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On the theory of electrons

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In his attempt to incorporate the *reaction force of radiation on a charged, moving particle* in quantum mechanics, and thus to perhaps get around the complications in quantum electrodynamics, the author has been led for some time ([2]) to a very remarkable connection between the force of radiation and *spin* that initially defied all attempts at interpretation. After various attempts to approach the subject from the classical ([3]) or quantum-theoretic ([4]) side, the problem could finally be put into a clearer form by posing the question ([5]) of *the existence and definition of commutation relations that would be compatible with a system of equations of motion for an electron that includes a force of radiation.* He arrived at the following conclusions:

1. One can count the energy that is emitted from a moving charge in the rest mass by an artifice of unforced simplicity. The system will then be conservative, and one can introduce a canonical impulse. Due to the influence of the radiation reaction, the canonical impulse will be a function of not only the velocity, but the acceleration. *Impulse and velocity will vary independently in that way.* That independence is an essential peculiarity of **Dirac**’s theory of spin.

2. *The Hamiltonian function assumes the Dirac form precisely.* That is possible, since for **Dirac**, only coordinates, impulses, and velocities appear, but not spin operators. Since the rest mass will be time-varying, due to the radiated energy that is included in it, one can see why it is assigned to a separate operator by **Dirac.**

3. If one seeks to introduce commutation relations that are compatible with the equations of motion then one will be led to the conclusion that *the square of the components of the velocity must be equal to the square of the speed of light.* That is another characteristic of **Dirac**’s theory.

([1]) Presently at Heidelberg.
([2]) **W. Wessel**, Zeit. Phys. **92** (1934), 407.
([3]) **W. Wessel**, Zeit. Phys. **110** (1938), 625.
([4]) **W. Wessel**, Naturwiss. **30** (1942), 606.
([5]) **W. Wessel**, Ann. Phys. (Leipzig) **43** (1943), 565.

Ultimately, all that remains are the commutation relations of the velocity components with each other. For us, they *cannot* emerge as simply as they do for **Dirac**, since the theories would no longer differ in the slightest then. In fact, here (and indeed *first* here) the great difference comes to light that exists between spin and radiation force kinematically (i.e., periodic-aperiodic perturbation) and dynamically (i.e., order of magnitude of the perturbation of terms compared to the fine-structure constant). Those relations would also be preserved approximately and in spirit then; they would thus *contain internal contradictions* in the latter conception of our theory.

Eliminating that difficulty is the main purpose of the present paper. One will succeed in the context of a *classical* theory, into which **Poisson** brackets enter in place of the commutation relations. That will now give us a viewpoint for the *interpretation* of the entire phenomenon, as well.

1. The moment as autonomous variable

It was already suggested at the conclusion of the previous publication that the solution was to be sought in the introduction of further variables. That also suggests that one must find an *ad hoc* basis for an overriding physical viewpoint. The expression that we used up to now for the reaction force of the radiation, which contains time derivatives up to the third, is true for only a strictly point-like electron ([6]). However, the development of quantum electrodynamics always leads back to the old necessity of giving the electron the quality of a spatial extent in some sense. The "electron radius" e^2 / mc^2 is the limit at which recent developments would also cease. If one now calculates the reaction force ([7]) for a charge that is distributed in the usual sense then one will find a series that progresses in higher derivatives of the coordinates with respect to time. It would then be entirely understandable that the radiation force should also contain higher derivatives for a finitely-extended particle, in an abstract sense, and perhaps a finite number of them. However, higher derivatives always translate into new variables in the analysis of equations of motion.

On the grounds of covariance, those variables will always combine into the components of a vector or tensor that then appears in the form of a new physical property of the particle. That then opens up the possibility of interpreting *the moment of the electron as being equivalent to its finite extent.* Namely, one assumes that its equations of motion are not third-order, as for point-like particles, but fifth-order. When one reduces it to a system of first-order equations, one will get $5 \cdot 3 = 15$, including the sixteen defining data in the aforementioned sense of varying rest mass. Of them, three are coordinates, three are velocity components, and four of them define the energy-impulse vector that the mass determines. Hence, six of them will remain, and that is precisely the number of components of an antisymmetric tensor. A fourth-order differential equation will lead to thirteen variables, and thus to three more, in addition to the present ones. There is no known interpretation for them, so the mechanical roster

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([6]) **P. A. M. Dirac**, Proc. Roy. Soc. (London) **167**A (1938), 148.

([7]) See, e.g., **H. A. Lorentz**, Enzykl. mathem. Wissensch., v. 2, Contribution 14, number III, or with newer tools: **F. Bopp**, Z. Naturforschg. **1** (1946), 53.

must be extended to either a four-vector or a six-vector ([8]). We will then first seek *to introduce the magnetic moment of the electron* (we shall always mean the corresponding six-vector by that) *as a new variable.*

In addition to the moment, **Dirac**'s theory knows of an entire series of further variables whose operators all arise from the ones that we have employed up to now – viz., the velocity and mass term – by multiplication. There are sixteen linearly-independent operators in all (their number coincides with the one above only incidentally), which correspond to the current-charge density four-vector j_i , a second four-vector k_i whose three spatial component define the spin, the six-vector M_{ik} of magnetic and electric moments, and two invariants. When one takes on additional elements in the classical theory, along with the four elements from which one constructs the remaining ones, it will seem irrefutable that *all of them* are now included. That would also happen entirely by itself, but the number of variables would *not* rise correspondingly. It would not be the operators – not even, say, the nabla operator in the usual wave mechanics – that would have physical meaning, but the matrices or density functions that they define. Hence, for example, not the transformation operator (four-rowed identity matrix) a_1 of velocity, but its (infinite) matrix $(a_1)_{mn}$ in a suitable representation or the current density:

$$\psi^* a_1 \psi = \psi_1^* \psi_4 + \psi_2^* \psi_3 + \psi_3^* \psi_2 + \psi_4^* \psi_1 ,$$

and *a series of algebraic relations* exists between those quantities. Some of them were already discovered by **Darwin** ([9]); one can also thank **Laporte** and **Uhlenbeck** ([10]). They imply that: The two invariants are determined by the two invariants of the antisymmetric moment tensor; the four-vector k is determined by the moment tensor, and the vector j and its square will lead to further invariants. That gives seven covariant relations, which reduces the number of independent variables from sixteen to nine. For that reason, we will reduce the time-like vector j by dividing it by its magnitude to produce a vector $u = j \,/\, |\, j\,|$ that has the character of a four-velocity with three independent components, moreover. In conjunction with the six-component moment tensor, it will then yield precisely nine independent defining data. For the moment, there is no reason to go beyond that number. Indeed, one can probably say (this thought was already always implicit in the entire search for the theory) that one should be able to describe more than just the pure spin phenomena with such an extensive apparatus.

The vectors j and k are perpendicular to each other and have the same magnitude. Since j is time-like, k will be space-like, and if we regard $u = j \mid j \mid$ as a subluminal velocity then $U = k \,/\, |\, k\,|$ will become a superluminal velocity. As such, it should have no physical meaning, and in fact, it is only a pure computational device, since k is indeed determined by the other variables. However, it will serve to simplify the equations when one employs it. When one goes from the four-vectors u_i and U_i to the usual velocities $\mathfrak{v}$ and $\mathfrak{V}$, it will follow from:

$$j_i\, k^i = 0 \tag{1.1}$$

([8]) **L. H. Thomas**, Phil. Mag. J. Sci. (7) **3** (1927), 1.
([9]) **C. G. Darwin**, Proc. Roy. Soc. (London) **120**A (1928), 621.
([10]) **O. Laporte** and **G. E. Uhlenbeck**, Phys. Rev. **37** (1931), 1380 and 1552.

that one has the conversion that will be found in Section 3:

$$\mathfrak{v}\ \mathfrak{B} = c^2 \tag{1.2}$$

(c = speed of light). When $\mathfrak{v}$ and $\mathfrak{B}$ are in the same direction, they will relate to each other like the phase velocity of **Schrödinger** waves to the group velocity of the particle. We will consider $\mathfrak{v}$ to be the velocity in the ordinary sense and refer to $\mathfrak{B}$ as the *associated* velocity.

