“Zur Invariantentheorie der Liniengeometrie,” Math. Ann. **37** (1890), 141-152.

On the invariant theory of line geometry.

By

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Hermann Grassmann represented a line segment as the external product of its end points; in that way, one will get the homogeneous coordinates p_{ik} of the straight line. As the product of second-degree determinants, this will be set equal to alternating products by *Grassmann* (*); namely, $p_{ik} = p_i\, p_k = -p_k\, p_i$. Similar statements are true for the coordinates of the elementary structures of higher geometry.

If one introduces these symbolic line coordinates p_i into some known line-geometric invariant constructions (**) then one will see that these invariant are composed of factors that are linear in the symbols, and thus, factors like:

$$p_1\, \xi_1 + p_2\, \xi_2 + p_3\, \xi_3 + p_4\, \xi_4\,,$$

in which the ξ_i mean plane coordinates. These elementary factors will be invariant in themselves, and e.g., the p will be contragredient to the ξ.

One will then also symbolize the linear and higher complexes and the behavior of symbols under linear transformations, and then define invariant elementary expressions and then, by aggregation, all invariants. Following through on this use of *Grassman's* symbolism and giving some applications of it is the purpose of the present paper.

1.

The symbols.

If x_i, y_i are the coordinates of two points on a line and ξ_i, η_i (i = 1, 2, 3, 4) are the coordinates of two planes through a line then it is known that the ray coordinates of this line will be the six quantities:

$$p_{ik} = x_i\, y_k - x_k\, y_i\,,$$

(*) Cf., also *Hankel*, *Theorie der complexen Zahlensysteme*, § 35, *et seq.*

(**) E. g., in the formulas that *Pasch* derived in his treatise “Zur Theorie der linearen Complexe,” Crelle's J., Bd. 75, pp. 106.

which are coupled with the *axis coordinates*:

$$\pi_{ik} = \xi_i \, \eta_k - \xi_k \, \eta_i$$

by the relations:

$$p_{23} = \rho \, \pi_{14}, \ldots, p_{34} = \rho \, \pi_{12}, \ldots$$

One has:

$$p_{ik} = - p_{ki} , \qquad \pi_{ik} = - \pi_{ki} .$$

One can introduce symbols in place of these coordinates by setting:

$$p_{ik} = p_i \, p_k = - p_k \, p_i , \qquad p_i \, p_i = 0,$$
$$\pi_{ik} = \pi_i \, \pi_k = - \pi_k \, \pi_i , \qquad \pi_i \, \pi_i = 0.$$

The p_i might be called *ray symbols*, while the π_i might be called *axis symbols*.
If:

$$\sum c_{ik} \pi_{ik} = 0 \qquad \text{or} \qquad \sum \gamma_{ik} \, p_{ik} = 0,$$

where $c_{23} = \rho \, \gamma_{14}$, etc., gives the equation of a linear complex, then the c_{ik} or γ_{ik} will be called its *coordinates*. One can again set:

$$c_{ik} = c_i \, c_k = - c_k \, c_i , \qquad \gamma_{ik} = \gamma_i \, \gamma_k = - \gamma_k \, \gamma_i .$$

Since:

$$\sum c_i c_k \pi_i \pi_k = \sum c_i \pi_i c_k \pi_k = \tfrac{1}{2}(c_1 \, \pi_1 + c_2 \, \pi_2 + c_3 \, \pi_3 + c_4 \, \pi_4)^2 = \tfrac{1}{2}(c \, \pi)^2,$$

the complex equation will then go to:

$$\tfrac{1}{2}(c \, \pi)^2 = 0 \qquad \text{or} \qquad \tfrac{1}{2}(\gamma p)^2 = 0.$$

If the line or complex coordinates enter into a form in a higher degree then one will introduce as many sequences of symbols $p = p^1 = p^2, \ldots; \pi = \pi^1 = \pi^2, \ldots$; etc., as that degree would imply, in order to avoid ambiguities.

2.

Linear transformations.

