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Electrons and gravitation. I

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Introduction. Relationship between the general theory of relativity to the quantum-theoretic field equations of the spinning electron: Mass, gauge invariance, teleparallelism. The expected modifications to **Dirac**’s theory.

I. Two-component theory: The wave function ψ has only two components.

§ 1. – Coupling the transformation of the ψ to the Lorentz transformation of the normal axis-cross in the four-dimensional world. Asymmetry of future and past from the right and left.

§ 2. – In the general theory of relativity, the metric at a world-point is established by a normal axis-cross. Components of vectors relative to the axes and the coordinates. Covariant differentiation of ψ.

§ 3. – General invariant conception of the **Dirac** action that is characteristic of the wave-field of matter.

§ 4. – The differential conservation laws for energy and impulse and the symmetry of the impulse tensor follows from the double invariance:

1. under coordinate transformations,
2. under rotations of the axis-cross.

Impulse and impulse-moment of matter.

§ 5. – **Einstein**’s classical theory of gravitation in the new analytical formulation. Gravitational energy.

§ 6. – The electromagnetic field. The indeterminacy of the gauge factor in ψ implies the necessity of introducing the electromagnetic potential. Gauge invariance and the conservation of electricity. The spatial integral of charge. Introduction of mass. Discussion and acceptance of another possibility, in which electricity appears to be a phenomenon that accompanies gravitation, not matter.

Introduction.

In this article, I shall develop a theory that encompasses gravitation, electricity, and mass in a thorough form that appeared as a brief sketch in Proc. Nat. Acad. Sci., April 1929. Various authors have commented on the connection between **Einstein**’s theory of teleparallelism and the spin theory of the electron (*). Despite certain similarities, my Ansatz differs radically in that I reject teleparallelism and base it upon **Einstein**’s classical relativistic theory of gravitation.

(*) **E. Wigner**, Zeit. Phys. **53** (1929), 592, and others.

On two grounds, the *adaptation of the Pauli-Dirac theory of the spinning electron to general relativity* promises to lead to fruitful results:

1. The **Dirac** theory, in which the wave field of the electron is described by a potential ψ with four components, gives twice too many energy levels. One should then revert to the two components of **Pauli**'s theory without abandoning the relativistic invariance. That prevents the mass m of the electron from being included as a factor in any term of the **Dirac** action. *However, mass is a gravitational effect*. There is hope for finding a replacement for that term in the theory of gravitation that will produce the desired correction.

2. The **Dirac** field equations for ψ, together with the **Maxwell** equations for the four potentials f_p of the electromagnetic field, have an invariance property that is formally the same as the one that I referred to as *gauge invariance* in my theory of gravitation and electricity from 1918. The equations will remain unchanged when one simultaneously replaces:

$$\psi \text{ with } e^{i\lambda}\cdot\psi \qquad \text{and} \qquad f_p \text{ with } f_p - \frac{\partial\lambda}{\partial x_p} ,$$

in which λ is understood to mean an arbitrary function of position in the four-dimensional world. The factor of $e / c\hbar$ has been absorbed into f_p in the last expression ($-e$ is the charge of the electron, c is the speed of light, and $\hbar = h / 2\pi$ is the quantum of action). The relationship between that "gauge invariance" and the conservation law for electricity remains inviolate. However, an essential difference that is experimentally significant is that the exponent of the factor that ψ takes on is not real, but purely imaginary. ψ now takes on the role that was played by **Einstein**'s *ds* in that older theory. Because of that, it seems to me that this new principle of gauge invariance, which arises necessarily, not from speculation, but from experiment, refers to the fact that *the electric field is a phenomenon that necessarily accompanies, not the gravitational field, but matter, which is represented by the wave field* ψ. Since gauge invariance includes an arbitrary function λ, it has the character of "more general" relativity and can naturally be understood only in that context.

I cannot believe in *teleparallelism* on several grounds. First of all, my mathematical sense resists accepting such an artificial geometry *a priori*. I find it difficult to comprehend the power that has frozen the local axis-crosses at the various world-points in their twisted positions into rigid bondage with each other. I believe that two compelling physical reasons must be added: It is due to precisely the fact that one loosens the connection between the local axis-crosses that the gauge factor $e^{i\lambda}$, which remains arbitrary in the quantity ψ, will necessarily be converted from a constant into an arbitrary function of position, i.e., the gauge invariance that actually exists will become understandable by just that loosening. Secondly, as we will see in what follows, the possibility of rotating the axis-crosses at different locations independently of each other is equivalent to the *symmetry of the energy-impulse tensor* or the validity of the conservation law for impulse-moment.

With each attempt at exhibiting quantum-theoretic field equations, one must keep in mind the fact that they cannot be compared directly with experiment, but it is only *their quantization* that will yield the basis for the statistical statements about the behavior of material particles and light quanta. The **Dirac-Maxwell** theory, in its form up to now, contains only the electromagnetic potential f_p and the wave field ψ of the *electron*. Undoubtedly, the wave field ψ' of the proton must be added, and indeed ψ, ψ', and f_p will be functions of *the same* four space-time coordinates in the field equations. Prior to quantization, one cannot demand, say, that ψ is a function of a world-point (t, x, y, z) and ψ' is a function of the world-point (t', x', y', z') that is independent of it. It is natural to expect that of the two component-pairs in the **Dirac** quantities, one of them belongs to the electron and the other, to the proton. Moreover, two conservation laws for electricity must appear that say (after quantization) that the number of electrons, like that of the protons, must remain constant. They must correspond to a two-fold gauge invariance that involves two arbitrary functions.

We first test the state of affairs in the special theory of relativity in regard to whether, and to what degree, the formal requirements of group theory (although still entirely apart from the dynamical differential equations that would bring about agreement with experiment) would make it necessary to raise the number of components of ψ from two to four. We will see that one will arrive at two components when *symmetry on the left and right* is removed.

Two-component theory.

§ 1. Transformation law for ψ. – If one introduces homogeneous projective coordinates x_α into the space with Cartesian coordinates x, y, z:

$$x = \frac{x_1}{x_0}, \qquad y = \frac{x_2}{x_0}, \qquad z = \frac{x_3}{x_0}$$

then the equation of the unit sphere will read:

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0 . \tag{1}$$

If one projects it from the South pole to the equatorial plane $z = 0$, which is considered to be the carrier of the complex variables:

$$x + i\,y = \zeta = \frac{\psi_2}{\psi_1},$$

then these equations will be true:

$$\left.\begin{aligned} x_0 &= \bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2, & x_1 &= \bar{\psi}_1 \psi_2 + \bar{\psi}_2 \psi_1, \\ x_2 &= i\,(-\bar{\psi}_1 \psi_2 + \bar{\psi}_2 \psi_1), & x_3 &= \bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2 . \end{aligned}\right\} \tag{2}$$

x_α are **Hermitian** forms of ψ_1, ψ_2. Only the *ratios* of the variables ψ_1, ψ_2, as well as the coordinates x_α, come under consideration here. A homogeneous linear transformation of ψ_1, ψ_2 (with complex coefficients) will produce a real linear transformation of the coordinates x_α: It represents a collineation that takes the unit sphere to itself and leaves the sense of rotation of the unit sphere unchanged. It is easy to show (as well as well-known) that one will obtain each such collineation once and only once in that way.

