"Das asymptotische Verteilungsgesetz der Eigenschwingungen eines beliebig gestalteten elastischen Körper," Rend. Circ. Mat. Palermo **34** (1914), 1-49.

The asymptotic distribution law for the eigen-oscillations of an elastic body of arbitrary form

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CHAPTER I.

THE GREEN TENSORS OF THE STATIC ELASTIC PROBLEM.

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§ 1. Introduction. The basic formulas of the theory of elasticity.

In order to solve the *static problem of the theory of elasticity* for a homogeneous, isotropic body of arbitrary form with the boundary condition of vanishing displacement, one appeals to two essentially different methods, one of which was developed by FREDHOLM, LAURICELLA, MARCOLONGO ([1]), while the other was developed by KORN and BOGGIO ([2]). Both have in common that they reduce the problem to a system of linear integral equations. However, since the method of KORN-BOGGIO leads to kernels with singularities that are difficult to discuss (in contrast to BOGGIO's assertion that they falsely prove to be "regular"), we will consider only the first path, which runs parallel to the NEUMANN-FREDHOLM solution of the corresponding problem in

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([1]) I. FREDHOLM, "Solution d'un problème fondamental de la théorie de l'élasticité," Arkiv för Matematik, Astronomi och Fysik, **2** (1906), no. 28, pp. 1-8.

G. LAURICELLA, "Sull'integrazione della equazioni dell'equilibrio dei corpi elastici isotropi," Atti della Reale Accademia dei Lincei **15** (1st semester 1906), 426-432.

R. MARCOLONGO, "La teoria delle equazioni integrali e le sue applicazioni alla Fisica-matematica," *ibid.* **16** (1st semester 1907), 742-749.

([2]) A. KORN, "Über die Lösung der ersten Randwertaufgabe der Elastizitätstheorie," Rend. Circ. Mat. Palermo **30** (2nd semester (1910), 138-152. KORN's previous treatises on the same subject are cited in this.

T. BOGGIO, "Nuova risoluzione di un problema fondamentale della teoria dell'elasticità," Atti della Reale Accademia dei Lincei **16** (1st semester 1907), 248-255.

potential theory; that path, which encounters no such complications, seems to me to be the only natural one. In order to be certain that the inhomogeneous integral equations in question possess a solution, it is known that one must establish that the corresponding homogeneous equations possess no other solutions besides the trivial solution 0; LAURICELLA ([3]) showed flawlessly that this is the case. However, it is important for us to make the construction independent of the proof of that fact in order to be able to adapt the method to other boundary conditions. In other words, we will show that the solution of the problem that is posed can be achieved when the homogeneous integral equations possess solutions, as well as when that is not the case. With that, a stone has been removed from the path that one often stumbles over in the conversion of the NEUMANN-FREDHOLM method to the general case ([4]).

On the basis of that argument and certain Ansätze of BOUSSINESQ, one will also succeed in solving the static problem of elasticity theory for the case in which not only the displacements, but also the stresses, are prescribed to be zero on the outer surface. To my knowledge, that case, which must be referred to as the natural one, has been treated only by BOGGIO using his method up to now ([5]). However, I must confess that, on the basis of what was suggested above, its method of proof does not seem to be completely convincing, and in any event will prove to be inadequate for our further purposes.

We shall consider a third type of boundary condition (in which $\mathfrak{u}$ is understood to mean the displacement):

$$\operatorname{div} \mathfrak{u} = 0, \qquad \mathfrak{u} \text{ normal} \qquad \text{(on the outer surface).}$$

It will be essential for us that it should bring about the transition from elasticity theory to potential theory according to the schema:

Elastic bodies → FRESNEL's *elastic æther* → *electromagnetic æther.*

In the present article, however, we will not treat the static problem, but the *oscillation problem.* For each of the three aforementioned boundary conditions, the force-free elastic bodies can exhibit an infinite sequence of eigen-oscillations whose oscillation numbers define the (discrete) spectrum of the bodies. *It shall be proved that the density with which those eigen-oscillations are distributed along the frequency scale of the spectrum in the region of the high oscillation numbers is asymptotically independent of the special form of the elastic body and depends upon only its volume and the two elastic constants.* More precisely, the result is formulated as follows: If t means time then the equation for the elastic oscillations will read:

([3]) *loc. cit.* ([1]).

([4]) Cf., the remarks of E. E. LEVI on this in "I problemi dei valori al contorno per le equazioni lineari totalmente ellittiche all derivate parziali," Memorie di Matematica e di Fisica della Società Italiana delle Science (detta dei XL) (3) **16** (190), 3-11; page 8 above and footnote.

([5]) T. BOGGIO, "Determinazione della deformazione di un corpo elastico per date tensioni superficiale," Atti della Reale Accademia dei Lincei **16** (2nd semester 1907), 441-449.

(1) $$\frac{\partial^2 \mathfrak{u}}{\partial t^2} = a \text{ grad div } \mathfrak{u} - b \text{ curl curl } \mathfrak{u}\,,$$

in which a and b are two (certainly positive) elastic constants ([6]). I will always denote the expression on the right-hand side, which plays the same role in elasticity theory that $\Delta\mathfrak{u}$ does in potential theory, and will go over to it when $a = b = 1$, moreover, by $\Delta^*\mathfrak{u}$. The eigen-oscillations are characterized by the fact that the time-dependency of $\mathfrak{u}$ is given by a periodic function $e^{i\nu t}$; the constant ν is its frequency. If J means the volume of the body then the number of eigen-oscillations whose frequency ν lies below the arbitrary limit ν_0 is asymptotically amounts to:

$$\frac{J\,\nu_0^3}{6\pi^2}\left\{\left(\frac{1}{a}\right)^{3/2} + 2\left(\frac{1}{b}\right)^{3/2}\right\}$$

in the limit as ν_0 goes to infinity. I shall obtain this result by now adapting my investigations into the problem of the oscillations of a membrane in Mathematischen Annalen and Crelle's Journal ([7]) to the elastic oscillations. At the same time, I hope that this paper will yield the opportunity that I desire of being able to present that theory once more in a clearer form, since it was developed before using various approaches, and its presentation was, as a result, afflicted with various gaps and sources of incompleteness.

For $a = b = 1$, the result will include the *asymptotic spectral law of cavity radiation* that I already proved before. The asymptotic law of elastic eigen-oscillations will be taken to be the starting point for the theory of specific heats of solid bodies (the law of DULONG-PETIT and its variations) ([8]) in a manner that is analogous to the way that it is used as the basis for the modern theory of radiation. However, as one convinces oneself in the theory of radiation, that law is derived for only the special case of cubic cavities (JEAN's cube), such that DEBYE (and indeed by explicit calculation) determined that elastic law only for a spherical body. That theory, which is based upon an application of the principles of thermodynamics and the quantum hypothesis, would itself contain an unreliable contradiction if the asymptotic law that was discovered in a special example could not lay claim to any general validity because it was not independent of the form of the cavity (body, resp.). One would therefore like to believe that the rigorous proof of

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([6]) If M is the elastic modulus, σ is the ratio of lateral contraction to length dilatation, and the mass density = 1 then one will have:

$$a = \frac{M(1-\sigma)}{(1+\sigma)(1-2\sigma)}\,, \qquad b = \frac{M}{2(1+\sigma)}\,.$$

([7]) *a*) "Das asymptotische Verteilungsgesetz linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlstrahlung)," Math. Ann. **52** (1912), 441-479.

b) "Über die Abhängigkeit der Eigenschwingungen einer Membran von deren Begrenzung," J. reine angew. Math. **141** (1912), 1-11.

c) Über das Spektrum der Hohlraumstrahlung," *ibid.* **141** (1912), 163-181.

d) "Über die Randwetaufgabe der Strahlungstheorie und asymptotische Spektralgesetze," *ibid.* **143** (1913), 177-202.

([8]) P. DEBYE, in his well-known paper: "Zur Theorie der spezifischen Wärmen," Ann. Phys. (Leipzig) (4) **39** (1912), 789-839.

this "LORENTZ postulate" ([9]) would be admittedly imperative for the mathematicians, and as good as irrelevant for the physicists. I can therefore assert: If one starts with a deeper foundation of those physical theories by which thermodynamics is replaced with *static* considerations then a mathematical proof of the asymptotic frequency law for bodies of arbitrary form that we spoke of will be absolutely essential. Of course, that has still not been achieved. In order to give, e.g., an electromagnetic-static basis for KIRCHHOFF's law of emission and absorption, one must derive that asymptotic spectral law for not only homogeneous, but also arbitrary inhomogeneous media, and one must further transfer it to the eigen*values* and eigen*functions*; i.e., consider them instead of the oscillation numbers and oscillation states, respectively. I hope to be able to go into that in more detail at another time. – The study of *integral equations* has embodied the mathematical essence of the theory of oscillations by way of the viewpoint that was most clearly emphasized by HILBERT that by ascertaining the eigen-oscillations of continuous media, one is *dealing with the transformation of an infinite-dimensional ellipsoid to its principle axes*, and that viewpoint shall be the prevailing one in any representation of that theory that strives for an actual incisive comprehension of matters. It is no wonder that the method of integral equations not only allows one to carry out the existence proof of the eigen-oscillations in the most transparent way, but also proves to be powerful enough to ascertain their asymptotic distribution over the spectrum.

We establish the following *conventions:* Let the body to be examined – hence, a region of space J of volume J that lies at finite points – be bounded by the outer surface $\mathfrak{O}$. p (as well as p', p'', …) means a variable point in J, o means a variable point on $\mathfrak{O}$, dp means the spatial element at the location p under integration, and do means the outer surface element at the location o. $\mathfrak{n} = \mathfrak{n}\,(o)$ is the unit vector in the direction of the interior normal that is applied to the point o of the outer surface. If $\mathfrak{v}$ is a vector then v_x, v_y, v_z; v_n will denote its components along the x, y, z axes and the normal $\mathfrak{n}$, resp., while $\mathfrak{v}_t = \mathfrak{v} - \mathfrak{n}\,(\mathfrak{v}\,\mathfrak{n})$ will denote the projection onto the tangential plane. x, y, z is a Cartesian coordinate system in this.

As is known, the GREEN formula that is crucial for all of *potential theory* reads:

$$(G) \qquad \int_J (\operatorname{grad} u \cdot \operatorname{grad} v + u\,\Delta v)\,dp = -\int_{\mathfrak{O}} u \frac{\partial v}{\partial n}\,do\,,$$

and immediately yields the result that:

$$(G') \qquad \int_J (u\,\Delta v - v\,\Delta u)\,dp = -\int_{\mathfrak{O}} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) do\,.$$

u and v are any two functions that are continuously differentiable in J and for which Δu, Δv exist. If u is a potential function ($\neq$ const.) and we set $u = v$ in (G) then that will yield the important inequality:

([9]) H. A. LORENTZ addressed that problem in the fourth of his invited talks to the WOLFSKEHL Foundation at Göttingen on April 1910 at the request of the mathematicians.

$$(G_0) \qquad - \int_{\mathfrak{O}} u \frac{\partial u}{\partial n} do = \int_J (\text{grad } u)^2 \, dp > 0.$$

There are several formulas in *elasticity theory* that are analogous to GREEN's formulas. If $\mathfrak{u}$, $\mathfrak{v}$ are any two vector fields that are continuously differentiable in J then one can bilinearly combine the first derivatives of $\mathfrak{u}$, $\mathfrak{v}$ with respect to the coordinates x, y, z of the point p into an expression $E(\mathfrak{u}, \mathfrak{v})$ that is symmetric in $\mathfrak{u}$ and $\mathfrak{v}$ and means twice the stress (i.e., potential energy density) at the point p that is produced by the displacement for $\mathfrak{u} = \mathfrak{v}$. One will then have $E(\mathfrak{u}, \mathfrak{u}) > 0$ when $\mathfrak{u}$ is not merely an infinitesimal motion with no change in form. The formula for $E(\mathfrak{u}, \mathfrak{u})$ reads:

$$E(\mathfrak{u}, \mathfrak{u}) = \left(a - \frac{4}{3}b\right)(\text{div}\, \mathfrak{u})^2 + \frac{2b}{3}\left\{\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_z}{\partial z}\right)^2 + \cdots + \cdots\right\} + b\left\{\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)^2 + \cdots + \cdots\right\}.$$

It emerges from this that the constants a, b fulfill the inequality:

$$(2) \qquad 3a > 4b > 0$$

for any elastic body. However, we will generally make only the assumptions that $a > 0$, $b > 0$ in order to also include the case of cavity radiation ($a = b = 1$) in this. The stress that is produced inside the body by the displacement is a *tensor* $\mathbf{\Pi} = \mathbf{\Pi}(\mathfrak{u})$ that depends upon the position of the surface element against which the pressure acts. The pressure vector that is directed against a surface element whose normal is the x-axis has the components:

$$(a - 2b) \text{ div } \mathfrak{u} + 2b \frac{\partial u_x}{\partial x},$$
$$b\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right),$$
$$b\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial x}\right).$$

We denote the pressure on the outer surface element do by $\mathfrak{P} = \mathfrak{P}(\mathfrak{u})$, and the pressure that the displacement produces on the outer surface by $\mathfrak{O} = \mathfrak{P}(\mathfrak{v})$. The BETTI *formula*, which is the first analogue of GREEN's formula, reads ([10]):

$$(B) \qquad \int_J \{E(\mathfrak{u}, \mathfrak{v}) + \mathfrak{u}\, \Delta^* \mathfrak{v}\}\, dp = - \int_{\mathfrak{O}} \mathfrak{u}\, \mathfrak{O}\, do$$

([10]) I employ the notations of the Enzyklopaedie der Mathematischen Wissenschaften for vector analysis; in particular, the square bracket means the vectorial product.

and leads directly to the *reciprocity law:*

$$(B') \qquad \int_J (\mathfrak{u}\,\Delta^*\mathfrak{v} - \mathfrak{v}\,\Delta^*\mathfrak{u})\,dp = -\int_{\mathfrak{O}} (\mathfrak{u}\,\mathfrak{O} - \mathfrak{v}\,\mathfrak{P})\,do\,.$$

However, for a vector field $\mathfrak{u}$ that satisfies the equation $\Delta^*\mathfrak{u} = 0$ in J, under the assumption (2), one will have:

$$(B_0) \qquad -\int_{\mathfrak{O}} \mathfrak{u}\,\mathfrak{P}\,do = \int_J E(\mathfrak{u},\mathfrak{u})\,dp \geq 0,$$

in which the equality sign is obtained only when the infinitesimal displacement $\mathfrak{u}$ is that of a rigid body.

The second analogue of GREEN's formula, which seems to have been little noticed up to now, is:

$$(C) \qquad \int_J (a\,\mathrm{div}\,\mathfrak{u}\,\mathrm{div}\,\mathfrak{v} + b\,\mathrm{curl}\,\mathfrak{u}\,\mathrm{curl}\,\mathfrak{v} + \mathfrak{u}\,\Delta^*\mathfrak{v})\,dp = -\int_{\mathfrak{O}} (a\,u_n\,\mathrm{div}\,\mathfrak{v} + b\,\mathrm{curl}\,\mathfrak{v}[\mathfrak{n},\mathfrak{u}])\,do\,.$$

If one switches $\mathfrak{u}$ and $\mathfrak{v}$ in this and subtracts the equation that is obtained from (C) in that way then that will yield an equation (C') that will convert the spatial integral:

$$\int_J (\mathfrak{u}\,\Delta^*\mathfrak{v} - \mathfrak{v}\,\Delta^*\mathfrak{u})\,dp$$

into an outer surface integral in the same way as (B'). For a field $\mathfrak{u}$ that satisfies the equation $\Delta^*\mathfrak{u} = 0$, one will have:

$$(C_0) \qquad -\int_{\mathfrak{O}} (a\,u_n\,\mathrm{div}\,\mathfrak{v} + b\,\mathrm{curl}\,\mathfrak{v}[\mathfrak{n},\mathfrak{u}])\,do = \int_J \{a\,(\mathrm{div}\,\mathfrak{u})^2 + b\,(\mathrm{curl}\,\mathfrak{u})^2\}\,dp \geq 0,$$

in which the equality sign is obtained only when $\mathfrak{u}$ is a field that is free from sources and vortices. The proof of (C) will be provided when one first makes the replacement:

$$\mathfrak{w} = \mathfrak{u}\,\mathrm{div}\,\mathfrak{v}$$

and then:

$$\mathfrak{w} = [\mathfrak{u}, \mathrm{curl}\,\mathfrak{v}]$$

in the GAUSS equation:

$$\int_J \mathrm{div}\,\mathfrak{w}\cdot dp = -\int_{\mathfrak{O}} w_n\,do\,,$$

multiplies the first equation that arises by a, the second one by b, and adds them.

Along with (B) and (C), one must also consider all equations that one obtains from the schema $\beta(B) + \gamma(C)$; i.e., in such a way that one multiplies (B) by a positive constant β, (C), by a positive constant γ, and adds them. In particular, we use the equation:

$$(D) = \frac{a}{a+b}(B) + \frac{b}{a+b}(C)$$

and the associated reciprocity equation (D') and the inequality (D_0). The latter does not require the assumption (2) for its validity, but only $a > b / 3 > 0$. The integrand of the spatial integral that appears will then read:

$$\frac{a}{a+b}\left(a - \frac{b}{3}\right)(\operatorname{div}\mathfrak{u})^2 + \frac{2ab}{3(a+b)}\left\{\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_z}{\partial z}\right)^2 + \cdots + \cdots\right\}$$

$$+ \frac{b^2}{a+b}(\operatorname{curl}\mathfrak{u})^2 + \frac{ab}{a+b}\left\{\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)^2 + \cdots + \cdots\right\}.$$

Something that is fundamental to all of potential theory is the *basic solution* to the potential equation that corresponds to a *point source* at the origin of the coordinate system:

$$\frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

If f is a given function of p that is non-zero only in a finite domain then those solutions of the equation:

$$\Delta u = -4\pi f$$

that vanishes at infinity will be given by:

$$u(p) = \int \frac{1}{r(p, p')} f(p')\, dp', \tag{3}$$

in which $r(p, p')$ means the distance from the "reference point" p to the "source point" p' ([11]).