2. Basic theorems and assumptions for the construction of the theory

We shall now deal with finding the correct equations of motion for the newly-introduced components of moment and improving the equations for velocity in such a way that the problem with symmetry that will appear at the end of the present paper will be avoided. The viewpoint will then always be this: Only the coordinates and their time derivatives (e.g., velocity, acceleration, change in acceleration, etc.) have any meaning to begin with. In particular, the radiation force will depend upon only them. All of the remaining variables (e.g., impulse, moment, and the invariants) represent those derivatives only for the purpose of introducing the commutation relations. *Their equations of motion must then be determined in such a way that the equation of motion of the particle under the influence of the radiation force would arise by their elimination* ([11]).

Admittedly, that problem is rather ill-defined. First of all, from the experiments with radiation damping, one knows everything with at most the accuracy that corresponds to the replacement of the electron with a point-like particle, and that will suffice when the generalized radiation force coincides with it in the first approximation. However, that agreement does not need to be complete when the reaction force contains reversible components ([12]) that average out over the course of the motion, in addition to the energy-consuming, irreversible component. It will obviously suffice that the *irreversible* component should properly appear in the first approximation.

We must then start from another viewpoint, and will put the question of the *quantizability* of our equations in the foreground. Quantum conditions mean commutation relations, and commutation relations correspond to **Poisson** brackets in the classical theory. It is known that one can derive the equations of motion from a

([11]) Equations of motion for a particle with a magnetic moment were presented by several authors already. Our problem is close to one that was treated in a paper by **H. J. Bhabha** and **H. C. Corben** [Proc. Roy. Soc. (London) **178**A (1941), 273], in which the equations of motion were derived from only the requirements of conservation of energy and angular impulse of the particle and field, and thus with consideration given to the radiation reaction. However, those authors calculated with a particle (that was point-like in regard to charge) that was assigned an additional inertial moment and angular impulse. As in the mechanics of rigid bodies, an equation for the center-of-mass must then arise, along with one for the moment. In contrast, we shall keep to precisely the variables in **Dirac**'s theory, so we will correspondingly have no inertial moment for the particle, but employ the twist (Ger: *Drall*) as the autonomous variable, and can basically reduce everything to *one* equation for the center-of-mass coordinates.

([12]) **G. A. Schott**, Phil. Mag. J. Sci. (6) **29** (1915), 49.

Hamiltonian function by using them. We then pose the problem of *finding a system of* ***Poisson*** *brackets and a* ***Hamiltonian*** *function such that the equations of motion satisfy the requirements that were specified in the foregoing.* The present paper follows through on that program on the basis of simple assumptions up to the presentation of the equations of motion and a *glimpse* into the form of the radiation reaction. Its precise determination demands very extensive calculations of elimination, by which the splitting-off of the reversible component will remain as a problem in its own right. We shall then content ourselves with a more physical consideration of the possibility of a non-conservative quantum mechanics that will assume an interesting new form through the appearance of two velocities.

Most likely, one would hardly have any way to get a handle on that problem if there were not close connection between our problem and **Dirac**'s theory. Above all, it would provide an answer to the closely-related question of *which canonical variables should be fundamental to the definition of the* ***Poisson*** *brackets.* As is well-known, one understands a "**Poisson** bracket" to mean the following: Suppose that a system of canonical variables $p_1, \ldots, p_F, q_1, \ldots, q_F$ is be given, along with two functions f and g of those variables. The **Poisson** bracket of f and g will then be ([13]):

$$(f, g) = \sum_{k=1}^{F} \left(\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q^k} - \frac{\partial f}{\partial q^k} \frac{\partial g}{\partial p_k} \right). \tag{2.1}$$

In particular, one will then have:

$$(p_i, q^k) = \delta_i^k . \tag{2.2}$$

In contrast to the commutation relations, which can be defined freely, the definition of the **Poisson** brackets assumes a system of canonical variables.

We pose the problem of finding **Poisson** brackets for the four-vector $j / | j |$ and the moment tensor M_k . Both of them are expressed bilinearly in terms of spinors ψ, χ: For example [cf., the first-cited paper by **Laporte** and **Uhlenbeck** ([10])], one has:

$$j_{\dot{m}l} = \psi_{\dot{m}} \psi_l + \chi_{\dot{m}} \chi_l . \tag{2.3}$$

One initially understands the ψ, χ to mean coordinate functions, so j will be a space-like density. It is now very tempting to characterize the classical electron by four spinors ψ_1, ψ_2, ψ_1, ψ_2, and their complex conjugates $\psi_{\dot{1}}$, $\psi_{\dot{2}}$, $\chi_{\dot{1}}$, $\chi_{\dot{2}}$, and to no longer consider the latter to be coordinate functions, but as supplementary parameters of motion that are introduced as canonical variables along with the usual coordinates and impulses. If, say, ψ and χ were canonically conjugate then one would have:

$$(\psi_\mu \chi_\nu) = \delta_\mu^\nu, \tag{2.4}$$

([13]) Here, we define it with a different symbol from the usual one, in order to have the usual sequence of factors from quantum mechanics in the definition of the equations of motion.

$$(\psi_{\dot{\mu}}\,\chi^{\dot{\nu}}) = \delta_{\dot{\mu}}^{\dot{\nu}}\,,$$

while all of the remaining brackets would vanish. If one now assembles the vector $j_{\dot{m}l}$ precisely as in (2.3) then the **Poisson** brackets of its components can be defined directly from the formula:

$$(j_{\dot{m}k}\ j_{\dot{n}l}) = \sum_{\lambda}\left(\frac{\partial j_{\dot{m}k}}{\partial\psi_{\lambda}}\frac{\partial j_{\dot{n}l}}{\partial\chi^{\lambda}} - \frac{\partial j_{\dot{m}k}}{\partial\chi^{\lambda}}\frac{\partial j_{\dot{n}l}}{\partial\psi_{\lambda}}\right) + \sum_{\dot{\lambda}}\left(\frac{\partial j_{\dot{m}k}}{\partial\psi_{\dot{\lambda}}}\frac{\partial j_{\dot{n}l}}{\partial\chi^{\dot{\lambda}}} - \frac{\partial j_{\dot{m}k}}{\partial\chi^{\dot{\lambda}}}\frac{\partial j_{\dot{n}l}}{\partial\psi_{\dot{\lambda}}}\right). \tag{2.5}$$

Obviously, another bilinear expression will then arise in that way. In general, the **Poisson** brackets of bilinear quantities will again lead to bilinear quantities. However, all bilinear combinations of spinors that arise in that way will correspond to vectors or tensors (it is a peculiarity of spin calculations that the invariants and antisymmetric tensors will also be have the same factor rank as vectors), *so one will always remain within the group of world tensors in the formation of* ***Poisson*** *brackets.* Spinors will drop out completely. The reducibility of the tensor relations to spinor ones means only a restriction of the possibilities in the former.

Such a restriction is desirable if a system of *commutation relations* between tensor components is to be still largely freely-chosen and is in no way established by the relativistic invariance alone. *However, for first-rank spinors, one can define only two invariant systems of* ***Poisson*** *brackets, and only with the system (2.4) will one remain inside of the* j_i *and* M_{ik}. We shall first show the invariance.

Let ψ', χ' be the spinors in a moving system, so, as is known, a **Lorentz** transformation of the rest system will be expressed by a binary transformation:

$$\psi'_k = a_{k\mu}\,\psi_\mu\,, \qquad \chi'_k = a_{k\mu}\,\chi_\mu\,, \qquad k, \mu = 1, 2 \tag{2.6}$$

with a coefficient determinant $|\,a\,| = 1$. The complex-conjugate formulas will always appear with these. One will then have (observe that $\psi^1 = \psi_2$, $\psi^2 = -\,\psi_1$):

$$\begin{aligned}
(\psi'_1\,\chi^{1'}) &= (\psi'_1\,\chi'_2)\\
&= (a_{11}\,\psi_1 + a_{12}\,\psi_2\,,\, a_{21}\,\chi_1 + a_{22}\,\chi_2)\\
&= \,a_{11}\,a_{21}\,(\psi_1\,\chi_1) + a_{11}\,a_{22}\,(\psi_1\,\chi_2)\\
&\quad + a_{12}\,a_{21}\,(\psi_2\,\chi_1) + a_{12}\,a_{22}\,(\psi_2\,\chi_2)\\
&= -\,a_{11}\,a_{21}\,(\psi_1\,\chi^2) + a_{11}\,a_{22}\,(\psi_1\,\chi^1)\\
&\quad - a_{12}\,a_{21}\,(\psi_2\,\chi^2) + a_{12}\,a_{22}\,(\psi_2\,\chi^1)\\
&= (a_{11}\,a_{22} - a_{12}\,a_{21})\cdot 1 = 1
\end{aligned}$$

