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On the system of forms of linear complexes in R_3

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The system of forms of linear complexes in three-dimensional space was presented by F. Mertens in the treatise: "Invariant Gebilde von Nullsystemen," Sitzungsberichte der Kaiserl. Akademie der Wissensch., Vienna, May, 1889, and here we shall also make use of a symbolic notation, at least, partially.

The same topic was treated in a paper by E. Waelsch: "Zur Invariantentheorie der Liniengeometrie," Sitzungsberichte der Kaiserl. Akademie der Wissensch., Vienna, December, 1889, but the results that were given in that paper were not exact. In regard to their rectification, cf., the remark that E. Waelsch submitted to the *Enzyklopädie der mathematischen Wissenschaften,* Jahresbericht der Deutschen Mathem.-Vereinigung, V. 19, Part 1, Section 2.

In my book, *Komplexe-Symbolik* (Sammlung Schubert, v. LVII, 1908), page 28, *et seq.*, I made a mistake that was similar to that of E. Waelsch, and among other things, stated that all invariants of a system of linear complexes are given by the quantities A_{ik} . Professor Study brought that mistake to my attention. When I recently addressed the questions in regard to it, that yielded an essential extension of the results that were obtained by F. Mertens as far as the representation of the invariants of such a system of linear complexes is concerned.

§ 1.

An arbitrary number of linear complexes are given by the symbolic equations:

$$(1) \qquad K_1 = (\pi\, a')^2 = 0, \qquad K_2 = (\pi\, b')^2 = 0, \;\ldots,\; K_m = (\pi\, m')^2 = 0.$$

In this, we mean, e.g.:

$$K_1 = (\pi\, a')^2 = (\pi'\, a)^2 = 2\sum_{ik} a_{ik}\pi'_{ik} = 2\sum_{ik} a'_{ik}\pi_{ik} = 2\sum a_{ik}\pi_{mn} = 2\sum a'_{mn}\pi'_{ik} ,$$

and we always set:

$$a_{ik} = a'_{mn} \qquad \text{and likewise} \qquad \pi_{ik} = \pi'_{mn} ,$$

such that the following pairs will belong together:

(12) and (34), (13) and (42), (14) and (23).

A point x is represented by a sequence of quantities x – hence, by the four ratios $x_1 : x_2 : x_3 : x_4$ – and a plane u' is represented by a sequence of quantities u', viz., $(u'_1 : u'_2, u'_3, u'_4)$. The coordinates of a varying line will be denoted by π_{ik} or ρ_{ik} in what follows.

§ 2.

One must deal with two types of symbolic factors in the complex-symbolic representation: viz., sums and determinants. The former has the form:

$$(u'\,x) = (x\,u') = u'_1 x_1 + u'_2 x_2 + u'_3 x_3 + u'_4 x_4 ,$$

which always couples a primed symbol with an unprimed one.

By contrast, only the same kind of symbol is present in a symbolic determinant – i.e., either all of them are primed or all of them are unprimed. One then has, e.g.:

$$(abcd) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \qquad \text{and} \qquad (u'\,v'\,w'\,s') = \begin{vmatrix} u'_1 & u'_2 & u'_3 & u'_4 \\ v'_1 & v'_2 & v'_3 & v'_4 \\ w'_1 & w'_2 & w'_3 & w'_4 \\ s'_1 & s'_2 & s'_3 & s'_4 \end{vmatrix} .$$

In the sequel, we will have different kinds of forms to consider that all have the property of invariance, and which we shall summarize with the concept of "invariant" when we are dealing with general considerations or when there can be no doubt as to the meaning of the expression. For a closer examination of the individual forms in regard to the types of variables that they include, we shall also use the following, otherwise-conventional names.

If an invariant expression contains only the coefficients of the given linear complex then it will be called an *invariant*, and if it contains only point coordinates x in addition to these coefficients then it will be called a *covariant*. If the coordinates of a plane u are present in place of the x then we will have a *contravariant*. If only varying ray coordinates enter along with the coefficients of the complex then we will be dealing with a *ray form*. Finally, if a form contains different sequences of varying coordinates then we shall speak of a *mixed form*. There are once more two different kinds of them to distinguish then, since three different kinds of coordinates can appear in three-dimensional space.