When one goes from the homogeneous standpoint to the inhomogeneous one, one will then regard x_α as coordinates in the four-dimensional world and (1) as the equation of the "light cone," and one will restrict oneself to those linear transformations U of ψ_1, ψ_2 whose determinant has the absolute value 1. U acts upon the x_α as a *Lorentz transformation*, i.e., a real homogeneous linear transformation that takes the form:

$$-\ x_0^2 + x_1^2 + x_2^2 + x_3^2$$

to itself. However, the formula for x_0 and our remark about the conservation of the sense of rotation on the sphere shows, with no further discussion, that of the Lorentz transformations, we will get only the one single Λ that is defined in a closed continuum and that:

1. Does not invert past and future, and
2. Has the determinant + 1, not – 1.

Of course, that is true without exception. The linear transformation U of ψ is not established uniquely by Λ, but one will be free to choose an arbitrary constant factor $e^{i\lambda}$ of absolute magnitude 1. One can normalize it by the demand that the determinant of U should be equal to 1, but even then a double-valuedness will still remain. In regard to the restriction 1, one would like to establish that one of the most promising aspects of the ψ-theory is that it can carry the *essential difference between past and future* in the calculations. The restriction 2 eliminates the equivalence of left and right. It is only that symmetry of right and left, which actually exists in nature, that will compel us (Part II) to introduce a second pair of ψ-components.

The **Hermitian** conjugate of a matrix $A = \| a_{ik} \|$ will be denoted by A^*:

$$a_{ik}^* = \bar{a}_{ki} \ .$$

Let S_α be the coefficient matrix of the **Hermitian** form of the variables ψ_1, ψ_2, by which the coordinates x_α will be represented in (2):

$$x_\alpha = \psi^* S_\alpha \psi \ . \tag{3}$$

Here, ψ means the column ψ_1, ψ_2 . S_0 is the unit matrix. The equations:

$$S_1^2 = 1, \qquad S_2 S_3 = i\, S_1 \tag{4}$$

are true, along with the ones that emerge from them by cyclically permuting the indices 1, 2, 3.

It is formally somewhat more convenient to replace the real time coordinate x_0 with the imaginary one $i\,x_0$. The Lorentz transformations then take the form of orthogonal transformations of the four quantities:

$$x\,(0) = i\,x_0\,, \qquad x\,(\alpha) = x_\alpha \qquad [\alpha = 1, 2, 3].$$

Instead of (3), one writes:

$$x\,(\alpha) = \psi^* S(\alpha)\psi\,. \tag{5}$$

The transformation law of the ψ-components consists of saying that under the influence of a transformation Λ of the world-coordinates x (α), they will be converted in such a way that the quantities (5) will suffer the transformation Λ. *A quantity of that kind represents the wave field of a material particle, as the phenomenon of spin would imply.* x (α) are the coordinates in a "normal axis-cross" **e** (α) : **e** (1), **e** (2), **e** (3) are real spatial vectors that define a left-handed Cartesian coordinate system, **e** (0) / i is a real, future-pointing, temporal world-vector. The transformation Λ describes the transition from one such normal axis-cross to another equivalent one that might be referred to briefly as a *rotation of the axis-cross* from now on. We will get the same coefficients $c(\alpha\,\beta)$ whether we express the transformation Λ in terms of the basis vectors of the axis-cross or the coordinates:

$$\mathbf{x} = \sum_\alpha x(\alpha)\,\mathbf{e}(\alpha) = \sum_\alpha x'(\alpha)\,\mathbf{e}'(\alpha)\,,$$

$$\mathbf{e}'\,(a) = \sum_\beta c(\alpha\,\beta)\,\mathbf{e}(\beta)\,, \qquad x'\,(\alpha) = \sum_\beta c(\alpha\,\beta)\,x(\beta)\,.$$

That follows from the orthogonal character of Λ.

In what follows, it will be necessary to calculate the infinitesimal transformation:

$$d\psi = dE \cdot \psi\,, \tag{6}$$

which corresponds to an arbitrary infinitesimal rotation $d\,\Omega$:

$$dx\,(\alpha) = \sum_\beta do(\alpha\,\beta)\cdot x(\beta)\,.$$

The do ($\alpha\,\beta$) define a skew-symmetric matrix. The transformation (6) is thought to be normalized such that trace of dE will be equal to 0. The matrix dE depends linearly and homogeneously upon the do ($\alpha\,\beta$). We then write:

$$dE = \tfrac{1}{3}\sum_{\alpha\,\beta} do(\alpha\,\beta)\cdot A(\alpha\,\beta) = \sum do(\alpha\,\beta)\cdot A(\alpha\,\beta)\,.$$

The last summation shall extend over only the pairs:

$$(\alpha\,\beta) = (0\ 1),\ (0\ 2),\ (0\ 3)\ ; \qquad (2\ 3),\ (3\ 1),\ (1\ 2)\ .$$

A $(\alpha\,\beta)$ Naturally, A $(\alpha\,\beta)$ depends upon α and β skew-symmetrically. One must not forget that the coefficients *do* $(\alpha\,\beta)$ are purely imaginary for the three pairs $(\alpha\,\beta)$ and real for the last three but are otherwise arbitrary. One finds that:

$$A\,(2\ 3) = -\frac{1}{2i}\,S(1), \quad A\,(0\ 1) = \frac{1}{2i}\,S(1)\ , \tag{7}$$

and two analogous pairs of equations that arise by cyclic permutation of the indices 1, 2, 3. In order to confirm that, one must merely calculate that the infinitesimal transformations dE:

$$d\psi = \frac{1}{2i}\,S(1) \quad \text{and} \quad d\psi = \frac{1}{2}\,S(1)$$

will produce the infinitesimal rotations:

$$dx\,(0) = 0\ , \quad dx(1) = 0\ , \quad dx\,(2) = -\,x\,(3)\ , \quad dx\,(3) = x\,(2)\ ,$$

and

$$dx\,(0) = i\,x(1)\ , \quad dx(1) = -\,i\,x(0)\ , \quad dx\,(2) = 0\ , \quad dx\,(3) = 0\ .$$

§ 2. Metric and parallel displacement. – We now move on to the *general theory of relativity*. We describe the *metric* at a world-point P by giving a *local normal axis-cross* **e** (α). Only the class of the normal axis-crosses, which are connected with each other by the group of rotations Λ, is determined by the metric. An individual element of that class will be selected arbitrarily. The laws are then *invariant* under *arbitrary rotations of the local axis-crosses.* In that way, the rotation of the axis-cross at the point P', which is different from P, will be independent of the rotation of P. Let ψ_1 (P), ψ_2 (P) be the components of the matter potential at the point P relative to the chosen local axis-cross **e** (α) there. A vector **t** at P can be written in the form:

$$\mathbf{t} = \sum_\alpha t\,(\alpha)\,\mathbf{e}\,(\alpha)\ .$$

The t (α) are its components with respect to the axis-cross.