$$\begin{aligned}
(\psi'_1\,\chi^{2'}) &= -(\psi'_1\,\chi'_1)\\
&= -\,(a_{11}\,\psi_1 + a_{12}\,\psi_2\,,\, a_{21}\,\chi_1 + a_{12}\,\chi_2)\\
&= -\,a_{11}^2\,(\psi_1\,\chi_1) - a_{11}\,a_{12}\,(\psi_1\,\chi_2)
\end{aligned}$$

$$- a_{12}\, a_{11}\, (\psi_2\, \chi_1) - a_{12}^2\, (\psi_2\, \chi_2)$$

$$= a_{11}^2\, (\psi_1\, \chi^2) - a_{11}\, a_{12}\, (\psi_1\, \chi^1) + a_{12}\, a_{11}\, (\psi_2\, \chi^2) - a_{12}^2\, (\psi_2\, \chi^1) = 0,$$

etc., which was to be proved. One sees directly that canonical conjugation of complex-conjugate spinors – say in the form $(\psi_1\ \psi^1)$ – will not come into question, since the transformation coefficients of complex-conjugate ψ are complex-conjugates of the ψ, and would not define a determinant of 1 then. By contrast, it would still be relativistically possible to demand that $(\psi_\mu\, \psi^\nu) = (\chi_\mu\, \chi^\nu) = \delta_\mu^\nu$ and to let all $(\psi\, \chi)$ vanish. However, as we will show in Section 4 by way of an example, the **Poisson** brackets to be defined in that way would emerge from the system of j and M components. Formulas (2.4) are then distinguished uniquely, and we would like to take the opportunity to draw upon their particular mathematical simplicity in order to construct the theory from them. Naturally, that principle is purely heuristic; it is crucial to know whether one will actually again find oneself in the first approximation to the equations of motion with the radiation force for a point-like particle by a suitable choice of **Hamiltonian** function.

3. Systematics of tensors

Here we would now like to summarize all of the quantities that come into play, along with their mutual relationships, in spinor, tensor, and vector form. We will then follow **Laporte** and **Uhlenbeck** throughout, whose notations we shall also include. The four-vectors j and k will then be written the same in spinor and tensor form, since the differ by the number of indices. By contrast, we shall write the moment tensor as a spinor with lower-case letters and as a tensor with upper-case ones, since it is doubly-indexed in both cases. We would also like to follow **Laporte** and **Uhlenbeck** and set the coefficients of the elements of the world-line to $g_{11} = g_{22} = g_{33} = 1$, $g_{44} = -1$. We thus leave behind the "pseudo-Euclidian metric" that we employed in our previous papers, since we are already calculating with complex spinors and propose to introduce commutation relations later, so it would seem desirable to avoid a further use of the imaginary unit. Summation over doubled indices will then go from 1 to 2 and from 1 to 4 for tensors.

From the four complex spinors ψ_1, ψ_2, χ_1, χ_2, and their complex conjugates, we can now define:

1. Two invariants:

$$\Delta = \psi_\lambda\, \chi^\lambda, \qquad \bar{\Delta} = \psi_{\dot\lambda}\, \chi^{\dot\lambda}. \tag{3.1}$$

Its real and imaginary parts are:

$$I = \frac{1}{2}(\Delta + \bar{\Delta}), \qquad J = \frac{1}{2i}(\Delta - \bar{\Delta}). \tag{3.2}$$

The invariants are also the single way of identifying the antisymmetric spinors of rank two:

$$\begin{aligned} \psi_r\,\chi^s - \chi_r\,\psi^s &= \delta_r^s\,\Delta\,, \\ \psi_{\dot r}\,\chi^{\dot s} - \chi_{\dot r}\,\psi^{\dot s} &= \delta_{\dot r}^{\dot s}\,\bar\Delta\,. \end{aligned} \tag{3.3}$$

II. The two four-vectors:

$$\begin{aligned} j_{\dot m l} &= \psi_{\dot m}\,\psi_l + \chi_{\dot m}\,\chi_l\,, \\ k_{\dot m l} &= \psi_{\dot m}\,\psi_l - \chi_{\dot m}\,\chi_l\,. \end{aligned} \tag{3.4}$$

They are perpendicular to each other:

$$k_{\dot\sigma\lambda}\,j^{\dot\sigma\lambda} = 0, \tag{3.5}$$

and its square is expressed in terms of Δ (observe that $\psi_\lambda\,\psi^\lambda = 0$!):

$$j_{\dot\sigma\lambda}\,j^{\dot\sigma\lambda} = 2\Delta\bar\Delta = -k_{\dot\sigma\lambda}k^{\dot\sigma\lambda}\,. \tag{3.6}$$

By the way, the two combinations of χ_l into $\chi_l \pm \chi$ in ψ that are still possible define a pair of vectors that are orthogonal to each other and to j and k. One can thus assemble precisely an orthogonal vierbein from the four spinors. Finally, we have:

III. The moment tensor:

$$\begin{aligned} m_{\dot r\dot s} &= \frac{\Lambda}{i}(\psi_{\dot r}\chi_{\dot s} + \psi_{\dot s}\chi_{\dot r})\,, \\ m_{rs} &= -\frac{\Lambda}{i}(\psi_r\chi_s + \psi_s\chi_r)\,. \end{aligned} \tag{3.7}$$

The length h / mc (viz., the **Compton** wave length) that appears for **Laporte** and **Uhlenbeck** is replaced with the symbol Λ here. The invariants of the tensors are:

$$\begin{aligned} m_{\rho\sigma}\,m^{\rho\sigma} &= 2\,\Lambda^2\,\Delta^2, \\ m_{\dot\rho\dot\sigma}m^{\dot\rho\dot\sigma} &= 2\,\Lambda^2\,\bar\Delta^2, \end{aligned} \tag{3.8}$$

and its contraction with j will yield k, and conversely:

$$\begin{aligned} j_l^{\dot\rho}\,m_{\dot\rho\dot s} &= -\frac{\Lambda}{i}\,k_{\dot s l}\,\bar\Delta \\ k_l^{\dot\rho}\,m_{\dot\rho\dot s} &= -\frac{\Lambda}{i}\,j_{\dot s l}\,\Delta\,. \end{aligned} \tag{3.9}$$

At this point, we would also like to account for the *dimensions* of our new variables. Since the moment has the dimension of charge times length, it will follow from (3.7) that the products of the ψ, χ must have the dimension of charge. *The invariants I and J will then have the dimension of charge.* As a result of (3.4), the ψ and χ must have the same dimensions. The vectors *also* seem to have the dimension of length then. We can once more conclude from this, as we already did in Section 1 from counting the independent variables, that in the context of these considerations (as opposed to wave mechanics) only the four-velocities that are defined from them are meaningful.

We would now like to repeat the algebraic relations in tensor notation. The well-known table of vectors:

$$
\begin{aligned}
j^1 &= \tfrac{1}{2}(j_{\dot{2}1} + j_{1\dot{2}}) = j_1, & \qquad j^3 &= \tfrac{1}{2}(j_{1\dot{1}} - j_{\dot{2}2}) = j_3, \\
j^2 &= \tfrac{1}{2i}(j_{\dot{2}1} - j_{1\dot{2}}) = j_2, & \qquad j^4 &= \tfrac{1}{2}(j_{1\dot{1}} + j_{\dot{2}2}) = -j_4,
\end{aligned}
\tag{3.10}
$$

will serve for the translation, along with the following table of the antisymmetric tensors, which was not given explicitly by **Laporte** and **Uhlenbeck** (the asterisk means the dual tensor):

$$
\begin{aligned}
M_{12} &= -\frac{1}{2i}(m_{12} - m_{\dot{1}\dot{2}}) = i\,M^*_{43}, \\
M_{23} &= \frac{1}{4i}(m_{11} - m_{\dot{1}\dot{1}} - m_{22} + m_{\dot{2}\dot{2}}) = i\,M^*_{41}, \\
M_{31} &= -\frac{1}{4i}(m_{11} + m_{\dot{1}\dot{1}} + m_{22} + m_{\dot{2}\dot{2}}) = i\,M^*_{42}, \\
M_{41} &= \frac{1}{4i}(m_{11} + m_{\dot{1}\dot{1}} - m_{22} - m_{\dot{2}\dot{2}}) = -i\,M^*_{23}, \\
M_{42} &= \frac{1}{4i}(m_{11} - m_{\dot{1}\dot{1}} + m_{22} - m_{\dot{2}\dot{2}}) = -i\,M^*_{31}, \\
M_{43} &= -\frac{1}{2}(m_{12} + m_{\dot{1}\dot{2}}) = -i\,M^*_{12}.
\end{aligned}
\tag{3.11}
$$