The points x of space will be subjected to the linear transformation:

$$x_i' = (\alpha^i \, x). \tag{1}$$

The coordinates of the transformed lines are then:

$$p'_{ik} = \alpha^i x \cdot \alpha^k y - \alpha^i y \cdot \alpha^k x = \sum \alpha^i_\lambda \alpha^k_\mu p_\lambda p_\mu ,$$

so:

(2) $$p'_{ik} = p'_i p'_k = (\alpha^i p) \cdot (\alpha^k p).$$

Now, if the form $F(p_{ik})$ is invariant under the transformation (1) then:

$$F(p'_{ik}) = F(p'_i p'_k) = F((\alpha^i p) \cdot (\alpha^k p)) = c\, F(p_{ik});$$

if one then applies the substitution:

$$p'_i = \alpha^i p$$

to the $F(p'_{ik})$ then one will arrive at the transformed form. It will then follow that:

The symbols p, c are cogredient in the coordinates, and therefore contragredient in the mutually cogredient π, γ, ξ.

Therefore, symbolic expressions such as:

$$(p\ \pi),\quad (p\ \gamma),\quad (p\ \xi),\quad (x\ \pi),\quad (x\ \gamma),\quad (p^1\ p^2\ p^3\ p^4),\quad (p^1\ p^2\ c\ x),\qquad (\gamma\, \pi\ \xi\ \eta),\qquad \text{etc.},$$

are invariant under linear transformations. One will then obtain invariant structures by aggregation of these expressions, the most important of which will be then ones for which the symbols enter in such degrees that the aggregate will have a non-symbolic meaning.

Thus, e.g., $(c\ g)^2$, $(c\ x)$, $(cx)(cy)$ will be invariants of linear complexes. The first one vanishes when the complex is singular; the last one will give the spatial reciprocity that is determined by the complex when it is annulled.

3.

Second-order basic forms.

The invariant form $F(c_{ik}, \gamma_{ik}, p_{ik}, \pi_{ik})$, which contains each sequence of symbols of the complex form c quadratically, is also an invariant structure when one introduces the quadratic form $(c\ \xi)^2$ in place of the complex in the system of given forms, and likewise, introduces the symbols of the quadratic forms $(\gamma x)^2$, $(p\ \xi)^2$, $(\pi\, x)^2$, in place of the symbols γ, p, π. Thus, any line-geometric invariant of a system in which linear complexes occur can be interpreted as an invariant of a system into which second-order forms – viz., *the basic forms (Bildformen)* – enter in place of these complexes.

Any invariant of the latter system is, however, a mere aggregate of the given elementary symbolic expressions. *One then also obtains all line-geometric invariants of the given system by aggregation of these elementary expressions.*

Example: Any invariant of the complex $(c\ \pi)^2 = 0$ that depends upon just c is the basis for an invariant of the surface $(c\ \xi)^2 = 0$, so only the single invariant $(c^1\ c^2\ c^3\ c^4)^2$ should be chosen as the basis now, since (cf., art. 5):

$$(c^1\ c^2\ c^3\ c^4)^2 = \tfrac{3}{2}((c\ \gamma)^2)^2,$$

so the linear complex will possess only the invariant $(c\ \gamma)^2$.

4.

n^{th}-degree complexes.

In vol. II of Math. Ann., Clebsch (*) showed that any complex of degree n can be represented symbolically as a power of a linear complex. If:

$$C \equiv \sum c_{ik,\,lm}\,,\ \ldots\ \pi_{ik}\ \pi_{lm}\ \ldots = 0$$

is the complex equation then one can, in fact, symbolically set:

$$c_{ik,\,lm}\,,\ \ldots = c_{ik}\ c_{lm}\ \ldots,$$

which will make:

$$C \equiv \left(\sum c_{ik}\pi_{ik}\right)^n.$$

We now define a new symbolic form by introducing symbolic line and complex coordinates into this. It becomes:

$$2^n\ C \equiv ((c\ \pi)^2)^n.$$

Therefore:

$$(c\ \pi)^{2n} = 0 \qquad \text{and} \qquad (\gamma p)^{2n} = 0$$

will be the symbolic equations of the complex. Moreover, the basis forms of order $2n$ (class $2n$, resp.) $(c\ \xi)^{2n}$ and $(\gamma x)^{2n}$, resp., will enter in place of the given complex, in the sense of the previous no., such that the theorems in that no. will still be true when the system contains complexes of degree n.

