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On the differential equations of mechanics

By

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It is known from the principle of virtual velocities that a system of material points is found in equilibrium when the virtual work that is done by the given forces vanishes for any virtual displacement of their points of application that is compatible with the mobility conditions of that point. In the foregoing, generally-specialized, way of conceptualizing the principle of virtual velocities, one cares to assume that these condition take the form of equations $\varphi_i = 0$ between the coordinates of the points of application, by whose variations $\delta\varphi_i = 0$, the conditions for the displacements δ will be obtained. *In fact, only the conditions for the vanishing of those variations will then enter into the treatment of the static problem.* However, in that, one recognizes that it is entirely irrelevant whether those variations do or do not come about by performing δ processes, $\delta\varphi_i = 0$. One thus has no basis for restricting the conception of the principle of virtual velocities above by any particular analytical formulation of it.

In order to give a more precise account of that fact, one first considers – say – the equilibrium of a material point under the influence of arbitrary forces with the components X, Y, Z for which the condition:

$$P\,\delta x + Y\,\delta y + R\,\delta z = 0$$

should exist. However, the latter is the expression for a *point-plane system*, and the structure that is defined by it can be regarded, in just the same sense, as a geometrically well-defined one, such as perhaps an ideal fixed surface on which the points is compelled to remain; any point will be associated with a well-defined plane by it, or even a well-defined surface in which the displacements must take place (*). As long as one first ignores friction reactions, etc., one can thus produce only normal reactions, and one will then obtain:

$$X - \lambda P = 0,$$
$$Y - \lambda Q = 0,$$
$$Z - \lambda R = 0$$

or

(*) For this way of looking at things, cf., my paper in volume XXIII of these Annalen, pp. 45, *et seq.* [Translator: In that article, he introduces the term "P-E-system" (P-E = *Punkt-Ebene*) for "point-plane-system.]

$$X\,\delta x + Y\,\delta y + Z\,\delta z = 0$$

as the equilibrium conditions. From that point onward, one will rise to the general case in which a certain number of conditions of the form:

$$\sum_{k=1}^{n} (p_{sk}\,\delta x_k + q_{sk}\,\delta y_k + r_{sk}\,\delta z_k) = 0, \quad s = 1, 2, \ldots, r$$

have been prescribed for the coordinate of the points of a system, perhaps in that same way that **Lagrange** did in *Théorie des fonctions* (*). However, the admissibility of the condition:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = 0$$

once more proves to be the criterion for equilibriuim in it (**).

With that remark, one arrives at a new viewpoint on the static problem, at least, formally. However, things seem to be different in *dynamics*. There, the assumption that there are *explicit* equations $\varphi_i = 0$ in the coordinates of the system points proves to be one that will have an essential effect on all of the further treatment of dynamical investigations. Indeed, it is practically due to the self-contained form that analytical mechanics has taken on since **Lagrange** that the term *differential equations of mechanics* refers to certain system of second-order differential equations that are soluble for the second differential quotients of the variables with respect to the independent variable t, in which a number of relations exist between the independent variables explicitly (which can also include t).

Naturally, there can be no question of the overarching importance of that aforementioned case. Thus, in what follows, I would like to at least *attempt* to verify that there also exists no essential basis in dynamics for excluding that other extended problem statement as inconceivable through the usual analytical conception of things, but that conditions of a general character can also very well have an entirely conceivable dynamical content, moreover.

In the formulation of the dynamical differential equations, in the sense of the customary viewpoint – which is, to my knowledge, assumed by all authors (***) – the case can already arise in which the relations $\delta\varphi = 0$ are not given in the form of *total variations*, although they can certainly lead to such things. I shall next discuss that case somewhat more closely, starting from which, an advance to more general assumptions will seem easily possible.

(*) **Lagrange**, *Théorie des fonctions*, Paris, 1813, pp. 350, *et seq.*

(**) Moreover, cf., the remark in **Jacobi's** *Vorelesungen über Dynamik*, pp. 15.

(***) Along with the representation in **Lagrange's** analytical mechanics itself, I shall mention only that of **Jacobi**, *loc. cit.*, pp. 52-57, and the one in **Kirchhoff's** *Mechank*, pp. 21, which probably goes further in the name of abstraction. The aforementioned assumption is itself preserved in the case in which one has been led to assume that the forces are functions of time and their differential quotients with respect to them.

The statements in the text *that the conditions can also depend upon the velocities*, even if one cannot exclude them, allow one to regard them as belonging to mechanics, and thus amount to explanations for the analytical processes that come to be employed with the differential equations of dynamics. Cf., the remark on pp. 9, moreover.

Namely, let r linear differential expressions with n variables $x_1, x_2, \ldots, x_n$ be given:

$$(1)\qquad \begin{gathered} \sum_{i=1}^{n} p_{1i}\, dx_i\,, \\ \sum_{i=1}^{n} p_{2i}\, dx_i\,, \\ \ldots. \\ \sum_{i=1}^{n} p_{ri}\, dx_i\,, \end{gathered}$$

in which the p_{ki} might contain only the x. One might then ask when they can be equivalent to just as many total differentials. In order for that to be true, it is requisite that total differentials should come about by multiplying the p_{ki} by certain multipliers λ_s and adding; i.e., equations of the form:

$$(2)\qquad \sum_{s=1}^{r} p_{si}\, \lambda_s = \frac{\partial \varphi}{\partial x_i}, \qquad i = 1, \ldots, n$$

must exist. Now, if the system of differential expressions (1) is assumed to be *linearly-independent* then not all partial determinants of degree r in the corresponding p_{ik} can vanish. One can then always substitute the values of λ that are calculated from, say, the first r equations in (2) into the last $n - r$ of them and thus obtain $n - r$ linear partial differential equations for the function φ. Now, should the system (1) be equivalent to r total differentials, that system of partial differential equations would then have to possess r mutually independent particular integrals:

$$\varphi_1, \varphi_2, \ldots, \varphi_r .$$

However, the conditions for that to be true have been known since **Jacobi** and **Clebsch**. In particular, they were put into a very suitable form for algebraic investigations by **Frobenius** (*). In view of the present purpose, I will meanwhile choose the following process for obtaining those conditions (**).

One brings the equations:

(*) **Frobenius**, "Ueber das Pfaff'sche Probleme," Journal v. **Borchardt** LXXXII, pp. 270, *et seq.*
(**) Cf., **Boole**, *Treatise on differential equations,* supplementary volume, pp. 74, *et seq.*

$$0 = \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1r} & \dfrac{\partial\varphi}{\partial x_1} \\ p_{21} & p_{22} & \cdots & p_{2r} & \dfrac{\partial\varphi}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} & \dfrac{\partial\varphi}{\partial x_r} \\ p_{l1} & p_{l2} & \cdots & p_{lr} & \dfrac{\partial\varphi}{\partial x_l} \end{vmatrix}, \quad l = r + 1, \ldots, n,$$

by means of the ever-possible division by the non-vanishing determinant:

$$P = \begin{vmatrix} p_{11} & \cdots & p_{1r} \\ \cdots & \cdots & \cdots \\ p_{r1} & \cdots & p_{rr} \end{vmatrix},$$

into the form:

$$0 = \frac{\partial\varphi}{\partial x_l} + \sum_{p=1}^{n} \frac{\partial\varphi}{\partial x_p} a_{lp} = (A_l)\ \varphi.$$

The existence of any two of the equations:

$$(A_l)\ \varphi = 0, \qquad (A_m)\ \varphi = 0$$

will then imply the new equations:

$$(A_m\, A_l - A_l\, A_m)\ \varphi = 0.$$

Since either $\partial\varphi\ /\ \partial x_i$ or $\partial\varphi\ /\ \partial x_m$ will enter into them, they cannot be a consequence of the remaining equations; i.e., all of their coefficients must vanish. Moreover, the conditions that emerge from that are necessary and sufficient for the system of differential equations (1) to be *complete* (*), or when the $n - r$ equations $(A_l)\ \varphi = 0$ should possess r integrals:

$$\varphi_1, \varphi_2, \ldots, \varphi_r$$

that are mutually-independent relative to the variables $x_1, \ldots, x_r$.

