu$ in the x, y, z directions, resp. The totality of stress vectors at a point defines a structure of "second order," such as a tensor or vector-dyadic. In complete analogy to the analytical definition of a vector, we define: *By the term* tensor *or* vector-dyadic, *we understand this to mean the totality of vectors that are associated with the directions of space by a certain linear law:*

$$\mathfrak{p}_\nu = \mathfrak{p}_x \cos(\nu, x) + \mathfrak{p}_y \cos(\nu, y) + \mathfrak{p}_z \cos(\nu, z). \tag{2}$$

With no further assumptions, it is clear that, in entirely the same way, a third order structure – or *vector-triadic* – can take the form of the totality of dyadics that are linearly associated with the direction ν, etc.

The simplest and most intuitive example of a vector structure of higher-order is given by the repeated differentiation of a space function. We have mentioned this above, and in what sense, the first derivative of a scalar function $f(x, y, z)$ is a vector. If one differentiates a given vector as a function of position in a given direction then one again obtains a vector. The totality of vectors that are defined by the derivatives in all space directions satisfies (from the theorem on "total" differentials) a relation of the form (2), so it is a dyadic. The second derivative of a vector, the third derivative of a scalar, or the first derivative of a dyadic will be represented in an entirely analogous way by the totality of dyadics that obey a linear law, so they are triadics, etc. Each differentiation then raises the order of the structure by a unit.

2. The motor-dyadic. In order to convert the concept of dyadic (or any other structure of higher order) from vector algebra to motor algebra, we must now let the notion of "unit motor" enter in place of that of "direction," which is indeed represented by a unit vector. In I, **10** we defined a unit motor $\mathfrak{E}$ to be one whose components E_1, E_2, …, E_6 satisfy either:

$$E_1^2 + E_2^2 + E_3^2 = 1 \qquad \text{or} \qquad E_1 = E_2 = E_3 = 0, \qquad E_4^2 + E_5^2 + E_6^2 = 1. \tag{3}$$

The invariant meaning of the unit motor has already been discussed above. Depending upon whether the first or the second of the alternatives (3) is the case, we would like to speak of a unit motor of the first or second type, respectively.

Now, if an arbitrary motor $\mathfrak{M}$ is given by its components M_1, M_2, …, M_6 then one can – always in analogy with vector algebra – define its "value for the unit motor $\mathfrak{E}$" to be the scalar product $\mathfrak{M} \cdot \mathfrak{E} = M_4 E_1 + M_5 E_2 + M_6 E_3 + M_1 E_4 + M_2 E_5 + M_3 E_6$. Conversely, when one has, say, established the notion of a unit motor, but not that of an arbitrary motor, one can assert: *By the term* motor, *we understand this to mean the* *totality of scalar numbers* $M_{\mathfrak{E}}$ *that are associated with the unit motors in space by a certain linear law:*

$$M_{\mathfrak{E}} = M_4\,E_1 + M_5\,E_2 + M_6\,E_3 + M_1\,E_4 + M_2\,E_5 + M_3\,E_6\,. \tag{4}$$

If a coordinate system is chosen then the motor is determined by the 6 numbers M_1 to M_6. The fact that this definition has not immediate practical significance follows from only the fact that there is no geometric structure that is associated with a unit motor that is as simple and intuitive as that of space direction for the unit vector.

Only the "analytical" definition of a motor that we formulated above will give us the possibility of arriving at the notion of motor-dyadic with no further complications. We assume that, instead of the six scalar numbers $M_1, M_2, \ldots, M_6$ that establish a motor $\mathfrak{M}$, we are now given six motors $\boldsymbol{\mathfrak{P}}_1, \boldsymbol{\mathfrak{P}}_2, \ldots, \boldsymbol{\mathfrak{P}}_6$. We define: *By the term* motor-dyadic Π, *we understand this to mean the totality of motors* $\boldsymbol{\mathfrak{P}}_{\mathfrak{E}}$ *that are associated with the unit motors in space by a definite linear law:*

$$\boldsymbol{\mathfrak{P}}_{\mathfrak{E}} = \boldsymbol{\mathfrak{P}}_1\,E_1 + \boldsymbol{\mathfrak{P}}_5\,E_2 + \boldsymbol{\mathfrak{P}}_6\,E_3 + \boldsymbol{\mathfrak{P}}_1\,E_4 + \boldsymbol{\mathfrak{P}}_2\,E_5 + \boldsymbol{\mathfrak{P}}_3\,E_6\,. \tag{5}$$

The motor-dyadic is therefore determined by 6 motors or by 12 vectors or by 36 scalars. We would like to denote the vector components of $\boldsymbol{\mathfrak{P}}_1, \boldsymbol{\mathfrak{P}}_2, \ldots$ by $\mathfrak{P}_1, \mathfrak{P}_{1\nu}; \mathfrak{P}_2, \mathfrak{P}_{2\nu}; \ldots$, and the scalar components by $P_{11}, P_{12}, \ldots, P_{16}; P_{21}, P_{22}, \ldots, P_{26}; \ldots$ If we would like to represent the dyadic Π by its scalar components then we would arrange them into a quadratic schema as follows:

$$\left\{\begin{matrix} P_{11} P_{12} \cdots P_{16} \\ P_{21} P_{22} \cdots P_{26} \\ \cdots\cdots\cdots\cdots \\ P_{61} P_{62} \cdots P_{66} \end{matrix}\right\}. \tag{6}$$

The quadratic matrix with 6 rows and columns decomposes into 4 squares with 3 rows and columns. They represent the ordinary vector-dyadic that we would like to denote by the series $\Pi_{11}, \Pi_{12}, \Pi_{21}, \Pi_{22}$, such that we can abbreviate (6) by also writing:

$$\left\{\begin{matrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{matrix}\right\}. \tag{6′}$$

The lack of an immediately intuitive interpretation for the unit motor makes itself felt in the definition of the motor-dyadic. Namely, since, from I.**10**, any motor takes the form of a scalar multiplicity of unit motors, one does not actually see why the unit motor was even used in the definition (6); one can just as well speak of linear functions of any motor. However, for the unified and unique conception of the component schemata (6) and (6′) it is better to stay with our definition. Before we go further, we would now like to employ the newly-obtained concept of motor-dyadic to a specific and very important example in mechanics.

3. The inertia dyadic. As is well-known, the inertia of a moving point can be described by a single scalar number – viz., the "mass" m – while the inertia of a rigid body demands further data for its complete description: the position of the center of mass, the magnitudes of the centrifugal and deviation moments. We will now show that all of the quantities define the elements of single motor-dyadic, in which all of the inertia-related properties thus find their expression.

Let $\mathfrak{B}$ be the velocity vector of a mass element dm, whose position is determined relative to a fixed origin o by the position vector $\mathfrak{r}$. The product $\mathfrak{B}\ dm$ is then called the *quantity of motion* or the *impulse* of dm and $\mathfrak{r} \times \mathfrak{B}\ dm$ is the *moment of the quantity of motion* or the *impulse moment*. When we think of $\mathfrak{B}$ as linked with the point $\mathfrak{r}$ – or at least with the line going through the point in the direction of $\mathfrak{B}$ – as the vector "attached" to the point, we arrive at the conception of the impulse as a "rod" or as a special kind of motor (viz., a "rotor"). Its resultant vector is $\mathfrak{B}\ dm$, while its second vector component relative to o is $\mathfrak{r} \times \mathfrak{B}\ dm$. The addition of impulses over all parts of the body delivers an impulse motor with the vector components:

$$\mathfrak{T} = \int \mathfrak{B}\ dm, \qquad \mathfrak{T}_o = \int \mathfrak{r} \times \mathfrak{B}\ dm. \tag{7}$$

In this, it is assumed that the distribution of masses is continuous; if only discrete mass points are present then corresponding summations appear in place of the integrals in (7).

The total velocity state of a rigid body will be described by a motor. If we assume that its unit motor is $\boldsymbol{\mathfrak{E}}$ then the velocity $\mathfrak{B}$ at the point $\mathfrak{r}$ is the moment of $\boldsymbol{\mathfrak{E}}$, hence, from I (1):

$$\mathfrak{B} = \mathfrak{E}_0 + (\mathfrak{E} \times \mathfrak{r}). \tag{8}$$

It is obvious that (7) and (8) define an association of motors $\boldsymbol{\mathfrak{T}}$ with the total unit motors. This association is certainly linear; then, if we introduce (8) into (7), the components of $\boldsymbol{\mathfrak{E}}$ always appear only as expressions that contain $\mathfrak{r}$ and m multiplied together. Thus, we have a motor-dyadic that we call the *inertia dyadic* $\mathbf{T}$ of the body and whose component matrix we would like to compute immediately.