5.

Identities.

In order to also agree with *Grassmann* formally, and for the sake of brevity in notation, we would like to set:

(*) "Ueber die Plücker'schen Complexe," pp. 1. Cf., also, *Salmon-Fiedler, Raumgeom. II,* pp. 503.

$$\tfrac{1}{2}(c\,\gamma)^2 = [c\,\gamma], \qquad \frac{1}{2^n}(c\,p)^{2n} = [c\,p]^n, \qquad x\,\gamma \cdot \gamma y = [x\,\gamma y],$$

$$\xi\,c^1 \cdot c^1\,\gamma^2 \cdot \gamma^2\,x = [\xi\,c^1\,\gamma^2\,x], \qquad \gamma^1\,c^2 \cdot c^2\,\gamma^3 \cdot \gamma^3\,c^4 \cdot c^4\,\gamma^1 = [\gamma^1\,c^2\,\gamma^3\,c^4], \qquad \text{etc.,}$$

and then derive a series of identities.

a. If one introduces complex symbols into the identity:

(1) $$(abcd) \cdot x\xi - (abcx) \cdot d\xi + (abdx) \cdot c\xi - (acdx) \cdot b\xi + (bcdx) \cdot a\xi = 0,$$

such that one sets:

$$a = c^1, \qquad b = c^1, \qquad c = c^2, \qquad d = c^2,$$

then since:

$$c\mu \cdot c\nu = -\,c\nu \cdot c\mu,$$

one will get the identity:

$$2\,(c^2\,c^2\,c^1\,x) \cdot c^1\xi + 2\,(c^2\,c^2\,c^1\,x) \cdot c^1\xi = 2\,(c^1\,c^1\,c^2\,c^2) \cdot x\xi\,.$$

Now, one has:

$$(cc'x) = 2c_2\,c_3\,(c_1'x_4 - c_4'x_1) + \ldots = \sum \gamma_i \gamma_k\,(c_i'x_k - c_k'x_i)$$
$$= c'\gamma \cdot \gamma x - x\gamma \cdot \gamma c',$$

so

(2) $$(cc'x) = 2\,[c'\gamma x] \qquad \text{or} \qquad (\gamma\gamma'\xi) = 2[\gamma' c\,\xi];$$

therefore, the last identity will go to:

I. $$[x\gamma^1 c^2\,\xi] + [x\gamma^2\,c^1\xi] = -\,[c^1\,\gamma^2] \cdot x\xi.$$

If the complexes 1 and 2 are identical then one will get (*):

II. $$[x\,\gamma c\,\xi] = -\tfrac{1}{2}[c\,\gamma] \cdot x\xi\,.$$

b. If one sets:

$$b = c^2, \qquad c = c^2, \qquad \xi = x, \qquad a = y$$

x = sub-determinant of the matrix $[c^1\;c^2\;c^3]$ in the identity (1) then one will get, with the use of (2):

(3) $$(x\,c^1\,c^2\,c^3)(y\,c^1\,c^2\,c^3) = (c^2\,c^3\,xy) \cdot c^2\,\gamma^1 \cdot \gamma^1\,c^3 + (c^3\,c^1\,xy) \cdot c^3\,\gamma^2 \cdot \gamma^2\,c^1 +$$
$$+ (c^1\,c^2\,xy) \cdot c^1\,\gamma^3 \cdot \gamma^3\,c^2,$$

or, with equality of the symbols $c^1 = c^2 = c^3$:

(*) Cf., *Pasch, loc. cit.*, § 4.

$$(x\, c^1\, c^2\, c^3)(\zeta\, c^1\, c^2\, c^3) = 3\,(c^1\, c^2\, x\zeta) \cdot c^1\, \gamma \cdot \gamma\, c^2,$$

and from that, when one employs formula II, and then (2), on the right-hand side:

III. $$(c^1\, c^2\, c^3\, x)(c^1\, c^2\, c^3\, y) = 3\,[c\, \gamma]\,[x\gamma y].$$

If one sets $x = y = c^4$ in this then one will have:

$$(c^1\, c^2\, c^3\, c^4)^2 = 6\,[c\, \gamma]^2 = (\gamma^1\, \gamma^2\, \gamma^3\, \gamma^4)^2$$

for a linear complex, and:

$$(p^1\, p^2\, p^3\, p^4)^2 = (\pi^1\, \pi^2\, \pi^3\, \pi^4)^2 = 6\,[p\, \pi]^2 = 24\,[p_{ik}]$$
$$= 24\,(p_{23}\, p_{14} + p_{31}\, p_{24} + p_{12}\, p_{34})^2$$

for line coordinates.

c. If one sets $c^1 = a$, $c^2 = b$, $c^3 = c$ in (3), and if these are the symbols of different complexes, then one will obtain:

$$(xbcd)(ybcd) = (xycd) \cdot c\beta \cdot \beta\, d + (xybd) \cdot b\gamma \cdot \gamma d + (xybc) \cdot b\delta \cdot \delta c\,.$$

If $x = y = a$ is the symbol of another complex then one will get, by applying formula (2):

$$-\tfrac{1}{2}(abcd)^2 = [\alpha\, c\, \beta\, d] + [\alpha\, b\, \gamma d] + [\alpha\, b\, \delta c],$$

and from this, with an application of identity I:

IV. $$-\tfrac{1}{2}(abcd)^2 = [\alpha\, b]\,[c\, \delta] + [\alpha\, c][\beta\, d] + [\alpha\, d][\beta\, c].$$

These formulas yield the following ones, when several of the symbols belong to the same complex:

(4) $$\begin{cases} \tfrac{1}{2}(abc^1c^2)^2 = [\alpha b][c\gamma] + 2[\alpha c][\beta c], \\ (ac^1c^2c^3)^2 = 6[\alpha c][\gamma c], \\ (c^1c^2c^3c^4)^2 = 6[c\gamma]^2, \end{cases}$$

and the latter formulas were already found above.

d. It follows from:

$$(c^1\, c^2\, c^3\, c^4) \cdot (\xi^1\, \xi^2\, \xi^3\, \xi^4) \cdot (c^1\, c^2\, c^3\, c^4) = (c^1\, c^2\, c^3\, c^4)^2 \cdot (\xi^1\, \xi^2\, \xi^3\, \xi^4)$$

and the last of formulas (4), when one equates the c symbols, that:

(5) $$4 \cdot c^1\xi^1 \cdot c^2\xi^2 \cdot c^3\xi^3 \cdot c^4\xi^4 \cdot (c^1\, c^2\, c^3\, c^4) = -\,[c\, \gamma] \cdot (\xi^1\, \xi^2\, \xi^3\, \xi^4).$$

With this:

(6) $$2 \cdot c^1\xi^1 \cdot c^2\xi^2 \cdot c^3\xi^3 \cdot (c^1\, c^2\, c^3\, c^4) = -[c\,\gamma] \cdot \gamma^4\, x^4 \cdot (\xi^1\, \xi^2\, \xi^3\, \xi^4),$$

(7) $$c^1\xi^1 \cdot c^2\xi^2 \cdot (c^1\, c^2\, c^3\, c^4) = \gamma^3\, x^3 \cdot \gamma^4\, x^4 \cdot (\xi^1\, \xi^2\, \xi^3\, \xi^4);$$

the last two formulas follow from (5) by a single (double, resp.) application of substitutions such as $\xi_i = x\gamma \cdot \gamma_i$ and inversion:

$$-\tfrac{1}{2}[c\,\gamma] : x_i = c\,\xi \cdot c_i\,,$$

as the latter is clear from II when one sets $\xi = \eta$ in it and $\xi_i = x\gamma \cdot \gamma_i$ on the left-hand side.