However, under the latter assumption, there will also be just as many systems of multipliers λ_s (which might be denoted by $\lambda_{s1}, \lambda_{s2}, \ldots, \lambda_{sr}$), whose determinant:

(*) In particular, cf., the proof that **A. Mayer** gave of this in his treatise in these Annalen, V, pp. 450, *et seq*. With the terminology that he introduced, when the expression (1) is equal to zero, it will define an *unrestricted integrable system* if the integrability conditions above are fulfilled.

(3) $$\Lambda = \begin{vmatrix} \lambda_{11} & \cdots & \lambda_{1r} \\ \vdots & \cdots & \vdots \\ \lambda_{r1} & \cdots & \lambda_{rr} \end{vmatrix}$$

cannot vanish. Namely, if one denotes the functional determinant of the φ relative to the $x_1, \ldots, x_r$ by F then it will follow immediately from the easily-proved relation:

$$P\,\Lambda = F$$

that Λ cannot be zero, either, since F does not vanish, from the assumptions that were made about the integrals φ above. However, if the completeness conditions for the system are not fulfilled then it can be possible for one to ascertain a smaller number of total differential equations by the given process, such that – say – the given expressions (1) can be replaced with the following ones:

$$d\varphi_1,\ d\varphi_2, \ldots, d\varphi_h,$$
$$\sum_{i=1}^{n} p_{k+1,i}\,dx_i, \ldots, \sum_{i=1}^{n} p_{ri}\,dx_i .$$

In this case, however, the possibility of a further reduction cannot be completely excluded, as long as one is dealing with the expressions (1) when they are set equal to zero. Then, since in the present examination the constants in the integrals:

$$\varphi_1 = \text{const.},\ \varphi_2 = \text{const.}, \ldots, \varphi_r = \text{const.}$$

have entirely specialized values (which might be determined from the initial positions of the system points), it can happen that by eliminating h variables with the help of the equations of the remaining differential expressions, a further treatment will once more be practicable. However, in every case, the aforementioned process will allow one to decide how the system of equations $\sum p_{ri}\,dx_i = 0$ can be replaced by explicit equations and other differential relations.

One then easily recognizes that in the event that a reduction (in one sense or the other), to r explicit equations:

$$\varphi_1 = 0,\ \varphi_2 = 0, \ldots, \varphi_r = 0$$

is possible at all, the differential equations:

$$m_i\,\frac{d^2x_i}{dt^2} = X_i + \sum_{s=1}^{r} \nu_s \frac{\partial \varphi_s}{\partial x_i}, \qquad i = 1, \ldots, n$$

can be replaced with the ones that are defined directly (*):

(*) For the sake of simplicity, I will denote all of the variables $x_i\ \ y_i\ \ z_i$ of the system by x_i, while the index i goes from 1 to n.

$$(4) \qquad m_i \frac{d^2 x_i}{dt^2} = m_i x_i'' = X_i + \sum_{s=1}^{r} \mu_s p_{si} .$$

By contrast, the conversion of the equations can no longer lead back to a proper variational problem, *and furthermore, the property of system* (1) *being complete defines a necessary and sufficient condition for that to be true.*

Namely, should the variation of the integral:

$$\int (T + U)\, dt$$

vanish, under the assumption of the conditions that belong to (1), which might be written in the form:

$$(5) \qquad \sum_{i=1}^{n} p_{si} x_i' = 0, \quad s = 1, \dots, r,$$

one would obtain in a known way, upon forming:

$$\delta \int \left(T + U + \sum_{i=1}^{n} \sum_{s=1}^{n} \lambda_s p_{si} x_i' \right) dt = 0,$$

the equation:

$$0 = \sum_{k=1}^{n} \left[\frac{\partial T}{\partial x_k} - \frac{d}{dt} \frac{\partial T}{\partial x_k'} + X_k - \sum_{s=1}^{r} \frac{d\lambda_s}{dt} p_{sh} \right] \delta x_k + \sum_{i=1,k}^{n} \sum_{s=1}^{r} \lambda_s (sik) x_i' \delta x_k ,$$

in which one has set:

$$(sik) = -(ski) = \frac{\partial p_{si}}{\partial x_k} - \frac{\partial p_{sk}}{\partial x_i} .$$

Now, in order for that to lead to the equations of mechanics, in the event that further relations between the variations δ are not present, the equations:

$$\sum_{i=1}^{n} (sik) x_i' = \sum_{l=1}^{r} \mu_l^s p_{lk} , \qquad \begin{array}{l} s = 1, \dots, r, \\ k = 1, \dots, n \end{array}$$

must be true as a result of the relations (5), since the λ_r can be subjected to no further restrictions. However, that will require the vanishing of the covariants:

$$\sum_{i=1}^{n} \sum_{k=1}^{n} (sik)\, x_i' y_k'$$

by means of the relations:

$$\sum_{i=1}^{n} p_{si} x_i' = 0, \quad \sum_{i=1}^{n} p_{si} y_i' = 0, \quad s = 1, \ldots, r.$$

Now, under that assumption, the system of differential expressions (1) will be complete (*), but, at the same time, one will have:

$$\frac{\partial T}{\partial x_h} - \frac{d}{dt}\left(\frac{\partial T}{\partial x_h'}\right) + X_k - \sum_{s=1}^{r} \mu_s p_{sh} = 0,$$

in the event that one sets:

$$\mu_s = \frac{d\lambda_s}{dt} - \sum_{h=1}^{r} \lambda_h \mu_s^h.$$

However, that was to be shown.

One can determine the multipliers μ_s in equations (4) in the usual way from the equations:

(6) $$\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial p_{si}}{\partial x_k} x_i' x_k' + \sum_{i=1}^{n} X_i \frac{p_{si}}{m_k} + \sum_{h=1}^{r} \mu_h (hs) = 0$$

that one obtains from differentiating them by t by means of (5), in which we have set:

$$(sh) = (hs) = \sum_{i=1}^{n} \frac{p_{hi} p_{si}}{m_i}.$$

One obtains the values of the μ_s from equations (6). The determinant of the (hs), as the sum of the squares of all partial determinants of degree r from the system of coefficients in equations (5), can vanish only when the latter are not mutually independent, which contradicts the assumption (**). Thus, the μ_s will be functions of the x and even, rational, entire functions of degree two of the x'. One can then ignore the relations (5) completely, and consider only the differential equations (4), from which the r integrable equations:

$$\frac{d}{dt}\sum_{i=1}^{n} p_{si} x_i' = 0,$$

and for a suitable determination of the constants, (5) itself, will follow once more. On the basis of this convention, it will be possible to determine all of the higher differential quotients of the x, as long as only the initial positions and velocities are assumed [the latter, according to equations (5)]; i.e., there will generally be equations of the form:

(7) $$x_i = x_{i_0} + t x_{i_0}' + \frac{t^2}{2} x_{i_0}'' + \ldots,$$

(*) **Frobenius**, *loc. cit.*
(**) **Jacobi**, *loc. cit.*, pp. 140.

in which the index 0 refers to the initial state.

The following theorem is connected with the property of the μ_s that they are even functions of the velocities:

If, at a well-defined moment, one simultaneously substitutes equal and opposite velocities to the ones that are present at all points of the system then, under the same circumstances, all points will traverse exactly the same paths along which they would have moved at that moment, but in the opposite order (*).

Namely, if one is given the equations:

$$\frac{d^2 x_i}{dt^2} = \psi_i$$

at all, in which the ψ_i are functions of x_i and even functions of the x_i', then the third differential quotients will become odd functions of the latter, the fourth, will be even, and so forth with the same alternation of signs. The differential quotients of even order will remain unchanged under a simultaneous change of sign of all x_i', but the ones of odd order will all assume opposite values. If one now assumes that the path from any moment with the index 0 onward can be represented in form (7) for each point then, one will get the same motion for negative t as when one simultaneously assigns the opposite signs to all x_i', with which, the theorem is proved.

