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Stress functions of the three-dimensional continuum

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Abstract: The conditions of equilibrium in a three-dimensional medium are satisfied by stresses that are obtained from the second derivatives of certain *stress functions* that are investigated in this report. The functions always form a tensor, and for a given state of stress, it is possible to find various tensors of functions whose differences will have the structure of a distortion tensor.

1. In what follows, I will treat the stress functions of a three-dimensional continuum – e.g., an elastic, plastic, or viscous body. As a convenient representation, I will employ a notation that does not deviate from the usual. The coordinates of a rectangular coordinate system will be x_1, x_2, and x_3 . All stresses will be denoted by τ with two indices. The first index will give the surface on which the stress acts, while the second index will give the direction of the stress. τ_{11} and τ_{12} will then denote the stresses on the surface x_1 = const.; thus, τ_{11} will point in the x_1-direction and τ_{12} will point in the x_2-direction. As a result, normal stresses will have two equal indices, while tangential (or shear) stresses will have two different indices.

In the absence of body forces and body moments, the following equilibrium conditions will be true:

$$\tau_{12} = \tau_{21}, \ldots, \tag{1}$$

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} = 0., \ldots \tag{2}$$

The ellipsis after any equation will suggest that two more equations will follow by cyclic permutation of the indices. In general, the equations will read:

$$\tau_{gh} = \tau_{hg}, \ldots, \tag{3}$$

$$\sum_{h=1,2,3} \frac{\partial \tau_{gh}}{\partial x_h} = 0, \ldots \tag{4}$$

The stresses can be expressed in different ways by derivatives of stress functions in such a way that the equilibrium conditions are fulfilled. Thus, it will be entirely irrelevant whether further conditions exist between the stresses (e.g., as a result of *Hooke’s* law).

There exists the following solution to the system of equations (1) and (2) that comes about with the help of the three functions F_{11}, F_{22}, and F_{33} :

$$\tau_{11} = \frac{\partial^2 F_{33}}{\partial x_2^2} + \frac{\partial^2 F_{22}}{\partial x_3^2}, \ldots, \tag{5}$$

$$\tau_{12} = -\frac{\partial^2 F_{33}}{\partial x_1 \, \partial x_2}, \ldots \tag{6}$$

I will refer to the functions F_{11}, F_{22}, and F_{33} as *stress functions of the first kind.*

If the stresses are fixed (e.g., by the loading of an elastic body) then there will be essentially one system of stress functions of the first kind for that that stress state. It will be found by integrating equations (6). The integration functions that appear in it will follow from equations (5).

There is also the following general solution to the system of equations (1) and (2) that comes about with the help of three other functions F_{12}, F_{23}, and F_{31} :

$$\tau_{11} = -\frac{2\,\partial^2 F_{23}}{\partial x_2 \, \partial x_3}, \ldots, \tag{7}$$

$$\tau_{12} = \frac{\partial^2 F_{23}}{\partial x_3 \, \partial x_1} + \frac{\partial^2 F_{31}}{\partial x_2 \, \partial x_3} - \frac{\partial^2 F_{12}}{\partial x_3^2}, \ldots \tag{8}$$

I will refer to the functions F_{12}, F_{23}, and F_{31} as *stress functions of the second kind.*

These functions are also essentially determined for a fixed stress state. They will be found by integrating equations (7). The integration functions will follow from equations (8).

A *general system of stress functions* can be defined from the stress functions of the first and second kind, in the form of the following symmetric matrix:

$$\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \qquad F_{gh} = F_{hg} .$$

The stresses τ_{gh} can be calculated from the formula:

$$\tau_{gh} = \frac{\partial^2 F_{g+1,h+1}}{\partial x_{g+2} \, \partial x_{h+2}} + \frac{\partial^2 F_{g+2,h+2}}{\partial x_{g+1} \, \partial x_{h+1}} - \frac{\partial^2 F_{g+1,h+2}}{\partial x_{g+2} \, \partial x_{h+1}} - \frac{\partial^2 F_{g+2,h+1}}{\partial x_{g+1} \, \partial x_{h+2}}, \ldots \tag{9}$$

for $g = h$, as well as for $g \neq h$.