§ 3.

In order to exhibit the complete system of forms of the given complexes, we must then define only sums and determinant factors of the following symbols and quantities:

$$(2) \qquad \begin{array}{c} a, \; b, \; c, \ldots, \; \pi, \; \rho, \ldots, x \\ a', b', c', \ldots, \pi', \rho', \ldots, u' \end{array}$$

In this, the symbols a, b, c, …, and likewise the symbols a', b', c', …, are complex symbols and are deduced from the coefficients of the given linear complexes. The symbols π, ρ, …, and likewise π', ρ', …, will be employed in the representation of the variable coordinates of lines. They are also complex symbols and can also be employed for the representation of coordinates of variable linear complexes.

The rules by which one calculates with the complex symbols now give us a simple means for overlooking the type of a determinant factor. If a complex symbol appears in such a determinant factor then one can always represent the expression as a sum of factors. For instance, if:

$$F = (abcd)\,(au') \cdot M,$$

in which b, c, and d have any sort of meaning, and M no longer includes the symbol a, then we will have ([1]):

$$F = [(a'\,c)\,(a'\,d)\,(b\,u') - (a'\,b)\,(a'\,d)\,(c\,u') + (a'\,b)\,(a'\,c)\,(d\,u')] \cdot M.$$

We can then ignore the determinant factors completely in this investigation and accordingly examine only aggregates of sums of factors.

§ 4.

We first treat the *invariants.* From the foregoing considerations, only sums of factors of type (ab') will appear in them. Now, the coefficients of the individual complexes can also enter to a higher degree than the first. We must then introduce a new symbol for the symbolic way of writing each such sequence of coefficients. We shall arrange this in such a way that we provide the individual symbols with indices and thus write, e.g.:

$$K_1 = (\pi a_1')^2 = (\pi a_2')^2 = (\pi a_3')^2 = \ldots = 0$$

for the first of these complexes.

In order to exhibit the general type of an invariant, we proceed as follows: We select an arbitrary factor (ab') and put in the first position, such that we then write:

$$J = (ab'), \ldots$$

Of the further factors, we take the ones that contain the symbol b' twice, which can happen for only one factor, since only two such symbols b' occur to begin with: let that factor be $(b'c)$. We connect it with the factor that was written already and get:

([1]) In regard to such calculations, I refer to the introduction to my aforementioned book. A more detailed presentation of these methods here would lead us too far afield.

$$J = (ab') (b'c), \ldots$$

We now further write down the factor that contains the symbol *c* twice – let that factor be (cd') – so we will have:

$$J = (ab') (b'c) (cd'), \ldots$$

We continue in that way. It is clear that we must then come once more to a sum of factors that include the second symbol *a*. The invariant *J* is also already written down completely then; i.e., all of its symbolic factors are accounted for. The trivial case in which *J* decomposes into several factors is included in this, and is thus inherently representable as a product of simple factors.

We then now have:

$$(3) \qquad J = (ab') (b'c) (cd') (d'e) (ef') \ldots (nm') (m'a) = [ab'cd' \ldots nm'].$$

We call such an expression a "chain." From the way that it was constructed, we can immediately conclude that *such an invariant can appear only for an even number of complexes.*

Since a linear complex seems to be determined by six numerical ratios, *in what follows, we can always restrict ourselves to the investigation of only six such complexes*, so every further linear complex can be represented in terms of these six in a linear way.

§ 5.

As the shortest chains, we come to the ones that consist of a sum of two factors. We will then have, for example:

$$(4) \qquad A_{12} = (ab') (b'a) = (ab')^2 = 2\sum a_{ik} b'_{ik} = 2\sum a_{ik} b_{mn}$$

as the simplest invariant of the two complexes K_1 and K_2 . The invariants of a single complex itself also belong to these shortest chains, so, e.g.:

$$(5) \qquad A_{11} = (a\,a'_1) (a'_1 a) = (a_1 a'_1)^2 = 2\sum a_{ik} a'_{ik} = 2\sum a_{ik} a_{mn} .$$

If A_{ii} vanishes then the complex K_i will be special.