The static problem of elasticity consists of integrating the equation:

$$\Delta^* \mathfrak{u} = -4\pi \boldsymbol{f},$$

in which $4\pi\boldsymbol{f}$ is a given vector field, namely, the (infinitely weak) force field that brings about the deformation of the elastic body. If we imagine that all of infinite space is filled with our elastic medium then that problem will be solved by a formula that is analogous to (3):

([11]) Confer my remarks in the paper ([7]), pp. 182, footnote, for the fact that with a natural interpretation of the operator Δ, a function (3) that satisfies the equation $\Delta u = -4\pi f$ might also be a continuous function f of the kind that was obtained.

$$\mathfrak{u}(p) = \int \mathbf{P}(p, p')\,\mathfrak{f}(p')\,dp'. \tag{4}$$

In this, **P** is a tensor (which was first determined by SOMIGLIANA) that is the GREEN *tensor of elasticity theory*. It is composed additively of two parts:

$$\mathbf{P} = \frac{1}{2a}\mathbf{P}_a + \frac{1}{2b}\mathbf{P}_b\,. \tag{5}$$

If we employ rectangular coordinates for which the source point p' is the origin, and if x, y, z are the coordinates of p then:

$$\mathbf{P}_a = \begin{Vmatrix} \frac{1}{r}-\frac{x^2}{r^3} & -\frac{xy}{r^3} & -\frac{xz}{r^3} \\ -\frac{yx}{r^3} & \frac{1}{r}-\frac{y^2}{r^3} & -\frac{yz}{r^3} \\ -\frac{zx}{r^3} & -\frac{zy}{r^3} & \frac{1}{r}-\frac{z^2}{r^3} \end{Vmatrix}$$

and

$$\mathbf{P}_b = \begin{Vmatrix} \frac{1}{r}+\frac{x^2}{r^3} & \frac{xy}{r^3} & \frac{xz}{r^3} \\ \frac{yx}{r^3} & \frac{1}{r}+\frac{y^2}{r^3} & \frac{yz}{r^3} \\ \frac{zx}{r^3} & \frac{zy}{r^3} & \frac{1}{r}+\frac{z^2}{r^3} \end{Vmatrix}.$$

The multiplication $\mathbf{P}(p, p')\,\mathfrak{f}(p')$ is understood to mean matrix multiplication, when one regards $\mathfrak{f} = (f_x, f_y, f_z)$ as a column vector that consists of those components. By contrast, $\mathfrak{f}(p)\,\mathbf{P}(p, p')$ is to be interpreted as a matrix multiplication in which $\mathfrak{f}$ means a row vector with those three components. Finally, if $\mathfrak{a}$, $\mathfrak{b}$ are any two vectors then I will understand $\mathfrak{a} \times \mathfrak{b}$ to mean the tensor that comes about under matrix multiplication when I regard $\mathfrak{a}$ as a column vector and $\mathfrak{b}$ as a row vector:

$$\mathfrak{a} \times \mathfrak{b} = \begin{Vmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{Vmatrix}.$$

Each of the three columns that **P** consists of is a solution of the equation $\Delta^*\mathfrak{u} = 0$, when it is considered as a vector. In that way, the two summands $\mathbf{P}_a$ and $\mathbf{P}_b$ will correspond to the two terms from which $\Delta^*\mathfrak{u}$ is composed, such that each column in the first summand will represent an irrotational vector field, and each column of the second one will represent a source-free one.

§ 2. Solution of the first boundary-value problem of elasticity theory.

However, our problem does not consist of integrating the "inhomogeneous problem":

$$\Delta^* \mathfrak{u} = -4\pi\mathfrak{f} \tag{6}$$

for all of infinite space, but only inside of a finite body J when one of the three boundary conditions that were enumerated in the introduction is prescribed on its outer surface $\mathfrak{O}$. We begin with the first one:

$$\mathfrak{u} = 0 \quad \text{on the outer surface}$$

and call it the (inhomogeneous) problem I. We will solve it in the form:

$$\mathfrak{u}(p) = \int_J \boldsymbol{\Gamma}(p, p')\,\mathfrak{f}(p')\,dp', \tag{7}$$

in which $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{\mathrm{I}}$ (viz., the GREEN tensor that belongs to I) is nothing but the solution to the static problem in the special case for which the exciting force $\mathfrak{f}$ is concentrated at the point p'. The corresponding homogeneous solution ($\mathfrak{f} = 0$) has the unique solution $\mathfrak{u} = 0$. According to the inequality (D_0), any solution of it is then a pure translation:

$$\mathfrak{u} = \text{const.} = \mathfrak{c},$$

and since $\mathfrak{u}$ must vanish on the outer surface, one must have $\mathfrak{c} = 0$. Of course, one assumes that $a > b / 3$ in this. If one did not wish for that to be true then one would have to start with the inequality (C_0) instead; one would arrive at the same result in that way. Using SOMIGLIANA's tensor $\mathbf{P}$, we set:

$$\boldsymbol{\Gamma} = \mathbf{P} - \mathbf{A}.$$

We denote the three columns of $\mathbf{P}$ by $\mathfrak{R}_x$, $\mathfrak{R}_y$, $\mathfrak{R}_z$, respectively; we employ corresponding notations for $\boldsymbol{\Gamma}$ and $\mathbf{A}$, and for any tensor that occurs at all. In order to determine $\mathbf{A}$, one must solve the problem I^0 of *determining a field* $\mathfrak{u}$ *that satisfies the homogeneous equation* (we say briefly: *a static field*) *and is given on the outer surface.* Namely, if one considers the source point p' to be fixed then $\mathfrak{A}_x$, for example, will be a static field as a function of p and will possess the same value as $\mathfrak{R}_x$ on the outer surface, and that is indeed well-known. We employ an Ansatz for I^0 that is analogous to the one that NEUMANN employed in order to solve the first boundary-value problem of potential theory, in which we let (D) (and not perhaps one of the other possible analogues) enter in place of GREEN's formula in potential theory. In order to formulate that Ansatz, we must define the expression:

$$\frac{a}{a+b}\mathfrak{P}(\mathfrak{u})\,\mathfrak{e} + \frac{b}{a+b}(a\operatorname{div}\mathfrak{u}\cdot e_n + b\operatorname{curl}\mathfrak{u}\,[\mathfrak{u},\mathfrak{e}])$$

in terms of an otherwise-unknown vectorial distribution $\mathfrak{e}(o)$ on the outer surface, in which we replace $\mathfrak{u}$ with the three column vectors of $\mathfrak{P}(p, p')$ in sequence (in which the source point p' is fixed). The three quantities that one obtains in that way are the components of a vector:

(8) $$\mathbf{\Lambda}(p', o)\,\mathfrak{e}(o)$$

($\mathbf{\Lambda}$ means a tensor), and the Ansatz that we have to make reads:

(9) $$\mathfrak{u}(p) = \frac{1}{2\pi}\int_{\mathfrak{O}}\mathbf{\Lambda}(p,o)\,\mathfrak{e}(o)\,do.$$

In order to perform the calculation, it is convenient to base it upon a coordinate system x, y, z whose origin is found at the point o of the outer surface and whose x-axis coincides with the normal at that point. If x, y, z are the coordinates of the source point p' then one will find the following values for the three components of (8):

(10) $$\begin{cases} \dfrac{1}{a+b}\left\{\left(2b\dfrac{x}{r^3}+(a-b)\dfrac{3x^3}{r^5}\right)e_x+(a-b)\dfrac{3x^2y}{r^5}e_y+(a-b)\dfrac{3x^2z}{r^5}e_z\right\}, \\ \dfrac{1}{a+b}\left\{(a-b)\dfrac{3x^2y}{r^5}e_x+\left(2b\dfrac{x}{r^3}+(a-b)\dfrac{3xy^2}{r^5}\right)e_y+(a-b)\dfrac{3xyz}{r^5}e_z\right\}, \\ \dfrac{1}{a+b}\left\{(a-b)\dfrac{3x^2z}{r^5}e_x+(a-b)\dfrac{3xyz}{r^5}e_y+\left(2b\dfrac{x}{r^3}+(a-b)\dfrac{3xz^2}{r^5}\right)e_z\right\}. \end{cases}$$

If I denote the angle that the vector $\overrightarrow{op} = \mathfrak{r}_{po}$ of length $r_{po} = |\mathfrak{r}_{po}|$ defines with the normal at the point o by ϑ_{po} then I will obviously get the expression:

$$\mathbf{\Lambda}(p, o)\,\mathfrak{e}(o) = \frac{2b}{a+b}\frac{\cos\vartheta_{po}}{r_{po}^2}\mathfrak{e}+\frac{3(a-b)}{a+b}\frac{\cos\vartheta_{po}}{r_{po}^4}\mathfrak{r}_{po}(\mathfrak{r}_{po},\mathfrak{e})$$

from this, which is liberated from any choice of special coordinate system by its vectorial notation. If $\mathbf{E}$ is the 3×3 identity matrix then one will find that:

(11) $$\mathbf{\Lambda}(p, o) = \frac{\cos\vartheta_{po}}{r_{po}^2}\left\{\frac{2b}{a+b}\mathbf{E}+\frac{3(a-b)}{a+b}\frac{\mathfrak{r}\times\mathfrak{r}}{r^2}\right\}.$$

From a remark of FREDHOLM ([12]), the vector field $\mathbf{\Lambda}(p, o)\,\mathfrak{e}(o)$ has this simple, intuitive meaning: If one lays the tangent plane to the outer surface through the point o on it and imagines that one of the two half-spaces into which that plane will divide the total space [namely, the one for which $\mathfrak{n}(o)$ is also its interior normal] as being filled with our elastic medium and applies the force $\mathfrak{e}(o)$ at the point o then that expression will represent the deformation of the elastic half-space when the displacement is assumed to be zero on the planar outer surface. The solution to the static problem for a half-space that is bounded by a plane was first given by CERRUTI and BOUSINESQ ([13]).

We assume that not only does the outer surface $\mathfrak{O}$ possess a continuous normal, but also that it satisfies a HÖLDER condition; i.e., there shall be a positive exponent $\alpha\,(\leq 1)$ such that the angle $\eta_{oo'}$ that the normals at two neighboring points o, o' make with each other satisfies an inequality:

$$|\,\eta_{oo'}\,| \leq \text{const.}\,(r_{oo'})^{\alpha}.$$

In particular, that will be the case ($\alpha = 1$) when the outer surface is curved continuously.

In order to determine the unknown distribution $\mathfrak{e}$, we get the following *integral equation* from the Ansatz (9):

$$\mathfrak{u}(o) = \mathfrak{e}(o) + \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{\Lambda}(o, o')\,\mathfrak{e}(o')\,do'. \tag{12}$$

Since:

$$\left|\frac{\cos\vartheta_{oo'}}{r_{oo'}^2}\right| \leq \frac{\text{const.}}{(r_{oo'})^{2-\alpha}},$$

the kernel of this integral equation will be infinite of order only less than two for $o = o'$, and as a result, FREDHOLM theory will be valid for the equation itself. On just those grounds, it would be necessary to start with the analogue (D) to GREEN's formula (and not any other one); that choice would have the consequence that x must appear as a factor everywhere in the expressions (10).

Should the homogeneous integral equation that corresponds to the inhomogeneous equation (12) admit no solution (other than the trivial one 0), then no matter how the vector field $\mathfrak{u}(0)$ might be given on the outer surface, (12) would always be soluble, and the construction of the GREEN tensor $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathrm{I}}$ would then be possible. If p, p' are any two distinct points in J that are excluded from the domain of integration by two infinitely-small balls then when the reciprocity formula (D'), as applied to:

$$\mathfrak{u}(p) = \mathbf{\Gamma}(p, p'), \qquad \mathfrak{v}(p) = \mathbf{\Gamma}(p, p''),$$

([12]) *loc. cit.* ([1]).

([13]) CERRUTI, "Ricerche intorno all'equilibrio dei corpi elastici isotropi," Atti della R. Accademia dei Lincei (Roma), Ser. III: Memorie della classe di Scienze Fisiche, Mathematiche e Naturali **13** (1882), 81-123.

will yield *the symmetry of the* GREEN *tensor* $\mathbf{\Gamma}$, which states that the tensors $\mathbf{\Gamma}(p', p'')$, $\mathbf{\Gamma}(p'', p')$ will go to each other when one converts the rows into columns.

However, the fact that (at least under the assumption that $a > b / 3 > 0$) any homogeneous equation that belongs to (12) will possess no solution can be shown in a manner that is completely analogous to potential theory. However, as was mentioned already in the introduction, we cannot be satisfied with this line of reasoning that was pursued by LAURICELLA ([14]) since it cannot be adapted to the case that we shall address later. In fact, the solubility of the inhomogeneous Problem I cannot be also made independent of the insolubility of the homogeneous integral equation that belongs to (12), but must be based upon the single condition that the homogeneous Problem I admits no solution besides 0. *We therefore now assume that the homogeneous integral equation:*

$$\mathfrak{e}(0) + \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{\Lambda}(o, o')\,\mathfrak{e}(o')\,do' = 0 \tag{13}$$

possesses a solution $\mathfrak{e}(0) \neq 0$. However, for the sake of simplicity, we take the case in which no further solutions exist besides that one that are linearly independent of it. The "transposed" homogeneous equation:

$$\mathfrak{d}(0) + \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{\Lambda}(o', o)\,\mathfrak{d}(o')\,do' = 0$$

will then also possess a unique solution $\mathfrak{d}(0) \neq 0$; the integral $\int_{\mathfrak{O}} \mathfrak{e}(o)\,\mathfrak{d}(o)\,do$ is non-zero and shall be taken to be equal to 1. The inhomogeneous equation (12) is soluble then only when the given left-hand side fulfills the condition:

$$\int_{\mathfrak{O}} \mathfrak{u}(o)\,\mathfrak{d}(o)\,do = 0.$$

In order to construct:

$$\mathfrak{A}_x(p, p') = \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{\Lambda}(p, o)\,\mathfrak{R}_x(o)\,do,$$

we must actually satisfy the equation:

$$\mathfrak{R}_x(o, p') = \mathfrak{K}_x(o) + \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{\Lambda}(o, o')\,\mathfrak{K}_x(o')\,do'.$$

However, that is not generally possible now, since in order for a solution $\mathfrak{K}_x(o)$ to exist, we must replace the left-hand side with:

$$\mathfrak{R}_x(o, p') - \mathfrak{e}(o)\int_{\mathfrak{O}} \mathfrak{R}_x(o, p')\,\mathfrak{d}(o)\,do.$$

([14]) *loc. cit.* ([1]).

The solution $\mathfrak{K}_x = \mathfrak{K}_x(o, p')$ will be normalized uniquely by the condition:

$$\int_{\mathfrak{O}} \mathfrak{K}_x(o, p')\,\mathfrak{d}(o)\,do = 0.$$

It is given by:

$$\mathfrak{K}_x(o, p') = \mathfrak{R}_x(o, p') - \int_{\mathfrak{O}} \bar{\mathbf{\Lambda}}(o, o')\,\mathfrak{R}_x(o', p')\,do',$$

in which the resolvent $\bar{\mathbf{\Lambda}}$ of $\frac{1}{2\pi}\mathbf{\Lambda}$ is understood in the modified sense. If we proceed correspondingly with y and z then we will get a tensor $\mathbf{K}(o, p')$ with the column vectors $\mathfrak{K}_x$, $\mathfrak{K}_y$, $\mathfrak{K}_z$, and:

$$\mathbf{A}^*(p, p') = \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{\Lambda}(p, o)\,\mathbf{K}(o, p')\,do$$

must initially enter in place of the GREEN "compensator" that must actually be constructed. The three rows of $\mathbf{A}^*$ are obviously static fields when considered to be functions of p'. The function $\mathbf{\Gamma}^* = \mathbf{P} - \mathbf{A}^*$ does not have the boundary value 0, but one will have:

$$\mathbf{\Gamma}^*(o, p) = \mathfrak{e}(o) \times \mathfrak{g}^*(p') \qquad \{\mathfrak{g}^*(p) = \int_{\mathfrak{O}} \mathbf{P}(p, o)\,\mathfrak{d}(o)\,do\}.$$

Of course, should $\mathfrak{g}^*$ be identically zero in J, then $\mathbf{\Gamma}^*$ would be the desired GREEN function $\mathbf{\Gamma}_{\mathrm{I}}$. Otherwise, I would set $c\mathfrak{g}^*(p) = \mathfrak{g}(p)$, in which I determine the constant c in such a way that I would have $\int_J \mathfrak{g}^2\,dp = 1$, and furthermore:

$$\mathfrak{f}(p) = \int_J \mathbf{\Gamma}^*(p, p')\,\mathfrak{g}(p')\,dp',$$

and define:

$$\mathbf{\Gamma}^{**}(p, p') = \mathbf{\Gamma}^*(p, p') - \mathfrak{f}(p) \times \mathfrak{g}(p').$$

That will make:

$$\mathbf{\Gamma}^{**}(o, p') = 0.$$

However, $\mathbf{\Gamma}^{**}$ (or rather, the column vectors that $\mathbf{\Gamma}^{**}$ consists of) will not satisfy the equation $\Delta^* = 0$ as a function of p, but one will have (one understands the meaning of the notation with no further clarification):

$$\Delta_p^*\,\mathbf{\Gamma}^{**} = 4\pi\,\mathfrak{g}(p) \times \mathfrak{g}(p').$$

The row vectors of $\mathbf{\Gamma}^{**}$ satisfy the homogeneous equation $\Delta^* = 0$ as functions of p', and the relation:

$$\int_J \mathbf{\Gamma}^{**}(p, p')\,\mathfrak{g}(p')\,dp' = 0$$

will come about.

If p', p'' are any two points in J then by the same line of reasoning by which we recognized the symmetry of $\mathbf{\Gamma}_{\mathrm{I}}$ in the case of the insolubility of the homogeneous integral equation, formula (D') will yield the result that:

$$\mathbf{\Gamma}_*(p', p'') = \mathbf{\Gamma}^{**}(p', p'') + \int_J \mathfrak{g}(p)\,\mathbf{\Gamma}^{**}(p, p')\,dp \times \mathfrak{g}(p'')$$

is symmetric. The rows vectors of $\mathbf{\Gamma}_*(p, p')$ are static fields with respect to p', so due to symmetry, the column vectors will also have that property as functions of p. When one sets:

$$\int_J \mathfrak{g}(p)\,\mathbf{\Gamma}^{**}(p, p')\,dp = \mathfrak{f}^*(p'),$$

the boundary values will be:

$$\mathbf{\Gamma}_*(o, p') = \mathfrak{f}^*(o) \times \mathfrak{g}(p') .$$

If one also shifts p' here to a boundary point $o' \neq o$ then that will yield the symmetry law:

$$\mathfrak{f}^*(o) \times \mathfrak{g}(o') = \mathfrak{g}\,(o) \times \mathfrak{f}^*(o'). \tag{14}$$

At this point, we now make use of the fact that the homogeneous Problem I possesses no solution besides 0, from which, we conclude that $\mathfrak{g}(o)$ cannot be identically = 0 on the outer surface, since otherwise $\mathfrak{g}(p)$ would have to vanish in all of J. As a result, (14) will imply the existence of a constant c^* such that:

$$\mathfrak{f}^*(o) = -\,c^*\mathfrak{g}\,(o),$$

so

$$\mathbf{\Gamma}(p, p') = \mathbf{\Gamma}_*(p, p') + c^*\,\mathfrak{g}\,(p) \times \mathfrak{g}(p')$$

will then have the boundary value 0, and the desired GREEN function will be $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathrm{I}}$. It can be left to the reader to carry out the analogous considerations in the event that the homogeneous integral equation (13) should possess more than one linearly-independent solution ([15]).