With that, we would next like to express the two invariants $M_{ik}\,M^{ik}$ and $M^*_{ik}\,M^{ik}$ of M in terms of I and J. Upon multiplication of (3.11) and keeping (3.8) and (3.2) in mind, we will get:

$$
\tfrac{1}{2} M_{ik}\,M^{ik} = \Lambda^2\,(I^2 - J^2), \qquad \tfrac{1}{2} M_{ik}\,M^{ik} = 4\,\Lambda^2\,I\,J. \tag{3.12}
$$

From (3.10) and (3.2), (3.6) will give:

$$-j_r j^r = I^2 + J^2 = k_r k^r, \tag{3.13}$$

and the relations (3.9) mean that:

$$\begin{aligned} j^r M_{rs} &= \Lambda J k_s, \\ k^r M_{rs} &= \Lambda J j_s, \end{aligned} \tag{3.14}$$

and upon multiplying this by M^{*st} and considering the fact that:

$$M_{rs} M^{*st} = M^*_{rs} M^{st} = -\tfrac{1}{4}\delta^s_r (M^* M) = i\,\delta^s_r \Lambda^2 I J,$$

one can also convert this into:

$$\begin{aligned} k_r M^*_{rs} &= i\Lambda I j_s, \\ j^r M^*_{rs} &= i\Lambda I k_s. \end{aligned} \tag{3.15}$$

With that, we have assembled seven conditions, one of which was spoken of in Section 1: The invariants I and J are expressed by (3.12), the first of eq. (3.13) connects j to the moment tensor, and (3.14) or (3.15) express k in terms of j and M.

As in Section 1, we now introduce the two four-vectors u and U and go to space vectors. To abbreviate, we set:

$$I^2 + J^2 = l^2, \tag{3.16}$$

such that $l = |j|$. We will then have:

$$u^i = j^i / l, \tag{3.17}$$

whose components are:

$$\mathfrak{u} = \frac{\mathfrak{v}/c}{\sqrt{1-v^2/c^2}}, \qquad u^4 = \frac{1}{\sqrt{1-v^2/c^2}} = -u_4, \tag{3.18}$$

and:

$$U^i = k^i / l, \tag{3.19}$$

whose components are:

$$\mathfrak{U} = \frac{\mathfrak{V}/c}{\sqrt{V^2/c^2-1}}, \qquad U^4 = \frac{1}{\sqrt{V^2/c^2-1}} = -U_4. \tag{3.20}$$

We have:

$$\begin{aligned} u_j u^j &= \mathfrak{v}^2 - u_4^2 = -1, \\ U_j U^j &= \mathfrak{U}^2 - U_4^2 = +1, \end{aligned} \tag{3.21}$$

and due to (3.5):

$$u_j U^j = 0, \tag{3.22}$$

or $\mathfrak{v}\mathfrak{B} / c^2 - 1 = 0$, which was proved already in Section 1.

Finally, we decompose the moment tensor into the magnetic and electric moments $\mathfrak{M}$ and $\mathfrak{P}$, resp., according to ([14]):

$$(M_{23}, M_{31}, M_{12}) = \mathfrak{M}, \quad (M_{14}, M_{24}, M_{34}) = \mathfrak{P}. \tag{3.23}$$

One can then write eq. (3.12) as:

$$\mathfrak{M}^2 - \mathfrak{P}^2 = \Lambda^2 (I^2 - J^2), \qquad \mathfrak{P}\,\mathfrak{M} = -\Lambda^2 I\, J, \tag{3.24}$$

and the first of eq. (3.14), (3.15), after dividing by l, and with the use of (3.17), (3.18), and eliminating the fourth one, respectively, will go to:

$$\frac{\mathfrak{B}}{c} = \frac{\mathfrak{P} + [\mathfrak{v}\mathfrak{M}]/c}{\mathfrak{v}\mathfrak{P}/c} = \frac{\mathfrak{M} - [\mathfrak{v}\mathfrak{P}]/c}{\mathfrak{v}\mathfrak{M}/c}. \tag{3.25}$$

In that way, the associated velocity is expressed in terms of the usual one. To invert the formula, one needs merely to switch $\mathfrak{B}$ and $\mathfrak{v}$, as one sees immediately from (3.14), (3.15).

The quantities $\mathfrak{M}$, $\mathfrak{P}$, and $\mathfrak{v}$, or also M_{ik} and u^j, are now our nine new variables then, along with coordinates and impulses. We still need to remark that they are subject to *one* invariant coupling. Indeed, we have constructed everything out of four complex spinors; hence, only eight quantities can actually be independent. The invariants of j and M were considered already; however, jM still remains, which is the square of (3.14), and in fact, it will lead back to j^2, due to the second half of equation (3.13). In terms of space vectors, the relation reads:

$$(\mathfrak{P} + [\mathfrak{v}\mathfrak{M}] / c)^2 - (\mathfrak{P}\mathfrak{v} / c)^2 = \Lambda^2 J^2 \left(1 - \frac{v^2}{c^2}\right). \tag{3.26}$$

That will always be a particular integral of our equations of motion. Further invariants cannot be defined from j and M, since products of more than two equal antisymmetric tensors can always be reduced to expressions with one or two factors ([15]).

4. Poisson brackets

With those preparations, we can go on to define the **Poisson** brackets. We can extend the Ansatz for them a bit by assuming that the formulas (2.4) [(2.5), resp.] are true for

---

([14]) In the two foregoing papers, we employed the symbol $\mathfrak{P}$ as an abbreviation for $\mathfrak{p} + e / c\, \mathfrak{A}$. With the meaning that is used here, it is, however, so commonplace to use the current notation that we prefer to alter the abbreviation.

([15]) **W. Wessel**, Zeit. Phys. **110** (1938), 625. Conclusion.

only certain primary spinors ψ_0, χ_0, such that the ψ, χ differ from them by a factor. Therefore, let:

$$
\begin{aligned}
(\psi_\mu\,\chi^\nu) &= \delta_\mu^\nu\,\Gamma\, e^{i\delta},\\
(\psi_{\dot\mu}\,\chi^{\dot\nu}) &= \delta_{\dot\mu}^{\dot\nu}\,\Gamma\, e^{-i\delta},
\end{aligned}
\tag{4.1}
$$

while all remaining brackets are zero. We have made the factor complex, since spinors are complex, and it will even emerge as pure imaginary. That has nothing to do with the appearance of the imaginary unit in the commutation relations. Here, we are forced to consider classical theory, and the **Poisson** brackets of real tensors must correspondingly be real in any event. That is also the case then in formula (2.5), in which one must now think of ψ_0, χ_0, in place of the ψ, χ, so they and their complex conjugates will occur symmetrically.

With the transition from (2.4) to (4.1), the two sums in (2.5) will take on the factor $\Gamma\, e^{i\delta}$ [$\Gamma\, e^{-i\delta}$, resp.]. As long as we are dealing with only bilinear expressions, we can also omit those formulas completely and calculate with (4.1) as we would with commutation relations. We will also refer to quantities whose **Poisson** brackets vanish as *commuting*, as we do in quantum mechanics. The easily-proved rules:

$$
\begin{aligned}
(\chi_\mu\,\psi^\nu) &= (\psi_\mu\,\chi^\nu),\\
(\chi_{\dot\mu}\,\psi^{\dot\nu}) &= (\psi_{\dot\mu}\,\chi^{\dot\nu})
\end{aligned}
\tag{4.2}
$$

are often applied in the following conversions, along with the known formula $\psi_\lambda\,\chi^\lambda = -\,\psi^\lambda\,\chi_\lambda$, as well as the one that follows from it, namely, $\psi_\lambda\,\psi^\lambda = 0$, along with its conjugate.