For the analytical representation, we will further require a *coordinate system* x_p . x_p are any four continuous functions of position in the world whose values allow one to distinguish the various world-points from each other. The laws are then *invariant under arbitrary coordinate transformations*. $e^p(\alpha)$ might be the components of **e** (α) in the coordinate system. Those 4×4 quantities $e^p(\alpha)$ describe the gravitational field. The contravariant components t^p of a vector **t** in the coordinate system are connected with its components t (a) in the axis-cross by the equations:

$$t^p = \sum_\alpha t_p(\alpha) \cdot e^p(\alpha) .$$

On the other hand, one can calculate $t(\alpha)$ from its covariant components t_p in the coordinate system by way of:

$$t(\alpha) = \sum_p t_p \cdot e^p(\alpha) .$$

Those equations govern the conversion of indices. I have written the Greek indices that relate to the axis-cross as arguments because there is nothing to distinguish between superscripts and subscripts here. The opposite conversion happens by means of the matrix $\| e_p(\alpha) \|$ that is inverse to $\| e^p(\alpha) \|$:

$$\sum_\alpha e_p(\alpha) \cdot e^p(\alpha) = \delta_p^q \qquad \text{and} \qquad \sum_p e_p(\alpha) \cdot e^p(\alpha) = \delta(\alpha, \beta) .$$

δ is equal to 0 or 1 according to whether the indices coincide or not, resp. The rule about dropping the summation sign will be followed for the Latin, as well as the Greek, indices from now on. Let ε be the absolute value of the determinant $| e^p(\alpha) |$. The division of a quantity that is denoted by Latin characters by ε will be indicated, as usual, by the conversion of the Latin symbols into the corresponding German ones; e.g.:

$$\mathfrak{e}^p(\alpha) = \frac{e^p(\alpha)}{\varepsilon} .$$

One can describe a vector and a tensor by their components with respect to either the coordinate system or the axis-cross. However, in regard to the quantities ψ, one can speak of only the components with respect to the axis-cross, because the transformation law for their components is governed by a representation of the rotation group that cannot be extended to the group of all linear transformations. Therefore, in the theory of matter, the gravitational field is represented in the way that was depicted here, instead of by the fundamental metric form (*):

$$\sum_{p,q} g_{pq}\, dx_p\, dx_q .$$

Moreover:

$$g_{pq} = e_p(\alpha)\, e_q(\alpha) .$$

The theory of gravitation must now be converted into this new analytical form. I shall begin with the formulas for the *infinitesimal parallel displacement* that is determined by the metric. It takes the vector $\mathbf{e}(\alpha)$ at the point P to the vector $\mathbf{e}'(\alpha)$ at the infinitely-close point P'. It defines

(*) Which agrees formally with **Einstein**'s recent papers on gravitation and electricity, Sitzungsber. Preuß. Akad. Wiss. (1928), pp. 217, 224; *ibid.* (1929), pp. 2. **Einstein** used the symbol h instead of e.

a normal axis-cross at the point P' that emerges from the local axis-cross there $\mathbf{e}(\alpha) = \mathbf{e}(\alpha\,;P')$ by an infinitesimal rotation $d\,\Omega$:

$$\delta\,\mathbf{e}\,(\beta) = \sum_{\gamma} do(\beta\,\gamma)\cdot\mathbf{e}(\gamma)\,,\quad \delta\,\mathbf{e}\,(\beta) = \mathbf{e}\,(\beta) - \mathbf{e}'\,(\alpha\,;P')\,. \tag{8}$$

$d\,\Omega$ depends linearly upon the displacement PP' or its components:

$$dx_p = (dx)^p = v^p = e^p(\alpha)\,v(\alpha)\;.$$

We then write:

$$d\,\Omega = \Omega_p\,(dx)^p\,,\qquad do\,(\beta\,\gamma) = o_p\,(\beta\,\gamma)\,(dx)^p = o\,(\alpha\,;\,\beta\,\gamma)\,v\,(\alpha)\,. \tag{9}$$

As one knows, the parallel displacement of the vector $\mathbf{t}$ with the components t^p will be described by an equation:

$$d\mathbf{t} = -\,d\,\Gamma\cdot\mathbf{t}\,,\qquad \text{i.e.,}\qquad dt^p = -\,d\,\Gamma^p_r\cdot t^r\,,\qquad d\,\Gamma^p_r = \Gamma^p_{rq}\,(dx)^q\,,$$

in which the quantities Γ^p_{rq}, which are independent of $\mathbf{t}$, as well as the displacement dx, are symmetric in r and q. We then have:

$$\mathbf{e}'\,(\beta) - \mathbf{e}\,(\beta) = -\,d\,\Gamma\cdot\mathbf{e}\,(\beta)\,.$$

Equation (8) is also true. Subtracting the two differences from the left-hand side will give the differential $d\,\mathbf{e}\,(\beta) = \mathbf{e}\,(\beta\,;\,P') - \mathbf{e}\,(\beta\,;P)$:

$$d\,\mathbf{e}^p(\beta) + d\,\Gamma^p_r\,e^r(\beta) = -\,do\,(\beta\,\gamma)\cdot e^p(\gamma)\,,$$

$$\frac{\partial e^p(\beta)}{\partial x_q}\cdot e^q(\alpha) + \Gamma^p_{rq}\,e^r(\beta)\,e^q(\alpha) = -\,o\,(\alpha\,;\,\beta\gamma)\cdot e^p(\gamma)\,.$$

Here, one can eliminate the o and obtain the known equations for determining Γ when one expresses the fact that $o\,(\alpha\,;\,\beta\gamma)$ is skew-symmetric with respect to β and γ. One eliminates the Γ and calculates o by making use of the fact that Γ^p_{rq} is symmetric with respect to r and q or:

$$\Gamma^p(\beta,\alpha) = \Gamma^p_{rq}\,e^r(\beta)\,e^q(\alpha)$$

is symmetric in α and β:

$$\frac{\partial e^p(\alpha)}{\partial x_q}\cdot e^q(\beta) - \frac{\partial e^p(\beta)}{\partial x_q}\cdot e^q(\alpha) = \{o\,(\alpha\,;\,\beta\gamma) - o\,(\beta\,;\,\alpha\gamma)\}\,e^p(\gamma) \tag{10}$$

or

$$o\,(\alpha\,;\,\beta\gamma) + o\,(\beta\,;\,\alpha\gamma) = [\mathbf{e}\,(\alpha), \mathbf{e}\,(\beta)]\,(\gamma)\,. \tag{11}$$

If one performs the three cyclic permutations of $\alpha\,\beta\,\gamma$ on that equation and adds the equations that arise in that way with the signs + – + then one will get:

$$2\,o\,(\alpha\,;\,\beta\gamma) = [\mathbf{e}\,(\alpha), \mathbf{e}\,(\beta)]\,(\gamma) - [\mathbf{e}\,(\beta), \mathbf{e}\,(\gamma)]\,(\alpha) + [\mathbf{e}\,(\gamma), \mathbf{e}\,(\alpha)]\,(\beta)\,.$$

$o\,(\alpha\,;\,\beta\gamma)$ is then, in fact, determined uniquely. The expression that is found satisfies all conditions because it is skew-symmetric in β and γ as is obvious with no further discussion.

In particular, for what follows, we will need the abbreviation:

$$o\,(\rho\,;\,\rho\alpha) = [\mathbf{e}\,(\alpha), \mathbf{e}\,(\rho)]\,(\rho) = \frac{\partial e^p(\alpha)}{\partial x_p} - \frac{\partial e^p(\rho)}{\partial x_q}\cdot e^q(\alpha)\,e_p(\rho)\,.$$

Since:

$$-\,\varepsilon\cdot\delta\left(\frac{1}{\varepsilon}\right) = \frac{\delta\varepsilon}{\varepsilon} = e_q(\rho)\cdot\delta e^q(\rho)\,,$$

that will give:

$$o\,(\rho\,;\,\rho\alpha) = \varepsilon\cdot\frac{\partial e^p(\alpha)}{\partial x_p}\,. \tag{12}$$