By applying formula (2), one obtains:

$$2\,(\gamma\xi^1\, \xi^2\, \xi^3)\; \gamma x = -\,|\, ccx \,| \cdot |\, \xi^1\, \xi^2\, \xi^3 \,| = -\sum \pm (c\,\xi^1 \cdot c\,\xi^2 \cdot x\,\xi^3),$$

so

(8) $$(\xi^1\, \xi^2\, \xi^3\, \gamma)\; \gamma x = [\xi^2\, c\, \xi^3] \cdot x\,\xi^1 + [\xi^3\, c\, \xi^1] \cdot x\,\xi^2 + [\xi^1\, c\, \xi^2] \cdot x\,\xi^3,$$

and furthermore:

(9) $$(\xi^1\xi^2\gamma^1\gamma^2)\; \gamma^1x^1 \cdot \gamma^2x^2 = \tfrac{1}{2}[c\;\gamma]\,(x^1\xi^2 \cdot x^1\xi^2 - x^1\xi^1 \cdot x^2\xi^2) + [x^1\gamma\,\xi^2]\,[\xi^1 c\;\xi^2],$$

as would follow from (8) for $\xi_i^3 = \gamma_i^2 \cdot \gamma^2\, x^2$ with the use of II.

6.

Linear complexes.

A linear complex has the equation:

$$(c\;\pi)^2 = [c\;\pi] = 0 \qquad \text{or} \qquad (\gamma p)^2 = [\gamma p] = 0,$$

and the invariant is $(c\;\gamma)^2 = [c\;\gamma]$. The equations:

$$c\;\xi \cdot c\;\eta = 0, \quad \gamma x \cdot \gamma y = 0$$

give a polar relationship that represents the null system that is coupled to the complex.

From formula III of the previous no., one has:

$$3\,[c\gamma] \cdot \gamma x \cdot \gamma y = (c^1\, c^2\, c^3\, x)(c^1\, c^2\, c^3\, y),$$
$$3\,[c\gamma] \cdot c\xi \cdot c\eta = (\gamma^1\, \gamma^2\, \gamma^3\, \xi)(\gamma^1\, \gamma^2\, \gamma^3\, \eta);$$

one will then have:

$$3\,[c\;\gamma] \cdot (\gamma x)^2 = (c^1\, c^2\, c^3\, x)^2$$

for the basis form $(\gamma x)^2$. The equation for the surface $(c^1\, c^2\, c^3\, x)^2$ is, however, $(c\ \xi)^2$ in plane coordinates, so it will be identical with other basic surfaces. Its invariant is:

$$(c^1\, c^2\, c^3\, c^4)^2 = 6\,[c\ \gamma]^2,$$

which implied, in no. 3, that $[c\ \gamma]$ was the only invariant of the linear complex.

Since, moreover, from formula (4) of the previous no., any invariant of a system of basic forms represents a linear complex as an aggregate of invariants $[c^i\ \gamma^k]$, the latter will define the complete system of invariants of a system of linear complexes.

If, for the moment, one lets $\hat{x}$ denote the null plane of the point x, and lets $\hat{\xi}$ denote the null point of the plane ξ then one will have:

$$\hat{\xi}_i = c\xi \cdot c_i\,, \qquad \hat{x}_i = \gamma x \cdot \gamma_i$$

for the coordinates.

The *relations* (*) that exist between the coordinates of four points and their null planes then follow from equations (5), (6), (7), of no. 5, and their duals:

$$4(\hat{\xi}\,\hat{\eta}\,\hat{\zeta}\,\hat{\vartheta}) = [c\ \gamma]^2 \cdot (\xi\,\eta\,\zeta\,\vartheta),$$
$$2(\hat{\xi}\,\hat{\eta}\,\hat{\zeta}\,x) = [c\ \gamma] \cdot (\xi\,\eta\,\zeta\,\hat{x}),$$
$$(\hat{\xi}\,\hat{\eta}\,x\,y) = (\xi\,\eta\,\hat{x}\,\hat{y}),$$
etc.

Let $\hat{x}$ be the null plane of the point x relative to a first complex, so the null point of $\hat{x}^1$ relative to a second complex will be:

$$\hat{x}^1\,\gamma^2 \cdot \gamma^2\,\xi = 0;$$

therefore:

$$[x\ \gamma^1\ c^2\ \xi] = 0$$

is the *special collineation* that arises by adding the null systems.