We now introduce a shorter notation for such chains by setting [cf., equation (3)]:

$$(6) \qquad J = (ab') (b'c) (cd') (d'e) (ef') (f'a) = [ab'cd'ef'],$$

in which we have written down a chain in which only six complexes occur here.

By applying the identity ([1]):

([1]) These important identities appear in different forms. Cf., e.g., E. Müller, "Beiträge zur Graßmannschen Ausdehnungslehre I," remark on equation (22).

$$(pq')\,(pu')\,(q'x) = -\,(qp')\,(qu')\,(p'x) + \tfrac{1}{2}\,(pq')^2\,(u'x)$$

or the one that is dual to it:

$$(pq')\,(pu')\,(q'x) = -\,(qp')\,(qu')\,(p'x) + \tfrac{1}{2}\,(pq')^2\,(u'x)$$

to an aggregate of three factors in J, we can now convert J in the various ways. We then have, e.g.:

$$J = (b'\,a)\,(b'\,c)\,(a\,f')\,(c\,d')\,(d'e)\,(e\,f'),$$

and if we apply the second of the identities that were given above to the first three factors then we will get:

$$J = (a'\,b)\,(a'\,c)\,(b\,f')\,(c\,d')\,(d'e)\,(e\,f') - \tfrac{1}{2}\,(a'\,b)^2\,(c\,f')\,(c\,d')\,(d'e)\,(e\,f),$$

so:

(7) $$J = [ab'cd'ef'] = -\,[ba'cd'ef'] - \tfrac{1}{2}\,A_{12}\,[cd'ef']\,.$$

In an analogous way, one gets the equation:

(8) $$J = [ab'cd'ef'] = -\,[ac'bd'ef'] - \tfrac{1}{2}\,A_{23}\,[ad'ef']\,.$$

These two equations (7) *and* (8) *show us that one can convert a chain in such a way that two successive symbols seem to be switched in it.* Yet another term will arise from such a conversion, in which, however, an actual factor A_{ik} has been split off. The other factor of that term will then be a chain that is shorter by two than the original chain. *We conclude from this fact that it is possible to bring about an arbitrary arrangement of the symbols in a chain.*

However, that possibility will likewise imply that *it suffices to consider only those chains in which the coefficients of each individual complex occur linearly.* If, e.g., the coefficients a_{ik} of the complex K_1 do not occur linearly in a chain J, so that chain J will contain symbols $a_1, a_2, \ldots$, etc., then we can convert that chain J in such a way that these equivalent (and therefore permutable) symbols a stand next to each other in such a way that we can write:

$$J = [a_1\,a_2'\,a_3, \ldots, b'\,cd'ef'].$$

However, if we apply such an expression to one of the equations (7) or (8) then we will obtain, e.g., by applying (7) to a_1 and a_2':

$$J = [a_1\,a_2'\,a_3, \ldots, b'\,cd'ef'] = -\,[a_2\,a_1'\,a_3, \ldots, b'\,cd'ef'] - \tfrac{1}{2}A_{11}\,[a_3, \ldots, b'\,cd'ef'].$$

They were given for the first time in the form above by E. Waelsch in the paper (1889) that was mentioned in the introduction.

Here, we can then switch a_1 with a_2 in the first term on the right, and then combine that term with the one on the left, which will give the equation:

(9) $$J = [a_1\, a_2'\, a_3, \ldots, b'\, cd'\, ef'] = -\tfrac{1}{4}[a_3, \ldots, b'\, cd'\, ef'] \cdot A_{11}\,.$$

One then splits off the actual factor A_{11} from such an invariant.

§ 6.