§ 3. Action quantity of matter. – Not only can the covariant derivative of a vector or tensor field be calculated with the help of parallel displacement, but also that of the ψ-field. Let $\psi_a\,(P)$, $\psi_a(P')$ $[a = 1, 2]$ be the components relative to the local axis-cross $\mathbf{e}\,(\alpha)$ at P (P', resp.). The difference $\psi_a(P') - \psi_a\,(P) = d\psi_a$ is the ordinary differential. On the other hand, we carry the axis-cross $\mathbf{e}\,(\alpha)$ from P to P' by parallel displacement: Let $\mathbf{e}'(\alpha)$; ψ'_a be the components of ψ at P' relative to the axis-cross $\mathbf{e}'(\alpha)$ there . ψ_a and ψ'_a depend upon only the choice of the axis-cross $\mathbf{e}\,(\alpha)$ at P. They have nothing to do with the local axis-cross at P'. Under a rotation of the axis-cross at P, the ψ'_a will transform just like the ψ_a, and similarly, the differences $\delta\psi_a = \psi'_a - \psi_a$. They are the components of the *covariant differential* $\delta\psi$ of ψ. $\mathbf{e}'(\alpha)$ emerges from the local axis-cross $\mathbf{e}\,(\alpha) = \mathbf{e}\,(\alpha\,;\,P')$ at P' by the infinitesimal rotation $d\,\Omega$ that was determined in § **2**. The corresponding infinitesimal transformation:

$$dE = \tfrac{1}{2}\,do\,(\beta\,\gamma)\cdot A\,(\beta\,\gamma)$$

takes $\psi_a(P')$ to ψ'_a, i.e., $\psi' - \psi(P')$ is equal to $dE\cdot\psi$. If one now adds $d\psi = \psi(P') - \psi(P)$ then one will get:

$$\delta\psi = d\psi + dE\cdot\psi\,. \tag{13}$$

Everything depends linearly upon the displacement PP'. It will be written:

$$\delta\psi = \psi_p\,(dx)^p = \psi\,(\alpha)\,v\,(\alpha)\,, \qquad dE = E_p\,(dx)^p = E\,(\alpha)\,v\,(\alpha)\,.$$

We then find that:

$$\psi_p = \left(\frac{\partial}{\partial x_p} + E_p\right)\psi \qquad \text{or} \qquad \psi\,(\alpha) = \left(e^p(\alpha)\frac{\partial}{\partial x_p} + E\,(\alpha)\right)\psi\,.$$

In that, we have:

$$E\,(a) = \tfrac{1}{2}\,o\,(\alpha\,;\beta\,\gamma)\,A\,(\beta\,\gamma)\,.$$

If ψ' is a quantity with the same transformation law as ψ then:

$$\psi^*\,S\,(\alpha)\,\psi'$$

will be the components of a vector relative to the local axis-cross. Hence:

$$v'\,(\alpha) = \psi^*\,S\,(\alpha)\,\delta\psi' = \psi^*\,S\,(\alpha)\,\psi' \cdot v\,(\beta)$$

will be a linear map $v \to v'$ of the vector space at P that is independent of the axis-cross. As a result, its trace:

$$\psi^*\,S\,(\alpha)\,\psi(\alpha)$$

will be a scalar, and the equation:

$$i\,\varepsilon\,\mathfrak{m} = \psi^*\,S\,(\alpha)\,\psi(\alpha) \tag{14}$$

will define a scalar density $\mathfrak{m}$ whose integral:

$$\int \mathfrak{m}\,dx \qquad (dx = dx_0\,dx_1\,dx_2\,dx_3)$$

can find employment as an action quantity.

In order to arrive at an explicit expression for $\mathfrak{m}$, we must calculate:

$$S\,(\alpha)\,E\,(\alpha) = \tfrac{1}{2}\,S\,(\alpha)\,A\,(\beta\,\gamma)\cdot o\,(\alpha\,;\beta\gamma)\,. \tag{15}$$

From (7) and (4), one gets that:

$$S\,(\alpha)\,A\,(\beta\,\gamma) = \tfrac{1}{2}\,S\,(\alpha) \qquad [\text{sum over } \alpha \neq \beta, \text{ but not } \beta\,!]$$

and

$$S\,(\beta)\,A\,(\gamma\,\delta) = \tfrac{1}{2}\,S\,(\alpha)\,,$$

when $\alpha\beta\gamma\delta$ is an even permutation of the indices 0123. The terms of the first and second kind then contribute the following multiple of $S(\alpha)$ to (15):

$$\tfrac{1}{2}\, o\,(\rho\,;\,\rho\alpha) = \frac{1}{2\varepsilon}\frac{\partial \mathfrak{e}^p(\alpha)}{\partial x_p}$$

or

$$o\,(\rho\,;\,\rho\alpha) + o\,(\gamma\,;\,\delta\beta) + o\,(\delta\,;\,\beta\gamma) = \frac{i}{2}\;\varphi(\alpha)\,,$$

resp. From (11), when $\alpha\beta\gamma\delta$ is an even permutation of 0123, one will have:

$$i\,\varphi(\alpha) = [\mathbf{e}\,(\beta), \mathbf{e}\,(\gamma)](\delta) + \ldots + (\text{cycl. permutation of } \beta\gamma\delta)$$

$$= \sum \pm \frac{\partial e^p(\beta)}{\partial x_q}\; e^q(\gamma)\, e_p(\delta)\,. \tag{16}$$

The sum extends over the six permutations of $\beta\gamma\delta$ with alternating sign (and naturally over p and q, in addition). With those notations, one will have:

$$\mathfrak{m} = \frac{1}{i}\left(\psi^*\,\mathfrak{e}^p(a)\,S(\alpha)\frac{\partial\psi}{\partial x_p} + \frac{1}{2}\frac{\partial\mathfrak{e}^p(a)}{\partial x_p}\cdot\psi^* S\,(\alpha)\psi\right) + \frac{1}{4\varepsilon}\cdot\varphi(\alpha)\,s(\alpha)\,. \tag{17}$$

The second part of that is equal to:

$$\frac{1}{4i\,\varepsilon}\left| e_p(\alpha), e^q(\alpha), \frac{\partial e^p}{\partial x_q}, s\,(\alpha)\right|$$

(summed over p and q). Each term is a determinant of four rows that one will obtain from the row that was written out when one sets $\alpha = 0, 1, 2, 3$, in succession:

$$s\,(\alpha) = \psi^*\,S(\alpha)\psi\;\;. \tag{18}$$

It is not the action integral:

$$\int \mathfrak{h}\,dx \tag{19}$$

that is meaningful in the laws of nature, but only its variation. Thus, it is not necessary for $\mathfrak{h}$ to be real, but it will suffice that the difference $\bar{\mathfrak{h}} - \mathfrak{h}$ is a divergence. In that case, we say that $\mathfrak{h}$ is *real in practice*. We must test how that works in regard to $\mathfrak{m}$. $e^p(\alpha)$ is real for $\alpha = 1, 2, 3$ and pure imaginary for $\alpha = 0$. Hence, $e^p(\alpha)\,S\,(\alpha)$ is a **Hermitian** matrix. Similarly, $\varphi(\alpha)$ is real for $\alpha = 1, 2, 3$ and pure imaginary for $\alpha = 0$. Hence, $\varphi(\alpha)\,S\,(\alpha)$ is also Hermitian. As a result:

$$\bar{\mathfrak{m}} = -\frac{1}{i}\left(\frac{\partial\psi^*}{\partial x_p}\mathfrak{S}^p\psi + \frac{1}{2}\frac{\partial\mathfrak{e}^p(\alpha)}{\partial x_p}\cdot\psi^* S(\alpha)\psi\right) + \frac{1}{4\varepsilon}\cdot\varphi(\alpha)\,s(\alpha)\,,$$

$$i(\mathfrak{m}-\bar{\mathfrak{m}}) = \psi^*\mathfrak{S}^p\frac{\partial\psi}{\partial x_p} + \frac{\partial\psi^*}{\partial x_p}\mathfrak{S}^p\psi + \frac{\partial\mathfrak{e}^p(\alpha)}{\partial x_p}\cdot\psi^* S(\alpha)\psi$$

$$= \frac{\partial}{\partial x_p}(\psi^*\mathfrak{S}^p\psi) = \frac{\partial\mathfrak{s}^p}{\partial x_p}\ .$$

$\mathfrak{m}$ is then, in fact, real in practice.