If we now once more assume that this integral equation is insoluble then we can raise the objection to the proof of the symmetry of the tensor $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathrm{I}}$ that was suggested on pp. 12: The proof assumed that the quantities:

$$S_n = \frac{a}{a+b}[P_n(\mathfrak{u}) + b \operatorname{div} \mathfrak{u}], \quad \mathfrak{S}_t = \frac{1}{a+b}\{a\,\mathfrak{P}_n(\mathfrak{u}) + b^2\,[\operatorname{curl} \mathfrak{u}, \mathfrak{u}]\},$$

([15]) For our argument, confer also E. E. LEVI, *loc. cit.* ([4]), pp. 11-14, and HILBERT, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig, B. G. Teubner, 1912, pp. 227-232.

which require single *differentiations* for their determination, exist on the outer surface, when one replaces $\mathfrak{u}$ with each of the columns of $\mathbf{\Gamma}_{\mathrm{I}}$. If we assume without proof, for the moment, that this is actually the case, and regard the three quantities $\mathfrak{S}_x$, $\mathfrak{S}_y$, $\mathfrak{S}_z$ that correspond to the column vectors $\mathfrak{u}$ of $\mathbf{\Gamma}_{\mathrm{I}}$ as the column vectors of a tensor $\mathbf{\Sigma}(o, p')$ then the solution of the Problem I will be given by the formula:

$$4\pi\,\mathfrak{u}(p) = \int_{\mathfrak{O}} \mathbf{\Sigma}(o, p)\,\mathfrak{u}(o)\,do,$$

which one sees when one applies equation (D') in such a way that one excludes p' from the domain of integration by an infinitely-small ball and takes $\mathfrak{v}$ to be each of the three column vectors of $\mathbf{\Gamma}_{\mathrm{I}}$ in turn. On the other hand, we have already solved this problem in the form:

(15) $$2\pi\,\mathfrak{u}(p) = \int_{\mathfrak{O}} \mathbf{\Theta}(o, p)\,\mathfrak{u}(o)\,do,$$

in which:

(16) $$\mathbf{\Theta}(p, o) = \mathbf{\Lambda}(p, o) - \int_{\mathfrak{O}} \mathbf{\Lambda}(p, o')\,\mathbf{\Lambda}(o', o)\,do'.$$

We must then have:

(17) $$\mathbf{\Sigma}(o, p) = 2\,\mathbf{\Theta}(p, o).$$

Under the assumption of the existence of $\mathbf{\Sigma}$, we can then calculate its value (= $2\mathbf{\Theta}$). Therefore, it is obvious that we can prove equation (17) directly by an altered arrangement of this train of thought, and also without the assumption of the existence of $\mathbf{\Sigma}$. This can come about perhaps as follows:

By construction, one has:

(18) $$\mathbf{A}(p, p') = \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{\Theta}(p, o)\,\mathbf{P}(o, p')\,do.$$

Any row of $\mathbf{A}(p, p')$ will then be a static field as a function of p'; as a result, one must have:

$$\mathbf{A}(p, p') = \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{A}(p, o')\,\mathbf{\Theta}(p', o')\,do',$$

and one will get the formula:

$$4\pi^2\,\mathbf{A}(p, p') = \int_{\mathfrak{O}}\int_{\mathfrak{O}} \mathbf{\Theta}(p, o)\,\mathbf{P}(o, o')\,\mathbf{\Theta}(p', o')\,do\,do',$$

which exhibits the symmetry of the tensor $\mathbf{A}$ with no further assumptions. One can then write:

$$\mathbf{\Gamma}(p, p') = \mathbf{P}(p, p') - \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{P}(p, o)\,\mathbf{\Theta}(p', o)\,do,$$

and the existence of $\mathbf{\Sigma}(o, p')$ can emerge from this equation with no doubt. One easily glimpses how that train of thought will be modified when the homogeneous integral equation (13) has solutions. In that way, perhaps in expectation of objections that point to the non-existence of certain quantities on the outer surface, the proof will also be easy to invalidate for the following boundary-value problem by a similar rearrangement. I shall go only so far into the question of the modifications that will perhaps be necessary for any further problems as a result of the solubility of the homogeneous integral equation, since I regard that point as having been likewise resolved by our argument above and would not like to get lost in the details (in this merely preparatory chapter).

Let one further point be mentioned here: One can say that the insolubility of the homogeneous Problem I:

$$\Delta^* \mathfrak{u} = 0 \quad \text{in } J, \qquad \mathfrak{u} = 0 \quad \text{on the outer surface}$$

is solved here under only the assumption that the expression $\mathfrak{S}$ that is formed from $\mathfrak{u}$ as above is finite on the outer surface. I do not think that it is necessary to discuss the extent to which I can free myself from that assumption; suffice it to say that at the single location where we make use of this law of the insolubility of the homogeneous problem [namely, where we conclude that $\mathfrak{g}(o)$ is not identically zero], we will, in fact, be dealing with a function $\mathfrak{u} = \mathfrak{g}(o)$ for which the expression $\mathfrak{S}$ exists on the outer surface and is continuous.

§ 3. Solution of the second boundary-value problem.

We now go on to Problem II:

$$\Delta^* \mathfrak{u} = -4\pi \mathfrak{f} \quad \text{in } J,$$

$$\operatorname{div} \mathfrak{u} = 0, \qquad \mathfrak{u}_t = 0 \quad \text{on the outer surface.}$$

The corresponding homogeneous Problem ($\mathfrak{f} = 0$) has no solution besides $\mathfrak{u} = 0$ when the space J is bounded by a single surface, which we would, in fact, like to assume (if only for the sake of simplicity). Namely, (C_0) will imply that a solution $\mathfrak{u}$ of the homogeneous problem will fulfill the identities:

$$\operatorname{curl} \mathfrak{u} = 0, \qquad \operatorname{div} \mathfrak{u} = 0$$

in all of J. If we set $\mathfrak{u}$ equal to identically zero outside of J then $\mathfrak{u}$ will be irrotational in all space, and due to the boundary condition $\mathfrak{u}_t = 0$, no surface vortices will exist on $\mathfrak{O}$, either. As a result:

$$\mathfrak{u} = \operatorname{grad} \varphi, \qquad \Delta\varphi = 0,$$

and the boundary condition $\mathfrak{u}_t = 0$ says that φ is a constant on $\mathfrak{O}$ (viz., the derivative of φ in the tangential direction is everywhere zero); hence, φ is equal to a constant in all of J, and $\mathfrak{u} = 0$. This result is nothing but the well-known fact that no electrostatic field can exist inside of a conductor ([16]).

Ansatz for the solution of the inhomogeneous problem:

$$\mathfrak{u}(p) = \int_J \boldsymbol{\Gamma}_{\mathrm{II}}(p, p')\,\mathfrak{f}(p')\,dp', \qquad \boldsymbol{\Gamma}_{\mathrm{II}} = \mathbf{P} - \mathbf{A}_{\mathrm{II}}.$$

In order to find the column vectors of $\mathbf{A} = \mathbf{A}_{\mathrm{II}}$, we must address the following problem: *Find a static field* $\mathfrak{u}$ *in J for which* div $\mathfrak{u}$, $\mathfrak{u}$ *are known on the outer surface.* The correct way will be given by formula (*C*). With the help of a scalar distribution $s(o)$ and a vectorial one $\mathfrak{e}(o)$, we then define:

$$-\,a\,u_n\,s(o) + b\,\operatorname{curl}\mathfrak{u}\,[\mathfrak{u}, \mathfrak{e}],$$

in which we replace $\mathfrak{u}$ with the three column vectors of $\mathbf{P}(p, p')$ in turn. If we introduce a rectangular coordinate system x, y, z with o as its origin in the same way as before (pp. 10) then we will get:

$$-\left(\frac{a+b}{2br} + \frac{a-b}{2b}\frac{x^2}{r^3}\right)s - \frac{y}{r^3}e_y - \frac{z}{r^3}e_z,$$
$$-\frac{a-b}{2b}\frac{xy}{r^3}s + \frac{x}{r^3}e_y,$$
$$-\frac{a-b}{2b}\frac{xz}{r^3}s + \frac{x}{r^3}e_z.$$

We now once more denote the point (x, y, z) by p, instead of p'. The divergence with respect to p of the vector whose x, y, z-components are the quantities that were just ascertained is:

$$= \frac{x}{r^3}\,s(o).$$

The vector itself can then be represented by:

$$-\frac{a-b}{2br}\mathfrak{n}\,s - \frac{a-b}{2b}\frac{\cos\vartheta}{r^2}\mathfrak{r}\,s + \frac{\cos\vartheta}{r^2}\mathfrak{e} - \frac{\mathfrak{n}}{r^3}(\mathfrak{r}\,\mathfrak{e}),$$

and our Ansatz will then read:

$$(19)\quad 2\pi\,\mathfrak{u}(p) = -\int_{\mathfrak{O}}\left(\frac{a+b}{2br_{po}}s(o) + \frac{(\mathfrak{r}_{po}\,\mathfrak{e}(o))}{r_{po}^3}\right)\mathfrak{n}(o)\,do + \int_{\mathfrak{O}}\frac{\cos\vartheta_{po}}{r_{po}^2}\left(\mathfrak{e}(o) - \frac{a-b}{2b}\mathfrak{r}_{po}\,s(o)\right)do.$$

([16]) If J is bounded by $h + 1$ surfaces then the homogeneous problem will have precisely h linearly-independent solutions. Cf., WEYL, *loc. cit.* ([7]), pp. 184 and pp. 188, *et seq.*

That will imply that:

$$(20)\qquad (\operatorname{div} \mathfrak{u})_o = s(o) + \frac{1}{2\pi}\int \frac{\cos\vartheta_{oo'}}{r_{00'}^2} s(o')\,do' .$$

From this equation, as is indeed known to be uniquely possible, one can imagine determining $s(o)$, and with its help:

$$(21)\qquad 4\pi b\ \mathfrak{v}(o) = (a + b) \int_{\mathfrak{O}} \frac{1}{r}\mathfrak{n}(o')\,s(o')\,do' + (a-b)\int_{\mathfrak{O}} \frac{\cos\vartheta_{oo'}}{r_{oo'}^2}\mathfrak{r}\,s(o')\,do' .$$

In order to write down the integral equation that is true for $\mathfrak{e}(o)$, one represents the projection $\mathfrak{a}_t$ of a vector $\mathfrak{a}(o)$ by:

$$\mathfrak{a}(o) - \mathfrak{n}(o)(\mathfrak{a}(o)\ \mathfrak{n}(o)),$$

and introduces the vector:

$$\mathfrak{n}(o') - \mathfrak{n}(o)(\mathfrak{n}(o)\ \mathfrak{n}(o')) = \mathfrak{n}_{oo'}$$

(which vanishes for $o = o'$), along with the tensor:

$$(22)\qquad \mathbf{\Lambda}_{\mathrm{II}} = \mathbf{\Lambda}(o, o') = \frac{\cos\vartheta_{oo'}}{r_{oo'}^2}\{\mathbf{E} - \mathfrak{n}(o) \times \mathfrak{n}(o)\} - \frac{\mathfrak{n}_{oo'} \times \mathfrak{r}_{oo'}}{r_{oo'}^3} .$$

That equation will then read:

$$(23)\qquad \mathfrak{e}_t(o) + \frac{1}{2\pi}\int \mathbf{\Lambda}(o,o')\,\mathfrak{e}(o')\,do' = \mathfrak{u}_t(o) + \mathfrak{v}_t(o).$$

The product $\mathbf{\Lambda}(o, o')\ \mathfrak{e}(o')$ depends upon only $\mathfrak{e}_t(o')$, since that is in the nature of our Ansatz. However, by this construction, one will have not only:

$$\mathbf{\Lambda}(o, o')\ \mathfrak{n}(o') = 0,$$

but also

$$\mathfrak{n}(o)\ \mathbf{\Lambda}(o, o') = 0.$$

We can then determine $\mathfrak{e}$ by means of the equation:

$$(24)\qquad \mathfrak{e}(o) + \frac{1}{2\pi}\int \mathbf{\Lambda}(o,o')\,\mathfrak{e}(o')\,do' = \mathfrak{a}(o) + \mathfrak{v}(o),$$

in which we take $\mathfrak{a}(o)$ to be any vector distribution on the outer surface with the property that $\mathfrak{a}_t = \mathfrak{u}_t$.

We cannot exclude the possibility that the homogeneous equation:

(25) $$\mathfrak{e}(o) + \frac{1}{2\pi}\int \mathbf{\Lambda}(o, o')\,\mathfrak{e}(o')\,do' = 0$$

is soluble *a priori*; that is realized, e.g., when the body J is a torus. However, as we know, that fact does not prevent us from implementing the construction of the tensor $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathrm{II}}$ by means of the inhomogeneous equation. It follows from formula (C') in a known way that this tensor is symmetric with respect to the source and reference point.

Since the column vectors of $\mathbf{\Gamma}_{\mathrm{II}}\,(p, p')$ must be normal to the outer surface when considered as functions of p, $\mathbf{\Gamma}_{\mathrm{II}}\,(o, p')$ must have the form:

$$\mathbf{\Gamma}_{\mathrm{II}}\,(o, p') = \mathfrak{n}(o) \times \mathfrak{g}(o, p').$$

If one poses the problem of determining a static field that is normal to the boundary and for which one is given that div $\mathfrak{u} = l(o)$ on the outer surface then an application of formula (C') to $\mathfrak{u}$ and each of the three column vectors of $\mathbf{\Gamma}_{\mathrm{II}}$ (in which p' must initially be excluded from the domain of integration by an infinitely-small ball) will imply that this solution can be only the following one:

(26) $$-4\pi\,\mathfrak{u}(p) = \int_{\mathfrak{O}} \mathfrak{g}(o, p)\,l(o)\,do\,.$$

On the other hand, the fact that [at least, under the assumption that the homogeneous equation (25) is insoluble] the problem that is posed in this form with the help of a certain vector $\mathfrak{g}$ always *can* be solved is implied directly by our existence proof above, which makes it possible to construct $\mathfrak{g}$ without appealing to the tensor $\mathbf{\Gamma}_{\mathrm{II}}$. The column vectors of:

(27) $$\left\{\begin{aligned} \mathbf{B}_{\mathrm{II}}(p, p') &= \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{\Theta}(p, o)(\mathfrak{n}(o) \times \mathfrak{g}(o, p'))\,do \\ &= \frac{1}{2\pi}\int_{\mathfrak{O}} (\mathbf{\Theta}(p, o)(\mathfrak{n}(o)) \times \mathfrak{g}(o, p'))\,do \end{aligned}\right.$$

are regular, static fields as functions of p (even at the location p'), and one has:

$$\mathbf{B}_{\mathrm{II}}(o, p') = \mathfrak{n}(o) \times \mathfrak{g}(o, p').$$

Both properties split $\mathbf{B}_{\mathrm{II}}$ into $\mathbf{\Gamma}_{\mathrm{II}} - \mathbf{\Gamma}_{\mathrm{I}}$, and one must then have:

$$\mathbf{B}_{\mathrm{II}} = \mathbf{\Gamma}_{\mathrm{II}} - \mathbf{\Gamma}_{\mathrm{I}}\,, \qquad \mathbf{\Gamma}_{\mathrm{II}} = \mathbf{\Gamma}_{\mathrm{I}} + \mathbf{B}_{\mathrm{II}}\,.$$

Naturally, it can be confirmed directly that the latter formula actually yields a tensor $\mathbf{\Gamma}_{\mathrm{II}}$ with the property that we demand.

