"Zur Theorie der allgemeinen Punktebenensysteme," Math. Ann. **23** (1884), 45-81.

On the theory of general point-plane systems

By

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The most general arrangement by which any point of a space belongs to a well-defined plane that goes through it is mediated by three functions p_i of the variables x_1, x_2, x_3 that assign the plane:

$$\sum (X_i - x_i)\, p_i = 0$$

to the point. In the absence of a suitable name, such a system of planes shall be called a *point-plane system*, or briefly a P-E-system (†); I would have preferred the term "null system," but that term is already generally employed for a special case of this association.

This association is the one by which the differential equation:

$$\sum p_i\, dx_i = 0$$

will assign an associated surface element. It seems that up to now one has always been restricted to a closer examination of the case in which the p_i satisfy the integrability condition. The planes of the system are then the tangent planes to a simply-infinite family of surfaces, so they define a *special P-E-system of the first kind* whose geometric character I emphasized some time ago in a note (*). In a more recent treatment of the general P-E-system, I realized that an entire series of properties whose understanding tends to form the domain of surface theory can be ascribed to such P-E-systems in a natural way. Since it seems not uninteresting to me to give a more precise explanation of that point, which (to my knowledge) has attracted no attention up to now (**), I might likewise be permitted to elaborate upon some of the viewpoints in my previous note, which were suggested only quite briefly and incidentally there, but define an essential moment for the present purpose.

At the same time, I have pursued the objective of giving a classification of the P-E-systems. Although it is based upon projective viewpoints, I have nonetheless preferred to avoid the use of projective coordinates. Thus, I shall take this opportunity to refer to the

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(†) Translator: From the German *Punkt-Ebenensystem.*

(*) These Annalen XVI, pp. 556, *et seq.*

(**) One finds considerations of that kind in the paper by **Kummer** in Crelle's Journal, Bd. 57, in regard to the theory of rectilinear ray systems.

next paper, in which the theory of algebraic surfaces and ray systems will be coupled with P-E-systems from the purely projective standpoint.

§ 1.

The projective association in P-E-systems.

If one can make the point x_1, x_2, x_3 correspond to the plane:

$$(1)\qquad (X_1 - x_1)\,p_1 + (X_2 - x_2)\,p_2 + (X_3 - x_3)\,p_3 = \sum (X_i - x_i)\,p_i = 0, \qquad i = 1, 2, 3,$$

which goes through it, then the point:

$$x_1 + dx_1, \qquad x_2 + dx_2, \qquad x_3 + dx_3,$$

for which the p_i will go to $p_i + dp_i$, will be assigned to the plane:

$$(2)\ (X_1 - x_1 - dx_1)(p_1 + dp_1) + (X_2 - x_2 - dx_2)(p_2 + dp_2) + (X_3 - x_3 - dx_3)(p_3 + dp_3) = 0.$$

The two planes intersect in the lines:

$$(3)\qquad \begin{aligned} &\sum (X_i - x_i)\,p_i = 0, \\ &\sum (X_i - x_i)\,dp_i = \sum p_i\,dx_i . \end{aligned}$$

If one now advances the system itself in the plane (1) then one will have:

$$(4)\qquad \sum p_i\,dx_i = 0,$$

and the two planes (3) intersect along a direction:

$$\delta x_1,\ \delta x_2,\ \delta x_3,$$

which satisfies the equations:

$$\sum p_i\,\delta x_i = 0, \qquad \sum dp_i\,\delta x_i = 0.$$

Any direction for which the increment of the x_i *is denoted by d thus corresponds projectively to a direction* δ. One can next distinguish the case according to whether that projectivity is *hyperbolic*, *parabolic*, or *elliptic*. In order to examine the characteristic association of neighboring planes of the system that follows from this more closely, one must determine the curves that are enveloped by the lines of intersection of plane that is assigned to a point of the plane (1) with the plane (1) itself.

In the vicinity of the point x_i, they will be characterized by differential equations:

$$d\xi = a_{11}\,\xi + a_{12}\,\eta,$$
$$d\eta = a_{21}\,\xi + a_{22}\,\eta,$$

for which the form of the integral curves in the neighborhood of the singular point $\xi = \eta = 0$ is known according to the behavior of the roots of the discriminant:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0.$$

I will therefore not go into questions of this nature; only the case of the *parabolic association* – i.e., *parabolic P-E-systems* – will be examined more closely from now on.

The projective relationship can also be a *special* one, in which case, one will call the P-E-system *special of the second kind.*

In order to arrive at the condition for the special projectivity, one sets:

$$dx_1, \quad dx_2, \quad dx_3\,;$$
$$\delta x_1, \quad \delta x_2, \quad \delta x_3$$

equal to the values:

$$\xi_1 + \lambda\,\xi_1', \qquad \xi_2 + \lambda\,\xi_2', \qquad \xi_3 + \lambda\,\xi_3';$$
$$\xi_1 + \mu\,\xi_1', \qquad \xi_2 + \mu\,\xi_2', \qquad \xi_3 + \mu\,\xi_3',$$

resp., which are proportional to them, and for which one must have:

$$\sum p_i\,\xi_i = \sum p_i\,\xi_i' = 0.$$

The bilinear relationship:

(5) $$A\,\lambda\,\mu + B\,\lambda + C\,\mu + D = 0$$

for the λ, μ will then arise, in which:

$$A = \sum \xi_i'\,\xi_k'\,\frac{\partial p_i}{\partial x_k},$$

$$B = \sum \xi_i'\,\xi_k\,\frac{\partial p_i}{\partial x_k},$$

$$C = \sum \xi_i\,\xi_k'\,\frac{\partial p_i}{\partial x_k},$$

$$D = \sum \xi_i\,\xi_k\,\frac{\partial p_i}{\partial x_k},$$

whose determinant $AD - BC$ must vanish.

The advance in the arbitrary direction λ will then continually correspond to the direction $\mu = -D / C$, while the particular direction $\lambda = -B / A = -D / C$ is assigned to that arbitrary direction.

However, the determinant $AD - BC$ will come about, up to a factor, when one multiplies the determinant:

(6)
$$\Delta = \begin{vmatrix} p_{11} & p_{12} & p_{13} & p_1 \\ p_{21} & p_{22} & p_{23} & p_2 \\ p_{31} & p_{32} & p_{33} & p_3 \\ p_1 & p_2 & p_3 & 0 \end{vmatrix},$$

in which one sets $p_{ik} = \partial p_i / \partial x_k$, by the product of the two determinants:

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1' & \xi_2' & \xi_3' \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}, \quad \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1' & \xi_2' & \xi_3' \\ \alpha_1' & \alpha_2' & \alpha_3' \end{vmatrix},$$

in which the α_i, α_i' mean arbitrary quantities. *Therefore,* $\Delta = 0$ *is the condition for of special projectivity. However,* Δ *will be, in turn, and up to a factor, the functional determinant of:*

$$\frac{p_1}{v}, \frac{p_2}{v}, \frac{p_3}{v}$$

when one sets:

$$v = \sum p_i x_i .$$

The identical vanishing of Δ *then expresses the idea that the planes of the P-E-system whose coordinates are:*

$$u_i = \frac{p_i}{v}$$

envelops a surface whose equation can be defined in inhomogeneous, **Hessian**, plane coordinates by perhaps:

$$F(u_1, u_2, u_3) = 0.$$

Here, all of the surface elements envelop *only one* surface – viz., the *order surface of the special P-E-system of the second kind* – while they will envelop a family of ∞^1 surfaces for special P-E-systems of the first kind. One must then emphasize the sub-case in which the order surface is a curve (developable, resp.), but I will not go further into these particular classes of P-E-systems.

The directions that go to themselves under the projective association are of particular importance. They are characterized by the fact that the associated plane of the system will rotate *around* the direction line in question for them, while in any other case they will wander around the plane that corresponds to projectively. Therefore, those directions might be called the *principal tangents of the system*. They are determined by the equations:

$$\sum p_i \, dx_i = 0,$$

(7)
$$\sum dp_i\, dx_i = \sum p_{ik}\, dx_i\, dx_k = 0,$$
or by:

(8)
$$\begin{aligned} \mu\, dx_1 &= p_2\, dp_3 - p_3\, dp_2\,, \\ \mu\, dx_2 &= p_3\, dp_1 - p_1\, dp_3\,, \\ \mu\, dx_3 &= p_1\, dp_2 - p_2\, dp_1\,, \end{aligned}$$

in conjunction with $\sum p_i\, dx_i = 0$. The elimination of the dx will lead to the quadratic equation:

(9)
$$\mu^2 - \mu\, G - \Delta = 0,$$

which I gave already, in which G represents the left-hand side of the integrability condition:
$$p_1\,(p_{23} - p_{32}) + p_2\,(p_{31} - p_{31}) + p_3\,(p_{12} - p_{21}) = 0.$$

Equations (7) can be solved in different ways, moreover, which I will discuss especially here, since the direct derivation of (9) would seem less elegant otherwise. If one replaces dx_i with ξ_i, to abbreviate, then:
$$p_{ik} + p_{ki} = 2q_{ik} = 2q_{ki}\,,$$
moreover, and one will have:

(10)
$$\sum p_i\, \xi_i = 0,$$
$$\varphi = \sum \xi_i\, \xi_k\, q_{ik} = 0,$$

instead of (7). One will then find the ratios of the ξ_i from the equations:
$$\begin{aligned} \lambda\, \xi_1 &= \xi_2 \varphi_3 - \xi_3 \varphi_2\,, \\ \lambda\, \xi_2 &= \xi_3 \varphi_1 - \xi_1 \varphi_3\,, \\ \lambda\, \xi_3 &= \xi_1 \varphi_2 - \xi_2 \varphi_1\,, \end{aligned}$$

in which one has set $\frac{1}{2}\frac{\partial \varphi}{\partial \xi_i} = \varphi_i$, and λ^2 is equal to the determinant of the q_{ik} that are edged by the p_i, which one will get immediately when one multiplies the i^{th} horizontal and vertical rows of that determinant by ξ_i, and then reduces it directly in a known way by an application of (10).