We will return to the special theory of relativity when we set:

$$e^0(0) = -i\,,\quad e^1(1) = e^2(2) = e^3(3) = 1\,,$$

and all remaining $e^p(\alpha) = 0$.

§ 4. Energy. – Let (19) be the action integral for matter in the broader sense (viz., matter + electric field), which is described by the ψ and the electromagnetic potential f_p . The laws of nature say that the variation is:

$$\delta\int\mathfrak{h}\,dx = 0$$

when the ψ and f_p are subject to arbitrary infinitesimal variations that vanish outside of a finite region of the world. The variation of ψ gives the material equations in the narrow sense, while the variation of the f_p gives the electromagnetic equations. Based upon that natural law, when one also subjects the $e^p(\alpha)$, which were fixed up to now, to an analogous infinitesimal variation, an equation will come about:

$$\delta\int\mathfrak{h}\,dx = \int\mathfrak{t}_p(\alpha)\cdot\delta e^p(\alpha)\cdot dx\,, \tag{20}$$

by which the tensor density of energy $\mathfrak{t}_p(\alpha)$ is defined.

(20) must vanish as a result of the invariance of the action quantity, when the variation $\delta e^p(\alpha)$ is produced in such a way that:

1. for a fixed coordinate system x_p, the local axis-cross **e** (α) suffers an infinitesimal rotation, or

2. for a fixed axis-cross, the coordinates x_p will be subject to an infinitesimal transformation.

The first process is described by the equations:

$$\delta e^p(\alpha) = o(\alpha\,\beta) \cdot e^p(\beta) .$$

The o $(\alpha\,\beta)$ in this is a skew-symmetric (infinitesimal) matrix that depends arbitrarily upon position, and the vanishing of (20) says that:

$$\mathfrak{t}\,(\beta, \alpha) = \mathfrak{t}_p\,(\alpha)\, e^p(\beta)$$

is symmetric in α and β. *The symmetry of the energy tensor is thus equivalent to the first invariance property*. However, the symmetry law is not fulfilled identically, but is a consequence of the material and electromagnetic laws, since for a fixed ψ-field, the components of ψ would indeed change under the rotation of the axis-cross!

The calculation of the variation $\delta e^p(\alpha)$ that is produced by the second process is somewhat more tedious. However, the arguments are familiar from the theory of relativity in its previous analytical conception (*). The point P with the coordinates x_p will have the coordinates:

$$x'_p = x_p + \delta x_p , \qquad \delta x_p = \xi^p(x)$$

in the transformed coordinate system. The point that has the same coordinates x_p in the new coordinate system that P had in the old one will be denoted by P'; it has the coordinates $x_p - \delta x_p$ in the old system. The vector **t** at P will possess the components:

$$\frac{\partial x'_p}{\partial x_q} \cdot t^q = t_p + \frac{\partial \xi^p}{\partial x_q} \cdot t^q$$

in the new coordinate system. In particular, the changes that the components $e^p(\alpha)$ of the fixed vector **e** (α) at the fixed point P undergo will be:

$$\delta' e^p(\alpha) = \frac{\partial \xi^p}{\partial x_q} \cdot e^q(\alpha) .$$

On the other hand, the difference between the vector **e** (α) at P' and P is given by:

$$de^p(\alpha) = -\frac{\partial e^p(\alpha)}{\partial x_q} \cdot \xi^q .$$

Hence, the variation that is generated by the coordinate transformation *for fixed coordinate values* x_p will be:

(*) Cf., say, **H. Weyl**, *Raum, Zeit, Materie*, 5th ed., pp. 233, *et seq.*, Berlin 1923 (cited at RZM).

$$\delta e^p(\alpha) = \frac{\partial \xi^p}{\partial x_q} \cdot e^q(\alpha) - \frac{\partial e^p(\alpha)}{\partial x_q} \cdot \xi^q .$$

The ξ^p in that are arbitrary functions that vanish outside of a finite region of the world. If we substitute that in (20) then we will get:

$$0 = \int \left\{ \frac{\partial \mathfrak{t}_p^q}{\partial x_q} + \mathfrak{t}_q(\alpha) \frac{\partial e^p(\alpha)}{\partial x_q} \right\} \xi^p \, dx$$

by a partial integration. The quasi-conservation law of energy and impulse is then given here in the form:

$$\frac{\partial \mathfrak{t}_p^q}{\partial x_q} + \frac{\partial e^p(\alpha)}{\partial x_q} \mathfrak{t}_q(\alpha) = 0 . \tag{21}$$

Due to the second term, it is only an actual conservation law in special relativity. In the general theory, it will first become such a thing when the energy of the gravitational field is added to it.

However, in the special theory of relativity, integrating with $d\xi = dx_1\, dx_2\, dx_3$ over the spatial cross-section:

$$x_0 = t = \text{const.} \tag{22}$$

will yield the temporally-constant components of the impulse (J_1, J_2, J_3) and energy ($-J_0$):

$$J_p = \int \mathfrak{t}_p^0 \, d\xi .$$

With the help of symmetry, one will further find the divergence equations:

$$\frac{\partial}{\partial x_q} (x_2 \, \mathfrak{t}_3^q - x_3 \, \mathfrak{t}_2^q) = 0 , \ldots,$$

$$\frac{\partial}{\partial x_q} (x_0 \, \mathfrak{t}_1^q + x_1 \, \mathfrak{t}_0^q) = 0 , \ldots$$

The three equations of the first kind show that the *impulse moment* (M_1, M_2, M_3) is temporally constant:

$$M_1 = \int (x_2 \, \mathfrak{t}_3^0 - x_3 \, \mathfrak{t}_2^0) \, d\xi , \ldots,$$

while the equations of the second kind include the theorem of the *inertia of energy*.

We calculate the energy density for the action quantity $\mathfrak{m}$ of matter that was exhibited above. We shall treat the two parts into which $\mathfrak{m}$ seems to be decomposed according to (17) separately. For the first part, we will get:

$$\int \delta\mathfrak{m} \cdot dx = \int \mathfrak{u}_p(\alpha)\,\delta\mathfrak{e}^p(\alpha) \cdot dx$$

after a partial integration, with:

$$i\,\mathfrak{u}_p(\alpha) = \psi^* S(\alpha)\frac{\partial \psi}{\partial x_p} - \frac{1}{2}\frac{\partial(\psi^* S(\alpha)\psi)}{\partial x_p}$$

$$= \frac{1}{2}\left(\psi^* S(\alpha)\frac{\partial \psi}{\partial x_p} - \frac{\partial \psi^*}{\partial x_p} S(\alpha)\psi\right).$$