Let us remark in passing only that the *principles of the motion of the center of mass and the motion of surfaces* are the same as in conventional mechanics, as long as corresponding assumptions about the coefficients in (5) are made. It seems to be more interesting that the *law of vis viva:*

$$T - T_0 = \sum_{i=1}^{n} \int_{t_0}^{t} X_i \, dx_i$$

(*) Cf., a remark of **Loschmidt** in the Abh. der Wiener Akademie LXXIII, v. II, pp. 128. However, the actual conditions for the possibility of such reversibility of a system are not stated there.

remains true (**), especially, the *principle of vis viva*, as well, as long as a *force function* U that depends upon only the coordinates x_i (in order to stay in the simplest case) exists. However, that also means that the general theorems that are immediate consequences of the latter (e.g., properties of the level surfaces, etc., …) must be true. In particular, one might care to count the *stability criterion* among them, by which stable equilibrium will exist only when the force function is a maximum for the equilibrium position of the system.

Admittedly, the theory of maxima and minima does not treat the case in which the function U of x is a maximum or minimum relative to the given differential relations of

(**) This is also true for the differential equations that arise from the variational problem that was treated above for any sort of relations (5); more generally, it will be true in the event that their left-hand sides are replaces with any forms that are homogeneous in the x_i'. On the basis of this property, one can imagine that any equations to which one arrives by means of the **Hamilton's** principle are equivalent to the ones above, in a certain sense. However, **Hamilton**'s principle is not at all a proper principle of mechanics, since it has – at least, to begin with – only the character of an analytical rule in mechanics that *also* yields the differential equations of motion. If one would not wish to deviate from the requirement that the surface element can experience only normal reactions then one would, in fact, have to take the position above. Moreover (let it be said here), the same thing will emerge from the *principle of least pressure.* Namely, according to it, the sum:

$$S = \sum_{i=1}^{n} \left[\frac{1}{m_i} x_i'' - X_i \right]^2$$

must be a maximum-minimum relative to all values of the x_i'' that satisfy the conditions:

$$F_i = \sum_{i,k=1}^{n} \frac{\partial p_{si}}{\partial x_k} x_i' x_k' + \sum_{i=1}^{n} p_{si}\, x_i'' = 0, \qquad s = 1, \ldots, r.$$

From the equations of condition:

$$\frac{\partial}{\partial x_k''} \left(S - \sum \lambda_s F_s \right) = 0,$$

however, the equations:

$$\frac{1}{m_i} x_i'' = X_i + \sum_{s=1}^{r} \nu_s p_{si}$$

will emerge immediately. In fact, the single analytical difference that exists between problems of this kind and those of ordinary dynamics consists of the fact that *in the former case, the conditions are not known explicitly at the outset, but they will be determined by means of the integration of simultaneous second-order systems and will play precisely the same role in regard to their dynamical effects as in the otherwise common investigations.*

If one considers the motion of a material point that is restricted by one differential relation then the integration of the equations of motion will yield its path in the form of a curve that is covered with surface elements – i.e., a *strip*, as one can say. If one now constructs a hypersurface $F = 0$ to which that strip belongs *then the same motion can also be produced in that way as when* $F = 0$ *is added to the equations of motion as a condition, as long as only the initial state remains unchanged. However, a similar argument will also be true (as far as I know) for an arbitrary point system*, in such a way that one can also say:

The processes of motion that are expressed by the extended form of the equations of motion are nothing but the ones that can also be described by means of the currently-customary way of expressing them. However, they will produce them in a fundamentally simpler representation, as long as, e.g., completely well-defined differential relations are introduced into the investigation in place of arbitrary surfaces on which only strips come under considerations.

the form (5). In fact, such a demand would not generally make any sense, either. However, the function U might very well have that property in regard to all paths from an equilibrium position that can bring the system into accord with those conditions. If one then denotes the increase in any coordinate that is evaluated at the fixed moment (which might correspond to equilibrium) and given the index 0 by:

$$t\, x'_{i_0} + \frac{t^2}{2} x''_{i_0} + \ldots$$

then the increase ΔU in U, under the assumption of equilibrium conditions:

$$\left(\frac{\partial U}{\partial x_i}\right)_0 + \sum_{s=1}^{r} h_s\, p_{si_0} = 0, \qquad i = 1, \ldots, n,$$

will assume the form:

$$\Delta U = \frac{t^2}{2} \sum_{i=1}^{n} \left[\frac{\partial U}{\partial x_i} x''_i + \sum_{k=1}^{n} \frac{\partial^2 U}{\partial x_i \partial x_k} x'_i\, x'_k \right] + \ldots$$

However, since:

$$\sum_{i=1}^{n} p_{ri_0} x''_i + \sum_{i=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial p_{ri_0}}{\partial x_k}\right)_0 x'_i\, x'_k = 0,$$

one will get:

$$\Delta U = \frac{t^2}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \left[\frac{\partial^2 U}{\partial x_i \partial x_k} + \sum_{s=1}^{r} h_s \frac{\partial p_{si}}{\partial x_k} \right]_0 x'_{i_0}\, x'_{k_0} + \ldots$$

Now, provided that the quadratic form:

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \left[\frac{\partial^2 U}{\partial x_i\, \partial x_k} + \sum_{s=1}^{r} h_s \frac{\partial p_{si}}{\partial x_k} \right]_0 x'_i\, x'_k$$

or:

$$\sum \alpha_{ik}\, x'_i\, x'_k$$

that appears here has a negative-definite character in regard to the relations (5), *one can ascribe to U the maximum property in regard to all possible motions of the system that start from the equilibrium position.*

In connection with a known argument in theoretical mechanics, the paradox then seems to appear that, here as well, this definite character is necessary and sufficient for the stability of equilibrium. Thus, it is easy to convince oneself of the incorrectness of that statement. A remarkable consideration of **Dirichlet**'s proof (*), which must support

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(*) Journal v. Crelle XXXII, pp. 85; cf., also **Schell**, *Mechanik*, pp. 540. The analysis that was developed in this article does not seem superfluous to me, since, in fact, in the presentations that are introduced into the fundamental theorem of mechanics by the principal of energy, one simply works with that investigation, shows that *it assumes not only the maximum property, but additionally demands that a value can be established that is smaller than certain values of –* ΔU *that are defined in that proof, and independently of the existing negative values of* ΔU *for the vis viva in the initial position* (which can be envisioned to be the equilibrium position itself, here, for the same of greater clarity). However, that is not possible in the present case, in which ΔU is expressed (up to higher-order terms) in terms of a quadratic function of the initial velocities, multiplied by t^2.

A closer examination will show that the quadratic form (8) no longer has any decisive connection to the question of stability at all, as long as one drops the customary assumptions in mechanics (i.e., the relations (5) no longer define a complete system). Rather, one will obtain the true conditions for a stable motion by *examining the small oscillations of the system*.

Apropos of that, it will be assumed that the increases ξ in each coordinate $x = x_0 + \xi$ will be small enough that the second powers of them can be neglected in comparison to the first, and that the velocities x' have that order of smallness.

