"Ueber die Anwendung des Princips des kleinsten Zwanges auf die Elektrodynamik," Sitz. Kön. Bay. Akad. Wiss. **24** (1894), 219-230.

On the application of the principle of least constraint to electrodynamics

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(Received 5 May)

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A point m in a system of particles might move from a to b in time τ, if it were free, while its *actual* motion is represented by the line segment ac. It is known that the principle of least constraint that **Gauss** expressed will then say that $\sum m \cdot (bc)^2$ must be a minimum, or that one must always have $\sum m \cdot (bc)^2 < \sum m \cdot (bd)^2$, when ad is a virtual motion (i.e., one that is consistent with the conditions on the system). If x, y, z are the coordinates of the point m on which the forces $m\,X$, $m\,Y$, $m\,Z$ should act then any coordinate x will go to:

$$x + \frac{dx}{dt}\tau + \frac{1}{2}\frac{d^2x}{dt^2}\tau^2$$

in a very small time τ under the actual motion, and:

$$x + \frac{dx}{dt}\tau + \frac{1}{2}X\,\tau^2$$

for the free motion, such that the square of the deviation bc^2 or the square of the coordinate differences will be equal to $\frac{\tau^4}{4}\left[\left(\frac{d^2x}{dt^2} - X\right)^2 + \cdots\right]$. One then has an expression Z that shall be called the *constraint* (from the German *Zwang*) and that takes the form:

$$Z = \sum m\left[\left(\frac{d^2x}{dt^2} - X\right)^2 + \left(\frac{d^2y}{dt^2} - Y\right)^2 + \left(\frac{d^2z}{dt^2} - Z\right)^2\right],$$

in which the summation extends over all particles, and this is the function that must be minimized in regard to the various accelerations $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, …, which shall be written briefly as $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, … If one differentiates the condition equations for the system $\varphi_1 = 0$, $\varphi_2 = 0$, … twice with respect to time then, as would emerge from the derivation above [1], one must currently regard the coordinates x and their first differential quotients as *given*. The equation $\frac{d^2\varphi}{dt^2} = 0$ expresses only that the $\frac{\partial\varphi}{\partial t}\ddot{x} + \dots$ must possess *unvarying* values. For given values of x and dx / dt, the $\ddot{x}$ shall then be determined in such a way that Z will be a minimum. One will then get the known equations:

$$m(\ddot{x} - X) + \lambda_1 \frac{\partial\varphi_1}{\partial x} + \lambda_2 \frac{\partial\varphi_2}{\partial x} + \dots = 0, \dots$$

Now, n mutually-independent variables $p_1, p_2, p_3, \dots, p_n$ will be introduced in place of the coordinates $x, y, z, \dots$ such that the virtual work will be equal to $P_1\,\delta p_1 + P_2\,\delta p_2 + \dots + P_n\,\delta p_n$ and the *vis viva* will be equal to $T = \frac{1}{2}\sum_{\kappa,\lambda} a_{\kappa\lambda}\,\dot{p}_\kappa\,\dot{p}_\lambda$, in which the P_μ and the $a_{\kappa\lambda} = a_{\lambda\kappa}$ depend upon only the coordinates, and the Greek symbols go from 1 to n, as they always will from now on.

Then, as **Lipschitz** showed (*loc. cit.*, pp. 330), the constraint Z will be expressed by:

$$Z = \sum_{\mu,\nu} \frac{A_{\mu\nu}}{\Delta} \left\{ a_{1\mu}\,\ddot{p}_1 + a_{2\mu}\,\ddot{p}_2 + \cdots + \begin{bmatrix} 11 \\ \mu \end{bmatrix} \dot{p}_1\,\dot{p}_1 + \begin{bmatrix} 12 \\ \mu \end{bmatrix} \dot{p}_1\,\dot{p}_2 + \cdots - P_\mu \right\}$$

$$\times \left\{ a_{1\nu}\,\ddot{p}_1 + a_{2\nu}\,\ddot{p}_2 + \cdots + \begin{bmatrix} 11 \\ \nu \end{bmatrix} \dot{p}_1\,\dot{p}_1 + \begin{bmatrix} 12 \\ \nu \end{bmatrix} \dot{p}_1\,\dot{p}_2 + \cdots - P_\nu \right\},$$