When one observes precisely the way by which the constants a and b enter into the tensor $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathrm{II}}$, one will see that:

(28) $$\mathbf{\Gamma}^* = \frac{1}{a}\mathbf{\Gamma}_a + \frac{1}{b}\mathbf{\Gamma}_b,$$

in which $\mathbf{\Gamma}_a$, $\mathbf{\Gamma}_b$ are completely independent of the constants a and b. We would like to show that *any column vector of* $\mathbf{\Gamma}_a$ *is irrotational, and any of the three column vectors of* $\mathbf{\Gamma}_a$ *is source-free, and one has:*

$$\mathbf{\Gamma}_a\,\mathbf{\Gamma}_b = 0, \qquad \mathbf{\Gamma}_b\,\mathbf{\Gamma}_a = 0,$$

moreover, when we perform the composition of the kernel matrices in the present way:

$$\mathbf{\Gamma}_a\,\mathbf{\Gamma}_b\,(p, p') = \int_J \mathbf{\Gamma}_a(p, p'')\,\mathbf{\Gamma}_a(p'', p')\,dp''.$$

In fact, we have:

(29) $$\left\{\begin{aligned} \mathfrak{u} &= \frac{1}{a}\mathfrak{u}_a + \frac{1}{b}\mathfrak{u}_b \\ &= \frac{1}{a}\int_J \mathbf{\Gamma}_a(p, p')\,\mathfrak{f}(p')\,dp' + \frac{1}{b}\int_J \mathbf{\Gamma}_b(p, p')\,\mathfrak{f}(p')\,dp' \end{aligned}\right.$$

as the solution to Problem II, in which $\mathfrak{u}_a$, $\mathfrak{u}_b$ are independent of the elastic constants. If one takes the divergence of both sides of the equation $\Delta^*\mathfrak{u} = -4\pi\,\mathfrak{f}$ then that will give:

(30) $$a \cdot \Delta(\text{div}\,\mathfrak{u}) = -4\pi \cdot \text{div}\,\mathfrak{f},$$

and the divergence of $\mathfrak{u}$ will then be independent of the constants, in light of the boundary condition div $\mathfrak{u} = 0$. One must then have:

$$\text{div}\,\mathfrak{u}_b = 0,$$

and the absence of sources for the column vectors of $\mathbf{\Gamma}_b$ will thus be proved. If follows further from (30) under the assumption that is div $\mathfrak{f} = 0$ identically that:

$$\text{div}\,\mathfrak{u} = 0,$$

and our equations can then be written:

$$\Delta\mathfrak{u} = -4\pi b\,\mathfrak{f}.$$

Since this differential equation, together with the boundary conditions, determines $\mathfrak{u}$ uniquely, $\mathfrak{u}$ will depend upon only b now, and one will then have:

(31) $$\int_J \mathbf{\Gamma}_a(p, p')\,\mathfrak{f}(p')\,dp' = 0,$$

in the event that div $\mathfrak{f} = 0$. In particular, if one applies this result to $\mathfrak{f} = \mathfrak{u}_b$ then it will follow that:

$$\boldsymbol{\Gamma}_a \boldsymbol{\Gamma}_b = 0,$$

and due to the symmetry of $\boldsymbol{\Gamma}_a$ and $\boldsymbol{\Gamma}_b$, one will also have $\boldsymbol{\Gamma}_a \boldsymbol{\Gamma}_b = 0$. The two components $\boldsymbol{\Gamma}_a$, $\boldsymbol{\Gamma}_b$ will then be orthogonal to each other. It follows from GAUSS's law:

$$\int_J \operatorname{div} \mathfrak{w} \cdot dp = - \int_{\mathfrak{O}} w_n \, do \,,$$

when we set $\mathfrak{w} = [\mathfrak{a}, \mathfrak{b}]$ and assume that $\mathfrak{a}$ is directed normal to the outer surface, that:

$$\int_J \operatorname{div} (\mathfrak{a} \operatorname{curl} \mathfrak{b} - \mathfrak{b} \operatorname{curl} \mathfrak{a}) \, dp = 0.$$

If we understand $\mathfrak{b}$ to mean an arbitrary vector field and set $\mathfrak{f} = \operatorname{curl} \mathfrak{b}$ in (31), and understand $\mathfrak{a}$ to be mean any of the three row vectors in $\boldsymbol{\Gamma}_a$ then this equation (in which we integrate over p', instead of p) will imply that the curl of $\mathfrak{a}$ that is taken with respect to p' will vanish. Due to its symmetry, each of the three column vectors of $\boldsymbol{\Gamma}_a$ will then be irrotational as a function of p, or (what amounts to the same thing) that:

$$\operatorname{curl} \mathfrak{u}_a = 0.$$

Since $\mathfrak{u}_a$ is directed normal to the boundary, in addition, we conclude that $\mathfrak{u}_a$ is the gradient of a scalar field φ_a that has a constant value on the outer surface and we can then take it to be equal to zero there. Equation (30) will then go to:

$$\Delta\Delta\varphi_a = - 4\pi \cdot \operatorname{div} \mathfrak{f}.$$

φ_a and φ_b vanish on the outer surface. If G denotes the usual GREEN function, and GG denotes its iteration:

$$GG(p, p') = \int_J G(p, p'') G(p'', p') \, dp''$$

then that will imply that:

$$- 4\pi \, \varphi_a(p) = \int_J GG(p, p') \operatorname{div} \mathfrak{f}(p') \, dp' .$$

If x, y, z are the coordinates of p, x', y', z' are the coordinates of p', and we introduce the tensor:

$$\mathbf{H} = \left\| \begin{matrix} \frac{\partial^2}{\partial x \partial x'} GG & \frac{\partial^2}{\partial x \partial y'} GG & \frac{\partial^2}{\partial x \partial z'} GG \\ \frac{\partial^2}{\partial y \partial x'} GG & \frac{\partial^2}{\partial y \partial y'} GG & \frac{\partial^2}{\partial y \partial z'} GG \\ \frac{\partial^2}{\partial z \partial x'} GG & \frac{\partial^2}{\partial z \partial y'} GG & \frac{\partial^2}{\partial z \partial z'} GG \end{matrix} \right\|,$$

which we can probably denote by:

$$\mathrm{grad}_p \, \mathrm{grad}_{p'} \, GG \, (p, p')$$

then it will turn out that:

$$4\pi \, \mathfrak{u}_a (p) = \int_J \mathbf{H}(p, p') \, \mathfrak{f}(p') \, dp',$$

and $4\pi \, \mathbf{\Gamma}_a$ will then be identical with the tensor **H**:

$$\mathbf{\Gamma}_a = \frac{1}{4\pi} \mathrm{grad}_p \, \mathrm{grad}_{p'} \, G \, G(p, p'). \tag{32}$$

What we have actually achieved with our method above is then the determination of the tensor $\mathbf{\Gamma}_b$. Since it is independent of the constants a, b, we can choose, e.g., $a = b = 1$ in order to find it, *and the coupling of elasticity theory with potential theory will be exhibited in that way.* Naturally, the possibility of such a decomposition (28) rests upon the special kind of boundary conditions II, and can in no way be adapted to Problems I and II.

§ 4. Solution of the third boundary-value problem.

In order for Problem III:

$$\Delta^* \mathfrak{u} = -4\pi \, \mathfrak{f} \quad \text{in } J, \qquad \mathfrak{P} = 0 \ \text{ on } \mathfrak{O}$$

to be soluble, the force field $4\pi \, \mathfrak{f}$ on the body must preserve its equilibrium as a rigid body, which is a requirement that is expressed by six linear integral conditions on $\mathfrak{f}$. If, for the sake of convenience, we employ a rectangular coordinate system whose origin lies at the center of mass of the body (of mass density 1) and whose coordinate axes coincide with the principle axes of inertia at the center of mass then the associated homogeneous problem will, in fact, have these six solutions:

$$\mathfrak{a}_1 = \left(\frac{1}{M}, 0, 0 \right), \qquad \mathfrak{a}_2 = \left(0, \frac{1}{M}, 0 \right), \qquad \mathfrak{a}_3 = \left(0, 0, \frac{1}{M} \right);$$

$$\mathfrak{a}_4 = \left(0, \frac{z}{R}, -\frac{y}{R}\right), \qquad \mathfrak{a}_5 = \left(-\frac{z}{S}, 0, \frac{x}{S}\right), \qquad \mathfrak{a}_6 = \left(\frac{y}{T}, -\frac{x}{T}, 0\right).$$

M^2 means the mass of J, while R^2, S^2, T^2 are the three principle moments of inertia. With that special choice of coordinate system, these six vectors will be mutually orthogonal and normalized:

$$\int_J \mathfrak{a}_i \, \mathfrak{a}_j \, dp = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases} \qquad (i, j = 1, 2, 3; 4, 5, 6).$$

When we set $\mathfrak{u}$ equal to one of the vectors $\mathfrak{a}_i$ in equation (B), but set $\mathfrak{v}$ equal to the desired solution $\mathfrak{u}$ of Problem III that will give:

(33) $$\int_J \mathfrak{f} \, \mathfrak{a}_i \, dp = 0 \qquad (i = 1, 2, \ldots, 6),$$

and that was our assertion. On the other hand, the solution of III cannot be unique, since one can add an arbitrary linear combination of the $\mathfrak{a}_i$ to $\mathfrak{u}$ without affecting equations III; it is only when one adds the normalized equations:

(34) $$\int_J \mathfrak{u} \, \mathfrak{a}_i \, dp = 0 \qquad (i = 1, 2, \ldots, 6)$$

that the solution will become unique. Our problem then consists of *finding the solution of* III *that satisfies the corresponding normalized equations* (34) *under the assumption of* (33). Formula (B_0) guarantees that no solutions to the homogeneous problem will exist in addition to the linear combinations of $\mathfrak{a}_i$; naturally, we now assume that $3a > 4b > 0$, which holds true for any elastic body. The inhomogeneous problem in the formulation that was just given shall now, in turn, be integrated by an equation:

$$\mathfrak{u}(p) = \int_J \mathbf{\Gamma}(p, p')\, \mathfrak{f}(p')\, dp'$$

that involves a GREEN tensor $\mathbf{\Gamma} = \mathbf{\Gamma}_{\text{III}}$ that must still be determined. We initially make $\mathbf{P}$ orthogonal to the $\mathfrak{a}_i$, while respecting its symmetry, and thus replace it with:

(35) $$\left\{ \begin{aligned} \dot{\mathbf{P}}(p, p') = \mathbf{P}(p, p') &- \sum_{i=1}^{6} \mathfrak{a}_i(p) \times \int_J \mathfrak{a}_i(p'')\, \mathbf{P}(p'', p')\, dp'' \\ &- \sum_{i=1}^{6} \int_J \mathbf{P}(p, p'')\, \mathfrak{a}_i(p'') \times \mathfrak{a}_i(p')\, dp'' \\ &+ \sum_{i,j=1}^{6} \mathfrak{a}_i(p) \times \mathfrak{a}_j(p') \int_J \int_J \mathfrak{a}_i(p)\, \mathbf{P}(p, p')\, \mathfrak{a}_j(p')\, dp\, dp'. \end{aligned} \right.$$

The column vectors of $\dot{\mathbf{P}}$ no longer satisfy the equation $\Delta^* = 0$ as functions of p, but rather when the process Δ^* is performed column-wise with respect to p, one will have:

$$\Delta^* \dot{\mathbf{P}} = 4\pi \sum_{i=1}^{6} \mathfrak{a}_i(p) \times \mathfrak{a}_i(p').$$

We make the Ansatz:

$$\mathbf{\Gamma}_{\text{III}} = \dot{\mathbf{P}} - \mathbf{A}_{\text{III}}$$

and seek to determine the column vectors:

$$\mathfrak{u}(p) = \mathfrak{A}_x(p, p'), \qquad \mathfrak{A}_y(p, p'), \qquad \mathfrak{A}_z(p, p')$$

of $\mathbf{A} = \mathbf{A}_{\text{III}}$ as static fields for which $\mathfrak{P}(u)$ assumes the same value on the outer surface as it does for:

$$\mathfrak{u} = \dot{\mathfrak{R}}_x(p, p'), \qquad \dot{\mathfrak{R}}_x(p, p'), \qquad \dot{\mathfrak{R}}_x(p, p').$$

The problem:

(36) $$\Delta^* \mathfrak{u} = 0 \quad \text{in } J \qquad \mathfrak{P}(u) = \text{given vector } \mathfrak{p}(o) \text{ on } \mathfrak{O}$$

can certainly be soluble only when:

$$\int_J \mathfrak{p}(o)\, \mathfrak{a}_i(o)\, do = 0 \qquad (i = 1, 2, \ldots, 6).$$

The proof is based upon formula (B) when one replaces $\mathfrak{u}$ with $\mathfrak{a}_i$, but $\mathfrak{v}$ with the solution $\mathfrak{u}$ of (36). However, those of the three vectors $\mathfrak{p} = \mathfrak{p}_x(o, p')$, $\mathfrak{p}_y(o, p')$, $\mathfrak{p}_z(o, p')$ for which the solution of the problem is required in order to determine the column vectors of $\mathbf{A}_{\text{III}}$ do, in fact, satisfy those linear integral conditions. One shows that when one replaces $\mathfrak{u}$ with $\mathfrak{a}_i$ in equation (B), but $\mathfrak{v}$ with each of the three column vectors of $\dot{\mathbf{P}}$; naturally, one must initially exclude p' from the domain of integration by a small ball. It was precisely in order to fulfill those conditions that we replaced $\mathbf{P}$ with $\dot{\mathbf{P}}$.

The closely-related Ansatz:

(37) $$\mathfrak{u}(p) = \frac{1}{2\pi} \int_{\mathfrak{O}} \mathbf{P}(p, o)\, \mathfrak{e}(o)\, do$$

for the solution of (36) is, as emerged from § **2**, not useful, since it will not lead to any regular integral equation for the unknown distribution $\mathfrak{e}(o)$. We refer to the vector (37) as the *elasticity vector that emerges from the outer surface distribution* $\mathfrak{e}$. Along with the outer surface distribution, we must mount an "antenna distribution" on $\mathfrak{O}$. I focus my attention upon a single location o of the outer surface and erect the exterior normal at o; the assumption shall be made initially that this normal does not re-enter the body when it is lengthened. I employ rectangular coordinates x, y, z for which o is the origin and the interior normal coincides with the positive x-axis. I then define the function:

$$V = x \ln(r + x) - r$$

and its derivatives:

$$\frac{\partial V}{\partial x} = \ln(r + x), \qquad \frac{\partial V}{\partial y} = -\frac{y}{r+x}, \qquad \frac{\partial V}{\partial z} = -\frac{z}{r+x}.$$

$\partial V / \partial x$ is the potential of an electromagnetic field that is generated by the exterior normal when I think of it as an antenna that is uniformly distributed with electricity. $\partial V / \partial x$ is the potential of a "dipole antenna" that we will obtain when we fold together two antennas that start from the point o in the xy-plane and lie symmetric to the x-axis, and are uniformly charged with equal and opposite electricity, and upon that passage to the limit, the densities of electricity increase simultaneously in such a way that the field will take on a finite magnitude. V itself will then be a potential function (as one also confirms easily by calculation). With its help, we now construct the tensor (p means the point x, y, z):

$$\mathbf{Y}(p, o) = \left\| \begin{matrix} \dfrac{\partial^2 V}{\partial x^2} & -\dfrac{\partial^2 V}{\partial x\,\partial y} & -\dfrac{\partial^2 V}{\partial x\,\partial z} \\ -\dfrac{\partial^2 V}{\partial x\,\partial y} & \dfrac{\partial^2 V}{\partial x^2} - \dfrac{\partial^2 V}{\partial z^2} & \dfrac{\partial^2 V}{\partial y\,\partial z} \\ \dfrac{\partial^2 V}{\partial x\,\partial z} & \dfrac{\partial^2 V}{\partial y\,\partial z} & \dfrac{\partial^2 V}{\partial x^2} - \dfrac{\partial^2 V}{\partial y^2} \end{matrix} \right\|.$$

The divergence of any column vector $\mathfrak{y}$ of $\mathbf{Y}$ is obviously = 0, and since one is dealing with nothing but potential functions, one will also have the equation curl curl $\mathfrak{y} = 0$ for every column vector then, and therefore $\Delta^* \mathfrak{y} = 0$. If one forms the pressure $\mathfrak{P}(\mathfrak{y})$ that acts upon a surface element parallel to the yz-plane and corresponds to the displacement field $\mathfrak{y}$ and employs the equation:

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{r}$$

for the calculation then one will get:

$$\frac{1}{b}\mathfrak{P}(\mathfrak{y}_x) = \left(2\frac{\partial}{\partial x}\frac{1}{r}, 2\frac{\partial}{\partial y}\frac{1}{r}, 2\frac{\partial}{\partial z}\frac{1}{r}\right),$$

$$\frac{1}{b}\mathfrak{P}(\mathfrak{y}_y) = \left(-2\frac{\partial}{\partial y}\frac{1}{r}, -\frac{\partial^3 V}{\partial x\,\partial y^3} + \frac{\partial^3 V}{\partial x^3} - \frac{\partial^3 V}{\partial x\,\partial z^3}, 0\right) \qquad \left\{= 2\frac{\partial^3 V}{\partial x^3} = 2\frac{\partial}{\partial x}\frac{1}{r}\right\},$$

and analogously for $\mathfrak{P}(\mathfrak{y}_z)$, such that the tensor that consists of these three pressure vectors will read:

$$-2b\left\|\begin{matrix} \frac{x}{r^3} & -\frac{y}{r^3} & -\frac{z}{r^3} \\ \frac{y}{r^3} & \frac{x}{r^3} & 0 \\ \frac{z}{r^3} & 0 & \frac{x}{r^3} \end{matrix}\right\|.$$

$$\mathfrak{u}(p) = \frac{1}{2\pi}\int_{\mathfrak{O}} \mathbf{Y}(p,o)\,\mathfrak{e}(o)\,do,$$

and I call it the *elasticity vector that emerges from the "antenna distribution"* $\mathfrak{e}(o)$. It can be constructed in the event that none of the exterior normals to the body J enter the body J except at their base points (which is the case for, e.g., convex bodies).

In order to solve our problem (36), *we combine an outer surface with an antenna distribution*, in which we write:

$$\mathfrak{u}(p) = \frac{1}{2\pi}\int_{\mathfrak{O}} \Xi(p,o)\,\mathfrak{e}(o)\,do, \tag{38}$$

where:

$$\Xi = \frac{1}{a-b}\left(\tfrac{1}{2}\mathbf{Y} - a\,\mathbf{P}\right).$$

Ξ is nothing but the solution that was first given by BOUSSINESQ ([17]) to the static problem of elasticity theory for a half-space that is bounded by the tangent plane at o when zero pressure acts upon its planar outer surface. If one then defines the pressure that is associated with the column vector $\mathfrak{x}$ of Ξ then one will get the tensor:

$$\left\|\begin{matrix} \frac{3x^3}{r^5} & \frac{3x^2y}{r^5} & \frac{3x^2z}{r^5} \\ \frac{3x^2y}{r^5} & \frac{3xy^2}{r^5} & \frac{3xyz}{r^5} \\ \frac{3x^2z}{r^5} & \frac{3xyz}{r^5} & \frac{3xz^2}{r^5} \end{matrix}\right\|,$$

and that will vanish for $x = 0$. From this, it is also clear that the boundary condition:

$$\mathfrak{P}(\mathfrak{u}) = \mathfrak{p}(o)$$

([17]) *Loc. cit.*, ([13]).

for the vector field (38) will go to an integral equation:

$$\mathfrak{e}(o) + \frac{1}{2\pi}\int \mathbf{\Lambda}(o, o')\,\mathfrak{e}(o')\,do' = \mathfrak{p}(o), \tag{39}$$

whose kernel tensor has the property ([18]):

$$|\,\mathbf{\Lambda}(o, o')\,| \leq \frac{\text{const.}}{(r_{oo'})^{2-\alpha}}\,.$$

Should the assumption that the external normals to the body meet nowhere not be fulfilled, then we would modify our Ansatz in such a way that we would cap off all of our antennas at a constant height that is chosen to be small enough that the capped antennas would no longer penetrate the body *J*. The analytical formulation of this idea can be implemented with no difficulty.

In each case, equation (39) is soluble only when:

$$\int \mathfrak{p}(o)\,\mathfrak{a}_i(o)\,do = 0 \qquad (i = 1, 2, \ldots, 6). \tag{40}$$

The associated homogeneous equation then has at least six linearly-independent solutions. If it possesses no others then the conditions (40) will be sufficient for its solubility, and the construction of the GREEN tensor $\mathbf{\Gamma} = \mathbf{\Gamma}_{\text{III}}$ will be completed. However, the construction of the GREEN tensor cannot break down, due to the fact that the homogeneous equation might possess even more solutions (and it is impossible to exclude that case from the outset), as we showed in § **2**. By a column-wise application of the process Δ^* with respect to the variable *p*, we will have the equation:

$$\Delta^* \mathbf{\Gamma}_{\text{III}} = 4\pi \sum_{i=1}^{6} \mathfrak{a}_i(p) \times \mathfrak{a}_i(p')\,. \tag{41}$$

We can take care of that in such a way that:

$$\int_J \mathfrak{a}_i(p)\,\mathbf{\Gamma}_{\text{III}}(p, p')\,dp \qquad (i = 1, 2, \ldots, 6). \tag{42}$$

The symmetry of the tensor $\mathbf{\Gamma} = \mathbf{\Gamma}_{\text{III}}$ can be deduced from (41) and (42) by an application of the Betti formula (B').