If an index becomes greater than 3 then one must subtract three units: e.g., for $g = 2$, one will have $g + 2 = 4 \equiv 1$. If one takes only the terms in the main diagonal (the off-diagonal terms, resp.) then one will get equations (5) and (6) [(7) and (8), resp.].

The general system can be defined in different ways for a given stress state. We first choose arbitrary functions F_{11}, F_{22}, and F_{33} , and then find the stresses from them by using equation (5) and (6). We regard the given stress state as the sum of these stress states thus-found and a second stress state; we then find the functions F_{12}, F_{23}, and F_{31} for it. Consequently, the manifold of general systems of functions will correspond to the manifold of all possible stress states.

2. Are there any more stress functions? The stresses shall be defined by differentiating with respect to the various variables twice. If one introduces the stresses that are expressed in that way into the equilibrium conditions then the functions that are differentiated with respect to the same variables must drop out. We take the stress τ_{gh} : The functions that produce it will be differentiated with respect to x_i and x_k ; the index i and the index k will assume all values 1, 2, and 3 in this. Naturally, it is irrelevant whether the function in question is first differentiated with respect to x_i and then with respect to x_k, or conversely. A function that will suffice for the definition of τ_{gh} will be given the upper indices (gi, ih); if it is differentiated with respect to x_i and x_k then it will further take on the indices i and k. We thus write:

$$\tau_{gh} = \sum_{i,k=1,2,3} \frac{\partial^2 F_{ik}^{(gh)}}{\partial x_i\, \partial x_k}, \quad \ldots \tag{10}$$

Since i and k commute, as well as g and h, it will then follow that:

$$F_{ik}^{(gh)} = F_{ik}^{(hg)} = F_{ki}^{(gh)} = F_{ki}^{(hg)}, \ldots \tag{11}$$

We must, moreover, investigate what relationship will exist between the individual functions on the grounds of the equilibrium conditions.

We introduce the Ansatz of equation (10) into the general equilibrium equation (4) and get:

$$\sum_{h,i,k=1,2,3} \frac{\partial}{\partial x_h} \frac{\partial^2 F_{ik}^{(gh)}}{\partial x_i\, \partial x_k} = 0, \ldots \tag{12}$$

In order for the functions $F_{ik}^{(gh)}$ to be stress functions, the third differential quotients with respect to the same variables must drop out.

We now evaluate the condition that the indices g, h, i, k can assume only the values 1 to 3. It will follow that at least two of the indices must be equal to each other. However, three or four indices can also be equal.

Here, we must examine different cases: We first take the functions $F_{hh}^{(gh)}$ and $F_{gh}^{(hh)}$; they appear in the expressions for the stresses τ_{gh} and τ_{hh} . Equation (9), with $\frac{\partial \tau_{gh}}{\partial x_h}$, contains the term $\frac{\partial}{\partial x_h} \frac{\partial^2 F_{hh}^{(gh)}}{\partial x_h^2}$; other terms with $\frac{\partial^3}{\partial x_h^3}$ are not present. Consequently, one will have $\frac{\partial^3 F_{hh}^{(gh)}}{\partial x_h^3} = 0$.

We then take equation (9) with $\frac{\partial \tau_{hg}}{\partial x_g} + \frac{\partial \tau_{hh}}{\partial x_h}$. It will contain the terms $\frac{\partial}{\partial x_g}\frac{\partial^2 F_{hh}^{(gh)}}{\partial x_h^2}$ and $\frac{\partial}{\partial x_h}\frac{\partial^2 F_{gh}^{(gh)}}{\partial x_g\,\partial x_h}$; other terms with $\frac{\partial^3}{\partial x_g\,\partial x_h^2}$ will not appear. As a result:

$$\frac{\partial^3}{\partial x_g\,\partial x_h^2}(F_{hh}^{(gh)} + F_{gh}^{(hh)}) = 0.$$

In order to satisfy both equations, we set $F_{hh}^{(gh)} = 0$ and $F_{gh}^{(hh)} = 0$. Naturally, our equations express only the fact that certain third differential quotients are zero. One can add irrelevant terms to the functions that we have set equal to zero that will drop out (persist, resp.) under differentiation. They will then have no influence upon the solution and will, in turn, be dropped. These irrelevant extra terms will also be tacitly omitted in the sequel.