in which Δ represents the determinant of the $a_{\kappa\lambda}$ and $A_{\mu\nu}$ represents the adjoint:

$$\left(A_{\mu\nu} = \frac{\partial\Delta}{\partial a_{\mu\nu}} \right),$$

and one sets:

$$\begin{bmatrix} \kappa\lambda \\ \mu \end{bmatrix} = \frac{1}{2}\left[\frac{\partial a_{\kappa\mu}}{\partial p_\lambda} + \frac{\partial a_{\lambda\mu}}{\partial p_\kappa} - \frac{\partial a_{\kappa\lambda}}{\partial p_\mu} \right].$$

The expression for Z will become clearer and more suited to (some) physical problems when the *vis viva* T is introduced. Namely, if one sets:

$$T_\mu = \frac{d}{dt}\frac{\partial T}{\partial \dot{p}_\mu} - \frac{\partial T}{\partial p_\mu}$$

then [2] one will also have:

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[1] **Lipschitz**, Borch. Journ., Bd. 82**, pp. 316** (**Rausenberger**, *Mechanik I*, pp. 166). **Gibbs**, Supplement IV, pp. 319.

[2] Cf., e.g., **Rayleigh**, *Sound*, pp. 111 (in the German translation). **Stäckel**, Borch. Jour. **107**, pp. 322.

$$T_\mu = a_{1\mu}\,\ddot{p}_1 + a_{2\mu}\,\ddot{p}_2 + \cdots + \begin{bmatrix} 11 \\ \mu \end{bmatrix} \dot{p}_1\,\dot{p}_1 + \begin{bmatrix} 1\,2 \\ \mu \end{bmatrix} \dot{p}_1\,\dot{p}_2 + \cdots,$$

and one will have:

$$Z = \sum_{\mu,\nu} \frac{A_{\mu\nu}}{\Delta} [T_\mu - P_\mu]\,[T_\nu - P_\nu], \tag{I}$$

i.e.:

$$Z = \frac{1}{\Delta}\left\{ \begin{array}{l} A_{11}(T_1 - P_1)^2 + A_{22}(T_2 - P_2)^2 + \cdots \\ \qquad + 2A_{12}(T_1 - P_1)(T_2 - P_2) + \cdots \\ \qquad\qquad + 2A_{23}(T_2 - P_2)(T_3 - P_3) + \cdots \end{array} \right\}.$$

Since this expression for the constraint Z would seem to be new, it would not be irrelevant to show that one comes to the Lagrange equations when one addresses the minimum condition for Z. Therefore, as a result of the remark above, one must regard the quantities p_1 and $\dot{p}_1$ as *given* or *fixed* in the differentiation of Z with respect to $\ddot{p}_1$, and make use of the relation:

$$\frac{\partial T}{\partial \ddot{p}_\kappa} = a_{\mu\kappa} = a_{\kappa\mu}\,.$$

One then gets:

$$\frac{\partial Z}{\partial \ddot{p}_1} = \sum_{\mu,\nu} \frac{a_{1\mu}\,A_{\mu\nu}}{\Delta}(T_\nu - P_\nu) + \sum_{\mu,\nu} \frac{a_{1\nu}\,A_{\mu\nu}}{\Delta}(T_\mu - P_\mu)\,,$$

or, since those sums are equal to each other:

$$\frac{\partial Z}{\partial \ddot{p}_1} = \frac{2}{\Delta} \sum_{\mu,\nu} a_{1\mu}\,A_{\mu\nu}(T_\nu - P_\nu)\,,$$

or

$$\frac{2}{\Delta}\sum_\nu (T_\nu - P_\nu)\sum_\mu a_{1\mu}\,A_{\mu\nu} = \frac{2}{\Delta}\sum_\nu (T_\nu - P_\nu)[a_{11}\,A_{1\nu} + a_{12}\,A_{2\nu} + \cdots]\,.$$

Now, from a property of the determinant that $a_{11}\,A_{1\nu} + a_{12}\,A_{2\nu} + \ldots = \Delta$ or zero according to whether $\nu = 1$ or $\nu > 1$, so it would follow from $\frac{\partial Z}{\partial \ddot{p}_1} = 0$ that one must also have $T_1 - P_1 = 0$; i.e., Lagrange's equation:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_1} - \frac{\partial T}{\partial p_1} = P_1\,.$$

One can find even more expressions for the constraint Z in a way that is entirely similar to the way that one exhibits *auxiliary forms* ([1]) for Lagrange's fundamental equation.