We therefore now consider only those invariants J that contain the coefficients of the individual complexes K_i linearly. Since, as we already remarked above, an even number of complexes must enter into J, we will then have to examine the chains J that contain two, four, or six complexes. The invariants thus-obtained, along with the six invariants A_{ii} that were already pointed out, will then give all possible invariants of the six complexes in question that there are.

For instance, for only two complexes K_1 and K_2, we will have the chain:

$$A_{12} = (ab')\,(b'a) = (ab')^2,$$

and we will then obtain fifteen such invariants A_{ik}, in all, for six complexes K_i (i = 1, 2, 3, 4, 5, 6).

For four complexes, we have the chain:

(10) $$J = [ab'cd'] = (ab')\,(b'c)\,(cd')\,(d'a).$$

When this chain is converted according to (7) and (8), it will give:

$$\begin{aligned}[ab'cd'] &= -[ba'cd'] - \tfrac{1}{2}A_{12}\,A_{34}\,,\\ &= +[bc'ad'] - \tfrac{1}{2}A_{13}\,A_{24} - \tfrac{1}{2}A_{12}\,A_{34}\,,\\ &= -[cb'ad'] - \tfrac{1}{2}A_{23}\,A_{14} + \tfrac{1}{2}A_{13}\,A_{24} - \tfrac{1}{2}A_{12}\,A_{34}\,,\end{aligned}$$

and since one has the equation:

$$[ab'cd'] = [cb'ad']\,,$$

as one easily confirms by rearranging the symbolic factors, one will have ([1]):

(11) $$[ab'cd'] = -\tfrac{1}{4}A_{12}\,A_{34} + \tfrac{1}{4}A_{13}\,A_{24} - \tfrac{1}{4}A_{14}\,A_{23}\,.$$

This equation shows that no other invariants than the A_{ii} and A_{ik} exist for four complexes, as well.

([1]) Confer the conclusion of § 3 in the treatise of F. Mertens that was mentioned in the introduction.

§ 7.

We now consider any chain that contains all six complexes to be the latter case. We then have:

$$R = [ab'cd''ef'''] = (ab')\,(b'c)\,(cd'')\,(d''e)\,(ef''')\,(f'''a). \tag{12}$$

A conversion of this invariant R that would lead to only the A_{ik} is no longer possible. The individual conversions that one can make out of R along then go back to the fact that one can perform an arbitrary reordering of the symbols a, b, c, d, e, and f in it. There are thus 6! such invariant R in all, and all of these different R are expressible in terms of one such chain – say, $[ab'cd''ef''']$ – whereby the A_{ik} will then be carried along in those equations.

We will then have, e.g.:

$$R = [ab'cd''ef'''] = -\,[ba'cd''ef'''] - \tfrac{1}{2}[cd''ef'''] ,$$

and $[cd''ef''']$ is once more representable in terms of the other A_{ik} by using (11).

We now have the result that all symbolically-represented invariants of the system of six complexes can be expressed in terms of A_{ii}, A_{ik}, and R, and in fact, rationally.

§ 8.

The invariants then seem to have been dealt with by the last theorem. Here, we have a difference between our complex-symbolic representation and the results that F. Mertens arrived at in the aforementioned paper. Namely, the following invariant:

$$D = \begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{34} & a_{42} & a_{23} \\ b_{12} & b_{13} & b_{14} & b_{34} & b_{42} & b_{23} \\ c_{12} & c_{13} & c_{14} & c_{34} & c_{42} & c_{23} \\ d_{12} & d_{13} & d_{14} & d_{34} & d_{42} & d_{23} \\ e_{12} & e_{13} & e_{14} & e_{34} & e_{42} & e_{23} \\ f_{12} & f_{13} & f_{14} & f_{34} & f_{42} & f_{23} \end{vmatrix} = (abcdef) \tag{13}$$