By contrast, if one would like to establish the p_{ik} then one could write φ in two ways:
$$\begin{aligned} \varphi &= \xi_1 \sum p_{1i}\, \xi_i + \xi_2 \sum p_{2i}\, \xi_i + \xi_3 \sum p_{3i}\, \xi_i \\ &= \xi_1 \sum p_{i1}\, \xi_i + \xi_2 \sum p_{i2}\, \xi_i + \xi_3 \sum p_{i3}\, \xi_i\,. \end{aligned}$$

If one now sets:

$$\psi_k(x) = \sum p_{ki}\, x_i\,, \qquad \chi_k(x) = \sum p_{ik}\, x_i$$

then by the same process that led to (8) will give two quadratic equations with the roots λ_1, λ_2; ρ_1, ρ_2, which correspond to the same system ξ_i, η_i, in such a way that:

$$\begin{aligned}
\xi_1\lambda_1 &= p_2\,\psi_3\,\xi - p_3\,\psi_2\,\xi, & \eta_1\lambda_2 &= p_2\,\psi_3\,\eta - p_3\,\psi_2\,\eta,\\
\xi_2\lambda_1 &= p_3\,\psi_1\,\xi - p_1\,\psi_3\,\xi, & \eta_2\lambda_2 &= p_3\,\psi_1\,\eta - p_1\,\psi_3\,\eta,\\
\xi_3\lambda_1 &= p_1\,\psi_2\,\xi - p_2\,\psi_1\,\xi, & \eta_3\lambda_2 &= p_1\,\psi_2\,\eta - p_2\,\psi_1\,\eta,
\end{aligned}$$

$$\begin{aligned}
\xi_1\rho_1 &= p_2\,\chi_3\,\xi - p_3\,\chi_2\,\xi, & \eta_1\rho_2 &= p_2\,\chi_3\,\eta - p_3\,\chi_2\,\eta,\\
\xi_2\rho_1 &= p_3\,\chi_1\,\xi - p_1\,\chi_3\,\xi, & \eta_2\rho_2 &= p_3\,\chi_1\,\eta - p_1\,\chi_3\,\eta,\\
\xi_3\rho_1 &= p_1\,\chi_2\,\xi - p_2\,\chi_1\,\xi, & \eta_3\rho_2 &= p_1\,\chi_2\,\eta - p_2\,\chi_1\,\eta.
\end{aligned}$$

The roots ρ, λ have a very simple connection to each other. In order to find it, one forms, e.g., $\xi_1^2 + \xi_2^2 + \xi_3^2$, and multiplies the result:

$$\lambda\,(\xi_1^2 + \xi_2^2 + \xi_3^2) = \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ p_1 & p_2 & p_3 \\ \psi_1 & \psi_2 & \psi_3 \end{vmatrix}$$

by the determinant:

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ p_1 & p_2 & p_3 \end{vmatrix},$$

whose value is $p_1^2 + p_2^2 + p_3^2$, when one assumes that the ratios of the ξ, η are determined in such a way that:

$$\begin{aligned}
p_1 &= \xi_2\,\eta_3 - \eta_2\,\xi_3\,,\\
p_2 &= \xi_3\,\eta_1 - \eta_3\,\xi_1\,,\\
p_3 &= \xi_1\,\eta_2 - \eta_1\,\xi_2\,.
\end{aligned}$$

One will then get:

$$\begin{aligned}
\lambda_1 &= -\sum \eta_i\, p_{ik}\, \xi_k\,,\\
\lambda_2 &= \sum \xi_i\, p_{ik}\, \eta_k\,,\\
\rho_1 &= -\sum \eta_i\, p_{ki}\, \xi_k\,,\\
\rho_2 &= \sum \xi_i\, p_{ki}\, \eta_k\,,
\end{aligned}$$

and therefore:

$$\lambda_1 + \rho_2 = 0 = \lambda_2 + \rho_1\,,$$
$$\lambda_1 + \lambda_2 = \sum (p_{ik} - p_{ki})\,(\xi_i\,\eta_k - \xi_k\,\eta_i) = G.$$

Finally, by means of the aforementioned conversion of Δ, one will get:

$$\lambda_1\rho_1 = -\,\lambda_1\rho_2 = -\,\rho_1\rho_2 = \Delta$$

directly, in which the λ, ρ are determined from the quadratic equation:

$$x^2 \mp xG - \Delta = 0,$$

in which the upper sign belongs to λ. *The integrability condition* $G = 0$ *is now, at the same time, the condition for the involution of the projective association.* $B = C$ is then the requisite notation for the latter, or:

$$(\xi_2\xi_3' - \xi_3\xi_2')(p_{23} - p_{32}) + (\xi_3\xi_1' - \xi_1\xi_3')(p_{31} - p_{13}) + (\xi_1\xi_2' - \xi_2\xi_1')(p_{12} - p_{21}) = 0,$$

so, by means of the relations:

$$p_1 : p_2 : p_3 = \xi_3\xi_1' - \xi_1\xi' : \xi_3\xi_1' - \xi_1\xi' : \xi_1\xi_2' - \xi_2\xi',$$

one will convert it into $G = 0$.

Curves whose tangents are all defined by the directions of advance in the associated planes shall be called *curves in P-E-systems.* The curves that have direction lines that correspond to themselves must then be referred to as *curves with principle tangents.* As one recognizes immediately, they have the property that their osculating planes coincide with the planes of the system. One then has the following theorem:

The planes of a P-E-system can be described as the osculating planes of a system of curves in two different ways, in which two associated planes will go through any point of space.

The principle tangents will coincide when the discriminant of (10), viz.:

$$\Delta' = \begin{vmatrix} q_{11} & q_{12} & q_{13} & p_1 \\ q_{21} & q_{22} & q_{23} & p_2 \\ q_{31} & q_{32} & q_{33} & p_3 \\ p_1 & p_2 & p_3 & 0 \end{vmatrix},$$

vanishes. *It is, at the same time, the discriminant of* (9), such that the relation:

(11) $$4\Delta' = G^2 + 4\Delta$$

will be true.

The surface $\Delta' = 0$ *is called the focal surface of the ray system of the principle tangent curves, and* $\Delta = 0$ *is the inflection surface of the P-E-system.* It will then follow from (11) that:

The focal surface and the inflection surface of the system contact each other at any of their common points.

The focal surface always has a certain number of *nodes.* A *first group* of them is characterized by the fact that all of the first sub-determinants of Δ' will vanish for them, or that the equations:

$$p_{ik} + p_{ki} = a_i\, p_k + a_k\, p_i$$

are true, in which the a_i are arbitrary quantities. At such a location, *any* direction of advance will be a principal tangent direction in the associated plane. If the stated conditions are fulfilled at that place then one will have the identity:

$$\sum \xi_i\, \xi_k\, p_{ik} = \sum \xi_i\, p_i \sum \xi_i\, a_i\,. \tag{12}$$

In order to recognize the circumstances under which it can be fulfilled, I set:

$$p_i = X\, q_i\,,$$

in which X might be a function of the x_i . One will then have:

$$\sum \xi_i\, \xi_k\, q_{ik} = \sum \xi_i\, q_i \left(a_i' - \frac{\partial \log X}{\partial x_i} \right) = \sum \xi_i\, q_i \sum \xi_i\, b_i\,.$$

Since one can dispose of X arbitrarily, one can also convert the arbitrary constants b_1, b_2, b_3 into zero. However, that means that the q_1, q_2, q_3 must be independent of x_1, x_2, x_3 if an equation of the form (12) is to exist at all. Moreover, it emerges from this that one must set the a_i equal to zero in equation (12). However, one will then have the equations:

$$p_{ik} + p_{ki} = 0,$$

whose integrals are:

$$p_1 = a_1 + b_2\, x_3 - b_3\, x_2\,,$$
$$p_2 = a_2 + b_3\, x_1 - b_1\, x_3\,,$$
$$p_3 = a_3 + b_1\, x_2 - b_2\, x_1\,.$$

One then finds the single "planar" P-E-system in which all curves of the system are principal tangent curves, namely, the *linear complex.* Moreover, it can also be recognized from the fact that when equation (12) is valid, a plane pencil of rectilinear principal tangents must emanate from every point of space.

A second *group of double points* is found at the places where the p_i vanish simultaneously; a point of that kind will also be a node of the inflection surface. The association will be completely undetermined here, and all lines that belong to a certain second-degree cone will be principal tangents.

For rectilinear ray systems, the focal surfaces will have their rays for double tangents. In the curve system of the principal tangent curves, this relationship to focal surfaces is no longer found to exist. The direction of the principal tangent at a point of it is then determined by the system of linear equations:

$$\begin{aligned}
\xi_1 q_{11} + \xi_2 q_{12} + \xi_3 q_{13} + \lambda p_1 &= 0, \\
\xi_1 q_{21} + \xi_2 q_{22} + \xi_3 q_{23} + \lambda p_2 &= 0, \\
\xi_1 q_{31} + \xi_2 q_{32} + \xi_3 q_{33} + \lambda p_3 &= 0, \\
\xi_1 p_1 + \xi_2 p_2 + \xi_3 p_3 &= 0,
\end{aligned}$$

from which, it will follow that:

$$\sum \xi_i \, \xi_k \, q_{ik} = \sum \xi_i \, \xi_k \, p_{ik} = 0.$$

By means of this, the equation of the tangent planes to the focal surface will assume the following form:

$$\begin{vmatrix}
q_{11} & q_{12} & q_{13} & \sum y_i p_{1i} - \left(\xi_1 \sum y_i q_{11i} + \xi_2 \sum y_i q_{12i} + \xi_3 \sum y_i q_{13i}\right) \\
q_{21} & q_{22} & q_{23} & \sum y_i p_{2i} - \left(\xi_1 \sum y_i q_{21i} + \xi_2 \sum y_i q_{22i} + \xi_3 \sum y_i q_{23i}\right) \\
q_{31} & q_{32} & q_{33} & \sum y_i p_{3i} - \left(\xi_1 \sum y_i q_{31i} + \xi_2 \sum y_i q_{32i} + \xi_3 \sum y_i q_{33i}\right) \\
p_1 & p_2 & p_3 & \left(\xi_1 \sum y_i q_{1i} + \xi_2 \sum y_i q_{2i} + \xi_3 \sum y_i q_{3i}\right)
\end{vmatrix} = 0$$

when one replaces the running coordinates with $X_i - x_i \, y_i$, as one recognizes immediately by means of simple determinant reductions, or, when the determinant of q_{ik} does not vanish, as one might assume (*):

$$\sum \xi_k \left(\xi_1 \sum y_i q_{k1i} + \xi_2 \sum y_i q_{k2i} + \xi_3 \sum y_i q_{k3i}\right) = 0.$$

The plane can therefore include the direction ξ_i only when:

$$\sum \xi_i \, \xi_k \, \xi_l \, q_{ikl} = 0.$$

However, in the next paragraph, it will be shown that in this way one can express the idea that two immediately-following tangents to the principal tangent curve must coincide at such a location, or that the *principal tangent must be stationary.* It will now emerge from this that:

The principal tangent curves have vertices at the points of the focal surface, since they can advance to real points on only one side of them. It is only at the points of the focal surface at which inflections of these curves are actually present that the focal surface will contact them. In particular, if the principal tangent curves are rectilinear then they must contact the focal surface as often as they meet it at associated points.

(*) If that determinant vanishes everywhere then Δ' will be a square; i.e., the focal surface will be counted twice, and one cannot naturally speak of contact then.

The curve along which the focal surface is contacted by the principal tangent will not itself contact the latter, in general. If that happens to be true then it will, at the same time, be a *singular principal tangent curve.* In particular, if this can occur at all points of the focal surface then there will exist ∞^1 singular curves of that kind – e.g., when the one family of principal tangent curves is rectilinear. Finally, in the special case of rectilinear ray systems, the tangent to the singular principal tangent curves will generate the tangents to the **Kummer** developables.

§ 2.

The special P-E-system of the second kind and the inflection surfaces.

