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Gravitation and electricity

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Translated by D. H. Delphenich

According to RIEMANN ([1]), geometry is based upon the following two facts:

1. *Space is a three-dimensional continuum*, so the manifold of its points can be represented by the system of values for three coordinates x_1, x_2, x_3 in a continuous way.

2. (*Pythagorean Theorem*). The square of the distance ds^2 between two infinitely-close points:

$$P = (x_1, x_2, x_3) \qquad \text{and} \qquad P' = (x_1 + d\,x_1\,,\, x_2 + d\,x_2\,,\, x_3 + d\,x_1) \tag{1}$$

is (with the use of arbitrary coordinates) a quadratic form in the relative coordinates dx_i :

$$ds^2 = \sum_{ik} g_{ik}\, dx_i\, dx_k \qquad (g_{ki} = g_{ki}). \tag{2}$$

We can express the second fact briefly by saying: Space is a *metric* continuum. In the spirit of modern local action physics, we assume that the Pythagorean Theorem is strictly valid only at infinity.

The special theory of relativity leads to the insight that time gets added to the three spatial coordinates as a fourth coordinate (x_0) with the same status as the others, so the stage on which material events play out – viz., *the world* – is then *a four-dimensional, metric continuum.* The quadratic form (2) that the world-metric establishes is not positive-definite, as in the case of three-dimensional spatial geometry, but has an index of inertia 3. RIEMANN already expressed the thought that it should be regarded as something physically real, since it reveals itself – e.g., in the centrifugal forces – as an agency that exerts real effects upon matter, and that, accordingly, one must assume that matter reacts to it, whereas up to now, all geometers and philosophers have been of the opinion that the metric of space is intrinsic to it, independently of the material content that fills it up. In our own time, EINSTEIN (independently of RIEMANN) has erected the grandiose edifice of his general theory of relativity upon those ideas, which RIEMANN could not have possibly followed through on. For EINSTEIN, the

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([1]) "Über die Hypothesen welche der Geometry zugrunde liegen," *Math. Werke* (2nd ed.), Leipzig, 1892, no. XIII, pp. 272.

phenomena of *gravitation* also enter into the calculation of the world-metric, and the laws by which matter acts upon the metric are nothing but the laws of gravitation; the g_{ik} in (2) define the components of the gravitational potential. Whereas the gravitational potential consists of an invariant *quadratic* differential form, the *electromagnetic phenomena* arise from a four-potential whose components ϕ_i combine into an invariant *linear* differential form $\sum \phi_i \, dx_i$. However, both realms of phenomena – viz., gravitation and electricity – have been completely isolated from each other, up to now.

From some recent presentations of LEVI-CIVITA ([1]), HESSENBERG ([2]), and the author ([3]), it has emerged with crystal clarity that a natural construction of RIEMANNian geometry that should be used as a basic concept is that of the infinitesimal parallel displacement of a vector. If P and P^* are any two points that are connected by a curve then one can displace a vector that is given at P along that curve from P to P^*. However, that vector translation from P to P^* is not integrable, generally speaking; i.e., the vector at P^* to which one will arrive will depend upon the path along which the displacement was carried out. Integrability is found only in Euclidian ("gravitationless") geometry. One has now obtained one last distance-geometric element in the RIEMANNian geometry that was characterized above, and as far as I can see, with no factual basis. It is only the coincidental genesis of this geometry from surface theory that seems to be at fault. Namely, the quadratic form (2) makes it possible to not only compare the lengths of two vectors at the same point, but also at any two points that are separated from each other. *However, a true local geometry allows one to know only a principle for translating a length from a point to an infinitely-close one*, and one must then assume from the outset that the problem of translating a length from one point to an infinitely-close point is integrable, just as the problem of the translation of direction has been found to be integrable. When one ignores the aforementioned inconsistency, a geometry will come about that will, surprisingly, explain *not only the gravitational phenomena, but also the electromagnetic field.* Both arise from the same source in that theory that emerges, and in fact *one cannot separate gravitation and electricity from each other arbitrarily, in general*. In that theory, *all physical quantities have a world-geometric meaning: In particular, the effective quantities appear in them as pure numbers from the outset. They lead to an essentially uniquely-determined world-law; indeed, they even allow one to grasp why the world is four-dimensional, in a certain sense.* I would now like to sketch out the construction of the corrected RIEMANNian geometry without any ulterior physical motives; the physical application then follow from it by themselves.