If the two complexes are singular and intersect their carrier then:

$$[x\ \gamma^1\ c^2\ \xi] = 0$$

will be the *equation of the intersection point* or the *connecting plane* of the two lines, according to whether one regards ξ or x as variable, respectively.

$$[x\ \gamma^2\ c^1\ \xi] = 0$$

is the collineation that is inverse to the one above, and from formula I:

$$[x\ \gamma^1\ c^2\ \xi] + [x\ \gamma^2\ c^1\ \xi] = 0$$

(*) Cf., *Pasch, loc. cit.*, § 4.

is the identity, since the expression on the left is proportional to $x\ \xi$; therefore:

$$[x\ \gamma^1\ c^2\ \xi] - [x\ \gamma^2\ c^1\ \xi] = 0$$

is the equation for the *collective involution* that is coupled with the two complexes.

7.

Three linear complexes.

Three linear complexes will have the covariant surfaces:

$$x\ \gamma^1 \cdot \gamma^1\ c^2 \cdot c^2\ \gamma^3 \cdot \gamma^3\ x = [x\ \gamma^1\ c^2\ \gamma^3\ x] = 0,$$
$$\xi\ c^1 \cdot c^1 \gamma^2 \cdot \gamma^2\ c^3 \cdot c^3\ \xi = [\xi\ c^2\ \gamma^1 c^3\ \xi] = 0.$$

These have the forms of equations of second-degree surfaces that will be defined by the common complex rays of the complex. Then:

$$\xi_i = x\ \gamma^1 \cdot \gamma_i^1, \qquad \xi_i^1 = x\ \gamma^3 \cdot \gamma_i^3$$

will be the coordinates of the null plane of the point x relative to the complexes 1 and 3; if these planes are to intersect in a ray of the complex 2 then one must have $\xi\ c^2 \cdot c^2\ \xi^1 = 0$.

In order to solve the converse problem of *determining the complex that contains the ruling of a surface:*

$$F_2 \equiv (\alpha\, x)^2 = 0,$$

we remark that two complexes with the desired property will be present in the pencil of two complexes c, c' that are polar to each other.

The equation:

$$(\alpha\,\beta\,\xi\,\eta)(\alpha\,\beta\,\xi'\,\eta') = 0$$

mediates the polar relationship between the lines that belong to F_2, so one will have:

$$c' \equiv [a\ c\ \beta] \cdot [\alpha\, p \beta] = 0;$$

thus, $\gamma_i' = \alpha c \cdot \alpha_i$ is the symbol of the complex c', and with the help of formula (2), of no. 5:

$$c' \equiv \tfrac{1}{2}(\alpha\,\beta\,\gamma\,\gamma) : [\alpha\, p \beta].$$

If one sets $\gamma = \gamma'$ in this then one will get the polar complex c'' to c':

$$c'' \equiv \tfrac{1}{2}(\alpha\,\beta\,\delta\,\varepsilon) : [\alpha\, p \beta] \cdot [\delta\, c\ \varepsilon] = \tfrac{1}{2} \cdot \tfrac{1}{24}(\alpha\,\beta\,\delta\,\varepsilon)^2\,(ppcc) = \Delta[\gamma p] = \Delta \cdot c.$$

The complex of the pencil:

$$\lambda\, c + c' = 0$$

then has the complex:

$$\lambda\, c' + \Delta\, c = 0$$

for its polar. Should both complexes be identical then it would follow that $\lambda^2 = \Delta$. The rulings of the surface $(\alpha x)^2 = 0$ are contained in the linear complexes:

$$\pm\sqrt{\Delta} \cdot [\gamma p] + [\alpha\, c\, \beta] \cdot [\beta\, p\, \alpha] = 0,$$

in which $[\gamma p] = 0$ is an arbitrary complex and $\Delta = |\, \alpha_{ik} \,|$ (*).

8.

Conversion principle for linear complexes.

If the plane ξ satisfies the equation:

$$[\xi\, c^1\, \gamma^2\, c^3\, \xi] = 0$$

then, from the above, its null points relative to the three complexes 1, 2, 3 will lie along a line. In fact, the linear expression in the determinant of the coordinates of the null point will go to the plane $x_4 = 0$ when one sets $\xi^1 = \xi^2 = \xi^3 = 0$.

From *Clebsch* (Crelle's Journal, Bd. 59), one knows that each invariant J of k points of a plane can be expressed in an entire and rational way in terms of the determinants (ikl) of the coordinates of any three of these points. If one then sets the determinant (ikl) equal to the expression:

$$(ikl) = [\xi\, c^i\, \gamma^k\, c^l\, \xi]$$

then one will obtain a covariant surface of the systems of linear complexes; any of its tangential planes will possess k null points relative to the complex for which $J = 0$.