From the foregoing, in the first row of the quadratic schema, one finds the six components of the motor $\mathfrak{T}_1$ that, as (5) shows, is associated with any unit motor $\boldsymbol{\mathfrak{E}}$ for which all of the components vanish, except for E_4, while E_4 equals 1. The vector components of this unit motor are $\mathfrak{E} = o$ and $\mathfrak{E}_o = \mathfrak{i}$, when we denote the unit vectors in the three directions of the coordinate cross by $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$, as usual. From (8), the velocity of an arbitrary point is independent of $\mathfrak{r}$ for the case considered, and indeed, $\mathfrak{B} = \mathfrak{i}$. If one substitutes this in (7) and writes x, y, z for the components of $\mathfrak{r}$ then one obtains the expressions $\mathfrak{T}_1 = \mathfrak{i} \int dm$, $\mathfrak{T}_{1o} = \int \mathfrak{r} \times \mathfrak{i}\ dm$ for the two components of $\boldsymbol{\mathfrak{T}}_1$, so the six scalar components are $T_{11} = \int dm$, $T_{12} = T_{13} = 0$, $T_{14} = 0$, $T_{15} = \int z\ dm$, $T_{16} = -\int y\ dm$. We call the total mass of the body m, the coordinates of the center of mass x^*, y^*, z^* then this gives the first row of the schema of **T** that corresponds to (6):

$$m,\ 0,\ 0,\ 0,\ mz^*,\ -my^*. \tag{9}$$

In exactly the same way, one calculate the second and third rows that arise from certain permutations of (9), namely:

$$0,\ m,\ 0,\ -mz^*,\ 0,\ mx^* \qquad \text{and} \qquad 0,\ 0,\ m,\ my^*,\ -mx^*,\ 0. \tag{9′}$$

In order to now obtain the fourth (and then the last two) rows, from (5), we must first choose the unit motor for $\mathfrak{C}$ that is characterized by $E_1 = 1$, $E_2 = E_3 = \ldots = E_6 = 0$, for which one then has $\mathfrak{C} = \mathfrak{i}$, $\mathfrak{C}_o = 0$. It then follows from (8) that $\mathfrak{B} = \mathfrak{i} \times \mathfrak{r}$ with the components $V_1 = 0$, $V_2 = -z$, $V_3 = y$. Substituting in (7) yields the first three components of $\mathfrak{P}_1$, which are the components $\int \mathfrak{B}\, dm = 0$, $-\int z\, dm = -mz^*$, and $\int y\, dm = my^*$; the last three components are those of the second expression in (7), so they are first $\int (y V_3 - z V_2)\, dm = \int (y^2 + z^2)\, dm$, then $\int (z V_1 - x V_3)\, dm = \int xy\, dm$, and finally $\int (x V_2 - y V_3)\, dm = -\int xz\, dm$. One sees how the inertia and deviation moments appear here, for which we would also like to use the abbreviations J_1, J_2, J_3 (D_1, D_2, D_3, resp.). The fourth row of the desired schema then reads:

$$0,\ -mz^*,\ my^*,\ J_1,\ -D_3,\ -D_2\,. \tag{9″}$$

The last two rows are obtained by the same argument (by the corresponding exchanges of:

$$mz^*,\ 0,\ -mx^*,\ -D_3,\ J_2,\ -D_1 \quad \text{with} \quad -my^*,\ mx^*,\ 0,\ -D_2\,,\ -D_1\,,\ J_3\,, \tag{9‴}$$

resp.)

If one introduces the static moment in place of the center-of-mass coordinates x^*, y^*, z^*, and – in connection with the previously-introduced abbreviations – sets:

$$\begin{aligned} m = \int dm,\ \ S_1 = x^* m = \int x\, dm,\quad S_2 = y^* m = \int y\, dm,\quad S_3 = z^* m = \int z\, dm,\\ J_1 = \int (y^2 + z^2)\, dm,\ \ J_2 = \int (z^2 + x^2)\, dm,\ \ J_3 = \int (x^2 + y^2)\, dm,\\ D_1 = \int yz\, dm,\qquad D_2 = \int zx\, dm,\qquad D_3 = \int xy\, dm, \end{aligned} \tag{10}$$

then the schema for the inertia dyadic **T** assumes the neat form:

$$\left\{\begin{matrix} m & 0 & 0 & 0 & S_3 & -S_2 \\ 0 & m & 0 & -S_3 & 0 & S_1 \\ 0 & 0 & m & S_2 & -S_1 & 0 \\ 0 & -S_3 & S_2 & J_1 & -D_3 & -D_2 \\ S_3 & 0 & -S_1 & -D_3 & J_2 & -D_1 \\ -S_2 & S_1 & 0 & -D_2 & -D_1 & J_3 \end{matrix}\right\}. \tag{11}$$

One sees that this matrix is symmetric about the diagonal, but also exhibits certain peculiarities that, together with the symmetry, reduce the number of distinct elements from 36 to 10. We will come back to the use of the inertia dyadic at a later place.

4. Addition and multiplication. The dyadic as transformation. The addition of dyadics and their multiplication by scalar numbers may be disposed of quite simply in connection with the definition of dyadics. We clarify that the sum of the dyadics **A** and **B** is a dyadic **Γ** that is defined by the fact that any unit motor $\mathfrak{C}$ is associated with the motor $\mathfrak{C}_{\mathfrak{c}} = \mathfrak{A}_{\mathfrak{c}} + \mathfrak{B}_{\mathfrak{c}}$; analogously, the product of the number λ and the dyadic **A** is the dyadic **B** for which $\mathfrak{B}_{\mathfrak{c}} = \lambda\, \mathfrak{B}$. One can then perform the "linear operations" of addition and multiplication by a scalar on the components – i.e., on the elements of the 36-element matrix. The integration and differentiation with respect to a scalar are likewise explained in this way.

However, the most important computational context into which a dyadic enters is simply multiplying the associated first-order structure – e.g., the vector (motor, resp.) – by it. This operation also leads to a somewhat altered conception of the dyadic itself. For the sake of intuitiveness, we would now like to discuss the vector-dyadic and the vector.

Let Π be a vector dyadic and let $\mathfrak{v}$ be an arbitrary vector. For a direction (unit vector) ν, Π has the value $\mathfrak{p}_\nu$ that is represented in the form of the expression (2). Now, if we let ν be chosen to be the direction of $\mathfrak{v}$ and let υ be the length of the vector $\mathfrak{v}$ then we call $\upsilon\, \mathfrak{p}_\nu$ the *product of the dyadic* Π *with the vector* $\mathfrak{v}$, and write:

$$\Pi\mathfrak{v} = \upsilon\,\mathfrak{p}_\nu = \mathfrak{p}_x\upsilon\,\cos(\nu x) + \mathfrak{p}_y\upsilon\,\cos(\nu y) + \mathfrak{p}_z\upsilon\,\cos(\nu z) = \mathfrak{p}_x\upsilon_x + \mathfrak{p}_y\upsilon_y + \mathfrak{p}_z\upsilon_z\,. \tag{12}$$

The latter form of the expression shows the analogy with the scalar product of two vectors; it is now just one factor in each of the three partial products of a vector, which the other is a scalar. The fact that an invariant – i.e., independent of the choice of axis direction – value exists comes from the first form of the definition. In words, it reads: *By the term* product of the (vector-)dyadic Π with the vector $\mathfrak{v}$, *we understand this to mean the product of the vector length* υ *in the direction of* $\mathfrak{v}$ *with the vector* $\mathfrak{p}_\nu$ *that is associated with the dyadic* Π. Here, if we replace the dyadic Π by a vector $\mathfrak{p}$ and accordingly set $\mathfrak{p}_\nu$ equals to the vector component p_ν then we obtain a new definition of the scalar product $\mathfrak{v}\mathfrak{p}$.

One achieves the transition to the altered interpretation of the dyadic when one applies eq. (12) to a unit vector $\bar{\nu}$. One obtains:

$$\Pi\bar{\nu} = \mathfrak{p}_\nu = \mathfrak{p}_x \cos(\nu x) + \mathfrak{p}_y \cos(\nu y) + \mathfrak{p}_z \cos(\nu z); \tag{12′}$$

i.e., the "value" of the dyadic for the direction ν is equal to its product with unit vector in this direction (like the "value" of the vector $\mathfrak{v}$ for the direction ν in relation to the product $\mathfrak{v}\bar{\nu}$). This is closely related to the interpretation of eq. (12) that *an arbitrary vector* $\mathfrak{v}$ *is "linearly associated" with a new vector* $\Pi\mathfrak{v}$ *by means of the dyadic* Π (while only the unit vectors $\bar{\nu}$ are associated with $\mathfrak{p}_\nu$ under the original definition). The 9 elements of the matrix that is determined by the dyadic likewise define the coefficient schema of three equations that express the components of the transformed vector in terms of the original one. We turn to this precisely, in order to explain the analogous definitions of the concepts in the context of motor analysis.