At every point, one finds a certain direction ξ_i for which the associated plane of the system remains *parallel* to its initial position. It will be determined by the equations:

$$\begin{aligned} \sum \xi_i\, p_{1i} &= \lambda\, p_1 , \\ \sum \xi_i\, p_{2i} &= \lambda\, p_2 , \\ \sum \xi_i\, p_{3i} &= \lambda\, p_3 . \end{aligned} \tag{1}$$

The vanishing of the determinant Δ or the existence of equations (1), along with $\sum \xi_i\, p_i = 0$, will then express the idea that a direction exists in the associated plane *along which the plane of the systems stays stationary.* As one sees, such a plane is, at the same time, *the inflection contact plane* of the corresponding principal tangent curve. *The inflection points of the latter will then lie on the inflection surface.* Here, as well, the principal tangents will not contact the inflection surface; moreover, that will happen only along a curve that is cut by the surface:

$$\begin{vmatrix} p_{11} & p_{12} & p_{13} & \sum \xi_i \xi_k \dfrac{\partial^2 p_1}{\partial x_i\, \partial x_k} \\ p_{21} & p_{22} & p_{23} & \sum \xi_i \xi_k \dfrac{\partial^2 p_2}{\partial x_i\, \partial x_k} \\ p_{31} & p_{32} & p_{33} & \sum \xi_i \xi_k \dfrac{\partial^2 p_3}{\partial x_i\, \partial x_k} \\ p_1 & p_2 & p_3 & 0 \end{vmatrix} = 0 \tag{2}$$

along the inflection surface in the event that one replaces the ξ_i in (2) with their values in (1). In particular, that curve can, at the same time, be a principal tangent curve, *so there will then exist a non-singular principal tangent curve in the system, and if equation* (2) *is fulfilled at all points of* $\Delta = 0$ *then a family of* ∞^1 *planar principal tangent curves will be present.* However, it is assumed in this that the principal tangent curves have well-defined osculating planes, If they are rectilinear then they will generate a family of (singular) enveloping curves on the inflection surface under the stated assumptions; this happens, in particular, when the p_i are linear functions of the x_i, so equation (2) will exist to begin with.

I shall now consider the case in which Δ vanishes identically – i.e., all planes of the system envelop the order surface F. *There will then exist a doubly-infinite family of planar principal tangent curves; in every tangent plane of F there will lie a planar principal tangent curve that is composed of the points for which the ratios of the p_i remain constant* (*).

Namely, let the equation of the plane associated with the point x_i be:

$$\sum X_i \, p_i = v, \qquad v = \sum x_i \, p_i$$

and let:

$$F\left(\frac{p_1}{v}, \frac{p_2}{v}, \frac{p_3}{v}\right) = 0 = F(u_1, u_2, u_3)$$

be the equation of the order surface.

One now chooses the point x_i^0 in the plane such that:

$$\sum x_i^0 \, p_i = v,$$

and at the same time, one has:

$$\frac{p_1^0}{p_2^0} = \frac{p_1}{p_2},$$

in which one will have:

$$F\left(\frac{p_1^0}{v^0}, \frac{p_2^0}{v^0}, \frac{p_3^0}{v^0}\right) = 0.$$

If one then sets:

$$\frac{p_3^0}{p_2^0} - \frac{p_3}{p_2} = \frac{1}{p_2}$$

then one will have:

$$\frac{p_1^0}{v^0} = \frac{p_1}{v + x_3^0 \lambda},$$

$$\frac{p_2^0}{v^0} = \frac{p_2}{v + x_3^0 \lambda},$$

$$\frac{p_3^0}{v^0} = \frac{p_3 + \lambda}{v + x_3^0 \lambda}.$$

Thus:

$$F\left(\frac{p_1}{v + x_3^0 \lambda}, \frac{p_2}{v + x_3^0 \lambda}, \frac{p_3}{v + x_3^0 \lambda}\right) = 0.$$

(*) There are ∞^2 principal tangent curves, which split into two families. In the case in the text above, one of the families will be planar.

If one now imagines that x_i^0 is sufficiently close to x_i then this equation will have only the root $\lambda = 0$. However, that shows that it will also follow from:

$$\frac{p_1^0}{p_2^0} = \frac{p_1}{p_2}$$

that:

$$\frac{p_3^0}{p_2^0} = \frac{p_3}{p_2},$$

and therefore the equation of planar principal tangent curve will arise:

$$p_1 : p_2 : p_3 = p_1^0 : p_2^0 : p_3^0 .$$

Let such a curve be denoted by C, so *all points of it will correspond to the plane of C as the associated one.* In order for any point x_i of it to determine the other principal tangent direction, one considers the equations:

$$\begin{aligned}
p_1\rho &= p_{11}\left(\frac{\partial F}{\partial u_1} - \rho x_1\right) + p_{21}\left(\frac{\partial F}{\partial u_2} - \rho x_2\right) + p_{31}\left(\frac{\partial F}{\partial u_3} - \rho x_3\right) \\
p_2\rho &= p_{12}\left(\frac{\partial F}{\partial u_1} - \rho x_1\right) + p_{22}\left(\frac{\partial F}{\partial u_2} - \rho x_2\right) + p_{32}\left(\frac{\partial F}{\partial u_3} - \rho x_3\right) \\
p_3\rho &= p_{13}\left(\frac{\partial F}{\partial u_1} - \rho x_1\right) + p_{23}\left(\frac{\partial F}{\partial u_2} - \rho x_2\right) + p_{33}\left(\frac{\partial F}{\partial u_3} - \rho x_3\right) \\
0 &= p_1\left(\frac{\partial F}{\partial u_1} - \rho x_1\right) + p_2\left(\frac{\partial F}{\partial u_2} - \rho x_2\right) + p_3\left(\frac{\partial F}{\partial u_3} - \rho x_3\right),
\end{aligned}$$

in which one has set:

$$v\,\rho = \sum p_i \frac{\partial F}{\partial u_i}.$$

One will get the same thing by differentiating the identity $F(u_1, u_2, u_3) = 0$ with respect to the x_i .

One then has, e.g.,:

$$\frac{\partial F}{\partial u_1} p_{11} + \frac{\partial F}{\partial u_2} p_{21} + \frac{\partial F}{\partial u_3} p_{31} = \frac{\partial v}{\partial x_1}\left[\frac{p_1}{v}\frac{\partial F}{\partial u_1} + \frac{p_2}{v}\frac{\partial F}{\partial u_2} + \frac{p_3}{v}\frac{\partial F}{\partial u_3}\right],$$

and thus, the first of the stated equations, when one sets:

$$\frac{\partial v}{\partial x_1} = p_1 + x_1\, p_{11} + x_2\, p_{21} + x_3\, p_{31}\,.$$

If one also writes ξ_i for $\frac{\partial F}{\partial u_i} - \rho\, x_i$, to abbreviate, then one will recognize that the equations:

$$\sum p_i\, \xi_i = 0,$$
$$\sum p_{ik}\, \xi_i\, \xi_k = 0$$

will be fulfilled; i.e., that the direction of the principal tangent will be determined by ξ_i . As one sees, however, it is the connecting line of the point x_i with the point $\frac{\partial F}{\partial u_i}$, in which the associated plane of x_i will contact the order surface F. It will then follow that:

The second family of the principal tangent curves of the special P-E-systems of the second kind will be defined by complex curves of the tangent complex of the order surface, and all points of C will be principal tangents that go through a fixed point S that lies in the plane of C that is the contact point of that plane with the order surface. Furthermore, the contact points of the tangents to the curve C that are drawn from S will then belong to the (doubly-counted) focal surface $4\Delta' = G^2 = 0$.

The curve C will not include the point S, in general, since that will be the case when S also belongs to the focal surface, which would emerge immediately from the theorems that were stated above. The curve of intersection of the order surface with the focal surface in this case will then be a *singular strip* of the differential equation that is linked with the association of the P-E-system.

If, e.g., $F(u_1, u_2, u_3) = 0$ comes about as a result of the linear equation:

$$a_1\, u_1 + a_2\, u_2 + a_3\, u_3 = 1$$

then the identity:

$$\sum p_i\, a_i - \sum p_i\, x_i = 0$$

will exist between the p_i . The principal tangent curves of the second kind have the differential equations:

$$dx_1 : dx_2 : dx_3 = a_1 - x_1 : a_2 - x_2 : a_3 - x_3\,;$$

i.e., they define the ray bundle through the representative point a_1, a_2, a_3 of the order surface.

§ 3.

Rectilinear principal tangent curves.

At the point x_i, one considers the associated plane:

$$\sum (X_i - x_i)\, p_i = 0$$

and chooses the direction ξ_i in it in such a way that:

$$\sum \xi_i\, p_i = 0. \tag{1}$$

The plane that belongs to the point $x_i + \lambda\xi_i$ has the equation:

$$\sum (X_i - x_i - \lambda\xi_i) \left(p_i + \lambda \sum p_{ik}\, \xi_k + \frac{\lambda^2}{2} \sum p_{ikl}\, \xi_k\, \xi_l + \cdots \right) = 0, \tag{2}$$

in which the higher differential quotients are denoted by the indices k, l. In order for the plane (2) that is associated with an infinitely-close point to go through the point x_i, the factor of λ^2 must vanish for $X_i = x_i$. That will give the condition for the principal tangent direction. If the factor of λ^3 also vanishes then that principal tangent will be stationary. For the latter case, it is necessary that equations (1) must be true and that:

$$\tag{3} \begin{aligned} &\sum \xi_i\, \xi_k\, p_{ik} = 0, \\ &\sum \xi_i\, \xi_k\, \xi_l\, p_{ikl} = 0. \end{aligned}$$

There is then a surface whose points are associated with principal tangents, in general, and on it, a curve (isolated points, resp.) at which that singularity is raised by one or two orders. A closer examination of these cases would correspond completely to the known questions that concern tangents that contact surface at multiple points (*).

The identical vanishing of the resultants of (1), (3) *is the condition for the principal tangents of the one family to define a rectilinear system of rays. In that case, which one can refer to as a skew P-E-system, the planes associated them with* ∞^2 *plane pencils;* that requires no further investigation.

It might now be assumed that the principal tangent directions coincide at every location. The P-E-system will then have no focal surface, so it is *parabolic*, and likewise an analogue of the developable surface, in the event that the integrability condition is fulfilled. I will show that *in this case, the entire system of principal tangents curves is rectilinear, so the parabolic P-E-system consists of* ∞^2 *plane pencils whose planes define an extraordinary self-conjugate system.*

(*) I refer to my following paper for these questions.

As a result of the identical vanishing of Δ', there are, in fact, functions ξ_i, λ that satisfy the system:

$$\text{(4)}\qquad \begin{aligned} \sum \xi_i\, q_{1i} + \lambda\, p_1 &= 0, \\ \sum \xi_i\, q_{2i} + \lambda\, p_2 &= 0, \\ \sum \xi_i\, q_{3i} + \lambda\, p_3 &= 0, \\ \sum \xi_i\, p_i &= 0 \end{aligned}$$

identically, and the direction of the principal tangent is determined by the associated ratios of the ξ_i, and thus, for every point. If one now differentiates equations (4) then what will come about is:

$$\text{(5)}\qquad \begin{aligned} \sum \xi_i\, dq_{1i} + \sum q_{1i}\, d\xi_i + \lambda\, dp_1 + p_1\, d\lambda &= 0, \\ \sum \xi_i\, dq_{2i} + \sum q_{2i}\, d\xi_i + \lambda\, dp_2 + p_2\, d\lambda &= 0, \\ \sum \xi_i\, dq_{3i} + \sum q_{3i}\, d\xi_i + \lambda\, dp_3 + p_3\, d\lambda &= 0, \\ \sum \xi_i\, dp_i + \sum p_i\, d\xi_i &= 0, \end{aligned}$$

and indeed these relations will be true for arbitrary values of the dx_i. If one now sets the latter proportional to the ξ_i – so $dx_i = dh\ \xi_i$ – then, from (4), one will have:

$$\sum \xi_i\, dp_i = 0,$$

and from (5):

$$\sum p_i\, d\xi_i = 0.$$

If one further multiplies the first three equations in (5) by the ξ_i, and the last one by λ and adds them then one will get:

$$\text{(7)}\qquad \sum \xi_i\ \xi_k\ dq_{ik} = 0.$$

However, that equation is, as a direct development will show immediately, identical with the second one in (3). That then says that there exists a system of rectilinear principal tangent curves. In fact, one can now also set the differentials of the ξ_i proportional to the ξ_i themselves. For $d\xi_i = \xi_i\ dk$, equations (5) then reduce to the following ones:

$$dh\left(\sum \xi_i \frac{dq_{ki}}{dh} + \lambda \frac{dp_k}{dh}\right) + (d\lambda - \lambda\, dk)\, p_k = 0, \qquad k = 1, 2, 3,$$

which one can satisfy with suitable values of the *dh*, *dk*, since the relation (7) exists between them, from which, it can be shown that the differentials of the ξ_i will remain continually proportional to these quantities themselves as long as one proceeds in the direction ξ_i itself.