The part of the energy that arises from this is then:

$$\mathfrak{t}_p(\alpha) = \mathfrak{u}_p(\alpha) - e_p(\alpha)\cdot \mathfrak{u}, \qquad \mathfrak{t}_p^q = \mathfrak{u}_p^q - \delta_p^q\,\mathfrak{u},$$

where $\mathfrak{u}$ means an abbreviation for $e^p(\alpha)\,\mathfrak{u}_p(\alpha)$. Those formulas are generally correct for non-constant $e^p(\alpha)$, as well. In the second part, however, we restrict ourselves to special relativity, for the sake of simplicity. In that case, we will have:

$$\int \delta\mathfrak{m} \cdot dx = \frac{1}{4i}\int \left| e_p(\alpha), e^q(\alpha), \frac{\partial(\delta e^p(\alpha))}{\partial x_q}, s(\alpha)\right| dx$$

$$= -\frac{1}{4i}\int \left| \delta e_p(\alpha), e_p(\alpha), e^q(\alpha), \frac{\partial s(\alpha)}{\partial x_q}\right| dx$$

$$\mathfrak{t}_p(0) = -\frac{1}{4i}\left| e_p(\alpha), e^q(\alpha), \frac{\partial s(\alpha)}{\partial x_q}\right|_{\alpha=1,2,3}.$$

$\mathfrak{t}_p^0$ arises from this upon multiplication by $-i$; therefore, $\mathfrak{t}_0^0 = 0$ and:

$$\mathfrak{t}_1^0 = \frac{1}{4}\left(\frac{\partial s(3)}{\partial x_2} - \frac{\partial s(2)}{\partial x_3}\right). \tag{23}$$

We combine both parts in order to determine the total energy, impulse, and impulse moment. From:

$$\mathfrak{t}_0^0 = -\frac{1}{2i}\sum_{p=1}^{3}\left(\psi^* S^p \frac{\partial \psi}{\partial x_p} - \frac{\partial \psi^*}{\partial x_p} S^p \psi\right),$$

we will get:

$$-J_0 = -\int \mathfrak{t}_0^0\, d\xi = \frac{1}{i}\int \psi^* \cdot \sum_{p=1}^{3} S^p \frac{\partial \psi}{\partial x_p} \cdot d\xi$$

after a partial integration that is applied to the subtrahend. That leads one to regard the operator:

$$\frac{1}{i}\sum_{p=1}^{3} S^p \frac{\partial}{\partial x_p}$$

as a representative of the energy of a free particle. Furthermore, one will have:

$$J_1 = \int \mathfrak{t}_1^0\, d\xi = \frac{1}{2i}\int \left(\psi^* \frac{\partial \psi}{\partial x_1} - \frac{\partial \psi^*}{\partial x_1}\psi\right) d\xi$$
$$= \frac{1}{i}\int \psi^* \frac{\partial \psi}{\partial x_1} d\xi\,.$$

The term (23) yields no contribution to the integral. The impulse will be represented by the operator:

$$\frac{1}{i}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right),$$

as it must be, according to **Schrödinger**. From the complete expression of:

$$x_2\, \mathfrak{t}_3^0 - x_3\, \mathfrak{t}_2^0\,,$$

one will ultimately get:

$$M_1 = \int \left\{\frac{1}{i}\,\psi^*\left(x_2 \frac{\partial \psi}{\partial x_3} - x_3 \frac{\partial \psi}{\partial x_2}\right) + \tfrac{1}{2}\, s\,(1)\right\} d\xi$$

with a suitable partial integration. Consistent with known formulas, M_1 is then represented by the operator:

$$\frac{1}{i}\left(x_2 \frac{\partial \psi}{\partial x_3} - x_3 \frac{\partial \psi}{\partial x_2}\right) + \tfrac{1}{2}\, S\,(1)\,.$$

Once one has inserted spin into the theory from the outset, it must naturally once more come to light here. However, it is still quite surprising and instructive how that comes about. The fundamental Ansätze of quantum theory will then have a less fundamental character than one would have probably assumed originally. They are coupled with special action quantity $\mathfrak{m}$. On the other hand, that connection confirms the irreplaceability of $\mathfrak{m}$ in its role as the action of matter. Only the general theory of relativity, which leads us to an arbitrary definition of energy by its free variability of the $e^p(\alpha)$, will allow us to close the circle of quantum theory in the way that was depicted.

§ 5. Gravitation. – We once more take up our transcription of **Einstein**'s classical theory of gravitation and next determine the **Riemann curvature tensor** (*). Draw the line elements d and δ through the point P to P_d and P_δ. The line element δ is shifted to P_d and d to P_δ in some way such that they meet at a common corner P^* of an infinitesimal "parallelogram" that lies opposite to P. The axis-cross **e** (α) at P is shifted to P^*, in one case, along the path $P P_d P^*$, and in the other case, along the path $P P_\delta P^*$. The two normal axis-crosses at P^* that one obtains in that way emerge from each other by an infinitesimal rotation:

$$P_{pq}\,(dx)^p\,(\delta x)^q = \tfrac{1}{2}\,P_{pq}\,(\Delta x)^{pq}\,,$$

where:

$$(\Delta x)^{pq} = (dx)^p\,(\delta x)^q - (\delta x)^p\,(dx)^q$$

are the components of the surface element that is spanned by dx and δx, and P_{pq} is skew-symmetric with respect to p and q. P_{pq} is a skew-symmetric matrix $\| r_{pq}\,(\alpha\,\beta) \|$; it is the **Riemann** curvature tensor.

The rotation that creates the axis-cross $\mathbf{e}^*$ (α) from the local axis-cross **e** (α) at P^* by parallel displacement to P^* along the first path is:

$$(1 + d\,\Omega)\,(1 + \delta\,\Omega\,(P_d))$$

with a notation that is easy to understand. The difference between that expression and the one that emerges from it by switching d and δ is:

$$= \{d\,(\delta\,\Omega) - \delta\,(d\,\Omega)\} + (d\,\Omega \cdot \delta\,\Omega - \delta\,\Omega \cdot d\,\Omega)\,.$$

Now:

$$d\,\Omega = \Omega_p\,(dx)^p,$$

$$\delta\,(d\,\Omega) = \frac{\partial\,\Omega_p}{\partial x_p}\,\delta x_p\,dx_p + \Omega_p\,\delta dx_p\,.$$

Because the parallelogram closes, $\delta\,dx_p = d\,\delta x_p$. Hence, one will finally have:

$$P_{pq} = \left(\frac{\partial \Omega_q}{\partial x_p} - \frac{\partial \Omega_p}{\partial x_q}\right) + (\Omega_p\,\Omega_q - \Omega_q\,\Omega_p)\,.$$

(*) Cf., RZM, pp. 119, *et seq.*

The scalar curvature:

$$r = e^p(\alpha)\, e^q(\beta)\, r_{pq}(\alpha\,\beta)$$

gets a contribution from the first, differentiated, component of:

$$(e^q(\alpha)\, e^p(\beta) - e^q(\beta)\, e^p(\alpha))\, \frac{\partial o_p(\alpha\,\beta)}{\partial x_q}\,.$$

When one neglects a complete divergence, it will contribute the two terms to $\mathfrak{r} = r / \varepsilon$:

$$-\ 2o(\beta, \alpha\,\beta)\, \frac{\partial\, \mathfrak{e}^q(\alpha)}{\partial x_q}$$

and

$$\frac{1}{\varepsilon}\, o_p\,(\alpha\,\beta) \left\{ \frac{\partial\, e^p(\alpha)}{\partial x_q}\, e^q(\beta) - \frac{\partial\, e^p(\beta)}{\partial x_q}\, e^q(\alpha) \right\}.$$

From (12), the first one is equal to:

$$-\,2\; o\,(\beta\,;\,\rho\,\beta)\; o\,(\alpha\,;\,\alpha\,\rho)\,,$$

and from (10), the second one is equal to:

$$2\; o\,(\alpha\,;\,\beta\,\gamma) \cdot o\,(\gamma\,;\,\alpha\,\beta)\,.$$

The result is the following expression for the action density $\mathfrak{g}$ of gravitation:

$$\varepsilon\, \mathfrak{g} = o\,(\alpha\,;\,\beta\,\gamma) \cdot o\,(\gamma\,;\,\alpha\,\beta) + o\,(\alpha\,;\,\alpha\,\gamma) \cdot o\,(\beta\,;\,\beta\,\gamma)\,. \tag{24}$$

The integral $\int \mathfrak{g}\, dx$ is not actually invariant, but only in practice, so $\mathfrak{g}$ differs from the scalar density $\mathfrak{r}$ by a divergence.