Obviously, the Δ and $\bar\Delta$ in (3.1) commute, since the ψ, χ, and their complex conjugates are; therefore, *the invariants I and J also commute.* The calculation of the remaining combination simplifies considerably when one begins with the moments between them, since they will give rise to a well-defined, simplified choice of phase δ in (4.1). We then define:

$$
\begin{aligned}
(m_{\mu\nu}\, m^{\rho\sigma}) &= -\,\Lambda^2\,(\psi_\mu\,\chi_\nu + \chi_\mu\,\psi_\nu,\ \psi^\sigma\chi^\sigma + \chi^\rho\psi^\sigma)\\
&= -\,\Lambda^2\,\Gamma\, e^{i\delta}\,\psi_\mu\,\delta_\nu{}^\rho\,\chi^\sigma + \psi_\nu\,\delta_{\nu\mu}{}^\sigma\,\chi^\rho + \text{etc.}\\
&= i\,\Lambda\,\Gamma\, e^{i\delta}\,\{\delta_\nu{}^\rho\, m_\mu{}^\sigma + \delta_\nu{}^\sigma\, m_\mu{}^\rho + \delta_\mu{}^\rho\, m_\nu{}^\sigma + \delta_\mu{}^\rho\, m_\nu{}^\sigma\},
\end{aligned}
\tag{4.3}
$$

along with the conjugate expression that arises in such a way that one puts dots on the indices and changes i into $-\,i$. Now, from (3.11), one has, e.g.:

$$
\begin{aligned}
(M_{23}\,M_{31}) &= -\frac{1}{16i}\,(m_{11} - m_{\dot1\dot1} - m_{22} + m_{\dot2\dot2},\ m_{11} + m_{\dot1\dot1} + m_{22} + m_{\dot2\dot2})\\
&= -\,\frac{1}{8i}\left\{(m_{11}\,m_{22}) - (m_{\dot1\dot1}\,m_{\dot2\dot2})\right\}\\
&= -\,\frac{1}{8i}\left\{(m_{11}\,m^{11}) - (m_{\dot1\dot1}\,m^{\dot1\dot1})\right\}.
\end{aligned}
\tag{4.4}
$$

However, as a result of (4.3), one will have:

$$
\begin{aligned}
(m_{11}\, m^{11}) &= \quad 4i\,\Gamma\Lambda\, e^{i\delta} m_1{}^1 = \quad 4i\,\Gamma\Lambda\, e^{i\delta} m_{12}\,, \\
(m_{\dot{1}\dot{1}}\, m^{\dot{1}\dot{1}}) &= -4i\,\Gamma\Lambda\, e^{-i\delta}\, m_{\dot{1}}^{\dot{1}} = -4i\,\Gamma\Lambda\, e^{-i\delta}\, m_{\dot{1}\dot{2}}\,,
\end{aligned}
\tag{4.5}
$$

and a result:

$$
\begin{aligned}
(M_{23}, M_{31}) &= -\Lambda\,\Gamma\left\{\frac{1}{2}(m_{12} + m_{\dot{1}\dot{2}})\cos\delta + \frac{i}{2}(m_{12} - m_{\dot{1}\dot{2}})\sin\delta\right\} \\
&= -\Lambda\,\Gamma\left\{M_{12}\sin\delta + iM_{12}^{*}\cos\delta\right\}.
\end{aligned}
\tag{4.6}
$$

Those **Poisson** brackets for the magnetic moment, along with the ones that extend them cyclically, correspond precisely to the one that is true for an angular impulse d_{ik} :

$$(d_{23}\, d_{31}) = -\, d_{12}, \qquad \text{etc.,} \tag{4.7}$$

when one sets cos $\delta = 0$. Since one imagines that the magnetic moment is directed oppositely to the proper angular impulse for a negative electron, we will direct it in such a way that we have:

$$(M_{23}\, M_{31}) = +\,\Gamma\,\Lambda\, M_{12}\,. \tag{4.8}$$

We will accordingly have to set:

$$\delta = -\,\pi / 2. \tag{4.9}$$

The fundamental relations (4.1) then go to:

$$
\begin{aligned}
(\psi_\mu\, \chi^\nu) &= \frac{\Gamma}{i}\delta_\mu^\nu\,, \\
(\psi_{\dot{\mu}}\, \chi^{\dot{\nu}}) &= -\frac{\Gamma}{i}\delta_{\dot{\mu}}^{\dot{\nu}}\,.
\end{aligned}
\tag{4.10}
$$

We can conclude the dimension of Γ from a comparison of (4.7) and (4.8) if we demand that our still-undetermined ψ_0, χ_0 should be such that *the equations of motion can all be collectively derived in the usual form by taking the* ***Poisson*** *brackets of the variables in question with the Hamiltonian function.* $\Lambda\Gamma$ will then take on the dimension of magnetic moment : angular impulse, and since Λ has the dimension of length, the moment, charge times length, and angular impulse, that of action, it will follow for Γ that:

$$[\Gamma] = \text{charge / action}. \tag{4.11}$$

One can also conclude: On the basis of the demand that was made above, the dimension of the product of the ψ_0, χ_0 , which is what one thinks of (4.10) being differentiated with respect to, must be that of a product of impulse and coordinates, and thus, an action, and since the ψ, χ-product corresponds to the charges that we spoke of before, it will follow from (4.10) that Γ again has the dimension of charge : action.

With that, we go on to the systematic construction of all possible **Poisson** brackets. The *commutability of the invariants I* and *J* was established before. What follow are then the combinations of *invariants and vectors*. We have, e.g.:

$$\begin{aligned}(j_{\dot{m}n}\,\Delta) &= (\psi_{\dot{m}}\,\psi_n + \chi_{\dot{m}}\,\chi_n, \psi_\lambda\,\chi^\lambda) \\ &= (\psi_{\dot{m}}\,\psi_n, \psi_\lambda\,\chi^\lambda) - (\chi_{\dot{m}}\,\chi_n, \psi^\lambda\,\chi_\lambda) \\ &= \frac{\Gamma}{i}\{\psi_{\dot{m}}\,\psi_n - \chi_{\dot{m}}\,\chi_n\} = \frac{\Gamma}{i}k_{\dot{m}n},\end{aligned} \tag{4.12}$$

or, since one can obviously just as well reason with the spinors on the vectors:

$$(j^l\,\Delta) = \frac{\Gamma}{i}k^l,$$

etc., and finally:

$$\begin{aligned}(j^l I) &= 0, \quad (j^l J) = -\Gamma k^l, \\ (k^l I) &= 0, \quad (k^l J) = -\Gamma j^l.\end{aligned} \tag{4.13}$$

I will then commute with the vectors, too. At this point, let it be remarked that one would preserve the **Poisson** bracket that was mentioned at the end of Section 2:

$$(j_{\dot{m}n}\,\Delta) = -\,\psi_{\dot{m}}\,\chi_n + \chi_{\dot{m}}\,\psi_n;$$

that is, one that was stated in connection with (3.6) between perpendicular vectors *j* and *k* and is ascribed no known meaning.

The *vectors with each other* define:

$$(k_{\dot{m}n}, j^{\dot{r}s}) = \frac{\Gamma}{i}\left\{\delta_n^s(\psi_{\dot{m}}\,\chi^{\dot{r}} - \chi_{\dot{m}}\,\psi^{\dot{r}}) - \delta_{\dot{m}}^{\dot{r}}(\psi_{\dot{n}}\,\chi^{\dot{s}} - \chi_{\dot{n}}\,\psi^{\dot{s}})\right\}. \tag{4.14}$$

Here, the antisymmetric spinors that were mentioned in (3.3) appear; we then have:

$$(k_{\dot{m}n}, j^{\dot{r}s}) = \frac{\Gamma}{i}\delta_n^s\delta_{\dot{m}}^{\dot{r}}(\bar{\Delta} - \Delta) = -\,2\,\Gamma\,J\,\delta_n^s\delta_{\dot{m}}^{\dot{r}}. \tag{4.15}$$

The factor – 2 drops out in the calculations, and one will have:

$$(k_r\,j^s) = \delta_r^{\ s}\,\Gamma\,J. \tag{4.16}$$

The combination of *vector components with each other* proceeds in a similar fashion, and yields:

$$(k_i\, k_k) = \frac{\Gamma}{\Lambda} M_{ik} = -\,(j_i\, j_k). \tag{4.17}$$

Our initial Ansatz, which was chosen with a view to mathematical simplicity, again brings us into satisfactory agreement with **Dirac**'s theory, in which, from **Laporte** and **Uhlenbeck**, the spatial components of the k_i correspond to the mechanical angular impulse of the electron. Indeed, one does not have $(k_1, k_2) \sim k_3$, since that is not possible on the grounds of transformations, but probably:

$$(k_1, k_2) = \frac{\Gamma}{\Lambda} \mathfrak{M}_3\,; \tag{4.18}$$

i.e., it is proportional to the component of the magnetic moment in the direction of k_3 .