Before I go into a more precise analytical characterization of the parabolic P-E-systems, I would like to add some further remarks, which, at the same time, give a new proof of the theorem that was stated just now.

The theorem that was proved in § 1 that any P-E-system is endowed with a system of principal tangent curves whose osculating planes are the planes of the system can be expressed in yet another way. The directions of the principal tangents are, in general, coupled by an irreducible quadratic equation. If it were reducible then both curve systems would decompose into two more. This will always be the case then when one system of curves is given arbitrarily, such that any point in space is uniquely assigned a curve in a certain domain. The osculating planes of that curve will then define a P-E-system whose one family of principal tangent curves is given immediately. One then has the general theorem:

Any curve system of stated kind corresponds to a conjugate system that likewise has the osculating planes of the original one for its osculating planes.

This relation is, in general, a *reciprocal* one; *i.e., the one system is the conjugate to the other one.* An exception then appears only when the conjugate system is rectilinear. That is the case, e.g. – in order to not mention other examples – for the osculating planes of a system of common helices with the same axis for which the osculating planes always go through the points of the curves on the perpendiculars to the axis, so the conjugate curve system consists of just those normals.

The question now arises: Under what circumstances can a curve system be conjugate to itself? However, from the previous study, the theorem must be true:

The only self-conjugate curve systems are represented by the rays of certain rays systems.

Here, I shall give a direct proof of this theorem that seems to be worthy of interest, due to its generality.

A curve system of the stated kind will be represented by the differential equations:

$$\frac{dx_i}{dt} = p_i ,$$

in which the p_i are functions of the coordinates x_i of the point M. The osculating plane of the curve that goes through the point M has the equation:

$$\sum (X_i - x_i)\, A_i = 0,$$

in which:

$$A_1 = p_2\, P_3 - p_3\, P_2 ,$$
$$A_2 = p_3\, P_1 - p_1\, P_3 ,$$

$$A_3 = p_1 P_2 - p_2 P_1 ,$$

in which equations, the P_i mean the second differential quotients:

$$\frac{d^2 x_i}{dt^2} = P_i = \sum p_k \frac{\partial p_i}{\partial x_k} .$$

In order to find the direction of the curve of the conjugate system that goes through the point M, one must solve the equations:

$$(8) \qquad \begin{aligned} \sum \xi_i A_i &= 0, \\ \sum \xi_i \xi_k A_{ik} &= 0, \end{aligned}$$

in which one again sets:

$$A_{ik} = \frac{\partial A_i}{\partial x_k} .$$

For that, one sets:

$$\xi_i = \alpha p_i + \gamma P_i .$$

One will then obtain:

$$\sum (\alpha p_i + \beta P_i) \frac{\partial A_i}{\partial x_j} = -\alpha \begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ \frac{\partial p_1}{\partial x_j} & \frac{\partial p_2}{\partial x_j} & \frac{\partial p_3}{\partial x_j} \end{vmatrix} - \beta \begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ \frac{\partial P_1}{\partial x_j} & \frac{\partial P_2}{\partial x_j} & \frac{\partial P_3}{\partial x_j} \end{vmatrix} .$$

If one multiplies these relations by $\alpha p_j + \beta P_j$ and sums over j then the second of the conditions in (8) will come about:

$$\alpha\beta \begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ \sum P_j \frac{\partial p_1}{\partial x_j} & \sum P_j \frac{\partial p_2}{\partial x_j} & \sum P_j \frac{\partial p_3}{\partial x_j} \end{vmatrix}$$

$$+ \alpha\beta \begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ \sum p_j \frac{\partial P_1}{\partial x_j} & \sum p_j \frac{\partial P_2}{\partial x_j} & \sum p_j \frac{\partial P_3}{\partial x_j} \end{vmatrix}$$

$$+\beta^2 \begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ \sum P_j \frac{\partial P_1}{\partial x_j} & \sum P_j \frac{\partial P_2}{\partial x_j} & \sum P_j \frac{\partial P_3}{\partial x_j} \end{vmatrix} = 0.$$

As one would expect, one will get the root $\beta = 0$ here, in one case, so the desired direction will be given by:

(9) $$-\frac{\beta}{\alpha} = \frac{N}{S},$$

in which N means the sum of the two coefficients of $\alpha\beta$, and S means the sum of the coefficients of β^2. Now, should the direction thus-found again coincide with the original one, N must vanish. Since the examination of the differential equations does not seem simple, I prefer to resolve this question in the following way:

The condition for the direction p_i to be the only one that satisfies equations (8) when one replaces the ξ_i with it comes from the fact that a straight line contacts a conic section, i.e., it comes from the system:

$$2p_1 \frac{\partial A_1}{\partial x_1} + p_2\left(\frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1}\right) + p_3\left(\frac{\partial A_1}{\partial x_3} + \frac{\partial A_3}{\partial x_1}\right) = \lambda A_1, \quad \text{etc.}$$

or

$$\sum p_i \frac{\partial A_1}{\partial x_i} + p_i \frac{\partial A_i}{\partial x_1} = \lambda A_1,$$

$$\sum p_i \frac{\partial A_2}{\partial x_i} + p_i \frac{\partial A_i}{\partial x_2} = \lambda A_2,$$

$$\sum p_i \frac{\partial A_3}{\partial x_i} + p_i \frac{\partial A_i}{\partial x_3} = \lambda A_3 .$$

However, by forming the differential quotients of these, one will get the equations:

(10) $$\begin{aligned} p_2 M_3 - p_3 M_2 &= \lambda (p_2 P_3 - p_3 P_2) + S_1, \\ p_3 M_1 - p_1 M_3 &= \lambda (p_3 P_1 - p_1 P_3) + S_2, \\ p_1 M_2 - p_2 M_1 &= \lambda (p_1 P_2 - p_2 P_1) + S_3, \end{aligned}$$

in which one has set:

$$S_i = \begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ \frac{\partial p_1}{\partial x_i} & \frac{\partial p_2}{\partial x_i} & \frac{\partial p_3}{\partial x_i} \end{vmatrix}, \quad M_i = \sum p_k \frac{\partial P_i}{\partial x_k} .$$

One obtains the identity:

$$\sum p_i \, S_i = 0$$

from these equations by multiplying them by p_i and adding them, while multiplying by P_i will give:

$$(11) \qquad \begin{vmatrix} P_1 & P_2 & P_3 \\ p_1 & p_2 & p_3 \\ M_1 & M_2 & M_3 \end{vmatrix} = P_2 \, M_3 - P_3 \, M_2, \qquad N_i = \sum P_k \frac{\partial p_i}{\partial x_k},$$

and thus, the condition $S = 0$. I now differentiate the equations (10) in such a way that the operation:

$$\delta = \sum p_k \frac{\partial}{\partial x_k}$$

is performed. p_i then goes to P_i, P_i to M_i, and one then gets from the first of them:

$$P_2 \, M_3 - P_3 \, M_2 + p_2 \, \delta M_3 - p_3 \, \delta M_2 = \lambda \, (P_2 \, M_3 - P_3 \, M_2) + \delta\lambda \, (p_2 \, P_3 - p_3 \, P_2)$$

$$+ \begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ \sum p_l \dfrac{\partial^2 p_1}{\partial x_1 \, \partial x_l} & \sum p_l \dfrac{\partial^2 p_1}{\partial x_1 \, \partial x_l} & \sum p_l \dfrac{\partial^2 p_3}{\partial x_1 \, \partial x_l} \end{vmatrix}.$$

If one multiplies these equations by the p_i and again adds them then that will give:

$$(12) \qquad \begin{vmatrix} P_1 & P_2 & P_3 \\ p_1 & p_2 & p_3 \\ M_1 & M_2 & M_3 \end{vmatrix} = \begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ \sum p_i \, p_l \dfrac{\partial^2 p_1}{\partial x_i \, \partial x_l} & \sum p_i \, p_l \dfrac{\partial^2 p_1}{\partial x_i \, \partial x_l} & \sum p_i \, p_l \dfrac{\partial^2 p_3}{\partial x_i \, \partial x_l} \end{vmatrix}.$$

However, one now has:

$$\frac{d^2 x_i}{dt^2} = \sum \frac{\partial P_i}{\partial x_k} \, p_k = M_i \, ,$$

or, more thoroughly:

$$\frac{d^2 x_i}{dt^2} = \sum p_k \, p_l \frac{\partial^2 p_i}{\partial x_k \, \partial x_l} + \sum p_l \frac{\partial p_i}{\partial x_k} \frac{\partial p_k}{\partial x_l} ,$$

and further:

$$\sum P_k \frac{\partial p_i}{\partial x_k} = \sum \frac{\partial p_k}{\partial x_l}\frac{\partial p_i}{\partial x_k} p_l ,$$

so:

$$\sum p_k p_l \frac{\partial^2 p_i}{\partial x_k \partial x_l} = \frac{d^2 x_i}{dt^2} - \sum P_k \frac{\partial p_i}{\partial x_k} = M_i - \sum P_k \frac{\partial p_i}{\partial x_k} .$$

If one substitutes this in (12) then that will give:

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ N_1 & N_2 & N_3 \end{vmatrix} = 0,$$

and from (11):

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ P_1 & P_2 & P_3 \\ M_1 & M_2 & M_3 \end{vmatrix} = 0.$$

This last determinant is, however, that of the first, second, and third differential quotients of the x_i; the curves of the system must then all be plane curves. In this case, however, all of their osculating planes will define a special P-E-system of the second kind. Now, since a curve that is enveloped by principal tangents lies in any plane, and the direction of the second principal tangent always runs from the curve point to the contact point of its plane with the order surface that is associated with the system, the directions of the two principal tangents can coincide only when the curves of the system are themselves straight lines. However, the question that was posed before is then answered by that.

§ 4.

The determination of all parabolic P-E-systems.