Varying the $e^p(\alpha)$ in the total action integral:

$$\int (\mathfrak{g} + \kappa\, \mathfrak{h})\, dx$$

implies the *gravitational equations* (κ is a numerical constant).

One obtains the *gravitational energy* $\mathfrak{v}_p^q$ from $\mathfrak{g}$ when one performs an infinitesimal *displacement* in coordinate space (*):

(*) Cf., RZM, pp. 272, *et seq.*

$$x'_p = x_p + \xi^p , \qquad \xi^p = \text{const.}$$

The variation that is produced in that way is:

$$\delta\,\mathbf{e}\,(\alpha) = -\,\frac{\partial\,\mathbf{e}(\alpha)}{\partial x_p}\cdot\xi^p .$$

$\mathfrak{g}$ is a function of $e^p(\alpha)$ and the derivatives $e^p_q(\alpha) = \dfrac{\partial\, e^p(\alpha)}{\partial x_p}$. Let the total differential be denoted by:

$$\delta\,\mathfrak{g} = \mathfrak{g}_p(\alpha)\,\delta e^p(\alpha) + \mathfrak{g}^q_p(\alpha)\,\delta e^p_q(\alpha) .$$

For the variation that is produced by the infinitesimal translation in coordinate space, one must have:

$$\int \delta\mathfrak{g}\cdot dx + \int \frac{\partial\mathfrak{g}}{\partial x_p}\,\xi^p\cdot dx = 0 . \tag{25}$$

The integral extends over an arbitrary region in the world.

$$\int \delta\mathfrak{g}\cdot dx = \int \left(\mathfrak{g}_p(\alpha) - \frac{\partial\,\mathfrak{g}^q_p(\alpha)}{\partial x_q}\right)\delta e^p(\alpha)\cdot dx + \int \frac{\partial\left(\mathfrak{g}^q_p(\alpha)\,\delta e^p(\alpha)\right)}{\partial x_q}\,dx .$$

From the gravitational equations, the bracket in the first integral is equal to $\kappa\, t_p\,(\alpha)$, while the integral itself is equal to:

$$\kappa \int \mathfrak{t}_q(\alpha)\,\frac{\partial\, e^p(\alpha)}{\partial x_q}\cdot\xi^p dx .$$

One introduces:

$$\mathfrak{v}^q_p = \delta^q_p\,\mathfrak{g} - \frac{\partial\, e^r(\alpha)}{\partial x_p}\cdot\mathfrak{g}^q_r(\alpha) .$$

Equation (25) says that the integral of:

$$\left(\mathfrak{v}^q_p - \kappa\,\mathfrak{t}_q(\alpha)\,\frac{\partial\, e^q(\alpha)}{\partial x_p}\right)\xi^p$$

that is extended over an arbitrary region in the world is zero. The integrand must vanish everywhere then. Since the ξ^p are arbitrary constants, the factors of ξ^p are individually zero. (21) will then be converted into the pure divergence equation:

$$\frac{\partial(\mathfrak{v}_p^q + \kappa\,\mathfrak{t}_q)}{\partial x_q} = 0\,,$$

and $\mathfrak{v}_p^q / \kappa$ proves to be the gravitational energy.

In order to be able to formulate an actual differential conservation law of the impulse moment in the general theory of relativity along with that, one must specialize the coordinates in such a way that the cogredient rotation of axis-crosses will take the form of an orthogonal transformation. That is certainly possible, but I shall not go into the details of that here.

§ 6. Electric field. – We now come to the critical part of the theory. In my opinion, the origin and necessity of the electromagnetic field is based upon the following fact: In reality, the components ψ_1, ψ_2 are not determined uniquely by the axis-cross, but only to the extent that they can still be multiplied by an arbitrary "gauge factor" $e^{i\lambda}$ of absolute value 1. The transformation that the ψ experience under the influence of a rotation of the axis-cross is determined only up to such a factor. In the special theory of relativity, one must regard that gauge factor as a constant, because we then have a single axis-cross that is not coupled with a point. Things are different in the general theory of relativity: Every point has its own axis-cross and therefore its own arbitrary gauge factor, as well, in such a way that when one abandons the rigid coupling of the axis-crosses at different points, the gauge factor will necessarily become an arbitrary function of position. However, the infinitesimal linear transformation dE of ψ, which corresponds to the infinitesimal rotation $d\,\Omega$, will not be established completely either, but dE can be increased by an arbitrary pure imaginary multiple $i \cdot df$ of the unit matrix. In order to establish the covariant differential $d\psi$ of ψ uniquely, in addition to the metric in the neighborhood of the point P, we will need one such df for every line element $\overrightarrow{PP'} = (dx)$ that starts from P. In order for $\delta\psi$ to depend upon dx linearly, as before:

$$df = f_p\,(dx)^p$$

must be a linear form in the components of the line element. If we replace ψ with $e^{i\lambda}\cdot\psi$ then we must, at the same time, replace df with $df - d\lambda$, which would emerge from the formula for the covariant differential.

That has the consequence that the term:

$$\frac{1}{\varepsilon}\,f(\alpha)\,s\,(\alpha) = \frac{1}{\varepsilon}\,f(\alpha)\cdot\psi^*\,S(\alpha)\psi = f_p\cdot\psi^*\,\mathfrak{S}^p\,\psi \tag{26}$$

gets added to the action density $\mathfrak{m}$. Gauge invariance prevails, in the sense that the action quantity will remain unchanged when one replaces:

$$\psi \text{ with } e^{i\lambda}\cdot\psi\,, \qquad f_p \text{ with } f_p - \frac{\partial\lambda}{\partial x_p}\,,$$

and one understands λ to mean an arbitrary function of position. From experiments, the electromagnetic potential acts upon matter in precisely the way that is described by (26). We are therefore justified in identifying the quantities f_p that were introduced here with that potential. The proof will be complete when we show that, conversely, the f_p-field will also be affected by the same laws of matter that are true for the electromagnetic potential field, according to experiments.

$$f_{pq} = \frac{\partial f_q}{\partial x_p} - \frac{\partial f_p}{\partial x_q}$$

is a gauge-invariant skew-symmetric tensor, and:

$$\mathfrak{l} = \tfrac{1}{4} f_{pq} \mathfrak{f}^{pq} \tag{27}$$

is the scalar density that is characteristic of **Maxwell**'s theory. Upon variation, the Ansatz:

$$\mathfrak{h} = \mathfrak{m} + a\,\mathfrak{l} \tag{28}$$

(a is a numerical constant) will imply the **Maxwell** equations, with:

$$-\,\mathfrak{s}^p = -\,\psi^* \mathfrak{S}^p \psi \tag{29}$$

as the density of the electric four-current.