We now take the functions $F_{ii}^{(gg)}$ and $F_{gi}^{(gi)}$. We obtain from equation (12) that:

$$\frac{\partial}{\partial x_g}\frac{\partial^2 F_{ii}^{(gg)}}{\partial x_i^2} + \frac{\partial}{\partial x_i}\frac{\partial^2 F_{gi}^{(gi)}}{\partial x_g\,\partial x_i} = 0.$$

Further terms with $\frac{\partial^3}{\partial x_g\,\partial x_i^2}$ will not appear.

It will follow from this that:

$$\frac{\partial^3}{\partial x_g\,\partial x_i^2}(F_{ii}^{(gg)} + F_{gi}^{(gi)}) = 0.$$

We can then set $F_{ii}^{(gg)} = -F_{gi}^{(gi)} = F_{gg}^{(ii)}$.

As the last case, we assume that only two indices are equal to each other. We thus take the functions $F_{ik}^{(gg)}$, $F_{gk}^{(gi)}$, $F_{gi}^{(gk)}$, and $F_{gg}^{(ii)}$.

Equation (12) will then give the relations:

$$\frac{\partial}{\partial x_g}\frac{\partial^2 F_{ik}^{(gg)}}{\partial x_i\,\partial x_k} + \frac{\partial}{\partial x_i}\frac{\partial^2 F_{gk}^{(gi)}}{\partial x_g\,\partial x_k} + \frac{\partial}{\partial x_k}\frac{\partial^2 F_{gi}^{(gk)}}{\partial x_g\,\partial x_i} = 0$$

or

$$\frac{\partial}{\partial x_g}\frac{\partial^2 F_{gk}^{(ig)}}{\partial x_g\,\partial x_k} + \frac{\partial}{\partial x_k}\frac{\partial^2 F_{gg}^{(ik)}}{\partial x_g^2} = 0,$$

resp.

On the basis of the second equation, we can set $F_{gk}^{(ig)} + F_{gg}^{(ik)} = 0$.

It will follow immediately from this and a further exchange of the indices *i* and *k* that:

$$F_{gg}^{(ik)} = -F_{gk}^{(gi)} = -F_{gi}^{(gk)} .$$

On the basis of the first equation, we find that:

$$F_{ik}^{(gg)} = 2\ F_{gg}^{(ik)} .$$

The number of functions is greatly reduced with that. The remaining distinct functions will now be denoted as follows:

$$F_{kk} = F_{ii}^{(gg)} = F_{gg}^{(ii)} = -F_{gi}^{(gi)} , \quad F_{ik} = F_{gg}^{(ik)} = -\tfrac{1}{2} F_{ik}^{(gg)} = F_{gk}^{(gi)} .$$

What will then remain will be six distinct functions that correspond to the functions in our matrix. The equations that follow from this will agree with equation (9). We see that there will be no other solutions.

3. What happens to the stress functions under a rotation of the coordinate system? We would like to rotate the coordinate system around only the x_3-axis through an angle of α and denote the new axes with a prime. We will get:

$$\left.\begin{aligned} \bar{x}_1 &= x_1 \cos\alpha + x_2 \sin\alpha, \\ \bar{x}_2 &= -x_1 \sin\alpha + x_2 \cos\alpha, \\ \bar{x}_3 &= x_3. \end{aligned}\right\} \tag{13}$$

We also denote the stress for the new coordinate system with a prime. For the stress $\bar{\tau}_{11}$, we get:

$$\bar{\tau}_{11} = t_{11} \cos^2 \alpha + t_{22} \sin^2 \alpha + 2\ t_{12} \sin \alpha \cos \alpha . \tag{14}$$