Let Π be a motor dyadic and let $\mathfrak{P}_\mathfrak{e}$ be its value for the unit motor $\mathfrak{E}$. An arbitrary motor $\mathfrak{M}$ can always be regarded as a scalar multiple of a particular unit motor, so one perhaps sets $\mathfrak{M} = M\,\mathfrak{E}$. We then refer to the *product of the motor-dyadic* Π *with the motor* $\mathfrak{M}$ *when we mean the product of the number M and the motor* $\mathfrak{P}_\mathfrak{e}$ *of the dyadic* Π *that is associated with the unit motor* $\mathfrak{E}$ *of* $\mathfrak{M}$:

$$\Pi\,\mathfrak{M} = M\mathfrak{P}_\mathfrak{e} = M(\mathfrak{P}_4 E_1 + \mathfrak{P}_5 E_2 + \ldots + \mathfrak{P}_3 E_1) = M_1\mathfrak{P}_4 + M_2\mathfrak{P}_5 + \ldots + M_6\mathfrak{P}_3\,. \tag{13}$$

For the motor $\mathfrak{M}$, multiplication by the dyadic Π then means a transformation into a motor $\mathfrak{M}'$, which we write as $\Pi\,\mathfrak{M}$. One obtains the 6 transformation equations, which express the components of $\mathfrak{M}'$ in terms of the $\mathfrak{M}$, by decomposing (13) into components:

$$\left.\begin{aligned}
M_1' &= P_{11}M_4 + P_{21}M_5 + P_{31}M_6 + P_{41}M_1 + P_{51}M_2 + P_{61}M_3 \\
M_2' &= P_{12}M_4 + P_{22}M_5 + P_{32}M_6 + P_{42}M_1 + P_{52}M_2 + P_{62}M_3 \\
M_3' &= P_{13}M_4 + P_{23}M_5 + P_{33}M_6 + P_{43}M_1 + P_{53}M_2 + P_{63}M_3 \\
M_4' &= P_{14}M_4 + P_{24}M_5 + P_{34}M_6 + P_{44}M_1 + P_{54}M_2 + P_{64}M_3 \\
M_5' &= P_{15}M_4 + P_{25}M_5 + P_{35}M_6 + P_{45}M_1 + P_{55}M_2 + P_{65}M_3 \\
M_6' &= P_{16}M_4 + P_{26}M_5 + P_{36}M_6 + P_{46}M_1 + P_{56}M_2 + P_{66}M_3
\end{aligned}\right\} \tag{14}$$

Here, the 36 elements of the matrix described by (6) for the coefficient schema define the six linear equations that convert $\mathfrak{M}$ into $\mathfrak{M}'$. In it, the rows and columns must generally be switched, and one must observe that the row of the variable begins with M_4, not with M_1.

The application of the transformation property of dyadics (the multiplication that we are now treating, resp.) that is most important for us derives from the fact that *velocity motor of a body is transformed into the impulse motor* (viz., the motor of the quantity of motion) *by multiplication with the inertia dyadic*. We will speak of this more thoroughly in section III.

5. Dyadic product. In vector algebra, as well as in motor algebra, one can arrive at the concept of dyadic (if only a somewhat restricted form of it) by another process, and indeed, by a formal computation. Here, we would also like to discuss the argument that is associated with the vector theory and then that of the motor theory by analogy.

The concept of "dyadic product" originates with the demand that each factor in a triple product of vectors should be "distinguishable." For the product $\mathfrak{a} \cdot (\mathfrak{b} \times \mathfrak{c})$, the demand is fulfilled a result of the commutation rules (cf., I.**9**), with no further assumptions; for the product $\mathfrak{a} \times (\mathfrak{b} \times \mathfrak{c})$, the rule for development (I.**9**) gives no solution, so the product $\mathfrak{a}(\mathfrak{b} \cdot \mathfrak{c})$ remains to be resolved. If two arbitrary vectors $\mathfrak{a}$ and $\mathfrak{b}$ are given, one now defines their "Gibbs product dyadic" or Gibbs product to be the dyadic whose then value is equal to $\mathfrak{a}(\mathfrak{b} \cdot \overline{\nu})$ for the direction $\overline{\nu}$. One recognizes, with no further assumptions, the fact that this will actually establish a linear function of the direction as long as one writes the expression in the form $\mathfrak{a}\,[b_x \cos(\nu\, x) + b_y \cos(\nu\, y) + b_z \cos(\nu\, z)]$. Here, any direction $\overline{\nu}$ is associated with a vector that has the same direction as $\mathfrak{a}$ and whose length is a times the projection of $\mathfrak{b}$ onto the direction in question. With Gibbs, we denote the dyadic product by $\mathfrak{a}\,;\,\mathfrak{b}$ (written: *a* – prime – *b*). The nine components of this dyadic are:

$$\begin{Bmatrix} a_x b_x & a_y b_x & a_z b_x \\ a_x b_y & a_y b_y & a_x b_y \\ a_x b_z & a_y b_x & a_z b_z \end{Bmatrix}, \tag{15}$$

from which, one sees that its specific properties as a product of $\mathfrak{a}$ and $\mathfrak{b}$ come from, e.g., the fact that the components change under a variation of the length a or b by a factor that is proportionality to them. If one now denotes an arbitrary vector in the direction $\overline{\nu}$ by $\mathfrak{c}$, such that $\mathfrak{c} = c\overline{\nu}$, then one obtains, by the definition of the product of a dyadic with a vector (4):

$$(\mathfrak{a}\,;\,\mathfrak{b})\,\mathfrak{c} = \mathfrak{a}\,(\mathfrak{b} \cdot \overline{\nu}\,)\,\mathfrak{c} = \mathfrak{a}\,(\mathfrak{b} \cdot \mathfrak{c}). \tag{16}$$

With this, the formal objective is achieved of representing the vector defined by $\mathfrak{a}\,(\mathfrak{b} \cdot \mathfrak{c})$ as a product in which $\mathfrak{c}$ is a factor.

G. Jaumann has contrasted the Gibbs product dyadic with a second one that proves to be the analogue of the other ternary product that was defined above. We define the "Jaumann product dyadic," or Jaumann product, of the vectors $\mathfrak{a}$ and $\mathfrak{b}$ to be the dyadic whose value for the direction $\bar{\nu}$ equals $\mathfrak{a} \times (\mathfrak{b} \times \bar{\nu})$ and thus write $\mathfrak{a} \times \mathfrak{b}$ for it (read: *a* – cross – *b*). One again recognizes that one is dealing with a linear function of direction since the direction cosines enter into the expressions for the components of $\mathfrak{a} \times (\mathfrak{b} \times \bar{\nu})$ only linearly; e.g., the *x*-component is equal to $a_y\,[b_x \cos(\nu y) - b_y \cos(\nu x)] - a_x\,[b_z \cos(\nu x) - b_x \cos(\nu z)]$, etc. The nine-element matrix (whose first column is obtained from the present expression when one lets *n* coincide with *x*, *y*, *z*) then reads:

$$\begin{Bmatrix} -a_y b_z - a_z b_z & a_x b_y & a_x b_z \\ a_y b_x & -a_z b_z - a_z b_z & a_y b_z \\ a_z b_x & a_x b_y & -a_x b_x - a_y b_y \end{Bmatrix}. \tag{17}$$

When $\mathfrak{c}$ has the direction $\bar{\nu}$, the application of the product notion that was discussed in **4** yields:

$$(\mathfrak{a} \times \mathfrak{b})\,\mathfrak{c} = \mathfrak{a} \times (\mathfrak{b} \times \bar{\nu})\,\mathfrak{c} = \mathfrak{a} \times (\mathfrak{b} \times \mathfrak{c}), \tag{18}$$

with which, in fact, the desired objective is achieved. We will not go into the many applications of the dyadic product of vectors in various branches of mechanics and physics here, but turn to the definition of the analogous concepts in motor algebra.

We define the first *dyadic product of two motors* $\mathfrak{A}$ *and* $\mathfrak{B}$ to be a *motor dyadic* $\mathfrak{A}\,;\,\mathfrak{B}$ *whose value for the unit motor* $\mathfrak{C}$ *equals* $\mathfrak{A}\,(\mathfrak{B} \cdot \mathfrak{C})$. If one introduces the expression (3) in I for the scalar product $\mathfrak{B} \cdot \mathfrak{C}$ then one obtains the value of the dyadic from:

$$\mathfrak{A}\,B_4\,E_1 + \mathfrak{A}\,B_5\,E_2 + \mathfrak{A}\,B_6\,E_3 + \mathfrak{A}\,B_1\,E_4 + \mathfrak{A}\,B_2\,E_5 + \mathfrak{A}\,B_3\,E_6\,. \tag{19}$$

The 36-element schema of the dyadic, which must include the components of the motors $\mathfrak{A}\,B_1$, $\mathfrak{A}\,B_2$, …, $\mathfrak{A}\,B_6$ as its columns, then takes the form:

$$\begin{Bmatrix} A_1 B_1 & A_2 B_1 & \cdots\cdots & A_6 B_1 \\ A_1 B_2 & A_2 B_2 & \cdots\cdots & A_6 B_2 \\ \cdots & \cdots & \cdots\cdots & \cdots \\ A_1 B_6 & \cdots & \cdots\cdots & A_6 B_6 \end{Bmatrix}. \tag{19′}$$

Since any element in this contains a component of $\mathfrak{A}$ as a factor, it follows from this that multiplying $\mathfrak{A}$ will multiply the value of the dyadic to the same extent; moreover, the values behave additively when one replaces $\mathfrak{A}$ with a sum $\mathfrak{A}' + \mathfrak{A}''$. In other words: One has the rules for calculation:

$$(\lambda\,\mathfrak{A};\,\mathfrak{B}) = \lambda\,(\mathfrak{A};\,\mathfrak{B}), \qquad (\mathfrak{A}' + \mathfrak{A}'')\,;\,\mathfrak{B} = (\mathfrak{A}';\,\mathfrak{B}) + (\mathfrak{A}'';\,\mathfrak{B}), \tag{20}$$

and naturally the corresponding statements are true for $\lambda\,\mathfrak{B}$, ($\mathfrak{B}' + \mathfrak{B}''$, resp.). When $\mathfrak{C}$ refers to any motor, this yields:

$$(\mathfrak{A};\,\mathfrak{B})\;\mathfrak{C} = \mathfrak{A}\,(\mathfrak{B}\;\mathfrak{C}), \tag{21}$$

so the expression on the right in (21) emerges from (19), as long as one replaces the components $E_1, E_2, \ldots, E_6$ with $C_1, C_2, \ldots, C_6$, here.