Once we have found that tensor, we can assert that the Problem (36) can be solved by only the formula:

$$4\pi\,\mathfrak{u}(p) = \int_J \mathbf{\Gamma}_{\text{III}}(o, p)\,\mathfrak{p}(o)\,do$$

([18]) The absolute value $|\,\mathbf{\Lambda}\,|$ of a tensor $\mathbf{\Lambda}$ is the square root of the sum of the squares of its nine components.

when we demand that $\mathfrak{u}$ must be orthogonal to the six $\mathfrak{a}_i$. The proof of this will be simple to carry out when we apply the Betti formula (B') to $\mathfrak{u}$ and each of the three column vectors of $\mathbf{\Gamma}_{\mathrm{III}}$. However, once we have already recognized [under the assumption (40)] the existence of such a solution in the form:

$$2\pi\,\mathfrak{u}(p) = \int_{\mathfrak{O}} \mathbf{Z}(p,o)\,\mathfrak{p}(o)\,do\,,$$

we can conclude that the tensor $\mathbf{Z}$ that appears in it must (essentially) coincide with $-\frac{1}{2}\mathbf{\Gamma}_{\mathrm{III}}(o, p)$. If we introduce the kernel $\dot{\mathbf{\Gamma}}_{\mathrm{I}}(p, p')$ that arises from $\mathbf{\Gamma}_{\mathrm{I}}$ by the same process (35) that produces $\dot{\mathbf{P}}$ from $\mathbf{P}$ then:

(43)
$$\mathbf{\Gamma}_{\mathrm{III}} - \dot{\mathbf{\Gamma}}_{\mathrm{I}} = \mathbf{B}_{\mathrm{III}} \qquad \text{and} \qquad -\frac{1}{\pi}\int_{\mathfrak{O}} \mathbf{\Theta}(p,o)\,\mathbf{Z}(p',o)\,do$$

will be essentially identical; i.e., they will differ by only an expression of the form:

$$\sum_{i,j=1}^{6} a_{ij}\,\mathfrak{a}_i(p)\times\mathfrak{a}_i(p')$$

with constant coefficients a_{ij} .

§ 5. Behavior of the compensating Green tensors in the limit.

A series of estimates will be necessary for the asymptotic laws that will be developed in Part Two, which shall be summarized here.

(I)
$$\int_{\mathfrak{O}} \frac{|\cos\vartheta_{po}|}{r_{po}^2}\,do \leq \text{const.}$$

for all p inside of J. – The integrand is the spatial angle under which the outer surface element do appears from the point p outwards. For convex surfaces, for example, the validity of the inequality is clear with no further discussion. In order to prove it in general, we establish the point o_1 that has the least distance from p such that the point po_1 is perpendicular to the outer surface, and draw the tangent plane to the surface at the point o_1 . We compare the integral with the one that arises from it when we let the integration point o run through that tangent plane, rather than the given surface.

(II) *If $K(o, o')$ is a kernel that satisfies the inequality:*

$$|K(o, o')| \leq \frac{\text{const.}}{(r_{oo'})^{2-\alpha}} \qquad (0 < \alpha \leq 1)$$

then its resolvent will also fulfill an inequality that reads the same way.

For the iterated kernels, one finds successively:

$$| K^m(o, o') | \leq \frac{\text{const.}}{(r_{oo'})^{2-m\alpha}} \qquad (m = 1, 2, 3, \dots)$$

as long as the exponent $2 - m\alpha$ is positive. For that reason, there is a well-defined index n for which:

$$| K^n(o, o') | \leq \text{const.}$$

Its resolvent $\bar{K}^n$ will then be restricted in the same way that would emerge from FREDHOLM's theory. The resolvent $\bar{K}$ of the original kernel K is:

$$= (K + K^2 + \dots + K^{n-1}) + (\bar{K}^n + \bar{K}^n K + \bar{K}^n K^2 + \cdots + \bar{K}^n K^{n-1}) .$$

That elucidates the validity of our assertion. It is also clear how the consideration is to be extended when the concept of resolvent must be understood in a modified sense due to the solubility of the homogeneous integral equation.

(III) *If $f(p, o)$ is a function with the properties that:*

$$| f(p, o) | \leq \frac{\text{const.}}{r_{po}^2}, \qquad \int_{\mathfrak{O}} | f(p,o) | do \leq \text{const.},$$

and one has the estimate:

$$| g(o, o') | \leq \frac{\text{const.}}{(r_{oo'})^{2-\alpha}}$$

for the function $g(o, o')$ then the function:

$$F(p, o) = \int_{\mathfrak{O}} f(p,o)\, g(o',o)\, do' \tag{44}$$

will satisfy an inequality:

$$| F(p, o) | \leq \frac{\text{const.}}{(r_{oo'})^{2-\alpha}} .$$

In order to prove this, (if p and o are given) I decompose the outer surface into two parts: Part [1] consists of all points o' for which $r_{o'o} \leq \frac{1}{2} r_{po}$, while all of the other ones belong to the Part [2]. Since $\int | g(o', o) | do'$, when it is extended over the domain of the points o' for which $r_{o'o} \leq \varepsilon$, will be:

$$\leq \text{const. } \varepsilon^{\alpha}$$

for all e (> 0), the integral F will have the absolute value:

$$\leq \frac{\text{const.}}{r_{po}^2} \int_{[1]} |\, g(o', o) \,|\, do' \leq \frac{\text{const.}}{r_{po}^{2-\alpha}}$$

when it is taken over only the part [1] of the outer surface (in which one has $r_{po'} \geq \frac{1}{2} r_{po}$). However, when one integrates over Part [2], in which $r_{o'o} > \frac{1}{2} r_{po}$, that will yield an absolute value:

$$\leq \frac{\text{const.}}{r_{po}^{2-\alpha}} \int_{\mathfrak{O}} |\, f(p, o') \,|\, do' \leq \frac{\text{const.}}{r_{po}^{2-\alpha}}.$$

(IV) *If:*

$$|\, f(p, o) \,| \leq \frac{\text{const.}}{r_{po}}$$

in the rule (III) *then one will have:*

$$|\, F(p, o) \,| \leq \frac{\text{const.}}{r_{po}^{1-\alpha}}$$

(*in the event that* $\alpha < 1$).

In order to prove this, one must replace the outer surface $\mathfrak{O}$ with a plane; one will then be dealing with an estimate for:

$$F_*(p, o) = \int \frac{do'}{r_{po'} (r_{oo'})^{2-\alpha}},$$

in which the integration is extended over the entire plane. Let o_1 be the base point of the altitude that dropped from p to the plane. If:

$$r_{po_1} \leq r_{oo_1}, \qquad \text{so} \qquad r_{po} \leq \sqrt{5} \cdot r_{oo_1},$$

then one concludes that:

$$r_{oo_1} < \tfrac{1}{2} r_{po_1}, \qquad \text{so} \qquad r_{po} < \frac{\sqrt{5}}{4} r_{po_1},$$

If one then draws the circle $\mathfrak{K}$:

$$r_{o_1 o'} \leq r_{po_1}$$

around o_1, which is contained with the circle:

$$r_{oo'} \leq \tfrac{3}{2} r_{po_1},$$

and integrates over only $\mathfrak{K}$ then one will get a value that is:

$$\leq \frac{1}{r_{po_1}} \int_{\mathfrak{K}} \frac{do'}{(r_{oo'})^{2-\alpha}} \leq \frac{1}{r_{po_1}} \cdot \frac{2\pi}{\alpha} \left(\frac{3}{2} r_{po_1} \right)^{\alpha} \leq \frac{\text{const.}}{r_{po}^{1-\alpha}}.$$

For all points outside of $\mathfrak{K}$, one will have $r_{o_1 o'} \geq \frac{2}{3} r_{oo'}$, so the integral over that external region will be:

$$\leq \frac{3}{2} \int_{(r_{oo'} \geq \frac{1}{2} r_{po_1})} \frac{do'}{(r_{oo'})^{3-\alpha}} \leq \frac{\text{const.}}{r_{po}^{1-\alpha}},$$

and the proof is also achieved in that case.

(V) *If $f(p, o)$ has the property:*

$$|f(p, o)| \leq \frac{\text{const.}}{r_{po}^2}, \qquad \int_{\mathfrak{O}} |f(p, o)| \, do \leq \text{const.}$$

and one has:

$$|g(o, p)| \leq \frac{\text{const.}}{r_{po}}$$

then:

$$F(p, p') = \int_{\mathfrak{O}} f(p, o) \, g(o, p') \, do$$

will satisfy the estimate:

$$|F(p, p')| \leq \frac{\text{const.}}{R(p, p')},$$

in which $R(p, p')$ means the minimum of $r_{po} + r_{p'o}$ when o runs through the entire outer surface (hence, the light ray from p to p' with a single reflection from $\mathfrak{O}$).

I set $R(p, p') = \varepsilon$ and divide $\mathfrak{O}$ into two parts: The first one [1] consists of all points o for which one has $r_{p'o} \leq \varepsilon / 2$; one will also have $r_{po} \geq \varepsilon / 2$ there. Now, since the integral

$$\int_{(r_{po} \leq \varepsilon)} \frac{do}{r_{po}} \leq \text{const.}\ \varepsilon$$

for any positive value of ε (here, "const." means "independent of p and ε"), the integral F over that first part will have an absolute value that is:

$$\leq \frac{\text{const.}}{\varepsilon^2} \int_{[1]} \frac{do}{r_{p'o}} \leq \frac{\text{const.}}{\varepsilon}.$$

Integrating over the rest of the outer surface, for which $r_{p'o} > \varepsilon / 2$, yields:

$$\left| \int_{[2]} \right| \leq \frac{\text{const.}}{\varepsilon} \int_{\mathfrak{O}} | f(p,o) | do \leq \frac{\text{const.}}{\varepsilon}.$$

We now go on to the application of these estimates to our problem in elasticity. In § **2**, we learned how to integrate the equation $\Delta^* \mathfrak{u} = 0$ with given boundary values $\mathfrak{u}(o)$ in the form:

$$\mathfrak{u}(p) = \frac{1}{2\pi} \int_{\mathfrak{O}} \mathbf{\Theta}(p,o)\, \mathfrak{u}(o)\, do.$$

The problem was reduced to an integral equation with the kernel $\mathbf{\Lambda}(o, o')$. If we now denote the tensor $\mathbf{\Lambda}(p, o)$ that appeared there by $\mathbf{\Lambda}_{\mathrm{I}}(o, o')$, in order to avoid confusion, then we would have [eq. (16)]:

$$\mathbf{\Theta}(p, o) = \mathbf{\Lambda}_{\mathrm{I}}(p, o) - \int_{\mathfrak{O}} \mathbf{\Lambda}_{\mathrm{I}}(p,o)\, \bar{\mathbf{\Lambda}}_{\mathrm{I}}(o',o)\, do'.$$

Due to (I), in addition to:

$$| \mathbf{\Lambda}_{\mathrm{I}}(p, o) | \leq \frac{\text{const.}}{r_{po}^2}, \quad \text{one also has} \quad \int_{\mathfrak{O}} | \mathbf{\Lambda}_{\mathrm{I}}(p,o) |\, do \leq \text{const.},$$

and by means of (II) and (III), it will follow from this that:

(VI) $$| \mathbf{\Theta}(p, o) | \leq \frac{\text{const.}}{r_{po}^2}, \quad \int_{\mathfrak{O}} | \mathbf{\Theta}(p,o) |\, do \leq \text{const.}$$

The compensating GREEN function $\mathbf{A}_{\mathrm{I}}$ is expressed by equation (18), and we then conclude from (V) that:

$$| \mathbf{A}_{\mathrm{I}}(p, p') | \leq \frac{\text{const.}}{R(p, p')}.$$

We would like to adapt this estimate to the other two tensors $\mathbf{A}_{\mathrm{II}}$, $\mathbf{A}_{\mathrm{III}}$, as well. We will then show that:

(VII) *The compensating GREEN tensors:*

$$\mathbf{A} = \mathbf{A}_{\mathrm{I}}, \mathbf{A}_{\mathrm{II}}, \mathbf{A}_{\mathrm{III}}$$

satisfy an inequality ([19]):

$$| \mathbf{A}(p, p') | \leq \frac{\text{const.}}{R(p, p')}.$$

([19]) This inequality expresses the idea that **A** can only be infinite when p and p' converge to the same boundary point of J. It is, I believe, the natural, and at the same time sharpest, estimate that one can exhibit for the GREEN compensators in that regard.

I shall first speak of $\mathbf{A}_{\mathrm{III}}$. The tensor $\mathbf{Z}(p, o)$ that appeared in § **4**, which proves to coincide with $-\mathbf{\Gamma}_{\mathrm{III}}(o, p)$, was given by the formula:

$$\mathbf{Z}(p, o) = \mathbf{\Xi}(p, o) - \int_{\mathfrak{O}} \mathbf{\Xi}(p, o')\,\overline{\mathbf{\Lambda}}_{\mathrm{III}}(o', o)\,do',$$

in which I have replaced the symbol $\mathbf{\Lambda}$ that was employed there with $\mathbf{\Lambda}_{\mathrm{III}}$ for the sake of clarity ([20]). It emerges from (II) and (IV) that:

$$|\,\mathbf{Z}(p, o)\,| \leq \frac{\text{const.}}{r_{po}}.$$

According to (V), the tensor:

$$-\frac{1}{\pi}\int_{\mathfrak{O}} \mathbf{\Theta}(p, o)\,\mathbf{Z}(p', o)\,do,$$

which is essentially identical to $\mathbf{\Gamma}_{\mathrm{III}} - \dot{\mathbf{\Gamma}}_{\mathrm{III}} = \mathbf{B}_{\mathrm{III}}$, will have an absolute value that is $\leq \dfrac{\text{const.}}{R(p, p')}$, and the equality can then be asserted for $\mathbf{A}_{\mathrm{III}}$.

When this estimate is adapted to the one that was treated in § **3**, and one solves the problem:

$$\Delta^{*}\mathfrak{u} = 0 \quad \text{in } J; \qquad \mathfrak{u}_t = 0, \qquad \operatorname{div}\mathfrak{u} = l(o) \ \text{ on } \mathfrak{O}$$

by way of:

$$-4\pi\,\mathfrak{u}(p) = \int_{\mathfrak{O}} \mathfrak{g}(o, p)\,l(o)\,do,$$

that will create a certain difficulty. I shall now express $\mathfrak{u}(p)$ by means of the function $s(o)$ that was given by (20), instead of $l(o) = (\operatorname{div}\mathfrak{u})_0$. When I determine $\mathfrak{v}(o)$ by means of (21) and then $\mathfrak{e}(o)$ on the basis of equation (24), from which the term $\mathfrak{a}(o)$ is dropped, I will get:

$$\mathfrak{e}(o) = \int_{\mathfrak{O}} \mathfrak{h}(o, o')\,s(o')\,do',$$

in which one has the inequality:

$$|\,\mathfrak{h}(o, o')\,| \leq \frac{\text{const.}}{r_{oo'}}$$

for the vector $\mathfrak{h}$. If I introduce this into (19) then that will yield an expression:

$$2\pi\,\mathfrak{u}(p) = \int_{\mathfrak{O}} \mathfrak{j}(p, o)\,s(o)\,do.$$

The following terms then appear in $\mathfrak{j}(p, o)$:

([20]) And the overbar again means taking the resolvent.

$$-\frac{a+b}{2b}\frac{1}{r_{po}}\mathfrak{u}(o), \qquad -\frac{a-b}{2b}\frac{\cos\vartheta_{po}}{r_{po}^2}\mathfrak{r}_{po}, \qquad \int_{\mathfrak{O}}\frac{\cos\vartheta_{po'}}{r_{po'}^2}\mathfrak{h}(o',o)\,do'.$$

These are all $\leq \dfrac{\cos\vartheta_{po'}}{r_{po'}}$ in absolute value (the last one, from III). However, in addition, $2\pi\,\mathfrak{u}(p)$ will contain the term:

$$\int_{\mathfrak{O}}\frac{(\mathfrak{r}_{po}\,\mathfrak{e}(o))}{r_{po'}^3}\,\mathfrak{n}(o)\,do.$$

The difficulty originates in it. We write $\mathfrak{e}(o)$ in the form:

$$\mathfrak{e}(o) = \mathfrak{v}(o) - \frac{1}{2\pi}\int_{\mathfrak{O}}\mathbf{\Lambda}_{\mathrm{II}}(o,o')\,\mathfrak{e}(o')\,do',$$

and then [see eq. (21)] have to deal with the expressions:

a) $$\int_{\mathfrak{O}}\frac{(\mathfrak{r}_{po}\,\mathfrak{n}(o))}{r_{po'}^3}\frac{1}{r_{o'o}}\,\mathfrak{n}(o')\,do,$$

b) $$\int_{\mathfrak{O}}\frac{(\mathfrak{r}_{po}\,\mathfrak{r}_{oo'})}{r_{po'}^3}\frac{\cos\vartheta_{o'o}}{r_{o'o}^2}\,\mathfrak{n}(o')\,do',$$

c) $$\int_{\mathfrak{O}}\frac{\mathfrak{r}_{po'}\,\mathbf{\Lambda}_{\mathrm{II}}(o',o)}{r_{po'}^3}\times\mathfrak{n}(o')\,do' = \mathbf{K}\,(p,o).$$

I split off the term:

$$\int_{\mathfrak{O}}\frac{(\mathfrak{r}_{po'}\,\mathfrak{n}(o'))}{r_{po'}^3}\frac{1}{r_{o'o}}\,\mathfrak{n}(o')\,(\mathfrak{n}(o)\,\mathfrak{n}(o))\,do'$$

from *a*), which will have an absolute value that is $\leq \dfrac{\text{const.}}{r_{po}}$, according to (III). What will remain is:

$$\int_{\mathfrak{O}}\frac{(\mathfrak{r}_{po'}\,\mathfrak{n}_{o'o})}{r_{po'}^3}\frac{1}{r_{o'o}}\,\mathfrak{n}(o')\,do'.$$

The normal at the point o has the components (1, 0, 0); that integral will then read:

$$\int_{\mathfrak{O}}\frac{y_{po'}\,n_y(o')+z_{po'}\,n_z(o')}{r_{po'}^3}\frac{1}{r_{o'o}}\,\mathfrak{n}(o')\,do'.$$

Of the two terms in the sum, I shall examine only the first one, and perhaps its *x*-component:

$$\int_{\mathfrak{O}} \frac{y_{po'}}{r_{po'}^3} \frac{1}{r_{o'o}} n_x\, n_y(o')\, do'.$$