Moreover, when one sets $x = x_0 + \xi$, $\mu_s = \mu_s^0 + \mu_s$, the differential equations:

$$m_i x_i'' = \frac{\partial U}{\partial x_i} + \sum_{s=1}^{r} \mu_s \, p_{si}$$

will become:

$$m_i \, \xi_i'' = \left(\frac{\partial U}{\partial x_i}\right)_0 + \sum_{s=1}^{r} (\mu_s \, p_{si})_0 + \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial^2 U}{\partial x_i \, \partial x_k} + \sum_{s=1}^{r} \mu_s \frac{\partial p_{si}}{\partial x_k}\right)_0 \xi_i + \sum_{s=1}^{r} \eta_s \, p_{si_0} \, ,$$

while from the equations of condition:

$$\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial p_{si}}{\partial x_k} \xi_i' \xi_k' + \sum_{i=1}^{n} \frac{1}{m_i} \frac{\partial U}{\partial x_i} p_{si} + \sum_{h=1}^{r} \mu_s (sh) = 0,$$

in conjunction with the equations of equilibrium, it will follow that one must set:

$$\mu_s^0 = h_s \, ,$$

up to quantities that have the same smallness as ξ. However, with that, the system above will reduce to the following one:

$$m_i \, \xi_i'' = \sum_{k=1}^{n} \alpha_{ik} \xi_k + \sum_{s=1}^{r} \zeta_s \, p_{si_0} \, , \qquad i = 1, \ldots, n,$$

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the concept of the maximum. Cf., e.g., **Thomson** and **Tait**, *Treatise on natural philosophy*, vol. I, § 292 (1883). This way of looking at things is generally completely adequate, since, in fact, the force function will be completely independent of the x', and time will come under scrutiny.

in which the ζ are again quantities with the same order of smallness as before, in the event that one sets:

$$\alpha_{ik} = \left\{ \frac{\partial^2 U}{\partial x_i \, \partial x_k} + \sum h_s \frac{\partial p_{si}}{\partial x_k} \right\}_0 ,$$

which must now be solved in conjunction with the equations:

$$\sum_{i=1}^{n} \xi_i' \, p_{si_0} = 0, \quad s = 1, \ldots, r.$$

In order to represent the ξ by particular integrals of the form:

$$\xi_i = e^{\lambda t} y_i ,$$

one must solve the determinant equation of degree $n - r$:

$$\begin{vmatrix} \alpha_{11} - m_1\lambda^2 & \alpha_{12} & \cdots & \alpha_{1n} & p_{11} & \cdots & p_{r1} \\ \alpha_{21} & \alpha_{22} - m_2\lambda^2 & \cdots & \alpha_{2n} & p_{12} & \cdots & p_{r2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} - m_n\lambda^2 & p_{1n} & \cdots & p_{rn} \\ p_{11} & p_{12} & \cdots & p_{1n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rn} & 0 & \cdots & 0 \end{vmatrix} = 0.$$

When the quadratic form (8) is *negative-definite*, this equation will have roots, about which we can only assert that the *have negative real parts*, according to a well-known argument that goes back to **Cauchy**. In the present case, however, the existence of stable oscillations will demand that all roots must be real and negative (*). If that condition is fulfilled then one will obtain expressions for ξ and ξ' in a known way that continually remain small and, at the same time, exhibit the equilibrium state of the system when one considers the basic approximation.

Now, as one knows, all roots of the equation above will be real when the determinant of the α_{ik} is symmetric. That will take place only when all (*sik*) vanish. However, that symmetric arrangement can also be produced in the equation above when the differential relations (5) belong to a complete system. If one recalls the property of the determinant Λ that was remarked in regard to (3) one can then assume that the μ_s^0 have the form:

$$\mu_s^0 = \sum_{h=1}^{r} \lambda_{sh}^0 \nu_h ,$$

(*) Cf., also **Thomson** and **Tait**, *loc. cit.*, § 344, § 345.

in which the λ_{sh}^0 are the values of the multipliers of the complete system that correspond to the equilibrium position. However, one now has:

$$\sum_{h=1}^{r} \lambda_{hs}\, p_{hi} = \frac{\partial \varphi_s}{\partial x_i}, \qquad \sum_{k=1}^{r} \lambda_{ks}\, p_{hk} = \frac{\partial \varphi_s}{\partial x_k},$$

so one will also have:

$$\sum_{s=1}^{r}\sum_{h=1}^{r} \nu_s \left(\frac{\partial \lambda_{hs}}{\partial x_k} p_{hi} - \frac{\partial \lambda_{hs}}{\partial x_i} p_{hk} \right) + \sum_{s=1}^{r}\sum_{h=1}^{r} \nu_s \lambda_{hs} (hik) = 0, \qquad i, k = 1, \ldots, n$$

or

$$\sum_{h=1}^{r} \mu_h^0 \left(\frac{\partial p_{hi}}{\partial x_k} \right)_0 + \sum_{s=1}^{r}\sum_{h=1}^{r} \nu_s \left(\frac{\partial \lambda_{hs}}{\partial x_k} p_{hi} \right)_0 = \sum_{h=1}^{r} \mu_h^0 \left(\frac{\partial p_{hk}}{\partial x_i} \right)_0 + \sum_{s=1}^{r}\sum_{h=1}^{r} \nu_s \left(\frac{\partial \lambda_{hs}}{\partial x_i} p_{hk} \right)_0,$$

and as one easily sees, these relations can have the effect that the determinant above is converted into a symmetric one; one has only to multiply the last r rows of the horizontal and vertical boundaries with suitable factors and add them to the α_{ik}. However, all roots will then certainly be real, and the complete criterion for stability can thus also be ascertained *algebraically* from the p_{ik} and their differential quotients, with no deeper analysis (*).

When the *principle of Jacobi multipliers* is applied to the differential equations (4), (5), that will demand that one can give a (particular) integral for the partial differential equation:

$$\frac{d \log M}{dt} + \sum_{s=1}^{r}\sum_{i=1}^{n} \frac{1}{m_i} \frac{\partial (\mu_s p_{si})}{\partial x_i'} = 0.$$

We will obtain the r equations:

$$(9) \qquad \sum_{h=1}^{r} (sh) \frac{\partial \mu_s}{\partial x_i'} + 2 \frac{dp_{si}}{dt} + \sum_{k=1}^{n} (ski) x_k' = 0, \qquad s = 1, \ldots, r$$

for the differential quotients of the μ_s that enter into it from the partial differentiation of (6) that we learn from **Jacobi**.

If one now denotes:

$$\sum_{s=1}^{r} \frac{1}{m_i} \cdot \frac{\partial (\mu_s p_{si})}{\partial x_i'}$$

by H_i, to abbreviate, then the equation for the determination of these quantities will follow:

(*) For the latter case, the representation above includes a formally-extended representation of the theory of small oscillations that seems to have always been treated since the time of **Lagrange** only under the explicit assumption of independent coordinates.

$$(10) \qquad \begin{vmatrix} (11) & \cdots & (1r) & 2\dfrac{dp_{1i}}{dt} + \sum_{k=1}^{n} (1ki)x'_k \\ \vdots & \vdots & \vdots & \vdots \\ (r1) & \cdots & (rr) & 2\dfrac{dp_{ri}}{dt} + \sum_{k=1}^{n} (rki)x'_k \\ p_{1i} & \cdots & p_{ri} & -\mathsf{H}_i m_i \end{vmatrix} = 0.$$

If one denotes the determinant of the (rs) by Δ and its sub-determinants by Δ_{rs} then one will obtain from (10):

$$\Delta \mathsf{H}_i + \sum_{k,h=1}^{n} \sum_{s=1}^{r} \left\{ 2\frac{dp_{hi}}{dt} + (hki)x'_k \right\} \frac{p_{si}}{m_i} \Delta_{rs} = 0,$$

so when one sums over i and substitutes the value in the multiplier equation, one will get:

$$\Delta \frac{d \ln \mathsf{M}}{dt} = \sum_{h,s=1}^{r} \Delta_{hs} \frac{d(hs)}{dt} + \sum_{k,h=1}^{n} \sum_{s=1}^{r} (hki)\, x'_k \, \frac{p_{si}}{m_i}$$

or:

$$(11) \qquad \Delta \frac{d \ln \mathsf{M}}{dt} = \frac{d\Delta}{dt} + \mathsf{A},$$

in which one denotes the four-fold sum on the right by A. This will yield **Jacobi**'s result that $\mathsf{M} = \Delta$ (*) only when the form A vanishes identically or due to equations (5); i.e., when either all (hki) are zero or when there are r quantities h_s that satisfy the n equations:

$$\sum_{i=1}^{n} \sum_{s,h=1}^{r} (hki) \frac{p_{si}}{m_i} \Delta_{hs} = \sum_{s=1}^{r} h_s \, p_{sk} \, , \qquad k = 1, \ldots, n.$$