It is important for the applications to note that the constraint Z can be represented in such a way that *one* acceleration – e.g., $\ddot{p}_1$ – will appear in it detached from the remaining ones. One has: $Z \cdot \Delta = \frac{1}{2}(L_1\,\ddot{p}_1^2 + 2M_1\,\ddot{p}_1 + N_1)$, in which L_1, M_1, N_1 *do not* include $\ddot{p}_1$, and L_1 and M_1 can be easily found from the equation $\frac{\partial Z}{\partial \ddot{p}_1} = 0$.

If the virtual work $\sum_\mu P_\mu\,\delta p_\mu$ *and the vis viva* $T = \frac{1}{2}\sum_{\kappa,\lambda} a_{\kappa\lambda}\,\dot{p}_\kappa\,\dot{p}_\lambda$ *are given for a physical problem then the constraint on the system can be determined by means of equation* (I). *The minimum property of Z expresses a law for the system – which is certainly new in many cases – and entirely distinct from the fact that other, perhaps already known, laws would follow from Z by actual differentiation.*

That is how, e.g., **Boltzmann** has derived Maxwell's equations for electricity in a wonderfully simple way from Lagrange's equations in volume I of his lectures, and thus supported it by mechanical processes. It illuminated the fact that one could also start from the principle of least constraint in the form of equation (I) above as the main theorem, and once the virtual work and *vis viva* were given, one would have to arrive at Lagrange's equations by appealing to the minimum condition, along with Maxwell's equations – in which one now proceeds entirely as **Boltzmann** did. If it might also become very hard to simplify **Boltzmann**'s classical methods (especially in Part 2 of his lectures) any more in that way, nonetheless, the condition Z = minimum will, after all, express a *newly-recognized* truth.

As an example, one might consider the case of two cyclic coordinates $\dot{p}_1 = l_1'$ and $\dot{p}_2 = l_2'$ (the slowly-varying parameters k will be ignored temporarily). Here, one has:

$$T = \tfrac{1}{2}a_{11}\,\dot{p}_1^2 + \tfrac{1}{2}a_{22}\,\dot{p}_2^2 + \tfrac{1}{2}a_{12}\,\dot{p}_1\,\dot{p}_2 = \frac{A}{2}l_1'^2 + \frac{B}{2}l_2'^2 + C\,l_1'l_2',$$

when one introduces Boltzmann's notation. It will then follow that:

$$a_{11} = A, \qquad a_{22} = B, \qquad a_{12} = C,$$

$$\Delta = \begin{vmatrix} A & C \\ C & B \end{vmatrix} = AB - C^2,$$

$$A_{11} = B, \qquad A_{12} = -C, \qquad A_{22} = A,$$

$$T_1 = \frac{d}{dt}(A\,l_1' + B\,l_2'), \quad T_2 = \frac{d}{dt}(B\,l_2' + C\,l_1').$$

([1]) **Weinstein**, Wied. Ann. **15**. **Budde**, *Mechanik I*, pp. 397.

In addition, when there is friction or viscosity, the dissipation function (*loc. cit.*, pp. 108 and 109) that Rayleigh exhibited might be denoted by:

$$F = \tfrac{1}{2}\sum(\kappa_1\,\dot{x}_1^2 + \cdots) = \tfrac{1}{2}b_{11}\,\dot{p}_1^2 + \tfrac{1}{2}b_{22}\,\dot{p}_2^2 + \tfrac{1}{2}b_{12}\,\dot{p}_1\,\dot{p}_2\,.$$

As is known, one then adds $-\dfrac{\partial F}{\partial \dot{p}_\mu}$ to the forces $P_\mu = L_\mu$. It is also clear that $b_{12} = 0$, on intrinsic grounds.