This six-rowed determinant is not rationally-representable in terms of the A_{ii} and A_{ik}, but probably irrationally. Namely, if we introduce the primed a'_{ik}, b'_{ik}, … into D then we will also have:

$$D = (a'\,b'\,c'\,d''e'\,f''),$$

or, when written out:

$$D = \begin{vmatrix} a'_{34} & a'_{12} & a'_{23} & a'_{12} & a'_{12} & a'_{14} \\ b'_{34} & \cdots & \cdots & \cdots & \cdots & b'_{14} \\ c'_{34} & & & & & \vdots \\ d'_{34} & & & & & \vdots \\ e'_{34} & & & & & \vdots \\ f'_{34} & f'_{42} & f'_{23} & f'_{12} & f'_{13} & f'_{14} \end{vmatrix} = - \begin{vmatrix} a'_{12} & a'_{13} & a'_{14} & a'_{34} & a'_{42} & a'_{23} \\ b'_{12} & \cdots & \cdots & \cdots & \cdots & b'_{23} \\ c'_{12} & & & & & \vdots \\ d'_{12} & & & & & \vdots \\ e'_{12} & & & & & \vdots \\ f'_{12} & f'_{13} & f'_{14} & f'_{34} & f'_{42} & f'_{23} \end{vmatrix},$$

as one easily confirms by rearranging the columns.

The product:

$$(abcdef)\,(a'\,b'\,c'\,d'\,e'\,f')$$

now gives:

$$D^2 = -\frac{1}{2^6}H,$$

in which we have set:

$$(14) \qquad H = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & \cdot & \cdot & \cdot & \cdot & A_{36} \\ A_{41} & \cdot & \cdot & \cdot & \cdot & A_{46} \\ A_{51} & \cdot & \cdot & \cdot & \cdot & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{vmatrix} = |\,A_{ik}\,|.$$

We then have the equation:

$$(15) \qquad 2^6 \cdot D^2 + H = 0;$$

i.e., D is probably expressible in terms of the A_{ii} and A_{ik}, but not rationally; *D will be a root of the above quadratic equation, whose coefficients are entire and rational functions of the A_{ik}.*

§ 9.

This suggests the question of whether the invariant R that was defined in § 7 can also be represented in that way as a root of a quadratic equation. That is in fact the case.

To that end, we derive an identity by whose help we can effect that representation.

Let:

$$(16) \qquad L = (pa')\,(pb')\,(qc')\,(qd')$$

be a symbolic expression in which p and q shall be equivalent and commuting complex symbols, and a', b', c', and d' have any meaning at all. We will then also have:

$$2L = (p'^2 a' b')\,(c'\,q)\,(d'\,q),$$

and we will then convert $(p'^2 a' b')\,(c'\,q)$ here. We will have:

$$2L = -(c'p'a'b')(p'q)(d'q) + (c'p'^2 b')(a'q)(d'q) - (c'p'^2 a')(b'q)(d'q)$$
$$= -(p'q)(p'a'b'c')(qd') - (pa')(pd')(qb')(qc') + 2(pa')(pc')(qb')(qd').$$

If we apply the identity that was discussed in § 5 to the first term here then since p commutes with q we will have:

(17) $$L = (pa')(pc')(qb')(qd') = +\tfrac{1}{4}(pq')(a'b'c'd') - (pa')(pd')(qb')(qc') + (pa')(pc')(qb')(qd').$$

From equation (12), we now have:

$$R = (a_1 b_1')(b_1' c_1)(c_1 d_1')(d_1' e_1)(e_1 f_1')(f_1' a_1) = (a_2 b_2')(b_2' c_2)(c_2 d_2')(d_2' e_2)(e_2 f_2')(f_2' a_2),$$

and we can then set:

(18) $$R^2 = (b_1' a_1)(b_1' c_1)(b_2' a_2)(b_2' c_2)(c_1 d_1')(d_1' e_1)(e_1 f_1')(f_1' a_1)(c_2 d_2')(d_2' e_2)(e_2 f_2')(f_2' a_2).$$