If one multiplies them by the expressions:

$$\sum_{j=1}^{r} \frac{p_{jk}}{m_k} \Delta_{lj}$$

and sums over k then what will result is:

$$h_i \, \Delta = \sum_{i,k=1}^{n} \sum_{s,h,j=1}^{r} \Delta_{lh} \Delta_{js} \frac{p_{hk}\, p_{si}}{m_i m_k} (jik).$$

However, when one switches i and k and then h and s, the right-hand side will become:

(*) **Jacobi**, *loc. cit.*, pp. 132-141.

$$\tfrac{1}{2}\sum(\Delta_{lh}\Delta_{js} - \Delta_{ls}\Delta_{jh})\frac{p_{hk}\,p_{si}}{m_i m_k}(jik).$$

When one cancels the factor Δ on both sides, one will then get:

$$h_l = \tfrac{1}{2}\sum_{i,k=1}^{n}\sum_{h,j,s=1}^{r}\frac{p_{hk}\,p_{si}}{m_i m_k}(jik)\Delta_{lh,\,js},$$

by means of a well-known determinant formula, in which $\Delta_{lh,js}$ now means a second sub-determinant of Δ. One will then have the conditions for the existence of this **Jacobi** multiplier in an explicit form when one substitutes the value that was found for h_l into the n conditions above.

I shall now examine the form of the multipliers under the assumption that relations (5) define a complete system. Since the determinant Λ does not vanish, one can also set:

$$p_{si} = \sum_{h=1}^{r}\lambda'_{hs}\frac{\partial\varphi_h}{\partial x_i}, \tag{12}$$

in which the λ'_{hs} now mean the coefficients of the substitution that is reciprocal to the substitution λ_{hs}, and whose determinant is $\Lambda' = 1 / \Lambda$. Moreover, one will have:

$$(st) = \sum_{i=1}^{n}\frac{p_{si}\,p_{ti}}{m_i} = \sum_{j=1}^{n}\lambda'_{hs}\lambda'_{jt}\frac{\partial\varphi_h}{\partial x_i}\frac{\partial\varphi_j}{\partial x_i}\frac{1}{m_i} = \sum_{j,h=1}^{n}\lambda'_{hs}\lambda'_{jt}[hj],$$

such that when one denotes the determinant of:

$$\sum_{i=1}^{n}\frac{\partial\varphi_h}{\partial x_i}\frac{\partial\varphi_j}{\partial x_i}\frac{1}{m_i} = [hj] = [jh]$$

by Δ, one will have $\Delta = \Delta'\,\Lambda'^2$. If one now introduces the expression (12) into the form A (11) then that will give:

$$\mathsf{A} = \sum_{i,k=1}^{n}\sum_{h,j,s=1}^{r}\left\{(hki)\,x'_i\,\Delta_{hs}\lambda'_{js}\frac{\partial\varphi_j}{\partial x_i}\frac{1}{m_i}\right\}.$$

One further finds from (12) that:

$$(hki) = \sum_{l=1}^{r}\left(\frac{\partial\lambda'_{lh}}{\partial x_i}\frac{\partial\varphi_l}{\partial x_k} - \frac{\partial\lambda'_{lh}}{\partial x_k}\frac{\partial\varphi_l}{\partial x_i}\right),$$

so, since from (12) and (5), the $d\varphi_i / dt$ are all zero:

$$\sum_{k=1}^{n} (hki)\, x_i' = -\sum_{l=1}^{r} \frac{\partial \varphi_l}{\partial x_i} \frac{d\lambda_{lh}'}{dt}.$$

One then has:

$$\mathsf{A} = -\sum_{i=1}^{n} \sum_{h,s,j,t=1}^{r} \left\{ \frac{\Delta_{hs}}{m_i} \lambda_{js}' \frac{d\lambda_{lh}'}{dt} \frac{\partial \varphi_l}{\partial x_i} \frac{\partial \varphi_j}{\partial x_i} \right\},$$

or, when one switches s with h and sums over i:

$$\mathsf{A} = -\tfrac{1}{2} \sum_{h,s,j,t=1}^{r} \left\{ \Delta_{hs} [tj] \frac{d(\lambda_{js}' \lambda_{lh}')}{dt} \right\}.$$

However, one can easily convert the latter expression in such a way that the formula (11) for M becomes integrable. It will then follow from the equation:

$$\Delta = \Delta' \, \Lambda'^2$$

by total differentiation with respect to t that:

$$\Lambda'^2 \frac{d\Delta'}{dt} + \Delta' 2 \frac{\Lambda' d\Lambda'}{dt} = \sum_{h,s,j,t=1}^{r} \Delta_{hs} \lambda_{js}' \lambda_{lh}' \frac{d[tj]}{dt} + \sum_{h,s,j,t=1}^{r} \Delta_{hs} [tj] \frac{d(\lambda_{js}' \lambda_{lh}')}{dt}.$$

If one observes that in this equation the first part on the right must be equal to the first part on the left, since differentiating both of them does not affect the coefficients of the substitution at all, then one will have:

$$\Delta' 2 \frac{\Lambda' d\Lambda'}{dt} = \sum_{h,s,j,t=1}^{r} \Delta_{hs} [tj] \frac{d(\lambda_{js}' \lambda_{lh}')}{dt}$$

or:

$$\mathsf{A} = -\frac{d\Lambda'}{dt} \frac{\Delta}{\Lambda'},$$

so

$$\frac{d \log \mathsf{M}}{dt} = \frac{d \log \Delta}{dt} - \frac{d \log \Lambda'}{dt},$$

and:

$$\mathsf{M} = \frac{\Delta}{\Lambda'} = \Delta \, \Lambda$$

will be the multiplier, which clearly assumes a knowledge of the determinant Λ, which cannot be ascertained in general without integrating the total system.

However, the form of it gives rise to a further remark, which I believe can be of use in the investigation of systems of differential equations that have been transformed by multipliers.

The system of differential equations:

$$(13) \qquad m_i \frac{d^2 x_i}{dt^2} = X_i + \sum_{s=1}^{r} \nu_s \frac{\partial \varphi_s}{\partial x_i}, \qquad i = 1, \ldots, n,$$

with the conditions:

$$\varphi_1 = c_1, \ldots, \varphi_r = c_r,$$

in which the c might also be arbitrary constants, has the **Jacobi** multiplier $N = \Delta'$, which follows from the multiplier equation:

$$\frac{d \log N}{dt} + \sum_{i=1}^{n} \sum_{s=1}^{r} \frac{1}{m_s} \frac{\partial \nu_s}{\partial x_i'} \frac{\partial \varphi_s}{\partial x_i} = 0,$$

in case the quantities ν_s and its differential quotients are determined by the equations $\frac{d^2 \varphi_s}{dt^2} = 0$. Now, if one introduces the system (5) that is equivalent to the relations $d\varphi_s = 0$ in place of them by means of equations (12) then one will obtain the transformed multiplier equation:

$$(14) \qquad \frac{d \log \mathsf{M}'}{dt} + \sum_{i=1}^{n} \sum_{s=1}^{r} \frac{1}{m_i} \frac{\partial(\mu_s p_{si})}{\partial x_i'} = 0,$$

under the assumption that the partial differential quotients of the μ_s are determined from the transformed relations $\frac{d^2 \varphi_h}{dt^2} = 0$, or:

$$(15) \qquad \sum_{i=1}^{n} x_i' \frac{dp_{hi}}{dt} - \sum_{i=1}^{n} \sum_{s=1}^{r} \frac{\partial \varphi_s}{\partial x_i} x_i' \frac{d\lambda_{sh}'}{dt} + \sum_{s=1}^{r} \mu_s [sh] + \sum_{i=1}^{n} \frac{X_i}{m_i} p_{hi} = 0,$$

in the form:

$$\Delta \frac{d \log \mathsf{M}}{dt} = \frac{d\Delta}{dt} - 2\Delta \frac{d \log \Lambda'}{dt},$$

so one will get:

$$\mathsf{M}' = \frac{\Delta}{\Lambda'^2} = \Delta' = N.$$

The multiplier then remains *unchanged*, which was to be expected. However, things will be different when one drops the second term on the left in equations (15), which will vanish due to the equations $d\varphi_s = 0$. One will then get equations (9), and from them, as one can show, the new multiplier $\mathsf{M} = \Delta\Lambda$. This result, which seems paradoxical on first glance, is explained by the fact that the differential equations (13), with the conditions

$\frac{d^2\varphi_s}{dt^2} = 0$ will not become identical with (4), with the conditions $\frac{d}{dt}\sum_{i=1}^{r} p_{si} x_i' = 0$, by merely a transformation, although they are entirely equivalent to them.