Ultimately, the constraint Z will then be expressed by:

$$Z\cdot(A\,B - C^2) = B\left[\frac{d}{dt}(A\,l_1' + C\,l_2') - L_1 - b_{11}\,l_1'\right]^2$$

$$-2C\left[\frac{d}{dt}(A\,l_1' + C\,l_2') - L_1 - b_{11}\,l_1'\right]\left[\frac{d}{dt}(A\,l_2' + C\,l_1') - L_2 - b_{11}\,l_2'\right]$$

$$+A\left[\frac{d}{dt}(A\,l_2' + C\,l_1') - L_2 - b_{22}\,l_2'\right]^2,$$

and Z must be a minimum, in such a way that one will have $\dfrac{\partial Z}{\partial l_1'} = 0$ and $\dfrac{\partial Z}{\partial l_2'} = 0$. In this (Boltzmann I, pps. 34 and 35), l_1' and l_2' represent the current strengths in the two conductors, b_{11} and b_{22} are their resistances, L_1 and L_2 are the electromotive forces in them, A and B are the coefficients of self-induction and, and C is the mutual induction. If condensers (*loc. cit.*, I, pp. 35) are also included then terms of the form $d_1\,l_1$ and $d_2\,l_2$ will also enter into the brackets.

The condition Z = minimum then expresses a *basic electrodynamical law* and implies the theory of self-induction and mutual induction for current fluctuations that are not too fast. If one would also like to include ponderomotive forces then one would have to introduce a slowly-varying parameter k along with l_1 and l_2 as a third variable, exhibit the general expression for Z and construct the equation $\dfrac{\partial Z}{\partial \ddot{k}} = 0$, in which k and $\dot{k}$ are assumed to be constant. Only *afterwards* does one take $\dot{k} = 0$ and $\ddot{k} = 0$ and obtain, as Boltzmann did, the relation:

$$K = -\frac{\partial T}{\partial k} = -\frac{l_1'^2}{2}\frac{\partial A}{\partial k} - \frac{l_2'^2}{2}\frac{\partial B}{\partial k} - l_1'\,l_2'\,\frac{\partial C}{\partial k}\,.$$

Acoustics also serves as another field of applications for the principle above. Frequently, only purely-quadratic terms with constant coefficients appear in the expression for the *vis viva* in that context, which makes the equation for the constraint take an even simpler form;

If one introduces the abbreviation: $T_\mu - P_\mu = Q_\mu$ then the constraint Z will be given by:

$$Z \cdot D = \sum_{\mu,\nu} A_{\mu\nu}(T_\mu - P_\mu)(T_\nu - P_\nu) = \sum_{\mu,\nu} Q_\mu\, Q_\nu$$

$$\begin{aligned} = \; & A_{11}\, Q_1^2 + 2\, A_{12}\, Q_1\, Q_2 + \ldots + 2\, A_{1n}\, Q_1\, Q_n \\ & + A_{22}\, Q_2^2 + \ldots \qquad + 2\, A_{2n}\, Q_2\, Q_n \\ & \qquad\qquad\qquad + A_{nn}\, Q_n^2 . \end{aligned}$$

The condition $\dfrac{\partial Z}{\partial \ddot{p}_\rho} = 0$ can be replaced with $\dfrac{\partial Z}{\partial Q_\rho} = 0$. Namely, one has:

$$\frac{\partial Z}{\partial \ddot{p}_\rho} = \frac{\partial Z}{\partial Q_1}\frac{\partial Q_1}{\partial \ddot{p}_\rho} + \cdots + \frac{\partial Z}{\partial Q_n}\frac{\partial Q_n}{\partial \ddot{p}_\rho},$$

or, since:

$$\frac{\partial Q_\nu}{\partial \ddot{p}_\rho} = \frac{\partial T_\nu}{\partial \ddot{p}_\rho} = a_{\nu\rho},$$

one will have:

$$\frac{\partial Z}{\partial \ddot{p}_\rho} = a_{1\rho}\frac{\partial Z}{\partial Q_1} + a_{2\rho}\frac{\partial Z}{\partial Q_2} + \cdots + a_{n\rho}\frac{\partial Z}{\partial Q_n} \qquad (\rho = 1, \ldots, n).$$