Let $\overrightarrow{po_1}$ be the shortest line segment from p to $\mathfrak{O}$. We write this integral in the form:

$$\frac{n_x n_y(o_1)}{r_{po}} \int_{\mathfrak{O}} \frac{y_{po'}}{r_{po'}^3} do' + \int_{\mathfrak{O}} \frac{y_{po'}}{r_{po'}^3} \left\{ \frac{n_x n_y(o')}{r_{o'o}} - \frac{n_x n_y(o_1)}{r_{po}} \right\} do'.$$

The first integral that appears in this remains unrestricted for all p, which one shows most simply when one replaces the outer surface with the tangent plane at the point o_1. Since:

$$|\, n_x\, n_y(o_1)\,| \le |\, n_y(o_1)\,| \le \text{const.}\ (r_{oo_1})^\alpha \le \text{const.}\ r_{po}^\alpha,$$

the first term will have an absolute value that is:

$$\le \frac{\text{const.}}{r_{po}^{1-\alpha}}$$

as a result. In the second term, I decompose the term that is placed in curly brackets, in turn, into two:

$$\frac{n_x n_y(o') - n_x n_y(o_1)}{r_{o'o}} \qquad \text{and} \qquad n_x\, n_y(o_1)\left(\frac{1}{r_{o'o}} - \frac{1}{r_{po}}\right).$$

The first one has an absolute value:

$$\le \text{const.} \frac{(r_{o_1 o'})^\alpha}{r_{o'o}} \le \text{const.} \frac{r_{po'}^\alpha}{r_{o'o}},$$

and the integral that it defines will then be:

$$\le \text{const.} \int_{\mathfrak{O}} \frac{do'}{r_{po'}^{2-\alpha}\, r_{o'o}} \le \frac{\text{const.}}{r_{po}^{1-\alpha}} \qquad (\alpha < 1).$$

The absolute value of the second one is found to be:

$$\le \frac{r_{po'}}{r_{po'}^{2-\alpha}\, r_{o'o}} |\, n_y(o_1)\,|,$$

and the integral that it defines will then be:

$$\leq \frac{\text{const.}}{r_{po}^{1-\alpha}} \int_{\mathfrak{O}} \frac{do'}{r_{po'}\, r_{o'o}} \leq \frac{\text{const.}}{r_{po}^{1-\alpha}} \cdot \ln \frac{1}{r_{po}} .$$

a) is achieved with that. *b*) is treated in an analogous way, in which one has to observe that when o', o_1, o are any points on the outer surface ([21]):

$$| \cos \vartheta_{o'o} - \cos \vartheta_{o_1 o} | \leq \text{const.}\ (r_{o'o_1})^{\alpha} .$$

([21]) It emerges from this that one can see the validity of this inequality in the event that all three of the points o, o', o_1 lie close to each other. If I employ a coordinate system x, y, z) with o as its origin and whose x-axis is normal at o then the equation of the outer surface $\mathfrak{O}$ might read:

$$x = f(y, z), \qquad (y^2 + z^2 \leq c^2)$$

in the vicinity of o, and o', o_1 might belong to the neighborhood that this represents. The projections of the points o', o_1 onto the yz-plane might be called (y', z'), (y_1, z_1). If:

$$r_{oo_1} < 2 r_{o'o_1}, \qquad \text{so} \qquad r_{oo'} < 3 r_{o'o_1},$$

then the assertion will be correct, since:

$$| \cos \vartheta_{o_1 o} | \leq \text{const.}\ (r_{o_1 o})^{\alpha} \leq \text{const.}\ (r_{o'o_1})^{\alpha},$$

and an analogous inequality will be true for $\cos \vartheta_{o'o}$. In the other case ($r_{oo_1} \geq 2 r_{o'o_1}$), one will have:

$$r_{oo'} \geq r_{oo_1} - r_{o'o_1} \geq \frac{1}{2} r_{oo_1} .$$

One must estimate $\dfrac{x'}{r_{oo'}} - \dfrac{x_1}{r_{oo_1}}$. One has:

$$x' = y' \int_0^1 \frac{\partial f}{\partial y}(ty', tz')\,dt + z' \int_0^1 \frac{\partial f}{\partial z}(ty', tz')\,dt ,$$

$$x_1 = y_1 \int_0^1 \frac{\partial f}{\partial y}(ty_1, tz_1)\,dt + z_1 \int_0^1 \frac{\partial f}{\partial z}(ty_1, tz_1)\,dt .$$

In addition, I define:

$$x_1' = y' \int_0^1 \frac{\partial f}{\partial y}(ty_1, tz_1)\,dt + z' \int_0^1 \frac{\partial f}{\partial z}(ty_1, tz_1)\,dt .$$

Since the difference between the values of $\partial f / \partial y$ at the location y, z, whose distance is $= \varepsilon$, itself proves to be $\leq$ const. ε^{α}, one will have:

$$| x' - x_1' | \leq \text{const.}\ r_{oo'} (r_{o'o_1})^{\alpha} .$$

In addition:

$$| x_1 - x_1' | \leq \text{const.}\ r_{o'o_1} (r_{oo_1})^{\alpha} ,$$

but:

$$r_{o'o_1} (r_{oo_1})^{\alpha} = (r_{o'o_1})^{\alpha} (r_{o'o_1})^{\alpha} (r_{oo_1})^{\alpha} \leq (r_{o'o_1})^{\alpha} \cdot \left(\frac{r_{oo_1}}{2}\right)^{\alpha} \cdot r_{oo_1} = \frac{1}{2^{1-\alpha}} r_{oo_1} (r_{o'o_1})^{\alpha} \leq 2^{\alpha} r_{oo'} (r_{o'o_1})^{\alpha} .$$

In total:

$$| x_1 - x' | \leq \text{const.}\ r_{o'o_1} (r_{oo_1})^{\alpha} ,$$

$$\left| \frac{x'}{r_{oo'}} - \frac{x_1}{r_{oo_1}} \right| \leq \frac{| x_1 - x' |}{r_{oo'}} + \left| x_1 \left(\frac{1}{r_{oo'}} - \frac{1}{r_{oo_1}} \right) \right| .$$

The first term in the sum on the right is $\leq$ const. $(r_{o'o_1})^{\alpha}$, while the second one is:

$$\leq \frac{|x_1|\, r_{o'o_1}}{r_{oo'}\, r_{oo_1}} \leq \text{const.}\ \frac{(r_{oo_1})^{1+\alpha}\, r_{o'o_1}}{r_{oo'}\, r_{oo_1}} = C \frac{(r_{oo_1})^{1+\alpha}\, r_{o'o_1}}{r_{oo'}\, r_{oo_1}} \leq 2C \frac{r_{o'o_1}}{(r_{oo_1})^{1-\alpha}} \leq \frac{2C}{2^{1-\alpha}} (r_{o'o_1})^{\alpha} .$$

The inequality in the text is proved with that. On this subject, cf., WEYL, *loc. cit.* ([7]), *a*), pp. 476, *et seq.*

That shows that the expression *b*) also has an absolute value that proves to be:

$$\leq \frac{\text{const.}}{r_{po}^{1-\alpha}} \ln \frac{1}{r_{po}}.$$

One introduces (22) for $\mathbf{\Lambda}_{\text{II}}$ in *c*); one finds by a corresponding treatment that *c*) has the absolute value:

$$\leq \frac{\text{const.}}{r_{po}^{2-\alpha}} \ln \frac{1}{r_{po}}.$$

The term:

$$\int_{\mathfrak{O}} \mathbf{K}(p, o') \mathfrak{h}(o', o)\, do'$$

that occurs in $\mathfrak{j}(p, o)$ will then have an absolute value:

$$\leq \frac{\text{const.}}{r_{po}^{1-\alpha}} \ln \frac{1}{r_{po}}.$$

We have considered all terms with that, and we have:

$$|\, \mathfrak{j}(p, o) \,| \leq \frac{\text{const.}}{r_{po}}.$$

If $M(o, o')$ denotes the resolvent to $\frac{1}{2\pi} \frac{\cos \vartheta_{oo'}}{r_{oo'}^2}$ then one will finally have:

$$\tfrac{1}{2}\, \mathfrak{g}(o, p) = -\, \mathfrak{j}(p, o) + \int_{\mathfrak{O}} \mathfrak{j}(p, o') M(o', o)\, do',$$

and one will have, in turn:

$$|\, \mathfrak{g}(o, p) \,| \leq \frac{\text{const.}}{r_{po}}.$$

As a result of (V), one will then get the inequality:

$$|\, \mathbf{B}_{\text{II}}(p, p) \,| \leq \frac{\text{const.}}{R(p, p')}$$

for the tensor $\mathbf{B}_{\text{II}}$ that is calculated from formula (27).

We conclude from that result that for all three GREEN tensors, we will have:

$$|\, \mathbf{\Gamma}(p, p) \,| \leq \frac{\text{const.}}{r_{pp'}}. \tag{45}$$

The $\mathbf{\Gamma}$ then exhibit kernel matrices to which the FREDHOLM-HILBERT theory of integral equations is applicable. One has that they *possess infinitely many discrete eigenvalues* at the singularity. I shall denote the sum of the three elements of the principal diagonal of $\mathbf{B} = \mathbf{B}_{\mathrm{II}}$ or $\mathbf{B}_{\mathrm{III}}$ by $B(p)$ for $p = p'$. In the Part Two, we will prove that $B(p) \geq 0$. We would have to call the integral $\int_J B(p)\,dp$ the *integral trace* of the tensor $\mathbf{B}$, in the event that it is finite. Our inequalities imply that when we understand $r(p) = \frac{1}{2}R(p, p)$ to mean the shortest distance from the point o to the outer surface $\mathfrak{O}$, we will have:

$$|B(p)| \leq \frac{\text{const.}}{r(p)}.$$

Of course, the finitude of the integral trace of $\mathbf{B}$ cannot be concluded from that, but we will see at least that it is only logarithmically infinite. We formulate that precisely as follows:

If ε is understood to mean a small positive number then we cut out a thin shell J_ε from the body J that lies on the outer surface with the thickness ε (i.e., a region whose points p will be characterized by the inequality $r(p) \leq \varepsilon$), and the following inequality will be true for all ε (< 1) ([22]):

$$\text{volume of } J_\varepsilon \leq \text{const. } \varepsilon, \qquad \int_{J-J_\varepsilon} \frac{dp}{r(p)} \leq \text{const. } \ln \frac{1}{\varepsilon}.$$

(VIII) *We will then have:*

$$\int_{J-J_\varepsilon} B(p)\,dp \leq \text{const. } \ln \frac{1}{\varepsilon}. \tag{46}$$

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([22]) The second of these inequalities is a consequence of the first one. From it, the shell S_n that is characterized by inequality:

$$\frac{1}{2^{n+1}} \leq r(p) \leq \frac{1}{2^n}$$

will then have a volume that is ≤ const. $\frac{1}{2^n} = \frac{C}{2^n}$ (for all n). The integral $\int_{S_n} \frac{dp}{r(p)}$ is $\leq 2^{n+1} S_n \leq 2C$, and as a result:

$$\int_{S_1+S_2+\cdots+S_n} \frac{dp}{r(p)} \leq 2Cn,$$

and that was our assertion. We will get the proof of the first inequality when we sets out to show that a body that is enveloped by an outer surface with continuous normals will possess a well-defined volume. We employ fine cubic nets and count the cubes that have points in common with J_ε "frame-wise" (i.e., parallel to the x, y, and z axis).

CHAPTER II.

THE REGULARITIES IN THE "SPECTRUM" OF AN ELASTIC BODY THAT ARE INDEPENDENT OF FORM

§ 6. Three general theorems on integral equations.

Let:

$$K(x, \xi) \qquad (0 \le x, \xi \le 1),$$

like all the other kernels that we shall consider in these paragraphs, be symmetric and such that the usual FREDHOLM-HILBERT theory is valid. The system of reciprocal, positive eigenvalues of K, which are arranged in a sequence, will be denoted by l_n (n = 1, 2, 3, …), and the associated eigenfunctions that define an orthonormal system will be denoted by $u_n(x)$. If only finitely many l_n are present then we will extend that sequence to an infinite one by the addition of nothing but zeroes. For kernels K', K^*, etc., that differ by upper indices, we shall employ the same symbols to distinguish the quantities l_n and $u_n(x)$ that belong to them. The theorem upon which we shall base our further investigations into elastic oscillation, which is both simple and rich in consequences, reads:

THEOREM I. – *If K is the sum of two kernels K′ + K″ then the following relation will exist:*

$$l_{m+n+1} \le l'_{m+1} + l''_{n+1} \qquad (m, n = 0, 1, 2, \ldots). \tag{47}$$

Proof: For all functions $v(x)$ whose square integral $\int_0^1 v^2 dx \le 1$, one has the inequalities:

$$\left\{ \begin{aligned} &\int_0^1\int_0^1 \left\{ K'(x,\xi) - \sum_{i=1}^{m} l'_i u'_i(x) u'_i(x) \right\} v(x)\, v(\xi)\, dx\, d\xi \le l'_{m+1}, \\ &\int_0^1\int_0^1 \left\{ K''(x,\xi) - \sum_{i=1}^{n} l''_i u''_i(x) u''_i(x) \right\} v(x)\, v(\xi)\, dx\, d\xi \le l''_{n+1}. \end{aligned} \right. \tag{48}$$

Now, if $l_{m+n+1} \ne 0$ then the eigenfunctions $u_i(x)$ (i = 1, 2, …, $m + n$ + 1) will exist, from which we define:

$$v(x) = \sum_{i=1}^{m+n+1} b_i\, u_i(x)$$

as a linear combination. The constants b_i in this shall be determined in such a way that $v(x)$ is orthogonal to all of the functions:

$$u'_1(x),\ u'_2(x),\ \ldots,\ u'_m(x); \qquad u''_1(x),\ u''_2(x),\ \ldots,\ u''_n(x).$$

That gives $m + n$ linear, homogeneous equations for the unknowns b_i ; we can then take care to insure that $v(x)$ fulfills the normalization condition:

$$\int_0^1 v^2 dx \equiv b_1^2 + b_2^2 + \cdots + b_{m+n+1}^2 = 1.$$

For this special function $v(x)$ the left-hand sides of the two inequalities (48) will be equal to:

$$\int_0^1\int_0^1 K'(x,\xi)\,v(x)\,v(\xi)\,dx\,d\xi\,, \quad \int_0^1\int_0^1 K''(x,\xi)\,v(x)\,v(\xi)\,dx\,d\xi\,, \text{ resp.,}$$

so their sum will be:

$$\int_0^1\int_0^1 K(x,\xi)\,v(x)\,v(\xi)\,dx\,d\xi = \sum_{i=1}^{m+n+1} l_i\, b_i^2 \geq l_{m+n+1} \sum_{i=1}^{m+n+1} b_i^2 = l_{m+n+1}\,.$$

We emphasize this particular consequence of our main theorem:

THEOREM II. – *All eigenvalues will be lowered by the addition of a positive-definite kernel to an arbitrary one.*

One understands a positive-definite kernel $K^*(x, x)$ to mean one whose associated quadratic integral form:

$$\int_0^1\int_0^1 K^*(x,\xi)\,v(x)\,v(\xi)\,dx\,d\xi$$

does not prove to be negative for any function $v(x)$. As is known, such a kernel can also be characterized by saying that all of its eigenvalues are positive. We shall show: If $K = K' + K^*$, and therefore K^*, is positive definite then we will have:

$$l_{m+1} \geq l'_{m+1} \qquad (m = 0, 1, 2, \ldots).$$

In fact, if we write:

$$K' = K + (-K^*)$$

and imagine that the first reciprocal positive eigenvalue of $-K^*$ is already equal to zero then the inequality of Theorem I will show the validity of our assertion when we take $n = 0$.

If $f(x)$ is any function whose square integral $\int_0^1 f^2 dx = 1$ then one can convert the arbitrary kernel K into one $\dot{K}$ that is orthogonal to $f(x)$:

$$\int_0^1 \dot{K}(x,\xi)\, f(x)\, d\xi = 0,$$

while respecting its symmetry, but at the same time, for all functions $v(x)$ that are orthogonal to $f(x)$, the equation:

$$\int_0^1\int_0^1 \dot{K}(x,\xi)\,v(x)\,v(\xi)\,dx\,d\xi = \int_0^1\int_0^1 K(x,\xi)\,v(x)\,v(\xi)\,dx\,d\xi$$

will be fulfilled. One must set (cf., pp. 23):

$$\dot{K}(x,\xi) = K(x,\xi) - f(x)\int_0^1 f(y)K(y,\xi)\,dy - \int_0^1 K(x,\eta)\,f(\eta)\,d\eta\cdot f(\xi)$$
$$+ f(x)\,f(\xi)\int_0^1\int_0^1 K(x,\eta)\,f(y)\,f(\eta)\,dy\,d\eta\,.$$

If we apply the inequality:

$$\int_0^1\int_0^1 \left\{K(x,\eta) - \sum_{i=1}^n l_i\,u_i(x)\,u_i(\xi)\right\} v(y)v(\xi)\,dy\,d\xi \le l_{n+1}$$

to only those functions $v(x) = \dot{v}(x)$ that are orthogonal to $f(x)$ then we can replace each $u_i(x)$ in them with the function:

$$\bar{u}_i(x) = u_i(x) - f(x)\int_0^1 f(y)\,u_i(y)\,dy\,,$$

which is orthogonal to $f(x)$. For that $v = \dot{v}(x)$, we will then have:

$$\int_0^1\int_0^1 \left\{\dot{K}(x,\xi) - \sum_{i=1}^n l_i\,\bar{u}_i(x)\,\bar{u}_i(\xi)\right\} v(x)\,v(\xi)\,dx\,d\xi \le l_{n+1}\,. \tag{49}$$

Now, if $v(x)$ is once more an entirely arbitrary function whose square integral = 1 then the expression on the left-hand side of (49) will have the same value for it as it does for the function:

$$\dot{v}(x) = v(x) - f(x)\int_0^1 v(y)\,f(y)\,dy$$

(whose square integral is ≤ 1). Therefore, (49) is true in general. The reciprocal first positive eigenvalue of the kernel that is set in curly brackets on the left-hand side is then $\le l_{n+1}$, and on the basis of our main theorem, we conclude that $\dot{l}_{n+1} \le l_{n+1}$, since the $(n+1)^{\text{th}}$ reciprocal positive eigenvalue of $\sum_{i=1}^n l_i\,\bar{u}_i(x)\,\bar{u}_i(\xi)$ is zero. On the other hand, since the difference $K - \dot{K}$ is a kernel that possesses only *one* positive and *one* negative eigenvalue, one has that $l_{n+1} \le \dot{l}_{n+1}$, as a result of the same theorem. The result thus-obtained is the precise analogue of a theorem of STURM on quadratic forms with finitely many variables. Namely, if we interpret the quadratic integral form that belongs to a kernel K as a *second-order surface in an infinite-dimensional functional space* then $\dot{K}$ will be nothing but the intersection of K with the "plane" that goes through the center (viz., origin) whose altitude = $f(x)$.