We replace t_{11}, t_{22}, and t_{12} with the expressions in equation (9):

$$\left.\begin{aligned} \bar{\tau}_{11} = &\left(\frac{\partial^2 F_{22}}{\partial x_3^2} + \frac{\partial^2 F_{33}}{\partial x_2^2} - 2\frac{\partial^2 F_{22}}{\partial x_2\,\partial x_3} \right)\cos^2\alpha + \left(\frac{\partial^2 F_{33}}{\partial x_1^2} + \frac{\partial^2 F_{11}}{\partial x_3^2} - 2\frac{\partial^2 F_{31}}{\partial x_3\,\partial x_1} \right)\sin^2\alpha \\ &+ \left(\frac{\partial^2 F_{23}}{\partial x_3\,\partial x_1} + \frac{\partial^2 F_{31}}{\partial x_2\,\partial x_3} - \frac{\partial^2 F_{12}}{\partial x_3^2} - \frac{\partial^2 F_{33}}{\partial x_1\,\partial x_2} \right)\sin\alpha\cos\alpha. \end{aligned}\right\} \tag{15}$$

We now express the functions F_{gh} in terms of $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ using equation (13) and replace the differentiations with differentiations with respect to $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$. After combining various terms with the same F_{gh}, we will get:

$$\left.\begin{aligned}\bar{\tau}_{11} = \frac{\partial^2 F_{33}}{\partial \bar{x}_2^2} + \frac{\partial^2}{\partial \bar{x}_3^2}(F_{22}\cos^2\alpha + F_{11}\sin^2\alpha - 2F_{12}\sin\alpha\cos\alpha) \\ -2\frac{\partial^2}{\partial \bar{x}_2\,\partial \bar{x}_3}(F_{22}\cos\alpha - F_{12}\sin\alpha).\end{aligned}\right\} \tag{16}$$

On the other hand, we can write:

$$\bar{\tau}_{11} = \frac{\partial^2 \bar{F}_{33}}{\partial \bar{x}_2^2} + \frac{\partial^2 \bar{F}_{22}}{\partial \bar{x}_3^2} - 2\frac{\partial^2 \bar{F}_{23}}{\partial \bar{x}_2\,\partial \bar{x}_3} \tag{17}$$

for $\bar{\tau}_{11}$.

By comparing eqs. (16) and (17), we will get the transformation equations:

$$\left.\begin{aligned}\bar{F}_{22} &= F_{22}\cos^2\alpha + F_{11}\sin^2\alpha - 2F_{12}\sin\alpha\cos\alpha, \\ \bar{F}_{23} &= F_{23}\cos\alpha - F_{12}\sin\alpha, \\ \bar{F}_{33} &= F_{33}\end{aligned}\right\} \tag{18}$$

for the stress functions under a rotation of the coordinate system around the x_3-axis.

One obtains $\bar{F}_{11}$ and $\bar{F}_{13}$ from the stress $\bar{\tau}_{22}$ in the same way, and the remaining function $\bar{F}_{12}$ from the stress $\bar{\tau}_{12}$.

We see from the equations that we obtained that the functions transform like the components of a tensor.

The stress functions thus define a tensor.

4. The last question that we shall pose is: In what way do two systems of stress functions differ when they determine the same stress state? We saw that there are infinitely many systems of stress functions for a given stress state. If we take the difference between two such stress functions that we will get functions that would like to denote by D_{11}, D_{12}, etc. These functions correspond to stresses that *are equal to zero.*

If we substitute the functions D_{gh} in place of the functions $F_{gh}h$ in equation (9) then we will get the six equations:

$$\frac{\partial^2 D_{22}}{\partial x_3^2} + \frac{\partial^2 D_{33}}{\partial x_2^2} - 2\frac{\partial^2 D_{23}}{\partial x_2\,\partial x_3} = 0, \tag{19}$$

$$\frac{\partial^2 D_{23}}{\partial x_3\,\partial x_1} + \frac{\partial^2 D_{31}}{\partial x_2\,\partial x_3} - \frac{\partial^2 D_{12}}{\partial x_3^2} - \frac{\partial^2 D_{33}}{\partial x_1\,\partial x_2} = 0. \tag{20}$$

In order to be able to make further statements about the functions D_{gh}, we first take the lower equation in (20). We integrate it over x_3 :

$$\frac{\partial D_{23}}{\partial x_1}+\frac{\partial D_{31}}{\partial x_2}-\frac{\partial D_{12}}{\partial x_3}-\frac{\partial^2}{\partial x_1\,\partial x_3}\int D_{33}\,dx_3 \;=\; \frac{\partial^2 f_3(x_1,x_2)}{\partial x_1\,\partial x_2}. \tag{21}$$

We choose an arbitrary undetermined integral $\int D_{33}\,dx_3$ in this.