Since the determinant $D = |\, a_{\mu\nu} \,|$ does not vanish, it will generally follow from these n equations that:

$$\frac{\partial Z}{\partial Q_\nu} = 0.$$

If one actually differentiates the expression above then that will yield the n equations:

$$\frac{1}{2}\frac{\partial Z}{\partial Q_\nu} = A_{1\nu}\, Q_1 + A_{2\nu}\, Q_2 + \ldots + A_{\nu n}\, Q_n = 0 \qquad (\nu = 1, \ldots, n),$$

and since the determinant $|\, A_{\mu\nu} \,| = D^{n-1}$ can never be zero, that will, in turn, imply the Lagrange equations: $Q_1 = 0, \ldots, Q_n = 0$.

If the force P has a potential U, such that $P_\mu = -\,\partial U / \partial p_\mu$, and the conditions do not include time t explicitly then one will have, on the one hand, the *vis viva* in the form:

$$2T = \frac{\partial T}{\partial \dot{p}_1}\dot{p}_1 + \frac{\partial T}{\partial \dot{p}_2}\dot{p}_2 + \ldots$$

or

$$2\frac{dT}{dt} = \dot{p}_1 \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_1}\right) + \ddot{p}_1 \frac{\partial T}{\partial \dot{p}_1} + \ldots,$$

whereas, on the other hand, since T is also a function of p_1, $\dot{p}_1$, …, one will have:

$$\frac{dT}{dt} = \dot{p}_1 \frac{\partial T}{\partial p_1} + \ddot{p}_1 \frac{\partial T}{\partial \dot{p}_1} + \ldots$$

Since:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_\mu}\right) - \frac{\partial T}{\partial p_\mu} = T_\mu,$$

it will follow upon subtraction that:

$$\frac{dT}{dt} = \dot{p}_1 T_1 + \dot{p}_2 T_2 + \ldots,$$

to which, one adds:

$$\frac{dU}{dt} = \frac{\partial U}{\partial p_1}\dot{p}_1 + \ldots = -\dot{p}_1 P_1 - \dot{p}_2 P_2 + \ldots,$$

and since $T_\mu - P_\mu = Q_\mu$, one will ultimately arrive at the equation:

$$\frac{d(T+U)}{dt} = \dot{p}_1 Q_1 + \dot{p}_2 Q_2 + \cdots + \dot{p}_n Q_n = R.$$

One sees that for $Q_1 = 0, \ldots, Q_n = 0$, one will also have $R = 0$; i.e., $T + U$ must be equal to a constant, or that the principle of the conservation of energy must also hold, from the Lagrange equations that were found above. Both of them can be obtained *simultaneously* when one eliminates one of the quantities Q – e.g., Q_1 – with the help of the relation $R = \dot{p}_1 Q_1 + \ldots$ and presents the minimum conditions: $\frac{\partial Z}{\partial Q_1} = 0$, $\frac{\partial Z}{\partial Q_n} = 0$ afterwards. It is preferable to use the determinant form for the constraint Z in that. For example, for $n = 3$, one has:

$$-Z \cdot D = \begin{vmatrix} 0 & Q_1 & Q_2 & Q_3 \\ Q_1 & a_{11} & a_{12} & a_{13} \\ Q_2 & a_{21} & a_{22} & a_{23} \\ Q_3 & a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{\dot{p}_1^2} \begin{vmatrix} 0 & R & Q_2 & Q_3 \\ R & b_{11} & b_{12} & b_{13} \\ Q_2 & b_{21} & a_{22} & a_{23} \\ Q_3 & b_{31} & a_{32} & a_{33} \end{vmatrix},$$

which will make:

$$b_{11} = (a_{11}\,\dot{p}_1 + a_{21}\,\dot{p}_2 + a_{31}\,\dot{p}_3) + \ldots = \frac{\partial T}{\partial \dot{p}_1}\dot{p}_1 + \ldots = 2T,$$

$$b_{21} = b_{12} = a_{21}\,\dot{p}_1 + a_{22}\,\dot{p}_2 + a_{23}\,\dot{p}_3 = \frac{\partial T}{\partial \dot{p}_2},$$

$$b_{31} = b_{13} = a_{31}\,\dot{p}_1 + a_{32}\,\dot{p}_2 + a_{33}\,\dot{p}_3 = \frac{\partial T}{\partial \dot{p}_3}.$$