We can now define the still-outstanding combinations of the moments with the remaining quantities very simply, while avoiding the spinor calculation, when we express the moments in terms of $(k_i\, k_k)$ or $(j_i\, j_k)$ by using (4.17). The *moments and invariants* are defined thus: Firstly, since I commutes with all vectors:

$$(I\, M_{ik}) = 0, \tag{4.19}$$

and secondly, when one applies the **Jacobi** identity:

$$\begin{aligned}(J\, M_{ik}) &= \frac{\Lambda}{\Gamma}\big(J(k_i\, k_k)\big) \\ &= -\,\frac{\Lambda}{\Gamma}\big\{\big(k_k(J\, k_i)\big)+\big(k_i(k_k J)\big)\big\} \\ &= \Lambda\big\{\big(k_k\; j_i\,\big)+\big(k_i\; j_k\big)\big\} \\ &= -\,\Lambda\,\Gamma\, J\,(g_{ki} - g_{ik}) = 0.\end{aligned} \tag{4.20}$$

The moments and invariants are then completely commuting. The invariant I commutes with all quantities, and as a result, it is a constant of the motion.

For the combination of *moments and vectors*, we express M_{ik} with j in terms of $(k_i\, k_k)$, and with k in terms of $(j_i\, j_k)$, and thus find, again by applying the **Jacobi** identity:

$$\begin{aligned}(M_{ik}\, k^l) &= \Lambda\Gamma\big\{\delta_i^{\,l} k_k - \delta_k^{\,l} k_i\big\}, \\ (M_{ik}\; j^l) &= \Lambda\Gamma\big\{\delta_i^{\,l}\, j_k - \delta_k^{\,l}\, j_i\big\}.\end{aligned} \tag{4.21}$$

Finally, by drawing upon the aforementioned formulas, one will get the *moment components:*

$$(M_{ik}\, M^{\,rs}) = \Lambda\,\Gamma\,\big\{\delta_i^{\,r} M_k^{\;s} - \delta_k^{\,r} M_i^{\;s} - \delta_i^{\,s} M_k^{\;r} + \delta_k^{\,s} M_i^{\;r}\big\}, \tag{4.22}$$

of which, (4.8) is already a special case.

All of these variables shall be appended to the position coordinates and impulses, and commute with them with no restriction. Formula (2.5) is, correspondingly, to be thought as having been extended by a sum in which one differentiates with respect to the p_k , q^k .

5. The Poisson brackets of the u_i, U, and M_{ik}

In the context of our considerations, the four-vectors j and k are merely auxiliary quantities, as we have stressed several times, and of them, only the "unit vectors" $u_i = j_i / | j |$ and $U_i = k_i / | k |$ have any meaning. We now come to the ultimate relations. From the foregoing, the **Poisson** brackets (4.22) of the moments with each other, which are free of j and k, belongs to them already, so we assemble them here, when written in terms of $\mathfrak{M}$ and $\mathfrak{P}$:

$$\begin{gathered}(\mathfrak{M}_1\, \mathfrak{M}_2) = \Gamma\Lambda\, \mathfrak{M}_3 = -\,(\mathfrak{P}_1\, \mathfrak{P}_2)\\ (\mathfrak{M}_1\, \mathfrak{P}_2) = \Gamma\Lambda\, \mathfrak{P}_3,\\ (\mathfrak{M}_1\, \mathfrak{P}_1) = (\mathfrak{M}_2\, \mathfrak{P}_2) = (\mathfrak{M}_3\, \mathfrak{P}_3) = 0.\end{gathered} \tag{5.1}$$

The first three are extended by cyclic permutation of the indices. The relations (3.14) and (3.15) are solved for the u_i , U_i with no further assumptions by simply dividing by l:

$$\begin{aligned} u^r M_{rs} &= \Lambda\, J U_s, \qquad & u^r M^*_{rs} &= i\Lambda\, J\, U_s,\\ U^r M_{rs} &= \Lambda\, J u_s, \qquad & U^r M^*_{rs} &= i\Lambda\, J\, u_s.\end{aligned} \tag{5.2}$$

In order to define the **Poisson** brackets of the u_i , U_i , we must now go back to the differentiation formulas (2.5) (when extended by the factors Γ / i and $-\,\Gamma / i$). Hence, if X means any of the quantities that occur, and after recalling the definition (3.16) of l:

$$\begin{aligned}\left(\frac{1}{l}\, X\right) &= (I\, X)\, \frac{\partial}{\partial I}\left(\frac{1}{l}\right) + (J\, X)\frac{\partial}{\partial J}\left(\frac{1}{l}\right)\\ &= -\frac{J}{l^3}\,(J\, X),\end{aligned} \tag{5.3}$$

since I commutes with all quantities. It follows that:

$$\begin{aligned}(u_i\, X) = \left(\frac{j_i}{l}\, X\right) &= (I\, X)\, \frac{1}{l}(j_i X) + j_i\left(\frac{1}{l}\, X\right)\\ &= \frac{1}{l}(j_i\, X) - \frac{J}{l^3}\, j_i\,(J\, X)\end{aligned} \tag{5.4}$$

or

$$(u_i\, X) = \frac{1}{l}(j_i\, X) - \frac{J}{l^2}\, u_i\,(J\, X), \tag{5.5}$$

$$(U_i\,X) = \frac{1}{l}(k_i\,X) - \frac{J}{l^2}U_i\,(J\,X).$$

Due to (4.13), for $X = J$, one will have, e.g.:

$$\begin{aligned}(u_i\,J) &= -\Gamma U_i,\\ (U_i\,J) &= -\Gamma u_i.\end{aligned} \tag{5.6}$$

Moreover, one must define $(u_i\,j_k)$, $(u_i\,k_k)$, etc., in order to then go on to the $(u_i\,u_k)$, $(u_i\,U_k)$. We give just the final formulas:

$$\begin{aligned}(u_i\,u_k) &= \frac{J\,\Gamma}{l^2}(U_i\,u_k - U_k\,u_i) - \frac{\Gamma}{\Lambda l^2}M_{ik} = -(U_i\,U_k),\\ (u_i\,U_k) &= -\frac{J\,\Gamma}{l^2}\{g_{ik} + (u_i\,u_k - U_i\,U_k)\} = -(U_i\,u_k).\end{aligned} \tag{5.7}$$

Finally, we still have to define:

$$\begin{aligned}(u^l\,M_{ik}) &= -\Gamma\Lambda\{\delta_i^l\,u_k - \delta_k^l\,u_i\},\\ (U^l\,M_{ik}) &= -\Gamma\Lambda\{\delta_i^l\,U_k - \delta_k^l\,U_i\}\end{aligned} \tag{5.8}$$

by the same process.