We now define the *second dyadic motor product of the motors* $\mathfrak{A}$, $\mathfrak{B}$ *to be the motor dyadic* $\mathfrak{A} \times \mathfrak{B}$ (read: $\mathfrak{A}$ – cross – $\mathfrak{B}$), *whose value for the unit motor* $\mathfrak{C}$ *equals* $\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C})$. The first three scalar components of this, the motor associated with the unit motor $\mathfrak{C}$, are the components of the vector $\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C})$, thus, they are $A_2\,(B_1\,E_2 - B_2\,E_1) - A_3\,(B_1\,E_3 - B_1\,E_3)$, etc.; the other three are the components of the vector $\mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C}_o + \mathfrak{B}_o \times \mathfrak{C}) + \mathfrak{A}_o \times (\mathfrak{B} \times \mathfrak{C})$, of which, the first one reads: $A_2\,(B_1\,E_5 - B_2\,E_4 + B_4\,E_2 - B_5\,E_1) - A_3\,(B_3\,E_6 - B_1\,E_6 + B_6\,E_1 - B_6\,E_2) + A_5\,(B_1\,E_2 - B_2\,E_1) + A_6\,(B_3\,E_1 - B_1\,E_2)$, etc. One obtains the schema for the dyadic when one focuses on the factors of $E_1, E_2, \ldots, E_6$ in these expressions, as follows:

$$\begin{array}{cccccc}
0 & 0 & 0 & -(A_2B_2+A_3B_3) & A_1B_2 & A_1B_3 \\
0 & 0 & 0 & A_2B_1 & -(A_1B_1+A_3B_3) & A_2B_3 \\
0 & 0 & 0 & A_3B_1 & A_3B_2 & -(A_1B_1+A_2B_2) \\
-(A_2B_2+A_3B_3) & A_1B_2 & A_1B_3 & -(A_2B_5+A_3B_6+A_5B_2+A_6B_2) & A_1B_1+A_2B_2 & A_1B_1+A_2B_2 \\
A_2B_1 & -(A_3B_3+A_1B_1) & A_2B_2 & A_2B_4+A_5B_1 & -(A_3B_6+A_1B_4+A_6B_3+A_4B_2) & A_1B_1+A_2B_2 \\
A_3B_1 & A_3B_2 & -(A_1B_1+A_2B_2) & A_1B_4+A_6B_1 & A_3B_5+A_6B_2 & -(A_1B_4+A_2B_6+A_4B_1+A_5B_2)
\end{array}$$

Since any element again includes a factor of $\mathfrak{A}$ (as well as $\mathfrak{B}$) here, one has the rules of calculation that are analogous to (20):

$$(\lambda\,\mathfrak{A} \times \mathfrak{B}) = \lambda\,(\mathfrak{A} \times \mathfrak{B}), \qquad (\mathfrak{A}' + \mathfrak{A}'') \times \mathfrak{B} = (\mathfrak{A}' \times \mathfrak{B}) + (\mathfrak{A}'' \times \mathfrak{B}), \tag{23}$$

and naturally the analogous statements are true for the second factor $\mathfrak{B}$. Finally, one has, corresponding to (21), and from the product definition in **4**:

$$(\mathfrak{A} \times \mathfrak{B})\;\mathfrak{C} = \mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C}). \tag{24}$$

6. Product of two dyadics. The pursuit of the important question of how the 36 scalar components of a dyadic change when one chooses a new coordinate system leads to the definition of a product of two dyadics. We first give the explanation for the case of vector analysis.

Let A and B be two (vector-) dyadics whose values for a direction $\bar{v}$ were denoted by $\mathfrak{A}$ ($\mathfrak{B}_\nu$, resp.) up to now, and let Γ be a third dyadic with the components $\mathfrak{C}_\nu$. We denote the product of A and B by Γ and write:

$$\Gamma = A \times B, \qquad \text{if} \qquad \mathfrak{C}_\nu = B\,\mathfrak{A}_\nu; \tag{25}$$

i.e., therefore any value of Γ is the value of A after it has been transformed by B. Since the $\mathfrak{A}_\nu$ define a linear function of the direction cosines, the same is true for $B\,\mathfrak{A}_\nu$, such that, in fact, a dyadic is defined by the second of eq. (25). The value of Γ for the x-direction is $\mathfrak{C}_\nu = B\,\mathfrak{A}_x = A_{xx}\,\mathfrak{B}_x + A_{xy}\,\mathfrak{B}_y + A_{xz}\,\mathfrak{B}_z$. If one takes the x-component of this then one obtains $C_{xx} = A_{xx}\,B_{xx} + A_{xy}\,B_{yx} + A_{xz}\,B_{zx}$, likewise, the y-component is $C_{xy} = A_{xx}\,B_{xy} + A_{xy}\,B_{yy} + A_{xz}\,B_{zy}$, etc. The matrix of Γ thus takes a form in which the i^{th} row and the k^{th} column arise from a sum of three products whose first factor is in the i^{th} row of A and whose second factor is in the k^{th} column of B, where both sequences run from the beginning to the end. One sees that by this definition the product $B \times A$ is essentially different from the product $A \times B$. If one exchanges the two rows and columns in each of the given dyadics then the elements are transposed in such a way that the nearest neighbors remain nearest neighbors, and conversely, so the product that is defined after the transposition is, by definition equal to $B \times A$.

We would now like to pursue this argument somewhat more thoroughly in the domain of motor analysis. We consider two motor dyadics $\mathbf{A}$ and $\mathbf{B}$ with the motor components $\mathfrak{A}$ and $\mathfrak{B}$, and introduce a product $\boldsymbol{\Gamma}$ of $\mathbf{A}$ and $\mathbf{B}$ by the following definition: *The product of a dyadic* $\mathbf{A}$ *with the dyadic* $\mathbf{B}$ *is defined to be a dyadic* $\boldsymbol{\Gamma}$ *whose value* $\mathfrak{C}_{\mathfrak{c}}$ *for an arbitrary unit motor* $\mathfrak{C}$ *is equal to the corresponding value* $\mathfrak{A}_{\mathfrak{c}}$ *of the dyadic* $\mathbf{A}$ *after it has been transformed by the dyadic* $\mathbf{B}$*, so:*

$$\boldsymbol{\Gamma} = \mathbf{A} \times \mathbf{B}, \qquad \text{when} \quad \mathfrak{C}_{\mathfrak{c}} = \mathbf{B} \cdot \mathfrak{A}_{\mathfrak{c}}. \tag{28}$$

Since $\mathfrak{A}_{\mathfrak{c}}$ is a linear function of the components of $\mathfrak{C}$, the same is also true for $\mathfrak{C}_{\mathfrak{c}}$, such that a dyadic is actually determined by (28).

If we let $\mathfrak{C}$ be the unit motor with the scalar components 0, 0, 0, 1, 0, 0 then, from (5), one can write the motor $\mathfrak{A}_1$ for $\mathfrak{A}_{\mathfrak{c}}$, and one can write the corresponding $\mathfrak{A}_1$ for $\mathfrak{A}_{\mathfrak{c}}$ from the given definition. From (14), the components $C_{11}, C_{12}, \ldots, C_{16}$ are calculated when one now sets $B_{11}, B_{12}, \ldots, B_{16}$, in place of $P_{11}, P_{12}, \ldots, P_{16}$, and $A_{11}, A_{12}, \ldots, A_{66}$, in place of $M_{11}, M_{12}, \ldots, M_{16}$, so:

$$\left.\begin{aligned} C_{11} &= A_{14}B_{11} + A_{15}B_{21} + A_{16}B_{31} + A_{11}B_{41} + A_{12}B_{51} + A_{13}B_{61} \\ C_{12} &= A_{14}B_{12} + A_{15}B_{22} + A_{16}B_{32} + A_{11}B_{42} + A_{12}B_{52} + A_{13}B_{62} \\ &\cdots \end{aligned}\right\} \tag{29}$$