We now apply the identity that is dual to (17) to the first four terms of that expression and obtain:

$$R^2 = \tfrac{1}{4} A_{22}(a_1' c_1' a_2' c_2')(c_1 d_1')(d_1' e_1)(e_1 f_1')(f_1' a_1)(c_2 d_2')(d_2' e_2)(e_2 f_2')(f_2' a_2)$$
$$- (b_1' a_1)(b_1' c_2)(b_2' c_1)(b_2' a_2)(c_1 d_1')(d_1' e_1)(e_1 f_1')(f_1' a_1)(c_2 d_2')(d_2' e_2)(e_2 f_2')(f_2' a_2)$$
$$+ (b_1' a_1)(b_1' a_2)(b_2' c_1)(b_2' c_2)(c_1 d_1')(d_1' e_1)(e_1 f_1')(f_1' a_1)(c_2 d_2')(d_2' e_2)(e_2 f_2')(f_2' a_2),$$

or, when written more briefly, if we use the notation that was employed for chains for the last two expressions:

(19) $$R^2 = \tfrac{1}{4} A_{22} G - [a_1 b_1' c_2 d_2' e_2 f_2' a_2 b_2' c_1 d_1' e_1 f_1'] - [a_1 b_1' a_2 f_2' e_2 d_2' c_2 b_2' c_1 d_1' e_1 f_1'].$$

This chain can now be reduced by the process that was described above to shorter chains in which terms with the factors A_{ik} will then arise. The expression G can likewise be expressed in terms of the A_{ik} and the shorter chains.

Since we have now seen that only a single chain R that cannot be reduced further will exist for six complexes, we return to that chain R and the A_{ik} in the conversion of the expression on the right-hand side. After a rather lengthy conversion, we will obtain the following equation:

(20) $$R^2 + R \cdot N + M = 0.$$

In this:

$$N = \tfrac{1}{8}[+ A_{12} A_{34} A_{56} - A_{12} A_{35} A_{46} + A_{12} A_{45} A_{36} - A_{13} A_{24} A_{56} + A_{13} A_{25} A_{36}$$
$$- A_{13} A_{45} A_{26} + A_{14} A_{35} A_{26} - A_{14} A_{25} A_{36} + A_{14} A_{23} A_{56} - A_{15} A_{34} A_{26}$$
$$+ A_{15} A_{24} A_{36} - A_{15} A_{23} A_{46} + A_{16} A_{45} A_{23} - A_{16} A_{24} A_{36} + A_{16} A_{34} A_{25}],$$

and M is an expression of quite significant length that is asymmetric in the A_{ii} and A_{ik} . The term in the expression for M that consists of the collection of factors $A_{11}\,A_{22}\,\ldots\,A_{66}$ has the numerical factor $-\,2^{-9}$, which we would like to point out here.

It is easy to see that the expressions M and N are not symmetric in the complexes, since R itself is not symmetric in the coefficients of these complexes. The result of the exchange of two complexes in R will be given by the equation:

$$R = [ab'cd'ef'] = -\,[ba'cd'ef'] - \tfrac{1}{2}A_{12}\,[cd'ef'].$$

Therefore, not only will the signs change under a switch, but a term that decomposes into the A_{ik} will also split off. The exchange of two complexes changes only the sign of D, so D will therefore be different from R, which one can also see easily in a different way.

Furthermore, no linear connection exists between the two invariants D and R. If such a thing did exist then it would also have to be preserved when we considered the case in which the six complexes define a Kleinian system; i.e., in which all $A_{ik} = 0$ and either $A_{ii} > 0$ or $A_{ii} < 0$. N would then vanish, and from a remark that was made above, we would have $M = -\,2^{-9}P$, in which we have set:

$$P = A_{11}\,A_{22}\,A_{33}\,A_{44}\,A_{55}\,A_{66}\,,$$

so we would then have the equation for R:

(21) $$R^2 - 2^{-9}P = 0.$$

On the other hand, in this case, since we would now have $H = P$, from equation (15), we would get for D that:

(22) $$D^2 + 2^{-9}P = 0.$$

It would follow from these two equations that:

(23) $$8R^2 + D^2 = 0;$$

i.e., an irrationality would enter into that case when one would like to express D in terms of R.