Applying the minimum conditions will yield:

$$A_{11}\,R + Q_2\,[\dot{p}_1\,A_{12} - \dot{p}_2\,A_{11}] + Q_3\,[\dot{p}_1\,A_{13} - \dot{p}_3\,A_{11}] = 0,$$

$$A_{12}\,R + Q_2\,[\dot{p}_1\,A_{22} - \dot{p}_2\,A_{12}] + Q_3\,[\dot{p}_1\,A_{23} - \dot{p}_3\,A_{12}] = 0,$$

$$A_{13}\,R + Q_2\,[\dot{p}_1\,A_{23} - \dot{p}_2\,A_{13}] + Q_3\,[\dot{p}_1\,A_{33} - \dot{p}_3\,A_{13}] = 0,$$

or, since the determinant of this system, namely, $\dot{p}_1^2 \left| A_{\mu\nu} \right| = \dot{p}_1^2 \cdot D$ never vanishes, $R = 0$ and $Q_2 = 0$, $Q_3 = 0$; i.e., one has the law of energy and Lagrange's equations.

If one eliminates, say Q_1, from the general equation:

$$A_{1\nu}\,Q_1 + A_{2\nu}\,Q_2 + \ldots + A_{n\,\nu}\,Q_n = 0$$

by using the relation:

$$R = \dot{p}_1\,Q_1 + \ldots$$

then it will likewise follow naturally that:

$$R = 0, \quad Q_2 = 0, \ \ldots\ Q_n = 0.$$

Addendum

Concerning linear current branches.

If $p_1, \ldots, p_k$ are, in turn, cyclic coordinates, $T = \frac{1}{2} a_{11}\,\dot{p}_1^2 + \frac{1}{2} a_{22}\,\dot{p}_2^2 + \ldots + \frac{1}{2} a_{12}\,\dot{p}_1\,\dot{p}_2 + \ldots$ is the *vis viva*, and $F = \frac{1}{2} b_{11}\,\dot{p}_1^2 + \frac{1}{2} b_{22}\,\dot{p}_2^2 + \ldots + \frac{1}{2} b_{12}\,\dot{p}_1\,\dot{p}_2 + \ldots$ is the dissipation function that **Lord Rayleigh** introduced then the force $-\frac{\partial F}{\partial \dot{p}_\mu} = -(b_{1\mu}\,\dot{p}_1 + b_{2\mu}\,\dot{p}_2 + \ldots)$ will be added to any force P_μ, and that will yield the principle of least constraint, namely, that:

$$Z \cdot D = \sum A_{\mu\nu}\,Q_\mu\,Q_\nu = A_{11}\,Q_1^2 + A_{22}\,Q_2^2 + 2A_{12}\,Q_1\,Q_2 + \ldots \tag{1}$$

must be a minimum for any Q. Therefore, one has:

$$Q_\mu = \frac{d}{dt}\left[a_{1\mu}\,\dot{p}_1 + a_{2\mu}\,\dot{p}_2 + \cdots\right] - P_\mu + \left[b_{1\mu}\,\dot{p}_1 + b_{2\mu}\,\dot{p}_2 + \cdots\right], \tag{2}$$

$$D = |\,a_{\kappa\lambda}\,|, \qquad A_{\kappa\lambda} = \frac{\partial D}{\partial a_{\kappa\lambda}}.$$