The dual arguments are likewise valid.

Example: Should six points lie on a conic section then one would need to have (**):

$$(123)(345)(561)(246) = (456)(612)(231)(513);$$

one then obtains a surface of class eight whose tangential planes relative to the six complex have six null points that lie in a conic section.

Should this be the case for any plane, then *the complexes must lie pair-wise in involution*. The plane then contains a common ray of the complexes 1, 2, 3, so the conic

(*) Cf., *Gordan*, "Ueber eine das Hyperboloid betreffende Aufgabe," Schlömilch, Bd. 13, pp. 59, and *Pasch*, *loc. cit.*, § 8 and 11.

(**) *S. Hunyady*, "Ueber die verschiedene Formen der Bedingung, welche ausdruckt, dass sechs Punkte auf einem Kegelschnitt liegen," Crelle's Journ., Bd. 83, pp. 76.

section must decompose, and therefore a common ray of the complexes 4, 5, 6 must lie in the plane; correspondingly, when the plane satisfies the surface (123), one of the factors that appear in the right-hand side of the last equation will also vanish, and indeed, the factor (456), since otherwise the complexes would be dependent upon each other. The pencil of complexes (1, 2, 3) and (4, 5, 6) will then yield the two families of the same F_2; the complexes of the first pencil are in involution with those of the second one.

It is, in fact, known that the null point of a plane relative to a Kleinian system of six fundamental complexes lies on a conic section.

Here arises the problem of verifying the vanishing of the 10 invariants $[\gamma^i\ c^k]$ from the identical vanishing of the covariant of class 8.

9.

Conversion principle for higher complexes.

Any invariant J in a ternary domain is an aggregate of determinants of degree three, and one regards them as linear forms that belong to a system when one omits the condition of the equality of the symbols. If one then replaces the determinants with expressions of the form $[\xi\, c^1\ \gamma^2\ c^3\ \xi]$ then, from the last no., one will get those covariants of a system of linear complexes whose tangential planes have a group of null points with the invariant $J = 0$. If one lets the linear complexes that originate by equating ternary symbols coincide once more then one will arrive at covariants that are also of systems of complexes of higher order. One likewise arrives dually at covariant surfaces that are the loci of points whose complex cones have given invariant properties.

If a term of the invariant contains i third-degree determinants then it will follow that:

The $\left.\begin{matrix}\textit{planes}\\ \textit{points}\end{matrix}\right\}$ *whose complex* $\left.\begin{matrix}\textit{curves}\\ \textit{cones}\end{matrix}\right\}$ *relative to a number of complexes of arbitrary degree have a property that is expressed by the vanishing of an invariant of weight i will all lie on a surface of* $\left.\begin{matrix}\textit{class}\\ \textit{order}\end{matrix}\right\}$ *two.*

In particular: The $\left.\begin{matrix}\text{planes}\\ \text{points}\end{matrix}\right\}$ whose complex $\left.\begin{matrix}\text{curves}\\ \text{cones}\end{matrix}\right\}$ relative to a complex of degree n have a property that is expressed by the vanishing of an invariant of degree k will all lie on a surface of $\left.\begin{matrix}\text{order}\\ \text{class}\end{matrix}\right\}$ $2i = \dfrac{2\pi k}{3}$; this is a result that was first achieved by *Clebsch* (Math. Ann., Bd. V) by dropping out a factor $(x\ \xi)^{kn/2}$.

Examples:

a. The discriminant of a plane curve has degree $3(n-1)^2$, so the order and class of the singularity surface of an n^{th}-degree complex will be equal to $2n(n-1)^2$.

For the complex of degree 2, $[c\,x]^2 = 0$, its equation will be:

$$[\xi\, c^1\ \gamma c^2\ \xi]^2 = 0,$$

in plane coordinates, and:

$$[x\ \gamma^1\ c\ \gamma^2\ x]^2 = 0,$$

in point coordinates.

b. For the tact invariant (*Tactinvariante*) of two plane curves of orders n and m, one will have $i = n \cdot m\ (n + m - 2)$, so the order and class of the focal surface of the intersection congruence of two complexes of degree n and m will be equal to $2m \cdot n(n + m - 2)$. Its equations will be representable when this tact invariant is known.