The 6 expressions, the first of which will be written out here, appear in the first row of the 36-element schema for the dyadic $\mathbf{B}$. The elements $C_{21}, C_{22}, \ldots, C_{26}$ of the second row are the components of $\mathfrak{C}_2$ and are computed in the same way from the components $A_{21}, A_{22}, \ldots, A_{26}$ of $\mathfrak{C}_2$. In summary, one can write all of these expressions in the form:

$$C_{\iota\kappa} = A_{\iota 4}\, B_{1\kappa} + A_{\iota 5}\, B_{2\kappa} + A_{\iota 6}\, B_{3\kappa} + A_{\iota 1}\, B_{3\kappa} + A_{\iota 2}\, B_{5\kappa} + A_{\iota 3}\, B_{6\kappa}\,, \tag{29′}$$

or, more briefly, in the form:

$$C_{\iota\kappa} = \sum_{\rho=1}^{6} A_{\iota\rho} B_{\rho+3,\kappa}\,, \tag{29″}$$

in which naturally the indices that are higher than 6 are residues modulo 6; i.e., for 7, 8, 9, one sets 1, 2, 3. These conventions yield the following rule for the formation of the product of two dyadics: *One obtains the element in the* ι^{th} *row and the* κ^{th} *column of the product* $\mathbf{\Gamma}$ *as a sum of six products of two factors; the one factor is given by the sequence of elements in the* i^{th} *row of* $\mathbf{A}$*, while the other one is given by the elements of the* k^{th} *column of* $\mathbf{B}$*, after the sequence has been (cyclically) permuted by three places.*

The definition and formulas (28) and (29) tell one nothing more than the fact that the operation that we have now defined is associative and that multiplication by a scalar distributes with the addition of dyadics; i.e., one has:

$$\lambda(\mathbf{A} \times \mathbf{B}) = (\lambda\mathbf{A}) \times \mathbf{B}, \qquad \mathbf{\Gamma} \times (\mathbf{A} + \mathbf{B}) = \mathbf{\Gamma} \times \mathbf{A} + \mathbf{\Gamma} \times \mathbf{B}\,. \tag{30}$$

On the contrary, the commutative law is not valid. Then, when the roles of $\mathbf{A}$ and $\mathbf{B}$ are switched, the elements of the i^{th} row of $\mathbf{B}$ and the k^{th} column of $\mathbf{A}$ appear in (29′). One then has:

$$\mathbf{\Gamma} = \mathbf{A} \times \mathbf{B}, \qquad \text{and similarly} \qquad \mathbf{\Gamma}' = \mathbf{B}' \times \mathbf{A}', \tag{31}$$

in which the symbols with an accent denote the dyadics that come about under transposition – i.e., switching the rows and columns – of the ones without accents.

A second associative law of practical utility is expressed by the following formula:

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{\Gamma} = \mathbf{A} \times (\mathbf{B} \times \mathbf{\Gamma}). \tag{32}$$

In order to prove this, one only observes that, from (29″), the element with the indices ι, λ on the left-hand side is the double summation over ρ and κ of $A_{\iota\rho}\, B_{\rho+3,\kappa}\, C_{\kappa+3,\lambda}$; here, if one combines the second and third factors for a fixed ρ then this gives the element with the indices $\rho + 3$, λ of the product $\mathbf{B} \times \mathbf{\Gamma}$, such that the total sum also yields the right-hand side, with the addition of the first factor of the element with the indices ι, λ. The brackets may thus continue to be omitted in writing down the triple products.

7. Coordinate transformation. One finds applications for the product concept, above all, when one deals with the representation of the transition from one coordinate cross to another. Let – first, in the context of vector algebra – x', y', z' be the new (right-angled) axis directions and let Π be an arbitrary vector dyadic with the components (referred to the old axis) $\mathfrak{p}_x$, $\mathfrak{p}_y$, $\mathfrak{p}_z$ (p_{xx}, p_{xy}, …, p_{zz}, resp.). If we briefly denote the cosine of the angle between a new axis and an old one by symbols in brackets: (xx'), (xy'), etc., then we have the value of the dyadic in the x-direction in the form $\mathfrak{p}_x(xx') + \mathfrak{p}_y(yx') +$

$\mathfrak{p}_z(zx')$. In order to obtain the components of this vector – for example, for the y'-direction – we must we must add the three components in the x, y, and z-directions, when multiplied by (xy'), (yy'), and (zy'), resp., in sequence. One then obtains:

$$p_{x'y'} = (xy')[p_{xx}(xy') + p_{yx}(yx') + p_{zx}(zx')] + (yy')[p_{zx}(xx') + p_{yy}(yx') + p_{zy}(zx')] \\ + (zy')[p_{zx}(xx') + p_{yz}(yx') + p_{zz}(zx')], \tag{26}$$

and analogous expressions for the remaining components. If one now introduces the two vector dyadics Ω and Ω' that arise by transposition, and whose matrices look like:

$$\begin{Bmatrix} (xx') & (yx') & (zx') \\ (xy') & (yy') & (zy') \\ (xz') & (yz') & (zz') \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} (xx') & (xy') & (xz') \\ (yx') & (yy') & (yz') \\ (zx') & (zy') & (zz') \end{Bmatrix}, \tag{27}$$

then one finds that the three expressions in the square brackets define precisely the three elements of the first row of $\Omega \times \Pi$. If they exist as products whose second factor is always taken to be the first row of Ω, while the other factor comes from the first, second, and third column of Π, resp. Now, these expressions are multiplied in sequence by the elements of the second column of Ω'. Therefore, the expression (26) defines the element of the first row and second column of a product dyadic that one can write in the form $(\Omega \times \Pi) \times \Omega'$. However, since one also finds in the other combinations of the terms in (26) that they define the analogous elements in the product $\Omega \times (\Pi \times \Omega')$, one may also drop the parentheses and say:

One obtains the components of the dyadic Π *for the new axis cross* $x'y'z'$ *from the components of the product* $\Omega \times \Pi \times \Omega'$, *when referred to the old axes, where* Ω, Ω' *are the dyadics that were defined in* (27) *in terms of the direction cosines.*

In the context of motor algebra, the coordinate transformation means a transition from an axis cross with the origin o and the axis directions x, y, z to a new reference system with o' as origin and the axis directions x', y', z'. The vector that reaches from o to o', when referred to the old directions, has the components a, b, c, while the nine directions are established in terms of the old ones by the nine direction cosines (27). We ask: What do the scalar components $P'_{\iota\kappa}$ of a motor dyadic Π look like they are when referred to the nine axes if the old components are given by $P_{11}, \ldots, P_{66}$? In order to respond to this question, we consider six unit motors $\mathfrak{E}'_1, \ldots, \mathfrak{E}'_6$, of which, the first three, when regarded as dynames, are pure force couples with moment vectors of length 1 in the directions of the nine axes, and the last three are individual forces of equal magnitude and direction that act on o'. The components of these six unit motors, referred to the old axis cross, yield the following schema when written in successive columns:

$$\left\{\begin{array}{cccccc} 0 & 0 & 0 & (xx') & (yx') & (zx') \\ 0 & 0 & 0 & (xy') & (yy') & (zy') \\ 0 & 0 & 0 & (xz') & (yz') & (zz') \\ (xx') & (yx') & (zx') & b(zx')-c(yx') & c(xx')-a(zx') & a(yx')-b(xx') \\ (xy') & (yy') & (zy') & b(zy')-c(yy') & c(xy')-a(zy') & a(yy')-b(xy') \\ (xz') & (yz') & (zz') & b(zz')-c(yz') & c(zz')-a(yz') & a(yz')-b(xz') \end{array}\right\}, \tag{33}$$

For the values that the dyadic Π assumes in succession for the six unit motors $\mathfrak{E}'_1$ to $\mathfrak{E}'_6$, we must define the six components relative to the new axis cross and combine them: They are then the desired $P'_{\iota\kappa}$. Now, the value of Π for $\mathfrak{E}'_\iota$ is, from (5), equal to the sum taken from ρ = 1 to 6 of $\mathfrak{P}_{\rho+3}\,E'_{\iota,\rho}$ and its κ^{th} component (relative to the old axis cross) equals the sum of $P_{\rho+3,\kappa}\,E'_{\iota,\rho}$. From the six old components that this formula delivers for k = 1, 2, …, 6, one defines the component that corresponds to the unit motor from (4), when one multiplies these components by the components of $\mathfrak{E}'_\lambda$ in the sequence 4, 5, 6, 1, 2, 3, so the value $P'_{\iota\lambda}$ is represented as a double sum according to:

$$P'_{\iota\lambda} = \sum_{\rho,\kappa} E'_{\iota\rho} P_{\rho+3,\kappa} E'_{\lambda,\kappa+3}\,. \tag{34}$$

This expression agrees precisely with the one that was given above in connection with (32) for the general term for the components of a triple product, when one replaces the symbol B with P, and now writes $E'_{\iota\rho}$ for $A_{\iota\rho}$ and $E'_{\lambda\kappa}$ for $C_{\kappa\lambda}$. Thus, we have the theorem:

If a new axis cross is given by the displacement vector a, b, c, and the direction cosines (xx'), … (zz') *then one defines the motor dyadic* $\mathbf{\Omega}$ *whose schema is* (33), *and the one* $\mathbf{\Omega}'$ *that arises from transposition; the elements of the product* $\mathbf{\Omega}\times\mathbf{\Pi}\times\mathbf{\Omega}'$ *(when referred to the old axis) are then equal to the elements of* $\mathbf{\Pi}$, *when referred to the new axis cross.*

The complete analogy between this theorem and the one in vector analysis that was discussed above is immediate.