In a certain coordinate system, the relative coordinates dx_i of a point P' that is infinitely close to a point P [see (1)] are the components of the *infinitesimal displacement* $\overrightarrow{PP'}$. The transition from one coordinate system to another is expressed by continuous transformation rules:

$$x_i = x_i(x_1^*, x_2^*, \ldots, x_n^*) \qquad (i = 1, 2, \ldots, n)$$

that establish the connection between the coordinates of the same point in one system and the other. The linear transformation formulas:

([1]) "Nozioni di parallelismo…," Rend. del Circ. Matem. di Palermo **43** (1917).
([2]) "Vektorielle Begründung der Differentialgeometrie," Math. Ann. **78**(1917).
([3]) *Raum, Zeit, Materie*, Berlin, 1918. § 14.

(3) $$dx_i = \sum_k \alpha_{ik}\, dx_k$$

then exist between the components dx_i (dx_i^*, resp.) of the same infinitesimal displacement of the point P, in which α_{ik} are the values of the derivatives $\partial x_i / \partial x_k^*$ at the point P. A (contravariant) *vector* $\mathfrak{x}$ at the point P has n numbers ξ^i for its components relative to any coordinate system that transform under the transition to another coordinate system in precisely the same way (3) as the components of an infinitesimal displacement. I refer to the totality of vectors at the point P as the *vector space* at P. It is:

1. *Linear or affine*.

That is, the multiplication of a vector at P by a number and the addition of two such vectors will always produce a vector at P.

2. *Metric.*

A scalar product of any two vectors $\mathfrak{x}$ and $\mathfrak{y}$ with the components ξ^i and η^i:

$$\mathfrak{x} \cdot \mathfrak{y} = \mathfrak{y} \cdot \mathfrak{x} = \sum_{ik} g_{ik}\, \xi^i \eta^k$$

is associated with the symmetric bilinear form that belongs to (2) in an invariant way. However, with our way of looking at things, *this form is only defined up to a positive proportionality factor that remains arbitrary.* If the manifold of spatial points is represented by the coordinates x_i then the metric at the point P will be established only by the ratios of the g_{ik}. Only the ratios of the g_{ik} have an immediate, intuitive, physical meaning, as well. Namely, for a given starting point P, the equation:

$$\sum_{ik} g_{ik}\, dx_i\, dx_k = 0$$

is satisfied by those infinitely-close world-points P' through which a light signal that is emitted at P will go. For the purpose of analytical representation, we must:

1. Choose a well-defined coordinate system.
2. Fix the arbitrary proportionality factor that g_{ik} is endowed with at each point P.

The formulas that come about must correspondingly be invariant in two ways:

1. They must be *invariant under arbitrary coordinate transformations.*
2. The must remain unchanged when one *replaces the g_{ik} with $\lambda\, g_{ik}$*, in which λ is an arbitrary continuous function of position.

The appearance of this second invariance property is characteristic of our theory.

If P, P^* are any two points, and if any vector $\mathfrak{x}$ at P is associated with a vector $\mathfrak{x}^*$ at P^* in such a way that generally $\alpha\mathfrak{x}$ goes to $\alpha\mathfrak{x}^*$ and $\mathfrak{x} + \mathfrak{y}$ goes to $\mathfrak{x}^* + \mathfrak{y}^*$ (α is an arbitrary number), and the vector 0 at P is the only one that corresponds to the vector 0 at P^* then an affine or *linear map* of the vector space at P to the vector space at P^* will be effected. In particular, that map will be *a similarity* when the scalar product $\mathfrak{x}^* \cdot \mathfrak{y}^*$ of the image vectors at P^* is proportional to that of $\mathfrak{x}$ and $\mathfrak{y}$ at P for all pairs of vectors $\mathfrak{x}$, $\mathfrak{y}$. (Only this notion of a *similarity* has any objective sense in our opinion; up to now, the theory made it possible to present the sharper notion of a *congruence*.) What we mean by the *parallel displacement of a vector* at the point P to a neighboring point P' will be established by the following two axiomatic requirements:

1. A similarity of the vector space at the point P to the vector space at the neighboring point P' will be implemented by the parallel displacement of vectors at P to P'.

2. If P_1, P_2 are two points that are close to P, and if the infinitesimal vector $\overrightarrow{PP_2}$ at P goes to $\overrightarrow{P_1P_{12}}$ at P_1 by parallel displacement, but $\overrightarrow{PP_1}$ goes to $\overrightarrow{P_1P_{21}}$ at P_2 by parallel displacement then P_{12}, P_{21} must coincide (commutativity).