The **Poisson** brackets of the components of $\mathfrak{v}$ and $\mathfrak{B}$ are of especial simplicity and interest now. We next have to define:

$$\left(\frac{v_i}{c}\;\frac{v_k}{c}\right) = \left(\frac{u_i}{c}\;\frac{u_k}{c}\right) = \frac{1}{(u_4)^3}\{u_4\,(u_i\,u_k) + u_k\,(u_4\,u_i) + u_i\,(u_k\,u_4)\} \tag{5.9}$$

and

$$\left(\frac{V_i}{c}\;\frac{V_k}{c}\right) = \left(\frac{U_i}{U_4}\;\frac{u_k}{u_4}\right)$$

$$= \frac{1}{(u_4)^2(U_4)^2}\{u_4\,U_4\,(U_i\,u_k) - u_k\,U_4\,(u_4\,u_i) - U_i\,u_4\,(U_4\,u_k) + U_i\,u_k\,(U_4\,u_4)\}. \tag{5.10}$$

Upon substituting this in the expressions (5.7), the parentheses drop out cyclically, and when one draws upon (5.2), what will remain will be:

$$\begin{aligned}\left(\frac{V_i}{c}\;\frac{v_k}{c}\right) &= \Gamma\frac{J}{l^2}\frac{1}{u^4\,U^4}\left(\delta_{ik} - \frac{V_i}{c}\frac{v_k}{c}\right),\\ \left(\frac{v_1}{c}\;\frac{v_2}{c}\right) &= \Gamma\frac{I}{l^2}\frac{U^4}{(u^4)^3}\frac{V_3}{c},\end{aligned} \tag{5.11}$$

$$\left(\frac{V_1}{c}\,\frac{V_2}{c}\right) = -\,\Gamma\frac{I}{l^2}\frac{u^4}{(U^4)^3}\frac{v_3}{c}.$$

For the sake of clarity, we repeat the first of these formulas once more for the indices 1 and 2, while introducing u_4 and U_4 by using (3.18) and (3.20) and formally symmetrizing the right-hand side:

$$(V_2\, v_1) = -\,\Gamma\,\frac{J}{l^2}\sqrt{\frac{V^2}{c^2}-1}\sqrt{1-\frac{v^2}{c^2}}\cdot\frac{1}{2}\,(V_2\, v_1 + v_1\, V_2). \tag{5.12}$$

In fact, in the context of classical theory, those **Poisson** brackets will define the resolution of the contradiction to which we were led at the conclusion of the foregoing paper when we sought to present *commutation relations* that would be compatible with the equations of motion of a point-like particle under the influence of the radiation force. There, we found in (31):

$$i\,(v_2\, v_1 - v_1\, v_2) \div \frac{3}{2\gamma}\left(1-\frac{v^2}{c^2}\right)\frac{1}{2}\,(v_2\, v_1 + v_1\, v_2). \tag{5.13}$$

Here, the factor i corresponds to the commutation relation; γ is the fine-structure constant, and the sign $\div$ means that the right-hand side and the left-hand side *cannot* be set equal to each other on the grounds of their mutual symmetry. Naturally, such a symmetry anomaly can never occur in our **Poisson** brackets, which are derived by consistent (*einheitliche*) differentiation, and one sees that it comes about, *while maintaining the strictest analogy*, when one forms the **Poisson** bracket that corresponds to the desired commutation relation, not with the components of the velocity among themselves, but with those of the *associated* velocity (3.25). The introduction of the moment then, in fact, allows that fundamental anomaly to be eliminated, at least in the domain of the **Poisson** brackets. Naturally, the **Poisson** brackets of the $\mathfrak{v}$ and $\mathfrak{V}$ components with themselves must read in an essentially different way from the previous commutation relations, which were free of contradiction.

Formula (5.12) becomes interesting due to the fact that the analogy (5.13) will obviously become especially close if – and only if – $\mathfrak{v}$ and $\mathfrak{V}$ are as equal as possible. Since they are linked to each other by the condition (1.2), that means that they must be parallel and that both of them must be *almost equal to the speed of light*, which means that $\mathfrak{V}$ must lie just as far beneath it as $\mathfrak{v}$ does. In that case, the product of the roots in (5.12) must agree with the factor $1 - v^2 / c^2$ on the right-hand side of (5.13) in the first approximation. Now, as we showed already in the first paper on this topic, the entire tendency of the motion is to go there, while, by contrast, the velocity very quickly increases to the speed of light. That once more corresponds completely to the behavior of **Dirac**'s theory, in which it is known that the particle performs a high-frequency "Zitterbewegung" with the speed of light around the slowly-advancing center-of-mass. The prospect of approaching this little-clarified process, at least with a classical theory, will, unfortunately, be occluded by the fact that here the course of motion depends upon the numerous, arbitrary *initial conditions* of the many variables in an incalculable way.

Here, one must even appeal to the quantum conditions. Otherwise, one cannot specify the factor $\Gamma J / l^2$ in (5.12), either. It may not be replaced with perhaps the factor 3/2 γ of (5.13), even when one understands (5.12) to be a commutation relation and extends it with a factor of h, since J and I do not commute with all quantities, and are therefore not constants of the motion. In any event, due to the smallness of γ, it will correspond to a "large" factor – i.e., the v_i, V_k are essentially *anti-commutative.*

6. Hamiltonian function and equations of motion

Now that we have addressed a system of relativistically-covariant **Poisson** brackets in the foregoing sections, we can now derive a system of likewise-covariant equations of motion for all quantities X that occur once we have been given a **Hamiltonian** function H by using the formula:

$$\frac{dX}{dt} = (H\,X). \tag{6.1}$$

If we would like to differentiate with respect to the proper time τ then, from the fact that:

$$d\tau = \sqrt{1-\frac{v^2}{c^2}}\,dt = \frac{dt}{u^4}, \tag{6.2}$$

all equations must be multiplied by u^4. It is most convenient to write them with the use of the world-line element:

$$ds = c\,d\tau. \tag{6.3}$$

We will suggest differentiation with respect to time by a dot and differentiation with respect to the line element by a prime ([16]).

The construction of the **Hamilton** function will proceed inevitably when one regards $\mathfrak{v}$ as the velocity in the usual sense from now on. One must then have:

$$\dot{x}^i = (H\,x^i) = \frac{\partial H}{\partial p_i} = v^i \tag{6.1}$$

for the coordinates x^i (i = 1, 2, 3). Since $\mathfrak{p}$ and $\mathfrak{v}$ are independent of each other, H can have only the form $\mathfrak{p}\mathfrak{v}$ plus an expression that is independent of $\mathfrak{p}$. Due to its meaning as an energy:

$$p_l\,u^l = \mathfrak{p}\;\mathfrak{u} + p_4\,u^4 = -\,m\,c \tag{6.5}$$

must be an invariant then, in which m has the character of a mass. That already implies that H must have the form:

([16]) We would like to apologize for our repeated alternation of the notation: In the communication to "die Naturwissenschaften," the dot referred to proper time, while in the previous paper, the prime did.

$$H = c\, p^4 = -\, c\, p_4 = \mathfrak{p}\mathfrak{v} + \frac{mc^2}{u^4}\,, \tag{6.6}$$

as long as only kinetic energy comes into question; that is once more the **Dirac** form. The introduction of external forces that are characterized by a vector potential $\mathfrak{A}$ and a scalar potential V will now have the consequence that the usual equations of motion of a point charge will result by dropping the radiation force. As a result of the foregoing paper ([5]), eqs. (6) and (8) will remain the equations of the coordinates and impulse, utterly free from the influence of the radiation force. Its introduction comes about in the usual way when one replaces $\mathfrak{p}$ and p^4 with:

$$\begin{aligned} \mathfrak{g} &= \mathfrak{p} + \frac{e}{c}\mathfrak{A}, \\ g^4 &= p^4 + \frac{e}{c}V, \end{aligned} \tag{6.7}$$

resp. In place of eq. (6.5), one will then have:

$$g_l\, u^l = -\, m\, c. \tag{6.8}$$

We would also like to introduce a corresponding notation for the invariant that is defined by the U^i, initially for only the sake of brevity, by:

$$g_l\, U^l = -\, M\, c. \tag{6.9}$$

The two equations are understood in entirely different ways: For a given m, (6.8) is a defining equation for p_4, while conversely, for a given p_4, (6.9) is a defining equation for M. For $H = cp^2$, (6.8) now yields:

$$H = \mathfrak{g}\mathfrak{v} - eV + \frac{mc^2}{u^4}\,, \tag{6.10}$$

from which, by forming the **Poisson** brackets with the impulses, one will get:

$$\dot{p}_i = (H\, p_i) = -\frac{\partial}{\partial x^i}\, e\left(\mathfrak{A}\frac{\mathfrak{v}}{c} - V\right), \tag{6.11}$$

in agreement with the foregoing paper, and furthermore, with the introduction of the field tensor F_{ik} of the external forces, it will emerge that:

$$\dot{g}_i = -\,\frac{e}{c}\, F_{ik}\, v^k \tag{6.12}$$

or

$$g_i' = -\frac{e}{c} F_{ik}\, u^k . \tag{6.12}$$

Those are the usual equations. The fact that they are also true in **Dirac**'s theory was shown for the first time by **Fock** ([17]).