We once more remark expressly that not a single invariant of the six complexes will be given by R, but that $R = [ab'cd'ef']$ represents a type that belongs to 6! such invariants, in all, that one will obtain from R by permuting the symbols a, b, …, f. However, it will suffice to consider only one of these 6! invariants, since all of the other ones can be expressed in terms of it rationally and entirely with the help of the A_{ii} and A_{ik} .

§ 10.

We therefore now have the theorem that:

All invariants of a system of six linear complexes can be expressed rationally in terms of the six A_{ii}, the fifteen A_{ii}, and R and D.

In this, D is given by equation (13), and:

$$R = [ab'cd'ef'] = (ab')\,(b'c)\,(cd')\,(d'e)\,(ef')\,(f'a)$$

or non-symbolically:

$$R = \sum a_{ik} b'_{kl}\, c_{lm} d'_{mn} e_{no}\, f'_{oi}$$

in which one always sets $a_{ik} = a'_{mn}$.

However, if we also direct our attention to the irrational representation then we will get the theorem that every invariant of the system of six complexes can be represented in terms of the six A_{ii} and the fifteen A_{ik} alone. In particular, R and D, which are otherwise referred to as invariant, on the whole, seem to be irrational functions of those quantities in the definition of the quantities A_{ii} and A_{ik}, and for that reason, one can refer to them as "symbolically-irrational" invariants.

That terminology is especially out of place for D, since it is not possible to represent D rationally in terms of the complex symbols, while such an expression will first be possible for D^2, as has been shown.

§ 11.

One simultaneously deals with the *ray forms* by the same arguments that were applied to the invariants, since one only needs to regard the coefficients of one of the complexes to be variable, and those variables will then be represented in precisely the same way as those coefficients themselves.

By a simple conversion, we will therefore have the result that:

All ray forms of six complexes can be expressed rationally in terms of the aforementioned invariants and the basic forms K_i, as well as the following forms:

$$R' = [\pi b'c\,d'\,e\,f'] = (\pi b')\,(b'c)\,(d'\,e)\,(e\,f')\,(f'\pi),$$
$$D' = (\pi\, b\; c\; d\; e\; f) \qquad \text{[cf., equation (13)]}.$$

Six forms R' and D' are possible in this, according to which of the six complexes one leaves out.

In this, $D' = 0$ means the complex that is determined by five of the six given complexes by the requirement that it must be in involution with them.

One can also regard these ray forms R and D as "symbolically-irrational," since they satisfy quadratic equations whose coefficients are entire, rational functions of the six basic forms K_i and the identity ray form:

$$K_0 = (\pi\,\rho')^2 = 2\sum \pi_{ik}\pi'_{ik} = 2\sum \pi_{ik}\pi_{mn} = 4\,(\pi_{12}\,\pi_{34} + \pi_{13}\,\pi_{42} + \pi_{14}\,\pi_{23})\,.$$

One will have $K_0 \equiv 0$ when one is dealing with a line, as one knows.

§ 12.

We now turn to the *covariants*. In their construction, we have to consider not only factors of type (ab') that enter into the invariants, but ones of type $(a'x)$, as well. We select such a factor $(a'x)$ and write it in the first place, such that we will have:

$$J = (xa')$$

for a covariant J. We deduce the factors that contain the symbol a' for a second time from this factor $(a'x)$, etc., such that we can also define a chain here. We will then have:

$$J = (xa')\,(a'b)\,(bc')\,(c'd),$$

and it is clear that such a chain must once more terminate with x, since we are considering only one sequence of point coordinates. We will then have ([1]):

$$J = (xa')(a'b)(bc')(c'd)\,\dots\,(nm')(m'x) = [xa'bc'd\,\dots\,nm'x]. \tag{24}$$

Now, as one can easily verify, the same laws of conversion are true for these chains that are true for chains that represent only invariants that are given by equations (7) and (8). We conclude from this, as above, that such covariants are always linear in the coefficients of the individual complexes. Since we now have no more than six such linear complexes to consider, and an odd number of complexes must always appear in such a chain, the covariants will be easy to write down.