If $m = 1$ then one will have to take the condition for the contact of a line with a curve of order n, and thus, the equation of the curve in line coordinates. One obtains this when one writes the expression $(\alpha\beta\xi)$ for $(\alpha\beta)$ in the discriminant of the binary forms of order n. If one then introduces $[x\gamma c\ \mathfrak{g}\ x]$ for $[\alpha\beta]$ in this discriminant then one will obtain the equation of the focal surface of the intersection of the linear complexes $\mathfrak{g}$ and a complex of degree n.

Example: If $n = 2$ here then one will have:

$$[x\gamma c\ \mathfrak{g}\ x]^2 = 0$$

for the equation of the *Kummer* focal surface. For $n = 3$, one will have:

$$[x\gamma c\ \mathfrak{g}\ x]^2 \cdot [x\gamma^1 c^1\ \mathfrak{g}\ x]^2 \cdot [x\gamma c^1\ \mathfrak{g}\ x] \cdot [x\gamma^1\ c\ \mathfrak{g}\ x] = 0$$

as the equation of the focal surface.

If $\mathfrak{g}$ is a singular complex then one will obtain the equation of the complex surface for the given complex of degree n.

c. For the invariants whose vanishing asserts that three curves of order m, n, r, resp., intersect at a point, one will have $i = m \cdot n \cdot r$; therefore, the order and class of the ruled surface of the common rays of the complexes of order m, n, r, resp. will be equal to $2\ m \cdot n \cdot r$.

If $r = 1$ then one must substitute $[x\ \gamma c\ \mathfrak{g}\ x]$ for $[\alpha\beta]$ in the two binary forms of order m and n in the resultant in order to arrive at the equation of the ruled surface.

For $m = 2$, $n = r = 1$, one will have, for the two complexes $\mathfrak{g}$, $\mathfrak{g}^1$:

$$[x\ \gamma\mathfrak{g}\ \mathfrak{g}^1\ x]^2 = 0.$$

For $m = n = 2$, $r = 1$, it follows that:

$$[x\gamma c\ \mathfrak{g}\ x]^2\ [x\gamma^1 c^1 \mathfrak{g}\ x]^2 - ([x\gamma c^1 \mathfrak{g}\ x]^2)^2 = 0,$$

etc.

d. A question arises here: Which ternary invariants have the property that the substitutions $[x\gamma^1 c^2\gamma^3\ x]$ and $[\xi c^1\gamma^3\ c^3\xi]$ that were performed above yield the same surfaces?

This is known to be case for the discriminant of a plane curve, since the singularity surface of a complex is, at the same time, the locus of the singular points and planes, and for the tact invariant of two curves, since the locus of the focal points and focal planes of the rays of a congruence are identical; for the condition that three curves must intersect at a point, the three complexes correspond to a common ruled surface.

e. However, one can also perform the latter substitution in covariants and other invariants (cf., *Gundelfinger*, Math. Ann., Bd. 6); one then obtains, e.g., a spatial intermediate form from any covariant that assigns any plane with a surface that cuts out the covariant of the complex curve from it. Certain invariant properties of the complex curves can be represented by the identical vanishing of a covariant; if the property is equivalent to three conditions then one will arrive, by conversion, at a system of surfaces whose surfaces will contact planes of the desired property.

Example: Should a curve $(a\ \xi)^2 = 0$ of class 2 degenerate into a doubly-counted point, then $(a\ b\ x)^2$ would have to vanish identically. One would obtain, by substitution, $[\xi\ c\ \gamma x]^2 = 0$ as the relation that covariantly assigns any plane to a surface that cuts out the complex conic section from it. If one sets $x_i = p_i \cdot p\xi$ then one will obtain the equation of its line coordinates. Should this equation vanish for every p_{ik} for a certain ξ then one would obtain: The double planes of the complex (i.e., the double tangential planes of the singularity surface) will contact all complex surfaces.

Prague, in November 1889.