In what follows, we shall deal with the problem of *specifying all P-E-systems with coincident principal tangent directions.* Since it emerges from the investigation in the previous paragraph that the principal tangent curves define a rectilinear ray system, *the partial differential equation* $\Delta' = 0$, upon whose integration the solution to the question rests, *can lead to a linear partial differential equation with one unknown whose integration can be performed.* Although it would presumably not be difficult to integrate the equation $\Delta' = 0$ in a purely analytical way, it still seems to me that the geometric analysis that we shall make should have some interest.

One now considers:

$$(1) \qquad \begin{aligned} f(a, b, x - az, y - bz) &= 0, \\ \varphi(a, b, x - az, y - bz) &= 0 \end{aligned}$$

to be the equations of a ray system, or, when one sets:

(2) $$x - az = p, \qquad y - bz = q,$$

$f(a, b, p, q) = 0$, $\varphi(a, b, p, q) = 0$. For given values of a, b, and the ones that belong to p, q, according to (1), the respective ray of the system will then be the intersection of the two planes (2). Conversely, if one thinks of a, b as functions of x, y, z that are determined from (1) and then differentiates the identity relations (1) with respect to these variables then that will give:

(3) $$\begin{aligned}
&\left(\frac{\partial f}{\partial a} - z\frac{\partial f}{\partial p}\right)\frac{\partial a}{\partial x} + \left(\frac{\partial f}{\partial b} - z\frac{\partial f}{\partial q}\right)\frac{\partial b}{\partial x} + \frac{\partial f}{\partial p} = 0,\\
&\left(\frac{\partial f}{\partial a} - z\frac{\partial f}{\partial p}\right)\frac{\partial a}{\partial y} + \left(\frac{\partial f}{\partial b} - z\frac{\partial f}{\partial q}\right)\frac{\partial b}{\partial y} + \frac{\partial f}{\partial q} = 0,\\
&\left(\frac{\partial f}{\partial a} - z\frac{\partial f}{\partial p}\right)\frac{\partial a}{\partial z} + \left(\frac{\partial f}{\partial b} - z\frac{\partial f}{\partial q}\right)\frac{\partial b}{\partial z} - a\frac{\partial f}{\partial p} - b\frac{\partial f}{\partial q} = 0,
\end{aligned}$$

along with analogous equations for φ. If one multiplies them by the a, b, 1 then that will give:

(4) $$\begin{aligned}
&\left(\frac{\partial f}{\partial a} - z\frac{\partial f}{\partial p}\right)\left(a\frac{\partial a}{\partial x} + b\frac{\partial a}{\partial y} + \frac{\partial a}{\partial z}\right) + \left(\frac{\partial f}{\partial b} - z\frac{\partial f}{\partial q}\right)\left(a\frac{\partial b}{\partial x} + b\frac{\partial b}{\partial y} + \frac{\partial b}{\partial z}\right) = 0,\\
&\left(\frac{\partial \varphi}{\partial a} - z\frac{\partial \varphi}{\partial p}\right)\left(a\frac{\partial a}{\partial x} + b\frac{\partial a}{\partial y} + \frac{\partial a}{\partial z}\right) + \left(\frac{\partial \varphi}{\partial b} - z\frac{\partial \varphi}{\partial q}\right)\left(a\frac{\partial b}{\partial x} + b\frac{\partial b}{\partial y} + \frac{\partial b}{\partial z}\right) = 0.
\end{aligned}$$

However, since the determinant:

$$\Gamma = \left(\frac{\partial f}{\partial b} - z\frac{\partial f}{\partial p}\right)\left(\frac{\partial \varphi}{\partial b} - z\frac{\partial \varphi}{\partial q}\right) - \left(\frac{\partial f}{\partial b} - z\frac{\partial f}{\partial q}\right)\left(\frac{\partial \varphi}{\partial b} - z\frac{\partial \varphi}{\partial p}\right)$$

does not vanish identically, it will then follow that:

(5) $$\begin{aligned}
a\frac{\partial a}{\partial x} + b\frac{\partial a}{\partial y} + \frac{\partial a}{\partial z} &= 0,\\
a\frac{\partial b}{\partial x} + b\frac{\partial b}{\partial y} + \frac{\partial b}{\partial z} &= 0.
\end{aligned}$$

Moreover, the equation of the plane that goes through the point x, y, z is:

$$(X - x)\,A + (Y - y)\,B + (Z - z)\,C = 0.$$

In order for this to contain the ray:

$$X - aZ = x - az,$$
$$Y - bZ = y - bz$$

completely, one must have:

$$Aa + Bb + C = 0.$$

Therefore, one will have the following expressions for the three functions p_i of the P-E-system:

$$p_1 = A,$$
$$p_2 = B,$$
$$p_3 = -(Aa + Bb),$$

in which the a, b satisfy the partial differential equations (5), while the A and B can still be arbitrary functions of the a, b, p, q, x, y, z (*). One must now ascertain the determinant Δ', which might be assumed to take the form:

(6)
$$\Delta' \equiv \begin{vmatrix} 2p_{11} & p_{12}+p_{21} & p_{13}+p_{31} & p_1 \\ p_{12}+p_{21} & 2p_{22} & p_{23}+p_{32} & p_2 \\ p_{13}+p_{31} & p_{23}+p_{32} & 2p_{33} & p_3 \\ p_1 & p_2 & p_3 & 0 \end{vmatrix}.$$

The following equations are necessary for this to be true, which are obtained easily upon considering the identities (5):

(6)[*sic*]
$$a\,p_{11} + b\,p_{12} + p_{13} = a\left(\frac{\partial A}{\partial x}\right) + b\left(\frac{\partial A}{\partial y}\right) + \left(\frac{\partial A}{\partial z}\right) = \beta_1\,,$$
$$a\,p_{21} + b\,p_{22} + p_{23} = a\left(\frac{\partial B}{\partial x}\right) + b\left(\frac{\partial B}{\partial y}\right) + \left(\frac{\partial B}{\partial z}\right) = \beta_2\,.$$

The differential quotients in brackets on the right-hand side are understood to mean that one only partially differentiates with respect to x, y, z whenever these quantities enter explicitly into them or into the p, q. One likewise finds that:

(7)
$$-(a\,p_{31} + b\,p_{32} + p_{33}) = a\left[a\left(\frac{\partial A}{\partial x}\right) + b\left(\frac{\partial A}{\partial y}\right) + \left(\frac{\partial A}{\partial z}\right)\right] + b\left[a\left(\frac{\partial B}{\partial x}\right) + b\left(\frac{\partial B}{\partial y}\right) + \left(\frac{\partial B}{\partial z}\right)\right] = \beta_3\,.$$

(*) The representation of all skew P-E-systems will be given in that way.

In addition, let:

(8)
$$\begin{aligned} a\,p_{11} + b\,p_{21} + p_{31} &= -\left[A\frac{\partial a}{\partial x} + B\frac{\partial b}{\partial x}\right] = \alpha_1\,, \\ a\,p_{12} + b\,p_{22} + p_{32} &= -\left[A\frac{\partial a}{\partial y} + B\frac{\partial b}{\partial y}\right] = \alpha_2\,, \\ a\,p_{13} + b\,p_{23} + p_{33} &= -\left[A\frac{\partial a}{\partial z} + B\frac{\partial b}{\partial z}\right] = \alpha_3\,, \end{aligned}$$

in which the following relations exist between α, β:

(9)
$$\begin{aligned} a\,\beta_1 + b\,\beta_2 + \beta_3 &= 0, \\ a\,\alpha_1 + b\,\alpha_2 + \alpha_3 &= 0. \end{aligned}$$

One now multiplies the first two vertical rows of the determinant Δ' by a, b and then adds them to the third one; that row will then have the following elements:

$$\alpha_1 + \beta_1, \quad \alpha_2 + \beta_2, \quad \alpha_3 + \beta_3, \quad 0.$$

If one now proceeds similarly with the horizontal rows then, with the help of the relations (7), (8), (9), one will obtain the following value for Δ':

$$\Delta' \equiv \begin{vmatrix} 2p_{11} & p_{12}+p_{21} & \alpha_1+\beta_1 & A \\ p_{12}+p_{21} & 2p_{22} & \alpha_2+\beta_2 & B \\ \alpha_1+\beta_1 & \alpha_2+\beta_2 & 0 & 0 \\ A & B & 0 & 0 \end{vmatrix} = [(\alpha_1 + \beta_1)\,B - (\alpha_2 + \beta_2)\,A]^2.$$

From the fact that the expression Δ' is a square, it will follow that the rays of the system do not contact the focal surface in general, which must usually be the case from § 1. Understandably, this cannot happen either, since all rays are already double tangents of their real **Kummer** focal surface. The equation for Δ' now reduces to the linear partial differential equation:

$$a\left[B\left(\frac{\partial A}{\partial x}\right) - A\left(\frac{\partial B}{\partial x}\right)\right] + b\left[B\left(\frac{\partial A}{\partial y}\right) - A\left(\frac{\partial B}{\partial y}\right)\right] + B\left(\frac{\partial A}{\partial z}\right) - A\left(\frac{\partial B}{\partial z}\right) -$$
$$\left(\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right)AB - \frac{\partial b}{\partial x}B^2 + A^2\frac{\partial a}{\partial y} = 0.$$

If one then sets $A / B = \tan \mu$ then the following equation will arise:

$$(10)\qquad a\left(\frac{\partial\mu}{\partial x}\right)+b\left(\frac{\partial\mu}{\partial y}\right)+\left(\frac{\partial\mu}{\partial z}\right)+\left(\frac{\partial b}{\partial y}-\frac{\partial a}{\partial x}\right)\sin\mu\cos\mu-\frac{\partial b}{\partial x}\cos^2\mu+\frac{\partial a}{\partial y}\sin^2\mu = 0.$$

Finally, one can also set:

$$a\left(\frac{\partial\mu}{\partial x}\right)+b\left(\frac{\partial\mu}{\partial y}\right)+\left(\frac{\partial\mu}{\partial z}\right) = a\frac{\partial\mu}{\partial x}+b\frac{\partial\mu}{\partial y}+\frac{\partial\mu}{\partial z},$$

in which the differentiations in the differential quotients on the right-hand side are performed only when the x, y, z appear in μ explicitly, since as long as the p, q enter into μ explicitly, no contribution will be made to the three differential quotients, in the aggregate. Moreover, since it follows from (3) that:

$$\Gamma\frac{\partial b}{\partial x} = \frac{\partial f}{\partial p}\frac{\partial\varphi}{\partial a}-\frac{\partial f}{\partial a}\frac{\partial\varphi}{\partial p} = -h_2,$$

$$\Gamma\frac{\partial a}{\partial y} = \frac{\partial f}{\partial b}\frac{\partial\varphi}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial\varphi}{\partial b} = h_1,$$

$$\Gamma\left(\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\right) = \frac{\partial f}{\partial q}\frac{\partial\varphi}{\partial a}-\frac{\partial f}{\partial p}\frac{\partial\varphi}{\partial b}-\frac{\partial\varphi}{\partial q}\frac{\partial f}{\partial a}+\frac{\partial f}{\partial p}\frac{\partial\varphi}{\partial b} = 2h_3,$$

the a, b, p, q can be regarded as constants in the differential equation (10), and of the variables x, y, z, only z will enter into the quadratic form Γ explicitly. If one then sets:

$$\frac{d\mu}{2h_3\cos\mu\sin\mu+h_1\sin^2\mu+h_2\cos^2\mu} = d\lambda$$

then one will be dealing with the integration of:

$$(11)\qquad a\frac{\partial\lambda}{\partial x}+b\frac{\partial\lambda}{\partial y}+\frac{\partial\lambda}{\partial z}+\frac{1}{\Gamma} = 0.$$