The $\mathfrak{v}$, $\mathfrak{M}$, $\mathfrak{P}$ shall commute with the coordinates and impulses. They will then pertain to the formation of the **Poisson** bracket with H in terms of merely $\mathfrak{v}$, u^4, and m. In fact, we can still allow m to depend upon the invariants I and J – i.e., due to the constancy of I, upon J – and will naturally do that after we have counted the emitted energy in the rest mass in the foregoing paper and have made it time-varying.

Everything that pertains to the calculation of the equations of motion from (6.1) has been summarized in the foregoing section, such that it will suffice for us to give the results. They read:

I. For the invariants:

$$I' = 0, \qquad J' = \Gamma\, Mc. \tag{6.14}$$

II. For the four-vectors:

$$\begin{aligned} u_i' &= \frac{J\,\Gamma}{l^2}\left\{ mcU_i - Mc\,u_i - \frac{1}{\Lambda\, J} g^l M_{li} \right\} + \Gamma U_i \frac{\partial m c}{\partial J} \\ U_i' &= \frac{J\,\Gamma}{l^2}\left\{ mc\,u_i - McU_i - g_i \right\} + \Gamma u_i \frac{\partial m c}{\partial J} . \end{aligned} \tag{6.15}$$

III. For the moment tensor:

$$M_{ik}' = -\,\Gamma\,\Lambda\,\{g_i\, u_k - g_k\, u_i\}. \tag{6.16}$$

The first equation in (6.15) then enters in place of eq. (7) in the foregoing paper, and with our current notation, it will read:

$$u_i' = \frac{3c}{2e^2}\{mc\, u_i - g_i\}. \tag{6.17}$$

The second equation in (6.15) is only a consequence of the first one, on the grounds of (5.2). However, it is convenient to employ the equations together; (5.2) will then appear to be a particular integral. We then append the equation for the usual theory without radiation force:

$$mc\, u_i = g_i\, . \tag{6.18}$$

([17]) **V. Fock**, Zeit. Phys. **55** (1929), 127.

As is known, its validity is essential for the validity of the law of angular impulse. In a central field, one will have $\mathfrak{g} = \mathfrak{p}$, and the change in $\mathfrak{p}$ will be parallel to the radius vector $\mathfrak{r}$, so:

$$\frac{d[\mathfrak{r}\,\mathfrak{p}]}{ds} = [\mathfrak{u}\ \mathfrak{p}], \tag{6.19}$$

and that will vanish from (6.18). However, it will *not* vanish from (6.17), but it will go to:

$$\frac{d[\mathfrak{r}\,\mathfrak{p}]}{ds} = -\frac{2e^2}{3c}[\mathfrak{u}\ \mathfrak{u}']; \tag{6.20}$$

the radiation force *impair* the law of impulse. *However, in our new theory, it will enter again in the force for the sum of orbital angular impulse and spin*, and here it will stand in for the negative magnetic moment. In fact, (6.16) is true for $\mathfrak{A} = 0$, $\mathfrak{g} = \mathfrak{p}$:

$$-\frac{d\mathfrak{M}}{ds} = -\Gamma\Lambda\ [\mathfrak{p}\ \mathfrak{u}]; \tag{6.20}$$

therefore, since (6.10) and (6.19) persist, from (6.19) and (6.21):

$$\mathfrak{j} = [\mathfrak{r}\ \mathfrak{p}] - \frac{\mathfrak{M}}{\Gamma\Lambda} \tag{6.22}$$

will be constant in time now. That result is independent of the form of the function m (J).

7. The appearance of the radiation force

According to our plan, we must now eliminate all variables up to the position coordinates and their derivatives and to attempt to determine the function m (J) and the constant I in such a way that the equation of motion for a point-like particle with the radiation force:

$$s_i = \frac{2e^2}{3c}[u_i'' - u_i(u')^2] \tag{7.1}$$

will emerge from it in the first approximation. [Here and hereinafter, for brevity and for ease of reading, we shall omit the indices in products of two four-vectors, so we shall then write $(u)^2$ and gU, instead of $u_l\, u^l$ and $g_l\, U^l$, etc.] For the sake of mathematical complexity, we have first followed through on that problem for the case of vanishing external forces (F_{ik}). The expression for the radiation force must also remain valid then, since it does not depend directly upon F_{ik}, but only on u', u''. The result is that one cannot fulfill all of the conditions with just the one function m (J), which was assumed. At the very least, certain order-of-magnitude relations must be fulfilled, whose confirmation can be accomplished only by very detailed calculation when one considers the F_{ik} . Now, one should not assume that the solution to such a fundamental problem is hidden behind such complications. Rather, we would like to seek then in an entirely different direction. In the foregoing paper, the motion was made conservative by allowing the rest to increase according to the law:

$$m' = \frac{2}{3}\frac{e^2}{c^2}(u')^2 \tag{7.2}$$

and writing $(mc\,u_i)'$, instead of $mc\,n_i'$, in the equation of motion. That gimmick was intended in point 1 in the Introduction. In the present conception of the theory, the constancy of the **Hamiltonian** function is established independently of that, since the equations of motion are derived from it. In return, due to the presence of two velocities, we must once more have two quantities *m* and *M* that have the character of mass, and that is related to *holding m constant again*, and thus *inquiring about the variability of M*, in the hope of once more finding the emitted energy *in that*. One then has a clean break between rest mass *m* and field energy Mc^2. The constancy of *m* means that since *m* depends upon time only through *J*, $\partial m / \partial J = 0$ in eq. (6.15), and the following connection now exists: It follows from:

$$U_i' = \frac{\Gamma}{J}\{mc\,u_i - Mc\,U_i - g_i\}, \tag{7.3}$$

on the one hand, and from (6.8) and (6.9), that:

$$g\,U' = -\frac{\Gamma}{J}\{(mc)^2 - (Mc)^2 + g^2\}, \tag{7.4}$$

but on the other hand, due to the facts that $u^2 = -1$, $U^2 = +1$, $uU = 0$, and from the respective formula:

$$(U')^2 = \left(\frac{\Gamma}{J}\right)^2\{(mc)^2 - (Mc)^2 + g^2\}, \tag{7.5}$$

it will follow that:

$$g\,U' = -\frac{J}{\Gamma}(U')^2. \tag{7.6}$$

Now, from the definition (6.9):

$$M'c = -g\,U' - g'U; \tag{7.7}$$

hence, from (6.13) and (7.6):

$$M' = \frac{J}{\Gamma c}(U')^2 + \frac{e}{c^2}F_{ik}\,U^i u^k\,. \tag{7.8}$$

That would be in precise agreement with (7.2) if *u*, *u′* appeared in place of *U*, *U′*, resp. (observing that $F_{ik}\,u^i\,u^k = 0$), and one could set:

$$J = \frac{2}{3}\frac{e^2}{c}\Gamma, \tag{7.9}$$

and both of them are allowed, to a certain degree. On the one hand, as was already remarked in the conclusion to Section 5, one has, in fact, the tendency in the motion to make $\mathfrak{B}$ and $\mathfrak{v}$, and thus also U_i and u_i, equal to each other. In general, that will happen at the expense of their finitude, such that limiting value of $F_{ik}\, U^i\, u^k$ can also be non-zero. On the other hand, one has the equation of motion (6.14) for J:

$$J' = \Gamma\, Mc. \tag{7.10}$$

If M were to start out from zero then that would imply that J = constant, as (7.9) would require, at least in the initial phase. One could then view the situation in such a way that the **Hamiltonian** function H would not be the energy of particle, but a suitable quantity that would be referred to as a *term value*, while the *energy* would be given by the same expression in $m - M$, and therefore:

$$E = \mathfrak{g}\mathfrak{v} - eV + \frac{(m-M)\,c^2}{u^4}, \tag{7.11}$$

so that will go to $(m - M)\, c^2$ in the rest system of the particle ($\mathfrak{v} = 0$, $u^4 = 1$), up to energy of position.

A complete *proof* of these last suggestions would probably be quite difficult, and also hardly far-reaching, as a purely classical state of affairs. In the meantime, it might therefore suffice to establish that we remain very close to the current, consistent conception of the theory from the older starting point, in any event. Moreover, the advanced arguments should be devoted to *quantization*, so the constant Γ must also be determined. Our theory was created from that step onward by the fact that it is based completely upon **Poisson** brackets. However, due to the necessary symmetrizations and other generalizations, the conversion of brackets into commutation relations will require just such a special examination.

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