In fact, the general theory of **Jacobi** multipliers will imply in any case (and I would like to thank **A Mayer**, in a gracious communication, for this knowledge, along with the investigation that is given relative to it in what follows) *that both formulas lead to precisely the same values of the multipliers, by means of which the condition equations will reduce, and then necessarily lead to the mutually-identical systems* (13) *and* (4).

For this, one employs the well-known **Jacobi** theorem:

If k integrals for the n differential equations:

$$(A) \qquad \frac{dx_1}{dt} = X_1, \ldots, \frac{dx_n}{dt} = X_n$$

are known, namely:

$$(B) \qquad \varphi_1 = c_1, \ldots, \varphi_k = c_k,$$

such that when one calculates $x_1, \ldots, x_k$ from (B) and indicates the substitution of their values by enclosing things in brackets [ ], the use of these integrals will convert the system into $n - k$ differential equations:

$$(C) \qquad \frac{dx_{k+1}}{dt} = [X_{k+1}], \ldots, \frac{dx_n}{dt} = [X_n],$$

then any multiplier M of the system (A) will imply the multiplier:

$$M' = \left[\frac{M}{\sum \pm \frac{\partial\varphi_1}{\partial x_1} \cdots \frac{\partial\varphi_k}{\partial x_k}}\right]$$

of the system (C), and this theorem will also be unchanged when one assigns certain constant values to the arbitrary constants c (e.g., all of them equal to zero), and thus when one considers only the solutions of the given system (A) that satisfy the particular integrals:

$$\varphi_1 = 0, \ldots, \varphi_k = 0.$$

Now, **Jacobi**'s differential equations (13), or:

$$(16) \qquad \frac{dx_i}{dt} = x_i', \qquad \frac{dx_i'}{dt} = \frac{1}{m_i}\left\{X_i + \sum_{s=1}^{r} \nu_s \frac{\partial\varphi_s}{\partial x_i}\right\}, \quad i = 1, \ldots, n,$$

in which the ν_s are defined as functions of the x and x' by the equations that emerge from the r equations:

$$\text{(16}'\text{)} \qquad \frac{d^2\varphi_s}{dt^2} = 0, \qquad s = 1, \ldots, r$$

by the substitutions (16), possess the $2r$ integrals:

$$\frac{d\varphi_s}{dt} = c_s, \qquad \varphi_s - t\,\frac{d\varphi_s}{dt} = \gamma_s .$$

If one sets the c and γ equal to zero here then it will follow from the theorem that we just developed that:

If M is a multiplier of the system (16), and if one has then determined – say – $x_1, \ldots, x_r$ as functions of $x_{r+1}, \ldots, x_n$ from the r equations:

$$\varphi_1 = 0, \ldots, \varphi_k = 0,$$

and in such a way that the system (16) reduces to $n - r$ second-order differential equations in $t, x_{r+1}, \ldots, x_n$, then after substituting the values of $x_1, \ldots, x_r$ thus-obtained:

$$\text{(17)} \qquad M' = \frac{M}{\left(\sum \pm \frac{\partial \varphi_1}{\partial x_1} \cdots \frac{\partial \varphi_r}{\partial x_r}\right)^2}$$

will be a multiplier of that reduced system.

On the other hand, if one chooses r^2 arbitrary functions λ'_{hk} of $x_1, \ldots, x_n$ whose determinant Λ' does not vanish, and sets:

$$\text{(18)} \qquad \sum_{k=1}^{r} \lambda'_{hk} \frac{d\varphi_k}{dt} = \sum_{i=1}^{n} p_{hi} x'_i , \qquad h = 1, \ldots, r$$

then one can define the differential equations:

$$\text{(19)} \qquad \frac{dx_i}{dt} = x'_i , \qquad \frac{dx'_i}{dt} = \frac{1}{m_i}\left\{X_i + \sum_{s=1}^{r} \mu_s p_{si}\right\},$$

in which the μ_s are determined from the r equations that arise from equations:

$$\text{(19}'\text{)} \qquad \frac{d}{dt}\sum_{i=1}^{r} (p_{hi} x'_i) = 0, \qquad h = 1, \ldots, r$$

by the substitutions (19), and these equations (19), which are not identical to (16), will possess the r integrals:

(20) $$\psi_h = \sum_{k=1}^{r} \lambda'_{hk} \frac{d\varphi_k}{dt} = c_h \,.$$

If one denotes the substitution of the values of $x'_1, \ldots, x'_r$ in these integrals by [ ] then the system (19) can lead to the $2n - r$ differential equations:

(21) $$\begin{cases} \dfrac{dx_1}{dt} = [x'_1], \; \cdots, \; \dfrac{dx_r}{dt} = [x'_r], \\[2ex] \dfrac{dx_\tau}{dt} = [x'_\tau], \; \dfrac{dx'_1}{dt} = \dfrac{1}{m_\tau}\left[X_\tau + \displaystyle\sum_{s=1}^{r} \mu_s \, p_{s\tau} \right], \; \tau = r+1, \ldots, n. \end{cases}$$

Therefore, if M is a multiplier of the system (19) then one will again have:

$$N = \left[\frac{\mathsf{M}}{\sum \pm \dfrac{\partial \psi_1}{\partial x_1} \cdots \dfrac{\partial \psi_r}{\partial x_r}} \right] = \left[\frac{\mathsf{M}}{\Lambda' \sum \pm \dfrac{\partial \varphi_1}{\partial x_1} \cdots \dfrac{\partial \varphi_r}{\partial x_r}} \right],$$

when one takes all c in (20) to be zero from the outset. However, it will follow, in turn, from the particular integral $\psi_h = 0$ that $d\varphi_k / dt = 0$, and the reduced system (21) will itself once more take on the r integrals:

(22) $$\varphi_k = \gamma_k,$$

such that it will follows further that:

If M is a multiplier of (19), and one has then solved the r equations $\varphi_1 = 0, \ldots, \varphi_r = 0$ – perhaps for $x_1, \ldots, x_r$ – and thus converted the system (19) [or, what must amount to the same thing, the system (16)] into $n - r$ second-order differential equations in $t, x_{r+1}, \ldots, x_n$, under these conditions, then by substituting the solutions $x_1, \ldots, x_r$:

(23) $$\mathsf{M}' = \frac{\mathsf{M}}{\Lambda' \left(\sum \pm \dfrac{\partial \varphi_1}{\partial x_1} \cdots \dfrac{\partial \varphi_r}{\partial x_r} \right)^2}$$

will become a multiplier of that reduced system

After the substitution of the values of $x_1, \ldots, x_r$ that follow from (22), formulas (17) and (23) will yield any arbitrary multiplier of one and the same system of differential equations, as long as one sets M and M equal to suitable multipliers of the system (16) and (19). For a given M, that must necessarily give an M for which one has:

$$[\mathsf{M}'] = [M']$$

or

$$[\mathsf{M}] = [\Lambda' M],$$

and this identity will not cease to be true if one sets $\gamma_k = \varphi_k$ in it. Thus, it finally follows that:

Any multiplier M of the ***Jacobi*** *system* (16) *belongs to a multiplier of the equivalent system* (19) *that is determined from the formula:*

$$\mathsf{M} = \frac{M}{\Lambda},$$

and conversely. This theorem will agree with the result above in the case where $M = \Delta' = \Delta\Lambda^2$.