However, one then has to integrate the system:

$$dx : dy : dz : d\lambda = a : b : 1 : -\frac{1}{\Gamma},$$

whose integrals are:

$$x - a\,z = c_1,$$

$$y - b\,z = c_2,$$

$$\int\left(\frac{\partial z}{\Gamma}+d\lambda\right) = c_3.$$

If one denotes the coefficients of the quadratic form:

$$\Gamma = \frac{\partial f}{\partial a}\frac{\partial \varphi}{\partial b} - \frac{\partial \varphi}{\partial a}\frac{\partial f}{\partial b} + z^2\left(\frac{\partial f}{\partial p}\frac{\partial \varphi}{\partial q} - \frac{\partial \varphi}{\partial p}\frac{\partial f}{\partial q}\right) - z\left[\frac{\partial f}{\partial a}\frac{\partial \varphi}{\partial b} + \frac{\partial \varphi}{\partial a}\frac{\partial f}{\partial b} - \frac{\partial \varphi}{\partial p}\frac{\partial f}{\partial b} - \frac{\partial \varphi}{\partial a}\frac{\partial f}{\partial q}\right]$$

by k_2, k_1, $2k_3$ and sets $\tan \mu = A / B = \zeta$ then one will have:

$$\int \frac{dz}{k_2 + 2k_3 z + k_1 z^2} + \int \frac{d\zeta}{h_2 + 2h_3\zeta + h_1\zeta^2} = c_3 .$$

A direct calculation now shows that both forms in the integrals have the same discriminants; i.e., that:

$$k_3^2 - k_1\, k_2 = h_3^2 - h_1\, h_2 = \omega.$$

Now, since when $\Delta = b^2 - ac$, one will have:

$$\int \frac{dz}{a + 2bz + cz^2} = \frac{1}{2\sqrt{\Delta}} \log\left\{\frac{b + cz - \sqrt{\Delta}}{b + cz + \sqrt{\Delta}}\right\}$$

to begin with, one will get:

$$r = \frac{k_3 + k_1 z - \sqrt{\omega}}{k_3 + k_1 z + \sqrt{\omega}} \cdot \frac{h_3 + h_1\zeta - \sqrt{\omega}}{h_3 + h_1\zeta + \sqrt{\omega}} = \text{const.},$$

in place of the third integral.

The general integral of (11) will then become:

(12) $$r = F(p, q, a, b),$$

in which F means an arbitrary function of the arguments. Now, ζ determines the location of the point z along the plane that is associated with the ray a, b. From the fact that the relationship between z and ζ is a bilinear one, which would emerge from the form of (12), the following theorem will come about:

The planes of the parabolic P-E-systems are associated projectively with the points along the rectilinear principal tangent curves (*).

The projective relationship takes on an especially simple form for the rays for which $\omega = 0$, and the sum of two reciprocal linear entire functions in z (ζ, resp.) will appear in place of the logarithm. Now, $\Gamma = 0$ is the equation of the **Kummer** focal surface of the ray system, as long as one substitutes the a, b, p, q as functions of the x, y, z. By contrast, the equation $\Gamma = 0$ will determine the two focal points at which the associated ray will

(*) I hereby recall the analogous property that is true for skew surfaces.

contact that focal surfaces when one fixes a, b along that ray (*). One likewise sees from this that the ray will be a principal tangent to that surface when $\omega = 0$.

Finally, I would like to point a special solution of the equation $\Delta' = 0$. The equation $f = 0$ is, as is known, the expression for a ray complex. Every point of space through which a ray of the system goes will be associated with a plane by it, namely, the tangential plane to the complex cone that belongs to that point along that ray. The equation of that tangential plane is:

$$(X-x)\left(\frac{\partial f}{\partial a}-\frac{\partial f}{\partial p}z\right)+(Y-y)\left(\frac{\partial f}{\partial b}-\frac{\partial f}{\partial p}z\right)-(Z-z)\left[\left(\frac{\partial f}{\partial b}-\frac{\partial f}{\partial q}z\right)b+\left(\frac{\partial f}{\partial a}-\frac{\partial f}{\partial p}z\right)a\right]=0.$$

If one now sets:

$$A=\frac{\partial f}{\partial a}-\frac{\partial f}{\partial p}z,$$
$$B=\frac{\partial f}{\partial b}-\frac{\partial f}{\partial p}z$$

then one will have:

$$-(\alpha_1+\beta_1)=\frac{\partial f}{\partial p}+A\frac{\partial a}{\partial x}+B\frac{\partial b}{\partial x}=0,$$
$$-(\alpha_2+\beta_2)=\frac{\partial f}{\partial q}+A\frac{\partial a}{\partial y}+B\frac{\partial b}{\partial y}=0.$$

With that, the condition $\Delta' = 0$ will be satisfied everywhere.

One will thus obtain a parabolic P-E-system when one associates the points P of the line l in a ray system that defines the intersection of two arbitrary complexes with the respective tangential planes at the point P of the associated complex cone of f along the line l.

(*) If one would like to prove that all rays of the system contact that focal surface then it would be simplest to regard x, y, z as functions of a, b and assume that f, φ have the forms:

$$q-f(a,b)=0,\quad p-\varphi(a,b)=0.$$

$\Gamma = 0$ then takes on the form:

$$\left(z+\frac{\partial\varphi}{\partial a}\right)\left(z+\frac{\partial f}{\partial b}\right)-\frac{\partial f}{\partial a}\frac{\partial\varphi}{\partial b}=0,$$

and one has:

$$dx = z\,da + a\,dz + d\varphi,$$
$$dy = z\,db + b\,dz + df.$$

If one now chooses z such that:

$$z\,da + d\varphi = 0,$$
$$z\,db + df = 0,$$

which is possible, since $\Gamma = 0$, then one will have $dx : dy = a : b$, with which, we have proved that the ray contacts the focal surface at both focal points.

Finally, if one examines the determinant Δ for an arbitrary skew P-E-system then one will easily find that it decomposes into two factors:

$$\Delta = \left[B\left(A\frac{\partial a}{\partial x}+B\frac{\partial b}{\partial x}\right)-A\left(A\frac{\partial a}{\partial y}+B\frac{\partial b}{\partial y}\right)\right]\left[B\left(\frac{\partial A}{\partial x}a+\frac{\partial A}{\partial y}b+\frac{\partial A}{\partial z}\right)-A\left(\frac{\partial B}{\partial x}a+\frac{\partial B}{\partial y}b+\frac{\partial B}{\partial z}\right)\right]$$

The basis for this is easy to see. When the second factor is set to zero, the ratio A / B will yield a value that depends upon only a, b, p, q, but is constant along that ray. Its vanishing will then relate to the point at which the associated plane stays stationary along the ray. By contrast, the first factor refers to rays for which the planes that belong to neighboring rays will coincide; that can happen only when that plane is itself the focal plane of the **Kummer** focal surface of the ray system.

§ 5.

Curvature and lines of curvature of P-E-systems.

If one advances the plane that is associated with the point x_i infinitely little then the direction of advance dx_i / ds will define an angle with the normal to the plane:

$$\sum (x_i - \xi_i)\,(p_i + dp_i) = 0$$

whose cosine is:

(1) $$\frac{\sum dx_i dp_i}{ds\sqrt{p_1^2 + p_2^2 + p_3^2}}.$$

That value measures the magnitude of the normal curvature of the plane of the system along the respective direction. One will obtain the mutually-perpendicular directions of the principal curvatures from this, and they will simultaneously bisect the angle that is defined by the principal tangents. Any P-E-system is then associated with a system of curves that intersect perpendicularly everywhere, whose tangents are the *principal curvature directions.* The values of the normal curvatures are coupled to those of the principal curvatures by a relation that is entirely analogous to one of **Euler**'s. In particular, the curvature can also be constant in all directions at any location, which is the case for, e.g., the system:

$$\begin{aligned} p_1 &= a_1 + b_3\,x_2 - b_2\,x_3 + k\,x_1\,, \\ p_2 &= a_2 + b_1\,x_3 - b_3\,x_1 + k\,x_2\,, \\ p_3 &= a_3 + b_2\,x_1 - b_1\,x_2 + k\,x_3\,. \end{aligned}$$

On the other hand, one can ask about the curves along which these normals to the planes of the system define a developable, or along which the associated planes rotate around a direction that is perpendicular to the tangent to the curve. They shall be referred to as *lines of curvature.* If one sets:

$$N = \sqrt{p_1^2 + p_2^2 + p_3^2}$$

then the differential of:

$$x_i + \frac{\lambda}{N} p_i$$

must vanish. Since obviously one must set $d\lambda = 0$, one will get the conditions:

$$dx_i + \frac{\lambda}{N} dp_i + \lambda\, p_i\, d\frac{1}{N} = 0.$$

Accordingly, the *differential equations of the lines of curvature* will be:

$$\begin{vmatrix} dx_1 & dx_2 & dx_3 \\ p_1 & p_2 & p_3 \\ dp_1 & dp_2 & dp_3 \end{vmatrix} = 0, \qquad \sum p_i\, dx_i = 0, \tag{2}$$

while the two associated radii of curvature λ_1, λ_2 will be determined from the quadratic equation:

$$\begin{vmatrix} \mu + p_{11} & p_{12} & p_{13} & p_1 \\ p_{21} & \mu + p_{11} & p_{23} & p_2 \\ p_{31} & p_{32} & \mu + p_{11} & p_3 \\ p_1 & p_2 & p_3 & p_0 \end{vmatrix} = 0, \tag{3}$$

whose roots μ_1, μ_2 are related to λ_1, λ_2 by:

$$\frac{1}{\mu} = \frac{\lambda}{N}.$$

Moreover, one can also assume that $N = 1$, as I did in my previous note (*). I have preserved the terminology "lines of curvature" for the aforementioned curves, since the following theorem will justify its validity, as one will recognize immediately from the differential equation (2) (the definition of the curves, resp.):

If two P-E-systems intersect in a line of curvature that is common to both of them then they will intersect along it at a constant angle,

and in particular:

If a line of curvature is planar, but not rectilinear then the normals to the system along it will define a constant angle with their plane,

(*) *Loc. cit.*, pp. 557.

and finally:

If two P-E-system intersect at constant angle along a curve that is a line of curvature for one of them then it will also be a line of curvature for the other one.

