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The equations of motion of the electron and the principle of least action

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As is known, **Hölder** and **Voss** ([1]) have presented the principle of least action in the general form:

$$\int_{t_0}^{t_1} [\delta L \cdot dt + 2L \cdot d\delta t + \delta U' \cdot dt] = 0 ,$$

in which L represents the present energy and $\delta U'$ is the elementary external force, and time is also varied. That equation is true for *all purely-mechanical* processes, but breaks down as soon as the principle is applied to reversible, *not purely-mechanical*, processes, e.g., to thermodynamics, electrodynamics, etc. It would then be advisable to write the principle in the form:

$$\int_{t_0}^{t_1} [\delta H \cdot dt + (H + E) \cdot d\delta t + \delta U \cdot dt] = 0 , \tag{I}$$

in which H is now the *kinetic potential*, which is a function of the general coordinates p_i and $\dot{p}_i$, $\delta U = \sum P_i \, \delta p_i$ represents the elementary external work, and the quantity E, which later proves to be the *energy*, is defined by the relation:

$$E = \sum \dot{p}_i \frac{\partial H}{\partial \dot{p}_i} - H . \tag{II}$$

Namely, if one imagines that in our case the variation $\delta \dot{\psi}$ of a velocity $\dot{\psi}$ obeys the rule:

([1]) Göttinger Nachrichten, 1896 and 1900.

$$\delta\dot{\psi} = \frac{d\,\delta\psi}{dt} - \dot{\psi}\,\frac{d\,\delta t}{dt},$$

then one will find that:

$$\delta H \cdot dt = dt \cdot \left[\sum \frac{\partial H}{\partial p_i}\delta p_i + \sum \frac{\partial H}{\partial \dot{p}_i}\delta \dot{p}_i \right]$$
$$= dt \cdot \sum \frac{\partial H}{\partial p_i}\delta p_i + dt \cdot \sum \frac{\partial H}{\partial \dot{p}_i}\frac{d\,\delta p_i}{dt} - dt \cdot \sum \dot{p}_i \frac{\partial H}{\partial \dot{p}_i}\frac{d\,\delta t}{dt} ,$$

or when one recalls (II):

$$\delta H \cdot dt = dt \cdot \sum \frac{\partial H}{\partial p_i}\delta p_i + \frac{d}{dt}\left[\sum \frac{\partial H}{\partial \dot{p}_i}\delta p_i \right] dt - \sum \delta p_i \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{p}_i} \right) - [E + H]\frac{d\,\delta t}{dt} dt$$

and

$$\int_{t_0}^{t_1} [\delta H \cdot dt + (H + E) \cdot d\delta t + \delta U \cdot dt] = 0 = \int_{t_0}^{t_1} dt \sum \delta p_i \left[\frac{\partial H}{\partial p_i} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{p}_i} \right) + P_i \right] = 0 ,$$

since the term $\sum \frac{\partial H}{\partial \dot{p}_i}\delta p_i$ vanishes at the limits. That will then yield the generalized Lagrange equations in the form:

$$\frac{\partial H}{\partial p_i} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{p}_i} \right) + P = 0 .$$

If one multiplies those equations by $\dot{p}_1$, …, $\dot{p}_i$, … in succession, adds them, and integrates over time then (as is known) one will find the equation for the conservation of energy:

$$d\,E = \sum P_i\, dp_i ,$$

such that the E that is defined by (II) will actually represent the energy. One sees clearly from (II), as **Helmholtz** had already emphasized, that E is defined completely by H, but H cannot be determined uniquely from E. Namely, if F represents a *linear* function of $\dot{p}_i$, such that one has:

$$0 = \sum \dot{p}_i \frac{\partial F}{\partial \dot{p}_i} - F ,$$

then one will also have:

$$E = \sum \dot{p}_i \frac{\partial (H + F)}{\partial \dot{p}_i} - (H + F) ,$$

such that H can then be increased by the arbitrary F, and *the same* E will nonetheless emerge from that.

Conversely, if the kinetic potential H is known for a reversible process then E can be calculated uniquely. It is only for purely-mechanical processes that one has $H = L - \Phi$, $E = L + \Phi$, in which Φ represents the potential energy.

Knowing the kinetic potential H as a function of p_i, $\dot{p}_i$, and the external work $\delta U = \sum P_i \, \delta p_i = \delta E$ makes it possible to apply the principle of least action even in those cases of reversible processes that are foreign to mechanics. If the variation of time is excluded, i.e., $d\,\delta t = 0$, then one will get back to the case that **Helmholtz** treated. Examples of that kind were found by **M. Planck** in the "Acht Vorlesungen über theoretische Physik," pp. 100, *et seq.*

Here, all that shall be shown is how the equations of motion for an electron can be derived from the known kinetic potential H. One has:

$$H = -\,\mu c^2 \sqrt{1 - \frac{q^2}{c^2}} + \text{const.},$$

where $q^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, μ is the mass for $q = 0$, and c is the speed of light. One then finds the kinetic energy:

$$E = \dot{x}\frac{\partial H}{\partial \dot{x}} + \dot{y}\frac{\partial H}{\partial \dot{y}} + \dot{z}\,\frac{\partial H}{\partial \dot{z}} - H = q\,\frac{\partial H}{\partial q} - H = \frac{\mu c^2}{\sqrt{1 - \frac{q^2}{c^2}}} + \text{const.},$$

and easily obtains from the rules above that:

$$\delta H \cdot dt + (H + E)\, d\,\delta t = \frac{\mu c^2}{\sqrt{1 - \frac{q^2}{c^2}}}\left[\dot{x}\frac{d\,\delta x}{dt} + \cdots\right] = \frac{d}{dt}\left[\frac{\mu \dot{x}\,\delta x}{\sqrt{1 - \frac{q^2}{c^2}}}\right] - \delta x \frac{d}{dt}\left[\frac{\mu \dot{x}}{\sqrt{1 - \frac{q^2}{c^2}}}\right] + \cdots$$

If one sets $\delta U = \mathfrak{F}_x\,\delta x + \mathfrak{F}_y\,\delta y + \mathfrak{F}_z\,\delta z$ and integrates over t in the sense of (I) then that will give:

$$\frac{d}{dt}\left(\frac{\mu \dot{x}}{\sqrt{1 - \frac{q^2}{c^2}}}\right) = \mathfrak{F}_x, \ldots$$

In so doing, as is known:

$$\mathfrak{F}_x = e\mathfrak{E}_x + \frac{e}{c}[\dot{y}\,\mathfrak{H}_z - \dot{z}\,\mathfrak{H}_y]\ , \ldots,$$

in which $\mathfrak{E}$ is the electric field strength, $\mathfrak{H}$ is the magnetic field strength, and e represents the charge of the electron ([1]).

([1]) Cf., **Einstein**, Jahrb. der Radioaktivität (1907), pp. 433. – **Max Planck**, *loc. cit.*, pp. 123.