I shall point a case in which the multiplier can be given immediately. One can also write equations (9) in the form:

$$\sum_{h=1}^{r} (sh) \frac{\partial \mu_k}{\partial x_i'} + \sum_{h=1}^{r} \sum_{i=1}^{n} \left(\frac{\partial p_{hi}}{\partial x_k} + \frac{\partial p_{hk}}{\partial x_i} \right) x_i' = 0.$$

The multiplier can thus be taken to be equal to unity in any case, as long as one has:

$$\frac{\partial p_{hi}}{\partial x_k} + \frac{\partial p_{hk}}{\partial x_i} = 0$$

for all values of the indices. One will then have $\dfrac{\partial p_{hk}}{\partial x_i} = 0$ and $\dfrac{\partial^2 p_{hk}}{\partial x_i^2} = 0$; i.e., the p_{hi} must be linear functions of the x. One will obtain:

$$p_{hi} = \sum_{k=1}^{n} (a_{hik} x_k) + b_{hi},$$

in which the constants a must satisfy the conditions $a_{hik} + a_{hki} = 0$, and the equations = 0 will read:

$$\sum_{i,k=1}^{n} a_{hik} (x_k x_i' - x_i x_k') + \sum_{i=1}^{n} b_{hi} x_i' = 0$$

in that case. For the simplest assumption of three variables, one will have:

$$(a_1 + b_2\, x_3 - b_3\, x_2)\; x_1' + (a_2 + b_3\, x_1 - b_1\, x_3)\; x_2' + (a_3 + b_1\, x_2 - b_2\, x_1)\; x_3' = 0,$$

i.e., *motion in a linear complex* (*).

By the unavoidably abstract character that the dynamical problem assumes when one decides to extend the assumptions that have been conventional up to now regarding the

(*) Cf., these Annalen XXIII, pp. 52.

nature of the conditions of a system that is not free, in the previously-defined sense (*), one cannot expect to discuss a greater number of examples of such motions more thoroughly. In what follows, I will then first treat only the motion of a material point in a P-E-system:

$$p_1\, dx_1 + p_2\, dx_2 + p_3\, dx_3 = 0,$$

which is the case that shows the greatest agreement with the typical examples of theoretical mechanics.

If no external forces are present then the equations of motion of a point that moves with an arbitrary velocity c in the system will have the form:

$$x_i'' = \lambda\, p_i\,,$$

so the path will be described by the constant velocity c, and the normal to the P-E-system will always lie in its curvature plane; these curves likewise give the form of a stressed, completely-flexible, inextensible filament of the system that is found in equilibrium, but no longer a geodetic curve in the system, which is connected with the fact that the position can no longer be attained naturally by an actual stress of the filament that would exist in a hypersurface.

If its radius of curvature is ρ then one will obtain the value:

$$\rho = \frac{c^2}{\lambda\sqrt{p_1^2 + p_2^2 + p_3^2}}\,,$$

or, when one sets:

$$(p_1^2 + p_2^2 + p_3^2)\,\lambda = -\sum \frac{\partial p_i}{\partial x_k}\, x_i'\, x_k' = P,$$

the value:

$$\rho = \frac{c^2\sqrt{p_1^2 + p_2^2 + p_3^2}}{P}\,.$$

The denominator P will be equal to zero for the motion in a linear complex, and in fact, the rays of the complex itself must also be described. However, in general, the path curve will contain an inflection when the direction of the velocity coincides with one of the *principle tangents* of the system, for which P will vanish (**). One further obtains from the equation:

$$\begin{vmatrix} x_1' \\ x_1'' \\ x_1''' \end{vmatrix} = \lambda^2 \begin{vmatrix} x_1' \\ p_1 \\ dp_1 / dt \end{vmatrix}$$

(*) Cf., the statement in the remark on pp. 9

(**) *Loc. cit.*, pp. 49.

(only the first column of the determinant is suggested), the following expression for the radius of torsion of the curve:

$$\frac{1}{c^2(p_1^2+p_2^2+p_3^2)}\begin{vmatrix} x_1' \\ p_1 \\ dp_1/dt \end{vmatrix}.$$

The factor that appears here will determine the directions of the *lines of curvature* of the *P-E-system* when it is set equal to zero (*), and those directions must coincide with those of the velocity, which means that the path will contain a plane of inflection.

If the point is subjected to the effect of an arbitrary force with the components X, and one denotes the increment in its coordinates by ξ_i, then one will find, by a simple calculation, that:

$$\sum_{i=1}^{3}\xi_i\,p_i+\tfrac{1}{2}\sum_{i,k=1}^{3}\xi_i\xi_k\frac{\partial p_i}{\partial x_k}+\tfrac{1}{6}\sum_{i,k,l=1}^{3}\xi_i\xi_k\xi_l\frac{\partial^2 p_i}{\partial x_k\partial x_l}=\frac{t^2}{12}\sum_{i,k=1}^{3}(x_i'\,x_k''-x_k'\,x_i'')\left(\frac{\partial p_i}{\partial x_k}-\frac{\partial p_k}{\partial x_i}\right).$$

If one now denotes the left-hand side by F then any curve that can be described at all must osculate the surface $F = 0$, which will be contact of degree three, as long as the osculating plane of the path includes the direction whose cosines are proportional to the three differences:

$$\frac{\partial p_2}{\partial x_3}-\frac{\partial p_3}{\partial x_2},\qquad \frac{\partial p_3}{\partial x_1}-\frac{\partial p_1}{\partial x_3},\qquad \frac{\partial p_1}{\partial x_2}-\frac{\partial p_2}{\partial x_1}$$

or

$$\alpha_1,\ \alpha_2,\ \alpha_3\,.$$

The normal curvatures and the curvature of that surface give precisely the same expressions that I referred to as curvatures in my previous consideration of P-E-systems (**), and yield an entirely intuitive interpretation for those quantities.

By contrast, the equations of the geodetic lines are obtained from a proper variational problem in the form:

$$\text{(12)}\qquad \begin{aligned} x_1'' &= -\,p_1\lambda'+\lambda[\alpha_2x_3'-\alpha_3x_2'],\\ x_2'' &= -\,p_2\lambda'+\lambda[\alpha_1x_3'-\alpha_3x_1'],\\ x_3'' &= -\,p_3\lambda'+\lambda[\alpha_3x_1'-\alpha_1x_3'],\end{aligned}\qquad \lambda'=\frac{d\lambda}{dt},$$

and they will also be described with constant velocity (***).

If one proposes to examine the motion in a *linear complex* then its equation can be assumed to have the simplified form:

(*) *Ibidem*, pp. 71.
(**) *Loc. cit.*, pp. 70.
(***) Cf., pp. 9, remark.

(13) $$(x_2\,x_1' - x_1\,x_2') + a\,x_3' = 0.$$

The motion of a gravitating material point for which the direction of the acceleration of gravity is parallel to the axis of the complex will result from the equations:

$$x_1'' = +\lambda\,x_2\,,$$
$$x_2'' = -\lambda\,x_1\,,$$
$$x_3'' = g + \lambda\,a.$$

One will then have:

$$\lambda = -\frac{ag}{x_1^2 + x_2^2 + x_3^2}\,.$$

If one denotes $x_1^2 + x_2^2$ by r^2 then one will find that:

$$2g\,x_3 + \text{const.} = \frac{r^3 r''}{a^2} + r'^2 + rr''$$

by means of the principle of *vis viva*, while x_1 and x_2 are derived from the two differential equations:

$$x_1'' = -\frac{a\,g\,x_2}{r^2 + a^2}\,, \qquad x_2'' = +\frac{a\,g\,x_1}{r^2 + a^2}\,.$$

The motion in a linear complex under the influence of a force R whose line of direction is perpendicular to the axis and cuts it is determined from the equations:

$$x_1'' = R\frac{x_1}{r} + \lambda\,x_2\,,$$
$$x_2'' = R\frac{x_2}{r} - \lambda\,x_1\,,$$
$$x_3'' = \lambda\,a.$$

$\lambda = 0$ in this. Thus, helices will arise in the linear complex, and in particular, proper complex helices, as well, as long as R is proportional to the radius vector.