If the coordinate change is a pure translation (pure rotation, resp.) then the schema for Ω assumes the simpler form:

$$\begin{Bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & c & -b \\ 0 & 1 & 0 & -c & 0 & a \\ 0 & 0 & 1 & b & -a & 0 \end{Bmatrix}, \qquad \begin{Bmatrix} 0 & 0 & 0 & (xx') & (yx') & (zx') \\ 0 & 0 & 0 & (xy') & (yy') & (zy') \\ 0 & 0 & 0 & (xz') & (yz') & (zz') \\ (xx') & (yx') & (zx') & 0 & 0 & 0 \\ (xy') & (yy') & (zy') & 0 & 0 & 0 \\ (xz') & (xz') & (zz') & 0 & 0 & 0 \end{Bmatrix}, \text{ resp.} \tag{33′}$$

If one sets a, b, c equal to zero in the first case, or makes the angle between x and x' in the second one equal to zero, etc., then one arrives at a matrix that contains nothing but ones in the two three-rowed parallels to the main diagonal, but zeroes everywhere else. Multiplication by this dyadic leaves a motor, as well as any other arbitrary dyadic, unchanged; one is then justified in calling it the "identity dyadic," or unit dyadic.

The dyadic $\mathbf{\Omega}$ whose schema is (33) (its transpose, resp.) can also serve to represent the transformation of a motor to a new axis cross. One needs only to imagine that the components M'_ι of a motor $\mathfrak{M}$, when referred to the new axis cross, equal the values of $\mathfrak{M}$ for the unit motors $\mathfrak{E}'_\iota$, so:

$$M'_\iota = \sum_\rho E'_{\iota\rho} M_{\rho+3} = E'_{\iota 1} M_4 + E'_{\iota 2} M_5 + \ldots + E'_6 M_1 \;. \tag{34′}$$

If one compares this with (14) the one sees that the product $\mathbf{\Omega} \cdot \mathfrak{M}$ gives the motor with the components M'_ι:

One obtains the components of a motor $\mathfrak{M}$ relative to a new axis cross when one defines the old components of the product $\mathbf{\Omega}' \cdot \mathfrak{M}$.

8. Infinitesimal transformations. Application to the inertia dyadic. For the applications in mechanics, it is worthwhile to also consider continuous coordinate transformations, as would relate to a continuously-moving reference system. It is most convenient to introduce both the concepts of time and velocity here. We thus assume that the axis cross o'; x', y', z' coincides with o; x, y, z at time $t = 0$, but then moves with the velocity motor $-\boldsymbol{\mathfrak{G}}$, such that $-\mathfrak{G}_o$ is the velocity vector of the origin and $-\mathfrak{G}$ represents the rotational velocity vector of the new axes with respect to the old ones. The components of $-\mathfrak{G}_o\,dt$ serve as the displacement magnitudes a, b, c of **7**, while the three direction cosines (xx'), (yy'), (zz') differ from 1 only by quantities of second order in dt such that one must set them equal to 1. The value of $(x'y)$ is, as one easily sees, equal to the angle between the x-axis and the projection of the x-axis onto the xy-plane, up to terms of higher order, and this is $-\mathfrak{G}_3\,dt$. We would like to the subtract the unit dyadic $\mathbf{E}$ (i.e., drop the ones in the parallels to the main diagonal) from the dyadic $\mathbf{A}$ that is constructed from these values using (33), and divide what remains by dt. The dyadic that thus results, which we call $\mathbf{O}$, has the following schema:

$$\begin{Bmatrix} 0 & 0 & 0 & 0 & -G_3 & G_2 \\ 0 & 0 & 0 & G_3 & 0 & -G_1 \\ 0 & 0 & 0 & -G_2 & G_1 & 0 \\ 0 & -G_3 & G_2 & 0 & -G_6 & G_5 \\ G_3 & 0 & -G_1 & G_6 & 0 & -G_4 \\ -G_2 & G_1 & 0 & -G_5 & G_4 & 0 \end{Bmatrix}. \tag{35}$$

If one now transposes the rows and columns then the same thing results as when one switches the plus and minus signs everywhere – i.e., one has $\mathbf{O}' = -\mathbf{O}$, while one has $\mathbf{E}' = \mathbf{E}$ for the identity dyadic. The elements of a dyadic Π relative to the new axis cross are, from the results of **7**, equal to the elements of $(\mathbf{O}\,dt + \mathbf{E}) \times \mathbf{\Pi} \times (\mathbf{O}'\,dt + \mathbf{E})$ and, since $\mathbf{E} \times \mathbf{\Pi} = \mathbf{\Pi} \times \mathbf{E}$ is equal to:

$$(\mathbf{O} \times \mathbf{\Pi}\,dt + \mathbf{\Pi}) \times (\mathbf{O}'\,dt + \mathbf{E}) = (\mathbf{O} \times \mathbf{\Pi} + \mathbf{\Pi} \times \mathbf{O}')\,dt + \mathbf{\Pi}.$$

up to terms in second order. If we now ask what the change is that the elements of $\mathbf{\Pi}$ experience during the motion, and divide this change by the time unit, in order to obtain finite quantities, then we have to subtract $\mathbf{\Pi}$ from the present expression and then divide it by dt, and what finally remains is:

$$\frac{d\mathbf{\Pi}}{dt} = \mathbf{O} \times \mathbf{\Pi} + \mathbf{\Pi} \times \mathbf{O}' = \mathbf{O} \times \mathbf{\Pi} - \mathbf{\Pi} \times \mathbf{O}. \tag{36}$$

The fact that we have the derivative of $\mathbf{\Pi}$ with respect to time on the left-hand side is justified as follows: If we imagine that the motor dyadic $\mathbf{\Pi}$ has unchanging elements relative to an axis cross that moves with the velocity + $\mathfrak{G}$ (like, e.g., the inertia dyadic of a rigid body for a system that moves with the body) then in the time dt a reference system that is at rest in space acquires a displacement – $\mathfrak{G}\,dt$ compared to the moving one and (36) gives precisely the change – during the time interval dt – that $\mathbf{\Pi}$ has experienced, as seen from the system at rest. The dyadic $\mathbf{O}$ is naturally determined by the velocity motor $\mathfrak{G}$ alone. It is convenient to also express in the notation and – with the introduction of a dyadic product of a motor with a dyadic – write (36) as:

$$\frac{d\mathbf{\Pi}}{dt} = \mathfrak{G} \times \mathbf{\Pi}. \tag{36′}$$

Therefore, the agreement between (36′) and (36) serves as definition of this product.

The expression (36) enters the dyadic algebra in place of the expression $\mathfrak{G} \times \mathfrak{M}$ of motor algebra that was mentioned at the conclusion of I.**7**, which gives the temporal change of a motor $\mathfrak{M}$ that moves with the velocity $\mathfrak{G}$. One also easily convinces oneself that $\mathfrak{G} \times \mathfrak{M}$ is identical to $\mathbf{O}' \cdot \mathfrak{M}$, by examining the components.

The practical meaning of the formula (36) derives, above all, from the fact that allows one to write down the changes in the 36 components of $\mathbf{\Pi}$ in a purely schematic way by applying the multiplication rule for **6**. One has, e.g., when we write, for the sake of clarity, u, v, w for G_4, G_5, G_6 and ω_1, ω_2, ω_3 for G_1, G_2, G_3 :

$$\left.\begin{aligned}
\frac{dP_{11}}{dt} &= -\omega_3 P_{21} + \omega_2 P_{21} - \omega_3 P_{12} + \omega_2 P_{13}, \\
\frac{dP_{14}}{dt} &= -\omega_3 P_{24} + \omega_2 P_{34} - w P_{12} + v P_{13} - \omega_3 P_{15} + \omega_2 P_{16}, \\
\frac{dP_{41}}{dt} &= -\omega_3 P_{42} + \omega_2 P_{43} - w P_{31} + v P_{31} - \omega_3 P_{51} + \omega_2 P_{61}, \\
\frac{dP_{44}}{dt} &= -w P_{24} + v P_{34} - \omega_3 P_{54} + \omega_2 P_{64} - w P_{42} + v P_{43} - \omega_3 P_{45} + \omega_2 P_{46}.
\end{aligned}\right\} \tag{37}$$

The value of the general transformation formulas in the foregoing section becomes much clearer when we apply them to the example of the inertia dyadic. From (11), the inertia dyadic, when we place the origin of the reference system at the center of mass and the axis directions in the principal axes of inertia, consists of elements along the main diagonal exclusively: $P_{11} = P_{22} = P_{33} = m$, $P_{44} = J_1$, $P_{55} = J_2$, $P_{66} = J_3$. In the double sum (34), then, there are only non-zero summands, for which $\rho + 3 = \kappa$. Therefore, (34) must be replaced by the simple sum:

$$P'_{\iota\kappa} = \sum_{\chi} E'_{\iota,\chi+3} P_{\chi\chi} E'_{\lambda,\chi+3} . \tag{38}$$