THEOREM III – *If $\dot{K}$ is generated in such a way that one intersects the second-order surface K with an arbitrary plane through the center then the principal axes of $\dot{K}$ will be separate from those of K:*

$$l_1 \geq \dot{l}_1 \geq l_2 \geq \dot{l}_2 \geq l_3 \geq \dot{l}_3 \geq \ldots$$

§ 7. Exact spectral laws.

As we have investigated in the previous chapter, the equation for the displacement $\mathfrak{U} = \mathfrak{U}(p, t)$ {t = time} in an elastic medium in the absence of external forces reads:

$$\frac{\partial^2 \mathfrak{U}}{\partial t^2} = \Delta^* \mathfrak{U}.$$

In order to examine a simple oscillation, we make the following Ansatz for $\mathfrak{U}$:

$$\mathfrak{U}(p, t) = e^{ivt} \cdot \mathfrak{u}(p),$$

in which the amplitude $\mathfrak{u}(p)$ is a function of only position, while the number v is a constant, namely, the frequency of the oscillation under scrutiny; $i = \sqrt{-1}$. We then obtain the equation:

(50) $$\Delta^* \mathfrak{u} + \lambda\, \mathfrak{u} = 0 \qquad (\lambda = v^2)$$

for the amplitude. According to whether one poses the outer surface conditions I, II, or II for the elastic body, the solutions (λ, $\mathfrak{u}$) of this equation will coincide with the eigenvalues and eigenfunctions $\boldsymbol{\Gamma}_{\text{I}}$, $\boldsymbol{\Gamma}_{\text{II}}$, $\boldsymbol{\Gamma}_{\text{III}}$ ([23]); it only in the last case that the eigenvalues and eigenfunctions of $\boldsymbol{\Gamma}_{\text{III}}$ approach the six-fold eigenvalue $\lambda = 0$ of all six eigenvectors. *The eigenvalues that define a discrete point spectrum are all positive;* i.e., the kernel matrix has a positive-definite character. Namely, if one introduces a solution (λ, $\mathfrak{u}$) of (50) that satisfies the relevant outer surface conditions into the equations (D), (C), (B), resp., then that will give:

$$\int_J (\cdots - \lambda \mathfrak{u}^2)\, dp = 0,$$

in which the part of the integrand that is suggested by … is ≥ 0 in each of the three cases. That equation excludes the possibility that one can have $\lambda < 0$.

The following theorem is true for the eigenvalues of Problem I:

([23]) We tacitly think of all the GREEN tensors and functions of Chap. I as being provided with the factor $1 / 4\pi$ (without altering the notation).

THEOREM IV. – *If one imagines any finite number of bodies J', J'', …, J^h that do not mutually penetrate each other as being contained within the body J then at least as many eigenvalues (of Problem I) that belong to J will lie below an arbitrary limit as the total number of the eigenvalues that belong to the individual bodies.*

If we denote the GREEN tensors $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathrm{I}}$ that belong to the sub-bodies J', J'', … by $\mathbf{\Gamma}'$, $\mathbf{\Gamma}''$, …, resp., in which we also make the convention that, e.g., $\mathbf{\Gamma}'$ is set equal to zero when the source or reference point lies outside of J' then according to Theorem II, the proof of this theorem will be achieved when it can be shown that the kernel matrix:

$$\mathbf{\Gamma} - (\mathbf{\Gamma}' + \mathbf{\Gamma}'' + \ldots + \mathbf{\Gamma}^h)$$

is positive definite. We assume that the sub-bodies are bounded by outer surfaces $\mathfrak{O}'$, $\mathfrak{O}''$, …, which fulfill the same assumptions that we have postulated in regard to the boundary $\mathfrak{O}$ of J, and neither contact each other nor the external hull $\mathfrak{O}$. I shall let $J^{(h+1)}$ denote the part of J that remains when I think of all the sub-bodies J', J'', … as having been removed from J. Since the GREEN tensor $\mathbf{\Gamma}^{h+1}$ that belongs to J^{h+1} and corresponds to the boundary condition I has positive-definite type, as was mentioned above, the assertion to be proved will be true *a fortiori* when:

$$\mathbf{\Delta} = \mathbf{\Gamma} - (\mathbf{\Gamma}' + \mathbf{\Gamma}'' + \ldots + \mathbf{\Gamma}^h + \mathbf{\Gamma}^{h+1})$$

turns out to be a positive-definite matrix; i.e., when $\mathbf{\Delta}$ turns out to possess only positive eigenvalues.

Hence, let μ be an eigenvalue of $\mathbf{\Delta}$, and let $\mathfrak{v}$ be the associated eigenfunction:

$$\mathfrak{v}(p) = \mu \int_J \mathbf{\Delta}(p, p')\,\mathfrak{v}(p')\,dp'. \tag{51}$$

The inequality $\mu \geq 0$ proves to be a consequence of that assumption. $\mathfrak{v}(p)$ satisfies the equation $\Delta^* \mathfrak{v} = 0$ in all of J, except for the outer surfaces $\mathfrak{O}'$, $\mathfrak{O}''$, … As far as the behavior of the vectors on these outer surfaces is concerned, one initially specifies that they are continuous on them. Namely, for a point ω on $\Omega = \mathfrak{O}' + \mathfrak{O}'' + \ldots + \mathfrak{O}^h$:

$$\mathfrak{v}(\omega) = \mu \int_J \mathbf{\Gamma}(\omega, p')\,\mathfrak{v}(p')\,dp'. \tag{52}$$

The vector $\mathfrak{S}(\mathfrak{v})$ exists on both sides of such an outer surface $\mathfrak{O}'$:

$$S_n = \frac{a}{a+b}[P_n(\mathfrak{v}) + b \operatorname{div} \mathfrak{v}], \qquad \mathfrak{S}_t = \frac{1}{a+b}(a\,\mathfrak{P}_t(\mathfrak{v}) + b^2 [\operatorname{curl} \mathfrak{v}, \mathfrak{n}]).$$

However, the values of this vector on both sides do not agree; I let $\mathfrak{s} = \mathfrak{s}\,(\omega)$ denote the "jump," which is the difference between its values on one side and the other. $\mathfrak{v}$ vanishes on the outer surface $\mathfrak{O}$.

We shall now apply equation (D') in such a way that we substitute the tensor $\boldsymbol{\Gamma}(p, p')$ (that belongs to J) for $\mathfrak{u}$ and indeed, apply it to the bodies $J', J'', \ldots, J^{h+1}$ in succession (in which the point p' must first be excised from the body in which it lies by a small ball). Adding the $h + 1$ equations that one obtains will give:

$$\mathfrak{v}(p') = -\int_{\mathfrak{O}} \boldsymbol{\Gamma}(\omega, p')\,\mathfrak{s}(\omega)\,d\omega. \tag{53}$$

If we substitute this value in (52) then that will give:

$$\mathfrak{v}(\omega) = -\,\mu\int_{\mathfrak{O}} \boldsymbol{\Gamma}\boldsymbol{\Gamma}(\omega, \omega')\,\mathfrak{s}(\omega')\,d\omega',$$

in which $\boldsymbol{\Gamma}\boldsymbol{\Gamma}$ is the kernel that corresponds to $\boldsymbol{\Gamma}$ by iteration. Finally, we make use of the inequality (D_0), when replace $\mathfrak{u}$ with $\mathfrak{v}$ in it, and replace J with the bodies $J', J'', \ldots, J^{h+1}$, in succession. We add the inequalities thus-obtained, in turn, and find that:

$$-\int_{\mathfrak{O}} \mathfrak{v}(\omega)\,\mathfrak{s}(\omega)\,d\omega \geq 0,$$

so

$$\mu\int_{\mathfrak{O}}\int_{\mathfrak{O}} \mathfrak{s}(\omega)\,\boldsymbol{\Gamma}\boldsymbol{\Gamma}(\omega, \omega')\,\mathfrak{s}(\omega')\,d\omega\,d\omega' \geq 0. \tag{54}$$

On the other hand, when one squares the left-hand and right-hand sides of (53) and integrates over J, one will conclude that:

$$\mu\int_{\mathfrak{O}} \mathfrak{s}(\omega)\,\boldsymbol{\Gamma}\boldsymbol{\Gamma}(\omega, \omega')\,\mathfrak{s}(\omega')\,d\omega\,d\omega' = \int_J \mathfrak{v}^2(p)\,dp > 0.$$

The factor μ in (54) must then be > 0, since it is non-zero.

We point out yet another consequence of this line of reasoning. If J^o is a region that is surrounded by J then $\boldsymbol{\Gamma}_{J^0} - \boldsymbol{\Gamma}_J$ will be positive definite when we let the source and reference points vary inside of J^o. When we set $\mathfrak{u}$ equal to only those particular fields that are equal to zero outside of J, it will emerge from the inequality:

$$\int_{J^0}\int_{J^0} \mathfrak{u}(p)\{\boldsymbol{\Gamma}_{J^0}(p, p') - \boldsymbol{\Gamma}_J(p, p')\}\mathfrak{u}(p')\,dp\,dp' \geq 0$$

which is formulated on the basis of the previous fact, that $\boldsymbol{\Gamma}_{J^0} - \boldsymbol{\Gamma}_J$ will also be a positive-definite kernel matrix when the source and reference points run through only the region J. If we take J^o to be, say, a large sphere and let its radius increase to infinity for a fixed center then passing to the limit will yield the formula:

$$\int_J \int_J \mathfrak{u}(p)\{\mathbf{P}(p, p') - \boldsymbol{\Gamma}_J(p, p')\}\mathfrak{u}(p')\,dp\,dp' \geq 0.$$

It shows that:

THEOREM V. – *The kernel matrix* $\mathbf{P} - \boldsymbol{\Gamma}_{\mathrm{I}}$ *is positive definite.*

Whereas the validity of the law of the dependency of the eigenvalues of the region J on the boundary condition I that was formulated in THEOREM IV is restricted in scope to that case, the following analogue to THEOREM V exists for the boundary conditions II and III, and it is proved by the same method:

THEOREM VI. – *The kernel matrices:*

$$\mathbf{B}_{\mathrm{II}} = \boldsymbol{\Gamma}_{\mathrm{II}} - \boldsymbol{\Gamma}_{\mathrm{I}}, \quad \mathbf{B}_{\mathrm{III}} = \boldsymbol{\Gamma}_{\mathrm{III}} - \dot{\boldsymbol{\Gamma}}_{\mathrm{I}}$$

are positive definite. As a result, we know at least as many eigenvalues of the oscillation problem II, as well as III, that lie below an arbitrary limit as we do for Problem I.

Let μ, $\mathfrak{v}$ be an eigenvalue and the associated eigenvector of $\mathbf{B} = \mathbf{B}_{\mathrm{II}}$:

(55) $$\mathfrak{v}(p) = \mu \int_J \mathbf{B}(p, p')\,\mathfrak{v}(p')\,dp' .$$

If we let the point p in (55) move about the outer surface then we will find the boundary value of $\mathfrak{v}$:

$$v_n(o) = \mu \int_J \mathfrak{g}(o, p')\,\mathfrak{v}(p')\,dp' ,$$

in which we understand $\mathfrak{g}$ to mean the same quantity as on pp. 19. On the other hand, if $l(o)$ means the divergence of $\mathfrak{v}$ on the outer surface then we will have (cf., pp. 19):

(56) $$\mathfrak{v}(p') = -\int_{\mathfrak{O}} \mathfrak{g}(o, p')\,l(o)\,do .$$

By substituting this in the previous equation, we will get:

$$v_n(o) = -\mu \int_{\mathfrak{O}} \mathfrak{g}\mathfrak{g}(o, o')\,l(o')\,do'$$

$$\{\mathfrak{g}\;\mathfrak{g}\;(o, o') = \int_J \mathfrak{g}(o, p')\,\mathfrak{g}(o', p)\,dp\,\}.$$

The inequality (C_0) yields:

$$-\int_{\mathfrak{O}} v_n(o)\,l(o)\,do = \int_J \left\{\frac{b}{a}(\operatorname{curl}\mathfrak{v})^2 + (\operatorname{div}\mathfrak{v})^2\right\} dp \geq 0 ;$$

that is:

$$\mu \int_{\mathfrak{O}} \int_{\mathfrak{O}} \mathfrak{g}\,\mathfrak{g}(o, o')\, l(o)\, l(o')\, do\, do' \geq 0.$$

One arrives at the inequality from (56):

$$\int_{\mathfrak{O}} \int_{\mathfrak{O}} \mathfrak{g}\,\mathfrak{g}(o, o')\, l(o)\, l(o')\, do\, do' = \int_J \mathfrak{v}^2 dp > 0$$

by squaring and integrating. μ is positive, as a result.

If $\mathbf{B}$ in (55) means the matrix $\mathbf{B}_{\text{III}}$ then we conclude as follows: One has:

$$\mathfrak{v}(o) = \mu \int_J \mathbf{\Gamma}(o, p')\, \mathfrak{v}(p')\, dp' + \mathfrak{a}(o). \tag{57}$$

In this, $\mathbf{\Gamma}$ means the tensor $\mathbf{\Gamma}_{\text{III}}$, and $\mathfrak{a}(p)$ is a linear combination of the solutions to the homogeneous static problem III that were mentioned at the beginning of § **4**.

$$\mathfrak{v}(p') = -\mu \int_{\mathfrak{O}} \mathbf{\Gamma}(o, p')\, \mathfrak{q}(o)\, do \qquad \{\mathfrak{q} = \mathfrak{P}(\mathfrak{v})\}. \tag{58}$$

By substituting this equation in the previous one, one will get:

$$\mathfrak{v}(o) = -\mu \int_{\mathfrak{O}} \mathbf{\Gamma}\mathbf{\Gamma}(o, o')\, \mathfrak{q}(o')\, do' + \mathfrak{a}(o).$$

One has the inequality:

$$-\int_{\mathfrak{O}} \mathfrak{v}(o)\, \mathfrak{q}(o)\, do \geq 0,$$

since one necessarily has (cf., pp. 24):

$$\int_{\mathfrak{O}} \mathfrak{q}(o)\, \mathfrak{a}(o)\, do = 0\,;$$

that is:

$$\mu \int_{\mathfrak{O}} \int_{\mathfrak{O}} \mathfrak{q}(o')\, \mathbf{\Gamma}\mathbf{\Gamma}(o, o')\, \mathfrak{q}(o')\, do\, do' \geq 0.$$

One must, in turn, combine this result with the one that follows from (58):

$$\int_{\mathfrak{O}} \int_{\mathfrak{O}} \mathfrak{q}(o')\, \mathbf{\Gamma}\mathbf{\Gamma}(o, o')\, \mathfrak{q}(o')\, do\, do' = \int_J \mathfrak{v}^2 dp > 0.$$

One concludes that $\mathbf{\Gamma}_{\text{III}}$ possesses just as many eigenvalues below an arbitrary limit L as $\dot{\mathbf{\Gamma}}_{\text{I}}$ does. However, the number of eigenvalues of Problem III that lie below L is higher by six than it is for the kernel matrix $\mathbf{\Gamma}_{\text{III}}$ since $\lambda = 0$ enters into it as a six-fold eigenvalue. On the other hand, from Theorem III, the number of eigenvalues of $\dot{\mathbf{\Gamma}}_{\text{I}}$ below an arbitrary limit amounts to six less than the number of eigenvalues of the tensor $\mathbf{\Gamma}_{\text{I}}$. The statement of our theorem is therefore true for the boundary conditions III.

§ 8. The asymptotic spectral law of the "three-dimensional membrane problem."

Now that we have spoken of the most important *exact* spectral laws (which are expressed by inequalities) in the previous paragraphs, we shall now go on to the *asymptotic* laws (which are formulated as limit equations). We begin with Problem I and first focus upon the special case in which the constants a and b possess the value 1, so $\Delta^* \mathfrak{u}$ will go to the potential expression $\Delta \mathfrak{u}$. Therefore:

$$\boldsymbol{\Gamma}_{\mathrm{I}} = \begin{Vmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{Vmatrix},$$

if G means the usual GREEN function that belongs to the first boundary-value problem of potential theory. The eigenvalues of $\boldsymbol{\Gamma}_{\mathrm{I}}$ are the same as those of G, except that each eigenvalue of $\boldsymbol{\Gamma}_{\mathrm{I}}$ must be must be counted three times as often as for G. The following facts are known about the GREEN function G:

1) $0 < G < \dfrac{1}{r}$.

2) If the region J' is contained in J then $G_{J'} < G_J$.

It will then follow from this that a GREEN function will also belong to a region J when the assumption that it is bounded by a finite number of such outer surfaces that satisfy the requirements that were formulated on pp. 11 is *not* fulfilled. Namely, if J is any region that lies completely at finite points then we can associate a region J_ε with any sufficiently small $\varepsilon > 0$ such that:

1) Any J_ε is bounded by a finite number of outer surfaces that possess continuous tangent planes and continuous curvatures.

2) J_ε is contained completely in J_δ when $\delta < \varepsilon$.

3) $\lim\limits_{\varepsilon=0} J_\varepsilon = J$; i.e., for any point p inside of J there exists an ε such that p also lies inside of J_ε.

If G_ε is the GREEN function that belongs to J_ε then $\lim\limits_{\varepsilon=0} G_\varepsilon = G$ will exist, and indeed it will be *uniformly* true for all p, p' that belong to a fixed, closed spatial region that lies completely inside of J that one will have:

$$\lim_{\varepsilon=0} \{G(p, p') - G_\varepsilon(p, p')\} = 0.$$

Since G always remains smaller than $1 / r$, it will have discrete, positive eigenvalues λ_1, λ_2, λ_3, …

Here, we do not need to examine the question of whether, and in what sense, the GREEN function G also takes on the boundary value 0. It is essential for us only that this will be true when J is a *cube*. If one imagines that the cube sits on a horizontal plane E then the GREEN function that belongs to the cube will, in fact, be smaller than the GREEN function of the half-space that is determined by the plane E, and since the latter has the boundary value 0, G will also have the boundary value 0 on those of the sides of J that cover the horizontal plane. If the cube is given by $0 \leq x, y, z \leq c$ then:

$$\sin\frac{l\pi x}{c}\sin\frac{l\pi y}{c}\sin\frac{l\pi z}{c} \qquad (l, m, n = 1, 2, 3, \dots)$$

will be eigenfunctions of the GREEN function of the cube, and there will no other ones besides them (or rather, besides ones that can be composed of a finite number of the aforementioned functions with constant coefficients). All of the eigenvalues of G will then be provided by the expression:

$$\frac{\pi^2}{c^2}(l^2 + m^2 + n^2) \tag{59}$$

here when one lets l, m, n run through all positive numbers independently of each other.