An arbitrary function of x_1 and x_3 must be on the right-hand side that we will write in the given way.

If we consider equation (21) and two others that arise by cyclic permutation of the indices then we will see that once-differentiated functions D_{gh} ($g \neq h$) will be found in all of them. If we add two of these equations then we will get:

$$2\frac{\partial}{\partial x_1}D_{23}-\frac{\partial^2}{\partial x_1\,\partial x_2}\int D_{33}\,dx_3-\frac{\partial^2}{\partial x_3\,\partial x_1}\int D_{22}\,dx_2 = \frac{\partial^2 f_3(x_1,x_2)}{\partial x_1\,\partial x_2}+\frac{\partial^2 f_2(x_3,x_1)}{\partial x_3\,\partial x_1}. \tag{22}$$

We integrate the equation that was written down over x_1 :

$$2D_{23}-\frac{\partial}{\partial x_2}\int D_{33}\,dx_3-\frac{\partial}{\partial x_3}\int D_{22}\,dx_2 = \frac{\partial f_3(x_1,x_2)}{\partial x_2}+\frac{\partial f_2(x_3,x_1)}{\partial x_3}+g_1(x_2,x_3). \tag{23}$$

If we differentiate (23) with respect to x_2 and x_3 then (after a sign change) we will get:

$$-2\,\frac{\partial^2 D_{23}}{\partial x_2\,\partial x_3}+\frac{\partial^2 D_{22}}{\partial x_3^2}+\frac{\partial^2 D_{33}}{\partial x_2^2} = -\,\frac{\partial^2 g_1(x_2,x_3)}{\partial x_2\,\partial x_3}. \tag{24}$$

These equations almost coincide with equation (19). In order to obtain complete coincidence, we must set:

$$g_1(x_2, x_3) = \alpha_1 + 2\,b_1\,x_2 + 2\,c_1\,x_3\,.$$

Thus, we will get:

$$2D_{23}-\frac{\partial}{\partial x_2}\int D_{33}\,dx_3-\frac{\partial}{\partial x_3}\int D_{22}\,dx_2 = \frac{\partial f_3(x_1,x_2)}{\partial x_2}+\frac{\partial f_2(x_3,x_1)}{\partial x_3} + \alpha_1 + 2\,b_1\,x_2 + 2\,c_1\,x_3 \tag{25}$$

from eq. (23).

We now set:

$$\left.\begin{aligned}
&\int D_{33}dx_3+f_{33}(x_1,x_2)+a_1x_2+b_1x_2^2+c_2x_1^2=W,\\
&\int D_{11}dx_1+f_{11}(x_2,x_3)+a_2x_3+b_2x_3^2+c_3x_2^2=U,\\
&\int D_{22}dx_2+f_{22}(x_3,x_1)+a_3x_1+b_3x_1^2+c_1x_3^2=V.
\end{aligned}\right\} \tag{26}$$

We will then have:

$$\left.\begin{aligned}
D_{11} &= \frac{\partial U}{\partial x_1}, \qquad D_{12} = \frac{1}{2}\left(\frac{\partial U}{\partial x_2} + \frac{\partial V}{\partial x_1}\right), \\
D_{22} &= \frac{\partial V}{\partial x_2}, \qquad D_{23} = \frac{1}{2}\left(\frac{\partial V}{\partial x_3} + \frac{\partial W}{\partial x_2}\right), \\
D_{33} &= \frac{\partial W}{\partial x_3}, \qquad D_{31} = \frac{1}{2}\left(\frac{\partial W}{\partial x_1} + \frac{\partial U}{\partial x_3}\right).
\end{aligned}\right\} \tag{27}$$

The stress functions D_{gh} that give zero stresses will depend upon three basic functions U, V, W. The law of definition is the same as the one for defining the distortion tensor (ε_x, $\frac{1}{2}\gamma_{xy}$) from the three displacement components (u, v, w).

The system of equations (27) is a *necessary* condition for the functions D_{gh}; at the same time, is also a *sufficient* condition. If we then substitute the expressions that are defined by equation (27) into equations (19) and (20) then we will establish that is it satisfied.

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