Gauge invariance is closely related to the conservation law for electricity. Since $\mathfrak{h}$ is gauge-invariant, $\delta\int \mathfrak{h}\,dx$ must vanish identically when the ψ and f_p are varied according to:

$$\delta\,\psi = i\,\lambda \cdot \psi\,, \qquad \delta f_p = -\,\frac{\partial\lambda}{\partial x_p}$$

for fixed $e^p(\alpha)$; λ is an arbitrary function of position. That implies a relation between the material equations and the electromagnetic ones that is fulfilled identically. If we know that the material equations (in the narrow sense) are valid then it will follow that:

$$\delta\int \mathfrak{h}\,dx = 0$$

when only the f_p are varied according to the equation $\delta f_p = -\,\partial\lambda / \partial x_p$. On the other hand, the same thing will follow from the electromagnetic equations for the infinitesimal variation $\delta\psi = i\,\lambda \cdot \psi$ of the ψ alone. When $\mathfrak{h} = \mathfrak{m} + a\,\mathfrak{l}$, we will get:

$$\int \delta\mathfrak{h} \cdot dx = \pm \int \psi^* \mathfrak{S}^p \psi \cdot \frac{\partial \lambda}{\partial x_p} dx = \mp \int \lambda \frac{\partial \mathfrak{s}^p}{\partial x_p} dx$$

in both cases. We find that an analogous state of affairs prevails for the conservation laws for energy-impulse and impulse-moment. They couple the material equations in the broader sense with the gravitational equations and correspond to invariance under coordinate transformations (arbitrary independent rotations of the local axis-crosses at the various world-points, resp.).

From:

$$\frac{\partial \mathfrak{s}^p}{\partial x_p} = 0 , \tag{30}$$

one gets that the flux of the vector density $\mathfrak{s}^p$ through a three-dimensional cross-section of the world, in particular, through a cross-section (22):

$$l = \int \mathfrak{s}^0 \, d\xi , \tag{31}$$

is independent of the position of the cross-section (of t, resp.). Not only that integral, but also the individual integral element, has an invariant meaning. At any rate, the sign depends upon which sense of direction is counted as a positive traversal of the three-dimensional section. In order to be able to speak of $\mathfrak{s}^0 \, d\xi$ as a spatial probability density, the **Hermitian** form:

$$e^0(\alpha) \cdot \psi^* S(\alpha) \psi \tag{32}$$

in ψ_1, ψ_2 must be definite. One easily finds that this is the case when (22) is actually a spatial cross-section at P, so when the line elements that lie in it and start from P are space-like. In order for (32) to take the positive sign, the cross-sections x_0 = const. must be arranged in increasing x_0, in which the directions of the future that are indicated by the vector **e** (0) / i follow in succession. The sign of the flux is also fixed by those natural restrictions in the coordinate system, and the invariant (31) will be normalized by the condition:

$$l \equiv \int \mathfrak{s}^0 \, d\xi = 1 \tag{33}$$

in the usual way. The constant a, which combines $\mathfrak{m}$ and $\mathfrak{l}$ with each other, is a real number equal to ch / e^2 then (i.e., inverse fine-structure constant).

We treat ψ_1, ψ_2, f_p, $e^p(\alpha)$ as the quantities that must be varied independently of each other. The energy density t_p^q that arises from $\mathfrak{m}$ must be increased by:

$$f_p \, \mathfrak{s}^q - \delta_p^q (f_r \, \mathfrak{s}^r)$$

due to the extension term (26). In the special theory of relativity, that leads one to associate energy with the operator:

$$H = \sum_{p=1}^{3} S^p \left(\frac{1}{i} \frac{\partial}{\partial x_p} + f_p \right) ,$$

since its value is:

$$\int \psi^* \cdot H \psi \cdot d\xi \, .$$

Of course, the material equations will then read:

$$\left(\frac{1}{i} \frac{\partial}{\partial x_p} + f_p \right) \psi + H \, \psi = 0 \qquad \text{and not} \qquad \frac{1}{i} \frac{\partial \psi}{\partial x_0} + H \, \psi = 0 \, ,$$

as has been assumed up to now in quantum mechanics. Naturally, the electromagnetic energy must be added to it, for which the classical **Maxwellian** expressions will retain their validity.

As far as *physical dimensions* are concerned, in general relativity, it is natural to regard the coordinates x_p as real numbers. The quantities that appear are not only invariant under changes of yardsticks, but also under arbitrary transformations of the x_p . If all $\mathbf{e}(\alpha)$ are converted into $b \cdot \mathbf{e}(\alpha)$ upon multiplying by a constant b then if the normalization (33) is to be preserved, at the same time, ψ must be replaced with $b^{3/2} \cdot \psi$. $\mathfrak{m}$ and $\mathfrak{l}$ will not be changed in that way, so they are real numbers. By contrast, $\mathfrak{g}$ takes on the factor $1 / b^2$, such that κ will be the square of a length d. κ is not identical to the **Einstein** gravitational constant but will arise from it upon multiplying by $2h / c$. d lies far below the atomic scale and is ~ 10^{-32} cm. Gravitation will be meaningful for only astronomical problems here.

If we overlook the gravitational term then the field equations will not contain any dimensional atomic constant. There is no place in the two-component theory for an action quantity like the term in **Dirac**'s theory, which carries *mass* as a factor (*). However, one knows how mass can be introduced on the basis of the conservation laws. One assumes that in the "empty environment" of the particle, the $\mathfrak{t}_p^q$ will vanish, and the $e^p(\alpha)$ will assume the constant values of special relativity outside of a certain world-tube whose cross-section x_0 = const. of finite extent. The:

$$J_p = \int \left(\mathfrak{t}_q^p + \frac{1}{\kappa} \mathfrak{v}_p^0 \right) d\xi$$

will then be the components of a temporally-constant four-vector in the surrounding environment that is not influenced by the arbitrariness in the coordinate system and the local axis-cross. The normal coordinate system itself can be established more precisely by the condition that the impulse

(*) Proc. Roy. Soc. (A) **117**, 610.

(J_1, J_2, J_3) vanishes. – J_0 is then the invariant, and at the same time constant, mass of the particle. It must now be demanded that this mass must have a value m that is given once and for all.

Along with the theory of the electromagnetic field that was discussed here, and which I consider to be correct, since it arises so naturally from the arbitrariness of the gauge factor ψ and thus allows one to understand the connection between the gauge invariance that exists experimentally and the conservation law for electricity, there is yet another one that couples electricity with gravitation that presents itself. The term (26) has the same form as the second part of $\mathfrak{m}$, namely, formula (17); $\varphi(\alpha)$ plays the same role in the term (26) that $f(\alpha)$ plays in the second part of $\mathfrak{m}$. One might then expect that matter and gravitation [viz., ψ and $e^p(\alpha)$] would already suffice to explain electromagnetic phenomena by themselves when one appeals to the quantities $\varphi(\alpha)$ as electromagnetic potentials. Those quantities depend upon the $e^p(\alpha)$ and their first derivatives in such a way that invariance under arbitrary coordinate transformations will exist. However, as far as rotations of the axis-crosses are concerned, the $\varphi(\alpha)$ will transform like the components of a fixed vector in the axis-cross only when the axis-crosses are subject to *the same* rotation at all points. If one ignores that matter field and observes only the connection between electricity and gravitation then one will arrive at a theory of electricity that has precisely the same type that **Einstein** recently attempted. At any rate, teleparallelism was only an illusion here.

I have convinced myself that one will not arrive at the **Maxwell** equation by that perhaps initially tempting Ansatz. Furthermore, gauge invariance remains entirely enigmatic. The electromagnetic potential itself has physical meaning, and not merely the field strengths. Therefore, I believe that this idea will lead to a wrong turn, and that we must rather trust in the wisdom of gauge invariance: *Electricity is a phenomenon that accompanies the wave-field of matter, and not gravitation.*

Palmer Physical Laboratory, Princeton University, 19 April 1929.

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