Let J', J'', …, J^h be any regions that are contained in the entirely-arbitrary finite region J, but have no interior points in common. One asks whether the theorem that was spoken of in the previous paragraphs, namely, that the kernel:

$$G - (G' + G'' + \dots + G^h)$$

is positive definite, will also be true now that we distance ourselves from any assumption about the boundaries of the regions J', J'', … In order to resolve that issue, we must approximate J', J'', … from the inside out by regions J'_ε, J''_ε, … in a manner that is analogous to what we just did for J. In order to do that, we will insure that J'_ε, J''_ε, …, including their boundaries, lie completely inside of J_ε. We must then show that the inequality:

$$\lim_{\varepsilon=0}\left\{\int_{J_\varepsilon}\int_{J_\varepsilon} G(p, p')\,u(p)\,u(p')\,dp\,dp' - \sum_{j=1}^{h}\int_{J_\varepsilon^j}\int_{J_\varepsilon^j} G^j(p, p')\,u(p)\,u(p')\,dp\,dp'\right\} \geq 0 \tag{60}$$

exists for any function $u(p)$ that is continuous in the closed region J. I assume that $\int_J u^2\,dp \leq \frac{1}{2}$. Let $0 < \delta < \varepsilon$. I replace G, G', G'', … in the expression $\boldsymbol{G}_\varepsilon$ that is contained in curly brackets in the latter inequality with the regions J_δ, J'_δ, J''_δ, … that belong to the GREEN functions G_δ, G'_δ, G''_δ, ….; the quantities $\boldsymbol{G}_{\delta\varepsilon}$ will then arise from the $\boldsymbol{G}_\varepsilon$ in that way. One has:

$$\lim_{\delta=0} \boldsymbol{G}_{\delta\varepsilon} = \boldsymbol{G}_{\varepsilon}\,.$$

From the investigations in the previous paragraph, I know that $\boldsymbol{G}_{\delta\delta}$ is non-negative. An application of the so-called SCHWARZ inequality will yield the relation:

$$(\boldsymbol{G}_{\delta\delta} - \boldsymbol{G}_{\delta\varepsilon})^2 \le \iint\limits_{J_\delta J_\delta - J_\varepsilon J_\varepsilon} (G_\delta)^2\, dp\, dp' + \sum_{j=1}^{h} \iint\limits_{J_\delta^j J_\delta^j - J_\varepsilon^j J_\varepsilon^j} (G_\delta^j)^2\, dp\, dp'$$
$$\le = \iint \left(\frac{1}{r}\right)^2 dp\, dp' = A_\delta - A_\varepsilon\,,$$

in which the latter integral should be taken over the six-dimensional region:

$$(J_\delta J_\delta - J_\varepsilon J_\varepsilon) + \sum_{j=1}^{h} (J_\delta^j J_\delta^j - J_\varepsilon^j J_\varepsilon^j)\,.$$

It will then follow that:

$$\boldsymbol{G}_{\delta\delta} \ge -\sqrt{A_\delta - A_\varepsilon}\,.$$

When the finite quantity $\lim_{\delta=0} A_\delta$ is denoted by A, the passage to the limit lim $\delta = 0$ will yield:

$$\boldsymbol{G}_{\varepsilon} \ge -\sqrt{A - A_\varepsilon}\,,$$

and the passage to the limit lim $\varepsilon = 0$ will confirm the inequality (60), moreover, as was to be expected. The number of eigenvalues of G', G'', …, G^h below an arbitrary limit will never exceed the number of eigenvalues of G that lie below the same limit then.

Let J, in turn, be an arbitrary finite region, and let c be a (small) positive number. We draw a cubic net in space whose edge length is c and understand H to mean the number of cubes in that net that belong to J completely. The eigenvalues of the GREEN function that belong to an individual cube w are given by (59). As we know, as long as these eigenvalues lie below an arbitrary limit L, their number N_w can be deduced as follows: In a space with rectangular coordinates ξ, η, ζ, we consider the positive octant of the unit sphere:

$$\xi^2 + \eta^2 + \zeta^2 \le 1, \qquad \xi \ge 0, \quad \eta \ge 0, \quad \zeta \ge 0,$$

and construct the cubic net in that space whose edge length is $\pi / c\sqrt{L}$ and is oriented parallel to the coordinate axes (and to which the origin belongs as a vertex). N_w will then be identical to the number of cubes in that net that belong to the positive octant of the unit sphere. If we then multiply it by the volume of the individual cube $\left(\frac{\pi}{c\sqrt{L}}\right)^3$ then

$N_w\left(\frac{\pi}{c\sqrt{L}}\right)^3$ must be equal to the volume of the spherical octant for infinitely-large L, and we will get the asymptotic expression for the number itself from that:

$$N_w \sim \frac{c^3}{6\pi^2} L^{3/2}.$$

From Theorem IV, and with consideration given to the extensions that we have just added to that theorem, we have the following inequality for the number N_J of eigenvalues of the GREEN function G that belongs to J that lie below L:

$$N_J \geq H\, N_w\,,$$

so:

$$\liminf_{L=\infty} \frac{N_J}{L^{3/2}} \geq \frac{H\, c^3}{6\pi^2}\,.$$

If the body J possesses a well-defined volume J then J will be the limit of Hc^3 as the mesh width c becomes infinitely small, and it will follow that:

$$\liminf_{L=\infty} \frac{N_J}{L^{3/2}} \geq \frac{J}{6\pi^2}\,.$$

In order to find an upper bound for the lim sup, I imprison the body J in a cubic box W, and if $W - J$ means the remaining empty space then I will have:

$$N_W \geq N_J + N_{W-J}\,,$$

so

$$\liminf_{L=\infty} \frac{N_J}{L^{3/2}} = \liminf_{L=\infty} \frac{N_W}{L^{3/2}} - \liminf_{L=\infty} \frac{N_{W-J}}{L^{3/2}} \leq \frac{W}{6\pi^2} - \frac{W-J}{6\pi^2} = \frac{J}{6\pi^2}\,.$$

The limit equation:

$$\liminf_{L=\infty} \frac{N_J}{L^{3/2}} = \frac{1}{6\pi^2} J$$

is thus proved. If λ_1, λ_2, λ_3, … are the eigenvalues of the GREEN function G that belong to J then we can also write them as:

$$\lambda_n \sim \left(\frac{6\pi^2 n}{J}\right)^{2/3}. \tag{61}$$

THEOREM VII. – *One has the asymptotic law:*

$$\lambda_n \sim \left(\frac{6\pi^2 n}{J}\right)^{2/3}$$

for the eigenvalues $\lambda = \lambda_n$ *of the boundary-value problem:*

$$\Delta u + \lambda\, u = 0 \quad \text{in } J, \qquad u = 0 \quad \text{on } \mathfrak{O}$$

when they are arranged in an increasing sequence.

When $a = b = 1$, the number N of eigenvalues of the tensor $\Gamma = \Gamma_{\mathrm{I}}$ below a limit L asymptotically amounts to:

$$N \sim \frac{J}{2\pi^2} L^{3/2} . \tag{62}$$

§ 9. The asymptotic spectral law of elastic oscillations.

From that result, it is possible to go on to the other two problems II, III, at first, under the assumption that $a = b = 1$. However, Problem II possesses the peculiarity that that when the asymptotic distribution of eigenvalues is known in the special case $a = b = 1$, we can deduce it in the general case of arbitrary (positive) constants a and b with no further assumptions. Namely, from § **3**, we have [eq. (28)]:

$$\mathbf{\Gamma}_{\mathrm{II}} = \frac{1}{a}\mathbf{\Gamma}_a + \frac{1}{b}\mathbf{\Gamma}_b$$

in which the matrices $\mathbf{\Gamma}_a$, $\mathbf{\Gamma}_b$ are mutually-orthogonal and do not depend upon the constants a and b. If λ^a are all of the eigenvalues of $\mathbf{\Gamma}_a$, and λ^b are all the eigenvalues of $\mathbf{\Gamma}_b$, then all of the eigenvalues of $\mathbf{\Gamma}_{\mathrm{II}}$ will be given by:

$$a \cdot \lambda^a , \qquad\qquad b \cdot \lambda^b .$$

The λ^a are nothing but the eigenvalues of the GREEN function G that was considered in the previous problem. Namely, if λ is an eigenvalue, and u is an associated eigenfunction of G then λ will be an eigenvalue of $\mathbf{\Gamma}_a$ with the associated eigenvector grad u, and at the same time that construction will provide *all* of the eigenvalues and eigenfunctions of the matrix $\mathbf{\Gamma}_a$. That will emerge from the analysis of the tensor $\mathbf{\Gamma}_a$ that was carried out in § **3**, which had the result (32). Now, if we know that the asymptotic law (62) is valid in the case of $a = b = 1$ for not only Problem I, but also for II, then we can conclude: The number of $\lambda^b < L$ is asymptotically $\sim (J / 3\pi^2)\, L^{3/2}$, and the number of eigenvalues of $\mathbf{\Gamma}_{\mathrm{II}}$ will generally be:

(63)
$$\sim \frac{J}{6\pi^2} L^{3/2} \left\{ \left(\frac{1}{a}\right)^{3/2} + 2\left(\frac{1}{b}\right)^{1/2} \right\}.$$

One would also expect the same formula then for Problem I for arbitrary positive constants a, b. In that way, II will make it possible to complete the transition from $a = b = 1$ to arbitrary values of the constants.

The conjectures that were just mentioned can be derived on the basis of Theorem I from the single fact that the eigenvalues of $\mathbf{B}_{\mathrm{II}} = \mathbf{\Gamma}_{\mathrm{II}} - \mathbf{\Gamma}_{\mathrm{I}}$ are asymptotically distributed in a way that is sparser than would correspond to formula (61), and the proof of that fact, in turn, rests upon the estimate for $\mathbf{B}_{\mathrm{II}}$ that was achieved in § **5**:

$$|\,\mathbf{B}_{\mathrm{II}}\,(p, p')\,| \leq \frac{\text{const.}}{R(p, p')}.$$

The eigenvalues of $\mathbf{B}_{\mathrm{II}}$, which are arranged in increasing magnitude (and are, as we know, all *positive*) shall be called β_n , and the associated eigenvectors (which are mutually-orthogonal and normalized) shall be called $\mathfrak{v}_n\,(p)$. The kernel matrix:

$$\mathbf{B}_{\mathrm{II}}\,(p, p') - \sum_{i=1}^{n} \frac{1}{\beta_i}\; \mathfrak{v}_n\,(p) \times \mathfrak{v}_n\,(p')$$

has the eigenvalues β_{n+1} , β_{n+2} , … Since they are all positive, the associated quadratic integral form will be positive definite, and therefore *the three functions in the principal diagonal of the matrix must* ≥ 0 *for* $p = p'$ *itself*. If we add those three inequalities then that will give:

$$\sum_{i=1}^{n} \frac{1}{\beta_i} \mathfrak{v}_i^2(p) \leq \mathbf{B}_{\mathrm{II}}\,(p).$$

According to the inequality (46) in § **5**, the integration over $J = J_\varepsilon$ will yield the estimate:

(64)
$$\sum_{i=1}^{n} \frac{1}{\beta_i} \int_{J-J_\varepsilon} \mathfrak{v}_i^2\, dp \leq \text{const. } \ln \frac{1}{\varepsilon}.$$

On the other hand, we have:

$$\sum_{i=1}^{n} \frac{1}{\beta_i} \mathfrak{v}_i^2(p) \leq \int_J |\,\mathbf{B}_{\mathrm{II}}(p, p')\,|^2\, dp' \leq \text{const.} \int_J \frac{dp'}{r^2(p, p')} \leq \text{const.},$$

and upon integrating over the shell J_ε :

(65)
$$\sum_{i=1}^{n} \frac{1}{\beta_i} \int_{J_\varepsilon} \mathfrak{v}_i^2\, dp \leq \text{const. } \varepsilon.$$

When we replace "const." with $\frac{1}{2}C$, it will follow from (64), (65) *a fortiori* that:

$$\frac{1}{\beta_n}\sum_{i=1}^{n}\int_{J-J_\varepsilon}\mathfrak{v}_i^2\,dp \le \tfrac{1}{2}C\ln\frac{1}{\varepsilon}, \qquad \frac{1}{\beta_n}\sum_{i=1}^{n}\int_{J_\varepsilon}\mathfrak{v}_i^2\,dp \le \tfrac{1}{2}C\,\beta_n\,\varepsilon\,.$$

Addition gives:

$$\frac{n}{\beta_n} \le \frac{C}{2}\left(\ln\frac{1}{\varepsilon}+\beta_n\varepsilon\right).$$

We can best utilize this inequality when we take $\varepsilon = (\ln \beta_n) / \beta_n$. It will then follow (as long as $\beta_n > e$) that:

$$\frac{n}{\beta_n} \le C \ln \beta_n\,,$$

and from that (as long as $n > e^{1/C}$):

$$\beta_n \ge \frac{1}{C}\frac{n}{\ln n}\,. \tag{66}$$

An analogous argument can be made for the eigenvalues of $\mathbf{B}_{\mathrm{III}}$ and also for those of the likewise positive-definite matrix $\mathbf{A}_{\mathrm{I}}$. We formulate the result in:

THEOREM VIII. – *One has the estimates:*

$$\frac{1}{\alpha_n} \le \text{const.}\,\frac{\ln(n+1)}{n}, \qquad \frac{1}{\beta_n} \le \text{const.}\,\frac{\ln(n+1)}{n}, \qquad (n = 1, 2, 3, \ldots)$$

for the eigenvalues α_n *of* $\mathbf{A}_{\mathrm{I}}$ *and the eigenvalues* β_n *of* $\mathbf{B} = \mathbf{B}_{\mathrm{II}}$ ($\mathbf{B}_{\mathrm{III}}$, resp.).

I shall denote the eigenvalues of $\mathbf{\Gamma}_{\mathrm{I}}$, $\mathbf{\Gamma}_{\mathrm{II}}$, when arranged in magnitude, by λ_n^{I}, λ_n^{II}, resp. We have:

1) $$\lambda_n^{\mathrm{II}} \le \lambda_n^{\mathrm{I}} \qquad \text{(Theorem VI)},$$

2) $$\frac{1}{\lambda_n^{\mathrm{II}}} \le \frac{1}{\lambda_h^{\mathrm{I}}} + \frac{1}{\beta_s}\,,$$

if h, s are any two indices of the sum $n - 1$ (Theorem I). *We next consider the case of* $a = b = 1$, since, as we know, we have:

$$\lim_{n=\infty}\frac{\lambda_n^{\mathrm{I}}}{n^{2/3}} = \left(\frac{2\pi^2}{J}\right)^{2/3} = D. \tag{67}$$

If we substitute the largest number $h = n - 1 - s$ that is contained completely in $n^{5/6}\sqrt{\ln n}$ for s in the inequality 2) that was just cited then from Theorem VIII, one will have:

$$\frac{1}{\beta_1} \leq \text{const.}\ \frac{\ln s}{s} \leq \text{const.}\ \frac{\sqrt{\ln n}}{n^{5/6}},$$

$$\lim_{n=\infty} \frac{n^{2/3}}{\beta_1} = 0,$$

and

$$\lim_{n=\infty} \frac{n^{2/3}}{\lambda_h^{\mathrm{I}}} = \lim_{n=\infty} \frac{h^{2/3}}{\lambda_h^{\mathrm{I}}} = \frac{1}{D}.$$

That will give:

$$\lim_{n=\infty} \sup \frac{n^{2/3}}{\lambda_n^{\mathrm{II}}} \leq \frac{1}{D}, \qquad \lim_{n=\infty} \inf \frac{\lambda_n^{\mathrm{II}}}{n^{2/3}} \geq D.$$

However, in view of relation 1) and (67), it will follow from this that:

(68) $$\lim_{n=\infty} \frac{\lambda_n^{\mathrm{II}}}{n^{2/3}} = D = \left(\frac{2\pi^2}{J}\right)^{2/3}.$$

If we once more think of a and b as arbitrary constants then we can deduce, in the way that was given at the beginning of this paragraph, that:

(69) $$\lim_{n=\infty} \frac{\lambda_n^{\mathrm{II}}}{n^{2/3}} = \left\{\frac{J}{6\pi^2}(a^{-3/2} + 2b^{-3/2})\right\}^{-3/2} = D_{ab}.$$

I shall now employ the inequality 2) in order to deduce λ^{II} from λ^{I} in the form:

$$\frac{1}{\lambda_{n+s-1}^{\mathrm{II}}} \leq \frac{1}{\lambda_n^{\mathrm{I}}} + \frac{1}{\beta_s}$$

with an unchanged meaning for s; that will give:

$$\lim_{n=\infty} \inf \frac{n^{2/3}}{\lambda_n^{\mathrm{I}}} \geq \frac{1}{D_{ab}}, \qquad \lim_{n=\infty} \sup \frac{\lambda_n^{\mathrm{I}}}{n^{2/3}} \leq D_{ab}.$$

The relation 1) and (69) once more gives:

(70) $$\lim_{n=\infty} \frac{\lambda_n^{\mathrm{I}}}{n^{2/3}} = D_{ab}.$$

The oscillation problems I and II are thus dispatched. If we now understand β_n to mean the eigenvalues of $\mathbf{B}_{\mathrm{III}}$ then we will have:

$$\frac{1}{\lambda_n^{\mathrm{I}}} \leq \frac{1}{\lambda_n^{\mathrm{III}}} \leq \frac{1}{\lambda_h^{\mathrm{I}}} + \frac{1}{\beta_s},$$

$$\frac{1}{\beta_s} \leq \text{const.}\ \frac{\ln s}{s},$$

and conclude from this in the same way as before that the asymptotic law (69) [(70), resp.] can also be adapted to the boundary-value problem III. We formulate the final result as:

THEOREM IX. – *The number of eigen-oscillations that an elastic body J (of volume J) is capable of performing with the outer surface tension* 0 *up to the frequency limit v asymptotically amounts to:*

$$\frac{J}{6\pi^2}(a^{-3/2} + 2b^{-3/2}) \cdot \nu^3$$

(for lim $v = \infty$).

Ascertaining that law was the goal of the foregoing paper. If we were to carry out the proofs of the estimates rigorously then we would see that the error (i.e., the difference between the number to be determined and its asymptotic expression that was derived here) is certainly ([24]):

$$\leq \text{const.}\ v^3 \cdot \sqrt{\frac{\ln \nu}{\nu}}.$$

Zurich, 9 March 1914.

HERMANN WEYL

([24]) WEYL, *loc. cit.*([7]), pp. 196-199.