For only one complex, we will have the chain:

$$J_1 = (xa')\,(a'x) = (a'x)^2 \equiv 0,$$

since the a' are complex symbols. All that remain now are the cases in which three or five of the six given complexes appear. We will thus obtain twenty covariants of the type:

$$J_3 = [x\;a'\;b\;c'\;x] = (xa')\,(a'b)\,(bc')\,(c'x) \tag{25}$$

and six covariants of the type:

$$J_5 = [x\;a'\;b\;c'\;d'\;e'\;x] = (xa')\,(a'b)\,(bc')\,(c'd)\,(de')\,(e'x)\,. \tag{26}$$

([1]) These are the forms that F. Mertens denoted by $(a \cdot b \cdot c \cdot d \cdot \dots x)\,x_1 + \dots$

The geometric meaning of these covariants in easy to give when one looks more closely at the null systems that are given by the individual complexes.

As a result of the dual character of the system of six complexes that is being considered, the *contravariants* are also dealt with by using the considerations above. We get, dually, twenty contravariants of the type:

$$J_3' = [u'ab'cu'] = (u'a)\,(ab')\,(b'c)\,(cu') \tag{27}$$

and contravariants of the type:

$$J_5' = [u'ab'cd'\,eu'] = (u'a)\,(ab')\,(b'c)\,(cd')\,(d'e)\,(eu')\,. \tag{28}$$

If we attempt to exhibit quadratic equations for these covariants and contravariants, as we did for the invariants *R* and *D*, then we will always arrive at purely formal identities, and thus, obtain no such representation.

§ 13.

Finally, we come to the *mixed forms*. Here, as we already mentioned to begin with, we will have four types of genera to distinguish ([1]):

1. Ones that contain u' and x.
2. Ones that contain π_{ik} and x.
3. Ones that contain π_{ik} and u'.
4. Ones that contain π_{ik}, u', and x.

One splits off the identical forms:

$$K_0 = (\pi\pi')^2 \qquad \text{and} \qquad (u'x)$$

in this. If the coefficients of the six complexes also enter in then we will have to investigate the following types of factors:

$$(a'x),\quad (a\,u'),\quad (a\pi'),\quad (\pi\,a'),\quad (ab'),\quad (\pi\,u'),\quad (\pi'\,x),\quad \text{and} \quad (\pi\rho').$$

However, we can restrict ourselves to the coefficients of the complexes and the quantities u' and x in the search for the general types, since the variables π_{ik} behave precisely like those coefficients. We therefore first treat the mixed forms that contain a sequence of point coordinates x and a sequence of plane coordinates u'.

We then once more write down these forms as chain forms, as we have up to now. We will then have:

$$J = (xa')\,(a'b)(bc')\,(c'd)\,\dots\,(n'm)\,(mu') = [xa'bc'd\dots\,n'mu'].$$

([1]) Cf., Clebsch, *Über eine Fundamentalaufgabe der Invariantentheorie,* § 3, Göttingen, 1872.

We once more derive the same laws of conversion from this representation as we did for the invariants or covariants. We will then once more obtain the result that such a mixed form can only be linear in the coefficients of the individual complexes. Now, since an even number of complexes always appears in such a chain, we will get:

Fifteen mixed forms of the type:

$$J_2 = [xa'bu'] = (xa')\,(a'b)\,(bu')\,. \tag{29}$$

Fifteen mixed forms of the type:

$$J_4 = [xa'bc'du'] = (xa')\,(a'b)\,(bu')\,(c'd)\,(du'), \tag{30}$$

and finally, a mixed form of the type:

$$J_6 = [xa'bc'de'\,fu'] = (xa')\,(a'b)\,(bc')\,(c'd)\,(de')\,(e'f)\,(f\,u')\,. \tag{31}$$

The mixed forms that also contain variable ray coordinates will be inferred first from the covariants and contravariants that were discussed in the previous § when one first replaces a sequence of coefficients with variable π_{ik}, and secondly from the mixed forms that were given in (29), (30), and (31), when one also replaces a sequence of coefficients with variable π_{ik}.

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