In order to obtain a well-defined $P'_{\iota\kappa}$, one must multiply the elements of the ι^{th} and λ^{th} rows of the matrix (33) that are above or below each other and then add this to the corresponding term in the main diagonal of the inertia dyadic as a third factor. One easily convinces oneself that nothing remains in the first quadrant of $\mathbf{\Pi}$ ($\iota \leq 3$, $\lambda \leq 3$) except for the m in the main diagonal. For $\iota = 1$, $\lambda = 5$, one obtains:

$$\left.\begin{aligned}
m[(xx')\{b(zy') - c(yy')\} + (yx')\{c(xy') - a(zy')\} + (zx')\{a(yy') - b(xy')\}] \\
= -m[a(xz') + b(yz') + c(zz')].
\end{aligned}\right\} \tag{38$'$}$$

The equality follows from the well-known property of the matrix (27) that each of its elements is equal to the associated sub-determinants. The expression itself is obviously the static moment S_3 that, from (11), belongs to the inertia matrix at this location. If we now look for the element $\iota = \lambda = 4$ then when we consider the properties of (27) we obtain:

$$\left.\begin{aligned}
J'_1 = J_1(xx')^2 + J_2(yx')^2 + J_3(zx')^2 + m(b'^2 + c'^2) \\
\text{with} \quad b' = a(xy') + b(yy') + c(zy'), \quad c' = a(xz') + b(yz') + c(zz').
\end{aligned}\right\} \tag{38$''$}$$

One thus succeeds in the most general transformation formula for the inertia dyadic, and, when one chooses, perhaps, $\iota = 4$, $\lambda = 5$, the deviation moment. One obtains the so-called

Steiner theorem of parallel translation with $(xx') = 1, \ldots, (xy') = 0$ in the form $J_1' = J_1 + m(b^2 + c^2)$ and from this, one sees its actual source.

We would like to apply the infinitesimal transformation that is expressed by (37) to the general form of the inertia dyadic, for which no specialized origin-reference system will be assumed. When we replace the values in (11) for the $P_{\iota\lambda}$ in (37) this then shows immediately, that non-zero derivatives appear only at the locations in (11) that are not filled with zero of the value m; e.g., for $\iota = \lambda = 4$, $\iota = 5$, $\lambda = 6$, and $\iota = 2$, $\lambda = 6$:

$$\left.\begin{aligned}\frac{dJ_1}{dt} &= 2wS_3 + 2vS_2 + 2\omega_3 D_3 - 2\omega_2 D_2,\\ \frac{dD_1}{dt} &= \omega_1(J_1 - J_3) + \omega_3 D_2 - \omega_2 D_3 + wS_2 + vS_3,\\ \frac{dS_1}{dt} &= mu + \omega_2 S_3 - \omega_3 S_2.\end{aligned}\right\} \tag{39}$$

These formulas are important for the general Ansätze of the equations of motion of rigid bodies (cf., III, **1** and **2** for this).

9. Invariants of the dyadic. From the mathematical standpoint, it is of great interest to ask what expressions are defined by the elements of a dyadic(and therefore, what functions of these elements) that preserve their value when one replaces the original elements with the values of the elements that result from a transformation of the reference system. Here, the question of the "invariants" shall be treated only to the extent that is meaningful in the applications.

From the calculation in **7** and **8**, as well as the original definition of the motor dyadic, it emerges that the "first quadrant" of its matrix is transformed "into itself," and indeed in exactly the same way as the matrix of a vector dyadic. If we apply, say, the formula (34), which allows one to express a new element $P'_{\iota\lambda}$ in terms of the old one $P_{\iota\lambda}$, to a number pair $\iota\lambda$ that runs through the values from 1 to 3 then this shows that well-defined values of E' can only come from the first three rows of (33). However, they contain only direction cosines (xx'), (xy'), etc., and are zero whenever not only ρ, but also $\kappa + 3$ is greater than 3; this further implies that only values of P whose indices both lie between 1 and 3 (inclusive), have any influence on the P' under scrutiny. We thus first need to consider the invariants of the first quadrant of the entire matrix, which we regard as a vector dyadic, which also immediately shows us the way to deeper investigations.

In order to determine the invariants of a vector dyadic, we start with its definition, from which any direction in space will be associated with a vector by means of a certain linear eq. (21), and then ask what the directions v are whose associated vector falls in the $+ v$ or $- v$ direction. If we denote its length by λ, and likewise the elements of the vector dyadic by P_{11}, P_{12} , …, P_{33}, then the direction cosines of the desired direction υ must satisfy the three equations:

$$\left.\begin{aligned} \lambda\cos(vx') &= P_{11}\cos(vx) + P_{21}\cos(vy) + P_{31}\cos(vz), \\ \lambda\cos(vy') &= P_{12}\cos(vx) + P_{22}\cos(vy) + P_{32}\cos(vz), \\ \lambda\cos(vz') &= P_{13}\cos(vx) + P_{23}\cos(vy) + P_{33}\cos(vz), \end{aligned}\right\} \tag{40}$$

which arise from the component decomposition of (2). In order for these equations, which are linear and homogeneous in the cosines, to have a solution, it is, as is well-known, necessary and sufficient that the determinant of the coefficients vanish. This again delivers an equation of third degree in λ:

$$\lambda^3 - A_1\,\lambda^3 + A_2\,\lambda - A_3 = 0, \tag{41}$$

through whose three coefficients A_1, A_2, A_3 the value of λ can be determined. Since, by the definition of λ, the coordinate system can play no role, these A must be invariants. One easily computes that A_1 is the sum $P_{11} + P_{22} + P_{33}$ of the coefficients that occupy the main diagonal, P_3 is the entire determinant of P (when λ is set to zero), and P_2 is the sum of the elements of the main diagonal to the adjoint sub-determinants. One then obtain the three invariants of a vector dyadic, but likewise also a subset of the invariants of the motor dyadic $\mathbf{\Pi}$ in the form:

$$A_1 = P_{11} + P_{22} + P_{33}, \quad A_2 = \begin{vmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{vmatrix} + \begin{vmatrix} P_{33} & P_{21} \\ P_{13} & P_{11} \end{vmatrix} + \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix}, \qquad A_3 = \begin{vmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{vmatrix}. \tag{42}$$

We can now apply exactly the same reasoning in order to find invariants of the motor dyadics. In order to not revert to the unit screws, we use eq. (14) as the basis and ask which motors $\mathfrak{M}$ have associated motors $\mathfrak{M}'$ that differ from $\mathfrak{M}$ by a scalar factor λ. If we set the left-hand side of (14) equal to λM_1, λM_2 , …, λM_6 in sequence then this gives six linear, homogeneous equations for the M, whose solubility condition is expressed by the determinant equation:

$$\begin{vmatrix} P_{11} & P_{21} & P_{31} & P_{41}-\lambda & P_{51} & P_{61} \\ P_{12} & P_{22} & P_{32} & P_{42} & P_{52}-\lambda & P_{62} \\ P_{13} & P_{23} & P_{33} & P_{43} & P_{53} & P_{63}-\lambda \\ P_{63}-\lambda & P_{24} & P_{34} & P_{44} & P_{54} & P_{64} \\ P_{15} & P_{25}-\lambda & P_{35} & P_{45} & P_{55} & P_{65} \\ P_{16} & P_{26} & P_{36}-\lambda & P_{46} & P_{56} & P_{66} \end{vmatrix} = 0. \tag{43}$$

The six coefficients in the equation of sixth degree for λ, which is identical with the present one:

$$\lambda^6 - B_1\,\lambda^5 + B_2\,\lambda^4 - B_3\,\lambda^3 + B_4\,\lambda^2 - B_5\,\lambda + B_6 = 0, \tag{44}$$

are our new invariants. One sees, with no further assumptions, that B_6 is the determinant of all 36 elements, B_1 is the sum of the P that are next to the λ, B_2 is the sum of certain sub-determinants of second order, and in all:

$$B_1 = \sum_{\alpha} P_{\alpha,\alpha+3} \quad B_2 = \sum_{\alpha,\beta} \begin{vmatrix} P_{\alpha,\alpha+3} & P_{\alpha,\beta+3} \\ P_{\beta,\alpha+3} & P_{\beta,\beta+3} \end{vmatrix} \quad \cdots$$

$$B_5 = \sum_{\alpha,\beta,\gamma,\delta,\varepsilon} \begin{vmatrix} P_{\alpha,\alpha+3} & P_{\alpha,\beta+3} & \cdots & P_{\alpha,\varepsilon+3} \\ P_{\beta,\alpha+3} & P_{\beta,\beta+3} & \cdots & P_{\beta,\varepsilon+3} \\ \cdots & \cdots & \cdots & \cdots \\ P_{\varepsilon,\alpha+3} & P_{\varepsilon,\beta+3} & \cdots & P_{\varepsilon,\varepsilon+3} \end{vmatrix}, \quad B_6 = \begin{vmatrix} P_{14} & P_{15} & \cdots & P_{13} \\ P_{24} & P_{25} & \cdots & P_{23} \\ \cdots & \cdots & \cdots & \cdots \\ P_{64} & P_{65} & \cdots & P_{63} \end{vmatrix} \tag{45}$$

The summation symbols α, β, … run through all combinations without repetition of the numbers 1 to 6, where, as always, indices that become greater than 6 are replaced with the same number, reduced by 6.