The directions ξ_i of the lines of curvature are, as one sees, determined from the equations:

$$
(4)\qquad
\begin{aligned}
\xi_1 (\mu + p_{11}) &+ \xi_2 p_{12} + \xi_2 p_{13} = \rho p_1 , \\
\xi_1 p_{21} &+ \xi_2 (\mu + p_{22}) + \xi_3 p_{23} = \rho p_2 , \\
\xi_1 p_{31} &+ \xi_2 p_{32} + \xi_3 (\mu + p_{33}) = \rho p_3 , \\
\xi_1 p_1 &+ \xi_2 p_2 + \xi_2 p_3 = 0.
\end{aligned}
$$

If one denotes the values of the ξ that are associated with the roots μ_1, μ_2 by ξ' and ξ'', resp., then that will give:

$$
\begin{aligned}
(\mu_1 - \mu_2) \sum \xi_i' \xi_i'' &= hG, \\
h^2 N^2 &= 1 - \sum (\xi_i' \xi_i'')^2 .
\end{aligned}
$$

If one then calls the angle between the lines of curvature ω then one will have:

$$\cot \omega = \frac{G}{(\mu_1 - \mu_2) N} .$$

The lines of curvature can then be mutually perpendicular only if the integrability condition $G = 0$ *is fulfilled.*

Along with the lines of curvature, one can consider the system of curves that intersect them at right angles. Their directions η_i', η_i'' are given by the relations:

$$
\begin{aligned}
\eta_1 (p_{11} + \mu) &+ \eta_2 p_{21} + \eta_2 p_{31} = \sigma p_1 , \\
\eta_1 p_{12} &+ \eta_2 (p_{22} + \mu) + \eta_3 p_{32} = \sigma p_2 , \\
\eta_1 p_{13} &+ \eta_2 p_{23} + \eta_3 (p_{33} + \mu) = \sigma p_3 , \\
\eta_1 p_1 &+ \eta_2 p_2 + \eta_2 p_3 = 0.
\end{aligned}
$$

Finally, one will also get the principal tangents with the help of the directions of the lines of curvature. Namely, should:

$$\xi_i = \xi_i' + \alpha\, \xi_i''$$

be the direction of a principal tangent then since:

$$\sum \xi_i \, p_{ki} \quad = \rho_1 \, p_k - \mu_1 \xi_k' + \alpha (\rho_2 \, p_k - p_2 \; \xi_k''),$$
$$\sum \xi_i \; \xi_k \, p_{ki} = - \mu_1 \sum \xi_i'^2 - \mu_2 \alpha^2 \sum \xi_i''^2 - \alpha(\mu_1 + \mu_2) \sum \xi_i' \xi_i'',$$

one will get the following quadratic equation for the determination of α:

$$\mu_1 + \mu_2 \; \alpha^2 - \alpha \, (\mu_1 + \mu_2) \; k = 0,$$

when one sets (*):

$$\sum \xi_i' \xi_i'' = k, \qquad \sum \xi_i'^2 = \sum \xi_i''^2 = 1.$$

It emerges from the latter equation for α that:

The lines of curvature will bisect the angle that is defined by the directions of principal tangents only when either $k = 0$ – *i.e., the P-E-system is integrable – or* $\mu_1 + \mu_2 = 0$ – i.e., *when the sum of the two radii of curvature that are associated with the direction of the lines of curvature is equal to zero.* In that latter case, which represents the *analogue of the minimal surface*, if the principal tangents are also perpendicular to each other then one will have $a = \pm 1$. In particular, the lines of curvature can also coincide at any location. I shall thus content myself by proving their existence for a very simple P-E-system, since the integration of the partial differential equation that arises when one sets the discriminant of (3) equal to zero does not seem to take a simple form. If one sets:

$$p_1 = x_1, \qquad p_2 = x_2, \qquad p_3 = X,$$

in which X might be an arbitrary functions of x_1, x_2, x_3 (**), then one will obtain the equations:

$$\xi_1 \, (\mu + 1) = \lambda \, x_1,$$
$$\xi_2 \, (\mu + 1) = \lambda \, x_2,$$
$$\xi_1 \frac{\partial X}{\partial x_1} + \xi_2 \frac{\partial X}{\partial x_2} + \xi_3 \left(\frac{\partial X}{\partial x_3} + \mu \right) = \lambda \, X,$$
$$\xi_1 \, x_1 + \xi_2 \, x_2 + \xi_3 \, X = 0$$

for the determination of the directions of the lines of curvature.

One then has either $\mu = -1$, $\lambda = 0$. If one then replaces ξ_i with dx_i then one will have:

$$dX = dx_3 \, ,$$
$$x_1 \, dx_1 + x_2 \, dx_2 + X \, dx_3 = 0,$$

so:

$$X - x_3 = c,$$
$$x_1^2 + x_2^2 + x_3^2 + 2c x_3 = C_1 \, .$$

(*) This assumption is admissible as long as not one of the directions of the lines of curvature meets the imaginary circle. In that case, however, they will, at the same time, be principal tangents.

(**) This P-E-system define the analogue of a family of surfaces of rotation with a common axis. As one sees, all normals to the planes of the system will meet the x_3-axis.

The lines of curvature of the one kind are then spherical, so they will be cut out from a system of spheres from the family of surfaces $X - x_3 = c$.

By contrast, one has the expression for the other root μ:

$$\mu = \frac{X\left(x_1\frac{\partial X}{\partial x_1} + x_2\frac{\partial X}{\partial x_2}\right) - (x_1^2 + x_2^2)\frac{\partial X}{\partial x_3} - X^2}{x_1^2 + x_2^2 + X^2}.$$

Now, in order for the double root $\mu = -1$ to appear, X must satisfy the partial differential equation:

$$x_1\frac{\partial X}{\partial x_1} + x_2\frac{\partial X}{\partial x_2} - \left(\frac{\partial X}{\partial x_3} - 1\right)\frac{x_1^2 + x_2^2}{X} = 0,$$

whose general integral is:

$$f\left(X - x_3,\ \frac{x_1}{x_2},\ x_1^2 + x_2^2 + X^2\right) = 0,$$

in which f means an arbitrary function of the argument. With that, all systems of the designated kind with coincident lines of curvature will be found.

If one demands that the other root μ must have the constant value a to begin with then one will get the differential equation:

$$X\left(x_1\frac{\partial X}{\partial x_1} + x_2\frac{\partial X}{\partial x_2}\right) - (x_1^2 + x_2^2)\left(\frac{\partial X}{\partial x_3} + a\right) - X^2(1 + a) = 0,$$

which is to be integrated for an arbitrary function:

$$f(p, q, r) = 0,$$

in the event that one sets:

$$p = \frac{x_1}{x_2}, \qquad q = \frac{X^2 + x_1^2 + x_2^2}{(x_1^2 + x_2^2)^{a+1}}, \qquad r = x_3 + s, \qquad s = \frac{1}{2}\int\frac{dt}{\sqrt{q\,t^{a+1} - t}}, \qquad t = x_1^2 + x_2^2.$$

However, when one sets:

$$u_1 = \frac{x_1}{x_1^2 + x_2^2 + x_3 X},$$

$$u_2 = \frac{x_2}{x_1^2 + x_2^2 + x_3 X},$$

$$u_3 = \frac{X}{x_1^2 + x_2^2 + x_3 X},$$

this equation will be converted into:

$$f\left(\frac{u_1}{u_2}, \frac{u_3^2}{u_1^2+u_2^2}, u_3\right) = 0$$

or

$$F(u_1, u_2, u_3) = 0,$$

which depends upon merely u_1, u_2, u_3.

The P-E-system is then special of the second kind, as one would expect, moreover. By contrast, for $a = +1$, one will get a system that possesses equal and opposite principal curvature radii at every location.

Finally, should the radius of curvature for the second system of lines of curvature be a constant that is equal to a then one will have to integrate the equation:

$$0 = x_1\frac{\partial X}{\partial x_1} + x_2\frac{\partial X}{\partial x_2} - \frac{x_1^2+x_2^2}{X}\frac{\partial X}{\partial x_3} - \left\{X^2 + \frac{\left(\sqrt{X^2+x_1^2+x_2^2}\right)^3}{a}\right\}\frac{1}{X},$$

which can be accomplished in any case, in general.

One obtains the following particular system from the differential equations (2) of the lines of curvature immediately. Namely, if one sets:

$$T = p_1^2 + p_2^2 + p_3^2 = 0,$$
$$\sum p_i\,dx_i = 0$$

then equations (2) will be fulfilled. *The curves of the system that are described on the surface $T = 0$ will then define a system of ∞^1 lines of curvature.* If one then sets:

$$\sum dx_i^2 = 0,$$
$$\sum p_i\,dx_i = 0$$
$$\sum dp_i\,dx_i = 0$$

then equations (2) will again be fulfilled. However, the ratios of the dx determine the principal tangents, which meet the imaginary circle. One then sees the *existence of ∞^1 principal tangent curves, which are, at the same time, lines of curvature.* In particular, there is a system for which all principal tangent curves of the one kind are lines of curvature; one obtains the partial differential equation that the p_i must then satisfy by eliminating the dx_i from the foregoing three equations, etc. (*).

Finally, I shall emphasize a case in which the equations of the lines of curvature can be ascertained immediately. Namely, if:

(*) Both systems of principal tangents can, at the same time, be lines of curvature; that will be the case, e.g., for the system with constant curvature that mentioned on page 27.

$$p_i = \alpha_i\, f + x_i\,,$$

where f means an arbitrary function of the x_1, x_2, x_3, then the differential equations of the lines of curvature will be:

$$f\,\sum \alpha_i\,dx_i + \sum x_i\,dx_i = 0,$$

(4)

$$df \begin{vmatrix} x_1 & x_2 & x_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ dx_1 & dx_2 & dx_3 \end{vmatrix} = 0.$$

The one family of it is then $f = c$, from which, it follows that:

$$2c\alpha_x + (x^2 + y^2 + z^2) = c_1\,.$$

These are then spherical curves, which will be cut out of a family of spheres by a family of surfaces $f = c$. On the other hand, if one sets:

$$\sum \alpha_i\,\lambda_i = 0,$$
$$\sum \lambda_i\,x_i = 0$$

then the determinant in (4) will vanish, and one will have to integrate the first of equations (4) in conjunction with $\sum \lambda_i\,x_i = 0$. These curves of curvature will then be planar; their planes will all go through the coordinate origin. Now, if f has the form:

$$k\,(x_1^2 + x_2^2 + x_3^2) + a_x + b = 0$$

then the two system of lines of curvature will be planar, so the ones of the first kind will be cut out of the spheres f = const. by the planes $(2c\alpha_x - c_1)\,k + c - (\alpha_x + b) = 0$. Since any line in the plane is a line of curvature, one will then have an example of three P-E-systems (two of which are, admittedly, of a special nature) that intersect reciprocally in lines of curvature, and therefore everywhere at constant angles along them.

§ 6.

On a linear P-E-system.

If one sets the p_i equal to entire linear functions of the x_i, so:

(1) $$p_i = \sum a_{ik}\,x_k + a_{i4}, \qquad i, k = 1, 2, 3,$$

then since the second differential quotients of the p_i vanish in general, all principal tangent curves will be rectilinear, and they will define a second-order ray system whose lines are cut out by the planes:

$$\sum (X_i - \alpha_i)\, p_i = 0$$

from the lines of intersection of the conic section at infinity:

$$\sum a_{ik}\, x_1\, x_k = 0 \tag{2}$$

that go through the point x. This system defines the analogue of the generators of the pencil of second-order surfaces into which it will, in turn, go when the p_i satisfy the integrability condition; i.e., when the $a_{ik} = a_{ki}$. It is obvious that the projective association of the tangent planes of a second-degree surface with their contact points along a generator will then remain the same for the points of a ray and its associated plane.

If no linear relation with constant coefficients exists between the p_i then one can choose the coordinate origin in such a way that the a_{i1} vanish – i.e., the system is referred to its center. That might be assumed in what follows, as long as it will lead to no contradiction. One will easily deduce the properties of the system for which the center lies at infinity.

