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Geometrical interpretation of the differential equation $P\,dx + Q\,dy + R\,dz = 0$.

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1. By the equation:

$$P\,dx + Q\,dy + R\,dz = 0, \tag{1}$$

any point M (x, y, z) will be associated with the plane that goes through it:

$$(X - x)\,P + (Y - y)\,Q + (Z - z)\,R = 0. \tag{2}$$

One will then obtain a geometric conception of the surface elements that are determined by the differential equation when one examines the character of the general *null system* (2). If one goes from M in the plane (2) to the neighboring point M_1 along the direction δ_1 then M_1 will correspond to the plane:

$$(X - x - \delta_1\,x)\,(P + \delta_1\,P) + \ldots = 0, \tag{3}$$

while:

$$P\,\delta_1 x + Q\,\delta_1 y + R\,\delta_1 z = 0.$$

The planes (2) and (3) intersect along a direction δ_2 that is determined from the equations:

$$\delta_1 x : \delta_1 y : \delta_1 z = Q\,\delta_1 R - R\,\delta_1 Q : R\,\delta_1 P - P\,\delta_1 R : P\,\delta_1 Q - Q\,\delta_1 P, \tag{4}$$

by means of which, one finds a *projective relationship* between the directions δ_1, δ_2, whose *double elements* are determined by means of the equations:

$$\begin{cases} \mu\,\delta x = Q\,\delta R - R\,\delta Q, \\ \mu\,\delta y = R\,\delta P - P\,\delta R, \\ \mu\,\delta z = P\,\delta Q - Q\,\delta P, \end{cases} \tag{5}$$

or by means of the quadratic equation for μ:

$$\mu^2 - \mu\,G - H = 0, \tag{6}$$

in which:

$$G = P\,(Q_z - R_y) + Q\,(R_x - P_z) + R\,(P_y - Q_x),$$

$$H = \begin{vmatrix} P_x & P_y & P_z & P \\ Q_x & Q_y & Q_z & Q \\ R_x & R_y & R_z & R \\ P & Q & R & 0 \end{vmatrix},$$

when one sets $\partial P / \partial x = P_x$, etc., to abbreviate.

If one then advances from M along any line in the plane (1) then the associated neighboring plane of the null system will rotate around a projectively-corresponding line through M (*).

2. The simple infinitude of planes (3) defines a system whose curvature in a certain direction δ can be set equal to the angle between the associated planes – i.e., the square root of:

$$\delta x\,\delta P + \delta y\,\delta Q + \delta z\,\delta R\,.$$

One will then get the *mutually-perpendicular directions of the principal curvatures* by the formulation of a known maximum problem.

By contrast, one will be led to the determinant H when one seeks the neighboring plane (3) whose normal meets the normal to (2). From the condition that the differential of:

$$x + \mu P, \qquad y + \mu Q, \qquad z + \mu R$$

should vanish with (1), one will, in fact, get:

$$\begin{vmatrix} \frac{1}{\mu} + P_x & P_y & P_z & P \\ Q_x & \frac{1}{\mu} + Q_y & Q_z & Q \\ R_x & R_y & \frac{1}{\mu} + R_z & R \\ P & Q & R & 0 \end{vmatrix} = 0,$$

(*) If the P, Q, R vanish at M then M will be the vertex of a second-degree cone of directions:

$$dx\,dP + dy\,dQ + dz\,dR = 0,$$

which is distinguished by the fact that when one proceeds along one of its edges, the associated neighboring plane will rotate around the edge in question. Among these edges, one will find six distinguished ones, along which three consecutive planes will cut, etc.

and thus two normals that meet the original one, so the corresponding directions will be different from those of the principal curvatures, in general.

Now, if H vanishes at the point M then equation (6) will have a root $\mu = 0$; i.e., one will have:

$$\delta P : \delta Q : \delta R = P : Q : R,$$

so the plane (1) *will be stationary along the direction* δ *in question, and all other directions will belong to planes that have the line of direction that is associated with the other root of* (6) *for their common line of intersection. If H vanishes identically then the* ∞^3 *planes* (1) *will be grouped into* ∞^2 *of them that envelop a surface, while the* ∞^1 *points will define a curve to which each of them belong (viz., a special null system).* In fact, since H is, up to a factor, the *functional determinant* of:

$$\frac{P}{v}, \frac{Q}{v}, \frac{R}{v}; \qquad v = Px + Qy + Rz,$$

an identity relation will exist:

$$\psi\left(\frac{P}{v}, \frac{Q}{v}, \frac{R}{v}\right) = 0;$$

i.e., the planes (1) or $XP + YQ + ZR - v = 0$ will envelop a surface. The infinitely-close planes:

$$(X - x - dx)\,(P + dP) + \ldots$$

all go through the point X, Y, Z, which corresponds to the common solution of the equations:

$$\begin{aligned}
(X - x)\,P_x + (Y - y)\,Q_x + (Z - z)\,R_x &= P, \\
(X - x)\,P_y + (Y - y)\,Q_y + (Z - z)\,R_y &= Q, \\
(X - x)\,P_z + (Y - y)\,Q_z + (Z - z)\,R_z &= R, \\
(X - x)\,P \;\; + (Y - y)\,Q \;\; + (Z - z)\,R \;\; &= 0,
\end{aligned}$$

and finds the direction line that corresponds to the root $\mu = G$, and thus, the respective contact point of the enveloping plane. Conversely, if one sets P, Q, R proportional to constants a, b, c, resp., then one will get a curve that is planar, since:

$$\Psi\left(\frac{a}{u}, \frac{b}{u}, \frac{c}{u}\right) = 0, \qquad u = ax + by + cz,$$

and the direction lines that are associated with the points of that curve will go through a common point of that plane.

3. The projective relationship (4) is not *involutory*, in general. As one recognizes immediately from (5), it can have that character only when equation (6) is *pure* quadratic; i.e., when:

$$G = 0.$$

However, the neighboring planes (3) are then arranged in precisely the same way as the tangential planes in the neighborhood of a point of a surface; i.e., they define a surface element. We understand that to mean the totality of all infinitely-close contact planes that go through a point of a surface. Such an element will be assigned a certain normal curvature along any direction in it; the two mutually-corresponding directions will be the *principal tangents.*

If the equation $G = 0$ is fulfilled identically then the surface elements will group into the ∞^1 surfaces of a system of surfaces that is the *integral of the differential equation* (1). By contrast, if G is not identically zero then the surface $G = 0$ will seem to be covered by a family of curves along which the surface elements go to each other. They will contact each other only at isolated points with $G = 0$, in general. However, this can happen along an entire curve, in particular, namely, a *singular strip*, or in an entire subset of $G = 0$. In the latter case, that subset of G will be a *singular integral* of the differential equation. An example of this is defined by the assumption that:

$$P : Q : R = \mu\frac{\partial\psi}{\partial x}+\psi\frac{\partial\psi}{\partial y} \,:\, \mu\frac{\partial\psi}{\partial y}+\psi\varphi_2 \,:\, \mu\frac{\partial\psi}{\partial z}+\psi\varphi_3,$$

in which the μ, ψ, φ_1, φ_2, φ_3 are arbitrary functions of x, y, z.

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