If one goes over to electrodynamics and sets: $\dot{p}_1 = J_1$, $\dot{p}_2 = J_2$, …, as well as:

$$Q_\mu = \frac{d}{dt}\left[a_{1\mu}\,J_1 + a_{2\mu}\,J_2 + \cdots\right] - P_\mu + \left[b_{1\mu}\,J_1 + b_{2\mu}\,J_2 + \cdots\right], \tag{3}$$

then the *minimum property* (1) *will imply a property of a linear current branch*. In it, a_{11}, a_{22}, a_{33}, … are the coefficients of the self-inductions of the first, second loops, a_{12}, a_{13}, a_{23}, … are the coefficients of mutual induction, and P_μ is the constant electromotive force. Furthermore, b_{11} is the resistance of the entire first loop, b_{22} is that of the entire second loop, etc, and b_{12} as is the resistance of that piece of the conductor that is common to loops 1 and 2. b_{12} is positive when J_1 and J_2 have the same directions and negative when they have opposite directions. Applying the minimum condition will imply that $Q_1 = 0$, i.e.:

$$P_1 = b_{11}\,J_1 + b_{12}\,J_2 + \ldots + b_{1n}\,J_n + \frac{d}{dt}[a_{11}\,J_1 + a_{12}\,J_2 + \ldots], \quad \ldots \tag{4}$$

Those are (in a somewhat generalized form) the equations that **H. von Helmholtz** presented in 1851 (*Abhandlungen* I, pp. 435) for the induction in linear current branches, which one must think of as being decomposed into the smallest-possible number of simple loops.

For the work done by the retarding forces, one gets:

$$\frac{\partial F}{\partial \dot{p}_1}dp_1 + \ldots = \left[\frac{\partial F}{\partial \dot{p}_1}d\dot{p}_1 + \cdots\right]dt = 2F\,dt = [b_{11}\,J_1^2 + b_{12}\,J_2^2 + \cdots + 2b_{12}\,J_1\,J_2 + \cdots],$$

up to sign; i.e., the Joule heat, and all of the summands in the bracket are positive.

Now assume that $a_{12} = a_{23} = \ldots = 0$; $a_{11} = a_{22} = a_{23} = \ldots = a$, which is a case is not too difficult to realize experimentally. One will then have:

$$Q_\mu = a\frac{d}{dt}[J_1 + J_2 + \ldots] - P_\mu + b_{1\mu}\,J_1 + b_{2\mu}\,J_2 + \ldots,$$

and one must have that $Z \cdot a = Q_1^2 + Q_2^2 + \ldots$ is a minimum when a is taken to be small enough. For lim $a = 0$, the current strengths will be independent of time, thus constant, and it will follow from the condition $Z^1 = Q_1^2 + Q_2^2 + \ldots =$ minimum. $Q_\mu = b_{1\mu}\,J_1 + \ldots - P_\mu = 0$.

Since the determinant of the b does not vanish, the equation $\frac{\partial Z^1}{\partial Q_\mu} = 0$ can be replaced with $\frac{\partial Z^1}{\partial J_\mu} = 0$. For constant currents, one then has to minimize:

$$Z^1 = \sum_\mu [(b_{1\mu} J_1 + \cdots) - P_\mu]^2$$

for every J.

Remark: One can get an oft-mentioned minimum property of constant currents from the well-known equation:

$$\frac{d(T+U)}{dt} = -2F \qquad \text{or} \qquad -\left[\frac{dU}{dt} + F + \frac{dT}{dt}\right] = F, \tag{1}$$

in which $U = P_1\, p_1 + \ldots$ represents the potential of the constant force, and as above:

$$T = \tfrac{1}{2} a_{11}\, \dot{p}_1^2 + \ldots = \tfrac{1}{2} a_{11}\, J_1^2 + \ldots, \qquad F = \tfrac{1}{2} b_{11}\, \dot{p}_1^2 + \ldots = \tfrac{1}{2} b_{11}\, J_1^2 + \ldots$$

If the current strengths J_1, J_2, which are initially zero, attain their full strengths J_1', J_2', … for $t = \infty$ (since $\frac{dT}{dt} = \frac{\partial T}{\partial J_1}\frac{dJ_1}{dt} + \ldots$), and also $\frac{dT}{dt} = 0$, then the F that is found on the right-hand side of (I), which consists of nothing but positive terms, will attain its *greatest* value. Therefore, the negative left-hand side of (I); i.e.:

$$F + \frac{dU}{dt} = \tfrac{1}{2} b_{11}\, J_1^2 + \ldots - (P_1\, J_1 + P_2\, J_2 + \ldots),$$

must represent a minimum for any J.