The focal surface of the system is a second-degree conic surface whose vertex is the exceptional point that is situated at the coordinate origin, for which the p_i vanish. However, the inflection surface is also such a cone whose equation is given by (2). Both cones contact along two lines, along which the plane that is associated with the point simultaneously coincides with the common tangent planes to that cone; all generators of the inflection are, at the same time, principal tangent curves. The ray system of the latter is also of class two, since only two lines of that system can be found in any plane in general. Now, one can find infinitely many rays in any tangent plane of the inflection surface that all go through the contact point at infinity on it. The focal surface can vanish identically only when all sub-determinants of the $a_{ik} + a_{ki}$ are zero; i.e., when the equations are valid:

$$\begin{array}{lll} a_{11} = \alpha_1^2 , & a_{22} = \alpha_2^2 & a_{33} = \alpha_3^2 , \\ a_{12} = \alpha_1\alpha_2 + a_3, & a_{23} = \alpha_2\,\alpha_3 + a_1, & a_{13} = \alpha_1\alpha_3 - a_2 , \\ a_{21} = \alpha_2\alpha_1 - a_3, & a_{32} = \alpha_3\,\alpha_2 - a_1, & a_{31} = \alpha_3\alpha_1 + a_2 , \end{array}$$

from which, one will get:

$$\begin{aligned} p_1 &= a_1\,\alpha_x + a_3\,x_2 - a_2\,x_3 , \\ p_2 &= a_2\,\alpha_x + a_1\,x_3 - a_3\,x_1 , \\ p_3 &= a_3\,\alpha_x + a_2\,x_1 - a_1\,x_2 ; \end{aligned}$$

i.e., it will give a ray system of class and order one that is defined by the lines of intersection of the line at infinity:

$$\alpha_x = 0$$

and the lines $x_1 : x_2 : x_3 = a_1 : a_2 : a_3$.

In order to bring the general case of the p_i into a simple form, one sets:

$$\begin{aligned} a_{12} - a_{21} &= 2a_3 \,, \\ a_{31} - a_{13} &= 2a_2 \,, \\ a_{23} - a_{32} &= 2a_1 \,, \\ a_{ik} + a_{ki} &= 2\alpha_{ik} = 2\alpha_{ki} \,, \\ \psi &= \sum \alpha_{ik}\, x_i\, x_k \,. \end{aligned}$$

One will then have:

$$\begin{aligned} p_1 &= \frac{1}{2}\frac{\partial \psi}{\partial x_1} + a_3\, x_2 - a_2\, x_3 \,, \\ p_2 &= \frac{1}{2}\frac{\partial \psi}{\partial x_2} + a_1\, x_3 - a_3\, x_1 \,, \\ p_3 &= \frac{1}{2}\frac{\partial \psi}{\partial x_3} + a_2\, x_1 - a_1\, x_2 \,, \end{aligned}$$

and

$$\sum p_i\, dx_i = \tfrac{1}{2} d\psi + \begin{vmatrix} dx_1 & dx_2 & dx_3 \\ x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \end{vmatrix} . \tag{3}$$

If one now transform the form ψ to its principal axes then that will give:

$$\psi = \sum \lambda_i y_i^2 \,,$$

and upon multiplying by the determinant of the transformation, which might have the value + 1, one will get:

$$\sum p_i\, dx_i = \tfrac{1}{2} d\psi + \begin{vmatrix} dy_1 & dy_2 & dy_3 \\ y_1 & y_2 & y_3 \\ b_1 & b_2 & b_3 \end{vmatrix} .$$

One can then always bring the p_i into the form:

$$\begin{aligned} p_1 &= \lambda_1\, x_1 + a_3\, x_2 - a_2\, x_3 + c_1, \\ p_2 &= \lambda_2\, x_2 + a_1\, x_3 - a_3\, x_1 + c_2, \\ p_3 &= \lambda_3\, x_3 + a_2\, x_1 - a_1\, x_2 + c_3, \end{aligned} \tag{4}$$

in which the constants can be dropped as long as the center is at a finite point. Moreover, the equation of the focal surface will be:

$$\sum \frac{p_i^2}{\lambda_i} = 0 = \left(\sum \lambda_i x_i^2\right)(\lambda_1\lambda_2\lambda_3 + \lambda_1 a_1^2 + \lambda_2 a_2^2 + \lambda_3 a_3^2)\left(\sum a_i \lambda_i x_i\right)^2 , \tag{5}$$

which is the inflection surface:

$$\sum \lambda_i x_i^2 = 0.$$

Since the focal surface will be contacted by all principal tangents, there will exist a family of singular principal tangent curves; it will be generated by the **Kummer** developable of the ray system. The equations of these curves can be derived easily. A point x_i of the focal surface will then correspond to the direction of the principal tangent dx_1, dx_2, dx_3 :

$$\begin{aligned} \lambda_1\, dx_1 &= d\mu\, p_1 , \\ \lambda_2\, dx_2 &= d\mu\, p_2 , \\ \lambda_3\, dx_3 &= d\mu\, p_3 , \\ \sum p_i\, dx_i &= 0. \end{aligned} \tag{6}$$

Upon multiplying the first three equations by a_1, a_2, a_3 (x_1, x_2, x_3, resp.), since:

$$\begin{aligned} \sum a_i\, p_i &= \sum a_i\, \lambda_i\, x_i + \sum a_i\, c_i , \\ \sum x_i\, p_i &= \sum \lambda_i\, x_i^2 + \sum x_i\, c_i , \\ \frac{d}{d\mu} \sum \lambda_i\, a_i\, x_i &= \sum \lambda_i\, a_i\, x_i + \sum a_i\, c_i , \\ \frac{1}{2}\frac{d}{d\mu} \sum \lambda_i\, x_i^2 &= \sum \lambda_i\, x_i^2 + \sum x_i\, c_i . \end{aligned} \tag{7}$$

One will obtain the equation for the focal surface from this by a suitable choice of integration constants; in particular, one will get equation (5) when the c_i are equal to zero. The last of equations (6) will then seem superfluous. If one then multiplies the first three of (6) by the factors k_1, k_2, k_3 and sets:

$$\begin{aligned} \lambda_1\, k_1 + a_2\, k_3 - a_3\, k_2 &= \omega\, \lambda_1\, k_1 , \\ \lambda_2\, k_2 + a_3\, k_1 - a_1\, k_3 &= \omega\, \lambda_2\, k_2 , \\ \lambda_3\, k_3 + a_1\, k_2 - a_2\, k_1 &= \omega\, \lambda_3\, k_3 , \end{aligned}$$

from which, it will follow that:

$$\begin{vmatrix} 1-\omega & -a_3 & a_2 \\ a_3 & 1-\omega & -a_1 \\ -a_2 & a_1 & 1-\omega \end{vmatrix} = 0,$$

then one will get two other roots ω_1, ω_2, in addition to the root $\omega = 1$, which will yield two relations of the form:

$$\frac{d}{d\mu} \log \sum \lambda_i\, k_i'\, x_i = \omega_1 ,$$

$$\frac{d}{d\mu}\log \sum \lambda_i\, k_i'' x_i = \omega_2 ,$$

from which, one will immediately infer the missing integral:

$$\left(\sum \lambda_i\, k_i' x_i\right)^{\omega_2} = \text{const.} \left(\sum \lambda_i\, k_i'' x_i\right)^{\omega_1} .$$

By contrast, when an identity of the form:

$$\sum a_i\, p_i = 1$$

exists between the p_i , one will have, with no further analysis:

$$\frac{d}{d\mu}\sum \lambda_i\, a_i\, x_i = 1,$$

which is an equation that will provide the system of curves in conjunction with the first of (7) and that of the focal surface.

The method of integration that was developed for the second-degree surfaces no longer seems justified for the integration of the equations of the lines of curvature. Lines of curvature are curves of the system that are described on the cone:

$$\sum p_i^2 = 0.$$

A second system will be defined by the four planar pencils of rays of principal tangents whose centers lie at the points of intersection of the inflection surface with the imaginary circle. In addition, that easily implies the existence of three systems of lines of curvature that lie in three planes that go through the center. Namely, if one sets:

$$\begin{aligned} \alpha_1\lambda_1 + a_2\, \alpha_3 - a_3\, \alpha_2 &= \rho\, \alpha_1 , \\ \alpha_2\lambda_2 + a_3\, \alpha_1 - a_1\, \alpha_3 &= \rho\, \alpha_2 , \\ \alpha_3\lambda_3 + a_1\, \alpha_2 - a_2\, \alpha_1 &= \rho\, \alpha_3 \end{aligned}$$

then, by means of the cubic equation:

$$\begin{vmatrix} \lambda_1 - \lambda & -a_3 & a_2 \\ a_3 & \lambda_2 - \lambda & -a_1 \\ -a_2 & a_1 & \lambda_3 - \lambda \end{vmatrix} = 0,$$

that will provide three systems of quantities α_i, which might be denoted by α_i, β_i, γ_i, that correspond to the roots ρ_1, ρ_2, ρ_3, resp. If one now multiplies the equation:

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ dp_1 & dp_2 & dp_3 \\ dx_1 & dx_2 & dx_3 \end{vmatrix} = 0$$

by the determinant:

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ dx_1 & dx_2 & dx_3 \\ p_1 & p_2 & p_3 \end{vmatrix}$$

and the corresponding one in which one replaces the α_i with β_i and γ_i then the relations:

$$(8) \qquad \begin{aligned} dX_1 &= \rho_1 \frac{d\mathrm{T}}{\rho_1 - S}, \\ dX_2 &= \rho_2 \frac{d\mathrm{T}}{\rho_2 - S}, \\ dX_3 &= \rho_3 \frac{d\mathrm{T}}{\rho_3 - S} \end{aligned}$$

will come about, in which one sets:

$$X_1 = \log \sum \alpha_i x_i, \qquad X_2 = \log \sum \beta_i x_i, \qquad X_3 = \log \sum \gamma_i x_i,$$

$$\mathrm{T} = \log \sqrt{p_1^2 + p_2^2 + p_3^2}, \qquad S = \frac{\sum \lambda_i dx_i^2}{\sum dx_i^2}.$$

That will yield the equations:

$$\sum \alpha_i x_i = 0, \qquad \sum \beta_i x_i = 0, \qquad \sum \gamma_i x_i = 0$$

as particular integrals, from which, in conjunction with $\sum p_i \, dx_i = 0$, one can easily find the curves of curvature that lie in these planes.

A general integral is given only in the special case for which one of the roots ρ is equal to zero. From (5), the equation of the focal surface will then reduce to $\sum \alpha_i x_i \lambda_i = 0$, and a family of lines of curvature will be planar. I have not yet arrived at the integration of equations (8) in a symmetric form. Moreover, the second-order differential equation with constant coefficients will follow from them:

$$\begin{vmatrix} \rho_1 & \rho_2 & \rho_3 \\ \rho_1 \, dX_1 & \rho_2 \, dX_2 & \rho_3 \, dX_3 \\ dX_1 & dX_2 & dX_3 \end{vmatrix} = 0,$$

while the linear equation $\sum p_i \, dx_i = 0$ will be transformed into:

$$\sum P_i \, dX_i = 0.$$

The desired curves of curvature will then be mapped onto the curves of the latter P-E-system, whose tangents meet a fixed conic section at infinity.

Dresden, 8 June 1883.