The geometric lines of the linear complex are given by:

$$x_1'' = -\lambda' x_2 - 2\lambda\,x_2'\,,$$
$$x_2'' = +\lambda' x_1 + 2\lambda\,x_1'\,,$$
$$x_3'' = -\lambda'\,a.$$

If one denotes arbitrary constants by c_1, c_2, c_3 then that will imply that:

$$x_3'' = c_2 - \lambda\,a,$$

$$x_1'^2 + x_2'^2 + x_3'^2 = c_1^2,$$
$$\lambda = \frac{c_3}{r^2 + a^2}$$

or by the introduction of polar coordinates r, φ in place of x_1 and x_2 :

$$\frac{1}{2}\frac{d\rho}{\sqrt{c_1^2\rho - c_1^2 a^2 - \left(c_2 - c_3\frac{a}{\rho}\right)^2 \rho}} = dt,$$

$$d\varphi = \frac{a}{\rho - a^2}\left(c_2 - c_3\frac{a}{\rho}\right)dt,$$

$$dx_3 = \left(c_2 - c_3\frac{a}{\rho}\right)dt.$$

In the special case $\lambda = c_3$, $\lambda' = 0$, one will obtain the equations of the complex helices; for $\lambda = 0$, one will obtain the lines of the complex itself, as it must be. In general, t will become an elliptic integral of the second kind, while:

$$x_3 - c_2\, t = -\, c_3\,\frac{a}{2}\int\frac{d\rho}{\sqrt{\rho^3(c_1^2 - c_2^2) + \rho^2(2c_2c_3a - c_2^2a^2) - c_3^2a^2\rho}}$$

will be an elliptic integral of the first kind. We shall not go into a closer examination of these transcendental curves; we emphasize only the case $c_3 = c_2\, a$, in which:

$$d\varphi = a\, c_2\,\frac{dt}{\rho},$$

so

$$x_3 = c_2\, t - a\,\varphi + c_4 .$$

Finally, if $c_1^2 = c_2^2$, in addition, then one will have:

$$t = \frac{1}{2c_2a}\int\frac{d\rho\,\rho}{\sqrt{\rho(\rho - a^2)}}, \qquad d\varphi = \frac{1}{2}\int\frac{d\rho}{\sqrt{\rho(\rho - a^2)}},$$

and if one again sets $\rho = r^2 + a^2$ then one will have:

$$\frac{2r}{a} = e^{\varphi + c} - e^{-\varphi + c},$$

$$t = \frac{1}{2c_2 a}\left[r\sqrt{r^2+a^2} + a^2\varphi\right] + c_3,$$

by which, the equations of the path curves will be determined completely.

Finally, I shall mention another example. Let the equation:

$$(x_1 - x_2)\, x_1' + (y_1 - y_2)\, y_1' + (z_1 - z_2)\, z_1' = 0$$

be prescribed for the motion of two material points whose masses are both equal to unity (which is, moreover, irrelevant) and whose coordinates are x_1, y_1, z_1 and x_2, y_2, z_2, respectively. One must then set:

$$x_2 = at, \qquad y_2 = bt, \qquad z_2 = ct,$$

and for:

$$\xi = x_1 - at, \qquad \eta = y_1 - bt, \qquad \zeta = z_1 - ct,$$

one will have:

$$\text{(24)} \qquad \begin{aligned} \xi'' &= \lambda\, \xi, \\ \eta'' &= \lambda\, \eta, \\ \zeta'' &= \lambda\, \zeta, \end{aligned}$$

with the condition:

$$\text{(25)} \qquad \xi\xi' + \eta\eta' + \zeta\zeta' + a\xi + b\eta + c\zeta = 0.$$

It follows from (24) that:

$$\text{(26)} \qquad \begin{aligned} \eta\zeta' - \zeta\eta' &= c_1, \\ \zeta\xi' - \xi\zeta' &= c_2, \\ \xi\eta' - \eta\xi' &= c_3\,. \end{aligned}$$

Furthermore, from the principle of *vis viva*, one will have:

$$\text{(27)} \qquad \xi'^2 + \eta'^2 + \zeta'^2 + 2\,(a\xi + b\eta + c\zeta) + a^2 + b^2 + c^2 = \text{const.} = h^2.$$

If one sets, to abbreviate:

$$\begin{aligned} \xi^2 + \eta^2 + \zeta^2 &= r^2, & c_1^2 + c_2^2 + c_3^2 &= B^2, \\ a\xi + b\eta + c\zeta &= p, & ac_1 + bc_2 + cc_3 &= C, \\ a^2 + b^2 + c^2 &= A^2, \end{aligned}$$

$$q = \begin{vmatrix} a & b & c \\ c_1 & c_2 & c_3 \\ \xi & \eta & \zeta \end{vmatrix}, \qquad q^2 = r^2\,(A^2 B^2 - C^2) - p^2 B^2$$

then from (25) and (26), one will get:

$$r^2\, \xi' = -\,\xi\, p + c_2\, \zeta - c_3\, \eta,$$

(28)
$$r^2\,\eta' = -\,\eta\,p + c_3\,\xi - c_1\,\zeta,$$
$$r^2\,\zeta' = -\,\zeta\,p + c_1\,\eta - c_2\,\xi,$$

and from (24), (25), one will get:

$$\lambda\,r^2 + \xi'^2 + \eta'^2 + \zeta'^2 + a\xi' + b\eta' + c\zeta' = 0.$$

One finds from (27), (28) that:

$$r^2\,(a\xi' + b\eta' + c\zeta') = q - p^2,$$
$$\xi'^2 + \eta'^2 + \zeta'^2 = h^2 - A^2 - 2p,$$

so

(29)
$$\lambda\,r^2 + h^2 - A^2 - 2p = \frac{p^2 - q}{r^2},$$

while it will arise from (26) by squaring and adding that:

(30)
$$r^2\,[h^2 - A^2 - 2p] = B^2 + p^2.$$

Thus, p is a known function of r, as well as q and λ. The integration of equations (24) is then reduced to the examination of a central motion. The motion itself is that of a point x_1, y_1, z_1 that is attracted to a moving center x_2, y_2, z_2 by a certain law. Meanwhile, for the determination of the ξ, η, ζ, it is more convenient for one to appeal to the following formulas.

One will find t as a function of p with no further analysis from the equation:

$$r = \xi\xi' + \eta\eta' + \zeta\zeta' = -\,p$$

and the equation:

$$2r\frac{dr}{dt} = \frac{d}{dt}\left(\frac{B^2 + p^2}{h^2 - A^2 - 2p}\right),$$

which follows from (30), and thus one will find r^2, as well, from (30), and finally, q, from the equation:

$$q = p^2 + r^2\frac{dp}{dt};$$

i.e., one can calculate ξ, η, ζ, linearly from the latter equation (26) and the values of p, q, in which the five constants c_1, c_2, c_3, h, and the latter integration constant are deduced from the initial conditions.

I shall conclude with the following remark: Up to now, it was assumed that the p_{ik} in the given differential relations did not include time explicitly. However, the equations of motion will keep the same form if that restriction is dropped. In the simplest case, one will have a time-varying P-E-system, and by applying the principle of virtual velocities, one will have to consider it to be at rest in a well-known way. Moreover, from that point onward, one can go to the more general case in which equations of the form:

$$\sum_{i=1}^{n} p_{ri} dx_i + T_r \, dt = 0$$

are given, in which the p_{ri} and T_r are functions of x and time t. The equations of motions experience no alteration here, either. At this level of generality, one then has the case in which a certain number of first integrals that are linear in the differential quotients x_i' are prescribed for the problem in question. On the other hand, that linear character will be necessary if any sort of analogy to the equations of mechanics is to exist at all.

Dresden, beginning of September 1884.

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