10. Symmetric dyadics. In the applications, one is mostly involved with the case of symmetric dyadics. Therefore, it will now be assumed that the 36 elements of the matrix of Π satisfy the 15 conditions $P_{\iota\kappa} = P_{\kappa\iota}$, so the schema of the dyadic is symmetric with respect to the main diagonal. One can then arrive at the invariants (45), and likewise (42), by following a somewhat altered line of reasoning. If one forms the scalar product of a motor $\mathfrak{M}$ with the motor $\mathfrak{M}'$ that is associated with it by means of the dyadic Π then what arises is a form of second degree in the components M_1 to M_6 whose coefficients are the P. If one subtracts from this form, the expression $\lambda(M_1 M_4 + M_2 M_5 + \ldots + M_6 M_3)$, whose value, from I (3) is independent of the reference system, then this produces a quadratic form whose coefficient matrix is written down in (43). Setting the determinant equal to zero delivers the value of λ for which the positive-definite character of the form can vanish, which thus represents an invariant relationship under coordinate changes; the coefficients of eq. (46) are then invariants. However, one can now go a bit further and also subtract $\lambda'(M_1^2 + M_2^2 + M_3^2)$ from the original form; i.e., λ' times the square of the length of the first vector component of $\mathfrak{M}$. With the differences $P_{44} - \lambda'$, $P_{55} - \lambda'$, $P_{66} - \lambda'$, in place of P_{44} , P_{55} , P_{66} , what results from setting the determinant equal to zero is an equation in λ, λ' of the form:

$$\left.\begin{aligned} \lambda^6 - B_1\lambda^5 + (B_2 - A_1\lambda')\lambda^4 - (B_3 + B_3'\lambda')\lambda^3 + (B_4 - \lambda' B_4' + A_2\lambda'^2)\lambda^2 \\ -(B_5 - \lambda' B_5' + \lambda'^2 B_5'')\lambda + B_6 - B_6'\lambda' + B_6''\lambda'^2 - A_3\lambda'^3 = 0. \end{aligned}\right\} \tag{46}$$

The coefficients A_1 to A_3 and B_1 to B_6 are the ones that were given in (42) ((45), resp.), while the other ones represent six new invariants that can likewise be computed as certain determinants; e.g.:

$$B_6'' = \sum_{\alpha} \begin{vmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{vmatrix}, \qquad B_5'' = \sum_{\alpha,\beta,\gamma} \begin{vmatrix} P_{\alpha\gamma} & P_{\alpha\alpha} & P_{\alpha\beta} \\ P_{\beta\gamma} & P_{\beta\alpha} & P_{\beta\beta} \\ P_{\gamma+3,\gamma} & P_{\gamma+3,\alpha} & P_{\gamma+3,\beta} \end{vmatrix}, \qquad \text{etc.} \tag{47}$$

For the α, β, … one now takes only combinations of the numbers from 1 to 3, while considering the constraint that was imposed above for the other ones.

The evaluation of the question of whether the expressions that were given in (42), (45), and (47) are all independent invariants, at least for the symmetric case, leads to the following argument. We prove:

If λ_1, λ_2 are two different roots of (44) *then the associated motors $\mathfrak{M}$ and $\mathfrak{M}'$, which are defined by the equations:*

$$\lambda_1\, \mathfrak{M}' = \mathbf{\Pi} \cdot \mathfrak{M}', \qquad \lambda_2\, \mathfrak{M}'' = \mathbf{\Pi} \cdot \mathfrak{M}'', \tag{48}$$

are mutually orthogonal.

Namely, if one multiplies the first of the scalar equations by $\mathfrak{M}''$ and likewise, the second one by $\mathfrak{M}'$, then one obtains:

$$(\lambda_1 - \lambda_2)\, \mathfrak{M}' \cdot \mathfrak{M}'' = \mathfrak{M}''(\mathbf{\Pi} \cdot \mathfrak{M}') - \mathfrak{M}'(\mathbf{\Pi} \cdot \mathfrak{M}'').$$

Application of the multiplication formula (13) shows that the κ^{th} component of the product that appears in the first bracket on the right, is the sum of $P_{\iota+3,\kappa}\, M_\iota'$ over ι, so the first expression on the right is the sum of $M_\kappa'' P_{\iota+3,\kappa}\, M_\iota'$ over ι and κ. The exchange of M' and M'', which leads to the second expression on the right-hand side, is equivalent here to the exchange of the indices ι, κ; which does not change the sum, however, due to the symmetry $P_{\iota\kappa} = P_{\kappa\iota}$. With this, both expressions are exactly equal, so $\mathfrak{M}' \cdot \mathfrak{M}'' = 0$. Q. E. D.

The theorem:

$$\mathfrak{A}\, (\mathbf{\Pi} \cdot \mathfrak{B}) = \mathfrak{B}\, (\mathbf{\Pi} \cdot \mathfrak{A}) \qquad \text{for symmetric } \mathbf{\Pi} \tag{49}$$

will find an immediate application later on.

We now know that – except for the case of multiple roots of (44), which nevertheless presents no complications here – the six unit motors, which change only by numerical factors λ_1, λ_2, …, λ_6 under the linear transformation that is determined by the $P_{\iota\kappa}$, are pairwise orthogonal to each other. The entire transformation is obviously determined by the given of these unit screws and the λ that correspond to them (as a general "affinity"). Now, six unit screws possess 30 independent determining data. That gives $6 \cdot 5/2 = 15$ orthogonality conditions and 6 line segments, which establishes the entire system (as a rigid body) relative to the coordinate cross. There thus remain 9 quantities, in addition to the 6 values of the roots, so, in total, 15 determining data for the linear transformation. Since we now have also found 15 invariants in eq. (46) we may assume that their number is therefore exhausted.

An overview of the possible forms for the transformation that is the result of a symmetric dyadic must start with a discussion of the system of six orthogonal motors. If we then base the definition of a dyadic on such a "six-fold right-angled system" of the general form, instead of a Cartesian coordinate system then we obtain a very symmetric representation. Now, let us choose a very special case, which seems to dominate the applications (inertia dyadic in II.**3**, elasticity dyadic of a rod in III.**4**). Here, the six principal motors have an ordinary right-angled axis cross at any two edges; thus, any time the two coaxial motors are "orthogonal," from I.**8**, it is also necessary and sufficient that they have equal and opposite pitches. If we call the pitches $\pm\,\alpha$, $\pm\,\beta$, $\pm\,\gamma$ then the unit motor with the components 1, 0, 0, α, 0, 0 must go to λ_1, 0, 0, $\lambda_1\alpha$, 0, 0 under the transformation. If one substitutes this in (14) then one obtains:

$$\begin{aligned} \lambda_1 \;\; &= P_{11}\,\alpha + P_{41}\,, \qquad 0 = P_{11}\,\alpha + P_{41} = P_{12}\,\alpha + P_{42}\,,\\ \lambda_1\alpha &= P_{14}\,\alpha + P_{44}\,, \qquad 0 = P_{15}\,\alpha + P_{45} = P_{16}\,\alpha + P_{46}\,. \end{aligned}$$

Since $P_{14} = P_{41}$, the first and fourth of these equations shows that one must have $P_{44} = P_{11}\,a^2$, while the other ones show that it must follow that $P_{12} = P_{13} = P_{15} = P_{16} = P_{42} = P_{43} = P_{45} = P_{46} = 0$, since they also must be true for $-\,\alpha$. If one repeats the argument for the other two axes then one finds that the matrix that we are considering – the "completely symmetric dyadic," as we would like to say – appears as follows when referred to the principal axis cross:

$$\begin{matrix} P_{11} & 0 & 0 & P_{14} & 0 & 0 \\ 0 & P_{22} & 0 & 0 & P_{25} & 0 \\ 0 & 0 & P_{33} & 0 & 0 & P_{36} \\ P_{14} & 0 & 0 & \alpha^2 P_{11} & 0 & 0 \\ 0 & P_{25} & 0 & 0 & \beta^2 P_{22} & 0 \\ 0 & 0 & P_{36} & 0 & 0 & \gamma^2 P_{33} \end{matrix}$$

i.e., they now include non-vanishing elements only in the main diagonal and the two parallel "half-diagonals." From (46), the equation for λ is easy to solve, and it delivers, when we again write P_{44} instead of $\alpha^2\,P_{11}$, etc:

$$\lambda_1,\, \lambda_2 = P_{14} \pm \sqrt{P_{11}\cdot P_{44}}\,, \qquad \lambda_3,\, \lambda_4 = P_{25} \pm \sqrt{P_{22}\cdot P_{55}}\,, \qquad \lambda_5,\, \lambda_6 = P_{36} \pm \sqrt{P_{33}\cdot P_{66}}\,.$$

One sees that – as compared to the case in vector algebra – one cannot conclude the reality of λ from the symmetry of the dyadic.

We would now like to thoroughly consider a series of applications of the concept definitions and formulas that we developed here to the mechanics of rigid bodies, elasticity theory, and hydrodynamics [1]).

[1]) Section III appears in the next issue.