The part of the first requirement that states that parallel displacement is an affine transplantation of the vector space at P to one at P' can be expressed analytically as follows: Under displacement, the vector ξ^i at $P = (x_1, x_2, \ldots, x_n)$ goes to a vector:

$$\xi^i + d\xi^i \qquad \text{at} \qquad P' = (x_1 + dx_1, x_2 + dx_1, \ldots, x_n + dx_n)$$

whose components depend linearly upon ξ^i:

$$d\xi^i = -\sum_r d\gamma^i_r \, \xi^r . \tag{4}$$

The second requirement teaches us that the $d\gamma^i_r$ are linear differential forms:

$$d\gamma^i_r = \sum_r \Gamma^i_{rs} \, dx_s$$

whose coefficients possess the symmetry property:

$$\Gamma^i_{sr} = \Gamma^i_{rs} . \tag{5}$$

If two vectors ξ^i, η^i at P go to $\xi^i + d\xi^i$, $\eta^i + d\eta^i$ at P' by parallel displacement then the requirement of similarity that is posed in 1, which emerges from the affinity, states that:

$$\sum_{ik}(g_{ik}+dg_{ik})(\xi^i+d\xi^i)(\eta^k+d\eta^k) \text{ must be proportional to } \sum_{ik} g_{ik}\,\xi^i\,\eta^k .$$

If we let $1 + d\phi$ denote the proportionality factor that deviates infinitely little from 1 and define the lowering of an index in the usual way by the formula:

$$a_i = \sum_k g_{ik}\,a^k$$

then that will yield:

(6) $$dg_{ik} - (d\gamma_{ki} + d\gamma_{ik}) = g_{ik}\,d\phi .$$

It emerges from this that $d\phi$ is a linear differential form:

(7) $$d\phi = \sum_k \phi_i\,dx_i .$$

If that form is known then equation (6), or:

$$\Gamma_{i,kr} + \Gamma_{k,ir} = \frac{\partial g_{ik}}{\partial x_r} - g_{ik}\,\phi_r ,$$

together with the symmetry condition (5), will yield the quantities Γ. *The intrinsic metric connection of space then depends upon not only the quadratic form* (2) (which is determined only up to an arbitrary proportionality factor), *but also upon a linear form* (7). If we replace g_{ik} with $\lambda\,g_{ik}$ without changing the coordinate system then the quantities will not change, while dg_{ik} will take on the factor λ, and dg_{ik} will go to $\lambda\,dg_{ik} + g_{ik}\,d\lambda$. Equation (6) will then teach us that $d\phi$ goes to:

$$d\phi + \frac{d\lambda}{\lambda} = d\phi + d\ln\lambda .$$

Therefore, in the linear form $\sum \phi_i\,dx_i$, it is not perhaps a proportionality factor that remains undetermined, which must be fixed by an arbitrary choice of unit of measurement, but rather, the arbitrariness that it is endowed with consists of an *additive total differential*. For the analytical presentation of the geometry, the forms:

(8) $$g_{ik}\,dx_i\,dx_k,\ \phi_i\,dx_i$$

are equivalent to:

(9) $$\lambda \cdot g_{ik}\,dx_i\,dx_k,\ \phi_i\,dx_i + d\ln\lambda,$$

in which λ is an arbitrary positive function of position. *Therefore, it is the skew-symmetric tensor with the components:*

(10) $$F_{ik} = \frac{\partial \phi_i}{\partial x_k} - \frac{\partial \phi_k}{\partial x_i}$$

that has an invariant meaning; i.e., the form:

$$F_{ik}\, dx_i\, \delta x_k = \tfrac{1}{2} F_{ik}\, \Delta x_{ik}\ ,$$

which depends bilinearly upon two arbitrary displacements dx and δx at the point P, or even better, upon the surface element that is spanned by those two displacements, and whose components are:

$$\Delta x_{ik} = dx_i\, \delta x_k - dx_k\, \delta x_i\ .$$

The special case of the theory, up to now, in which the arbitrarily-chosen unit of length at a starting point can be translated by parallel displacement to all points of space in a manner that is independent of path, exists when the g_{ik} can be fixed absolutely in such a way that the ϕ_i vanish. The Γ^i_{rs} are nothing but the CHRISTOFFEL three-index symbols then. The necessary and sufficient invariant condition for that case to present itself then consists of the tensor F_{ik} vanishing identically.

It is therefore quite natural to interpret ϕ_i as the *four-potential* in the world-geometry, so the tensor F will then be the *electromagnetic field*. The absence of an electromagnetic field is then the necessary condition for the present theory of EINSTEIN, which yields only gravitational phenomena, to be valid. If one accepts that viewpoint then one will see that it is in the nature of the electric quantities that their characterization by numbers in a certain coordinate system does not depend upon the arbitrary choice of a unit of measurement. In order to address units of measurement and dimension, one must completely re-orient oneself in this theory. Up to now, one spoke of a quantity, e.g., as a tensor of degree 2 (of rank 2) when a single value of it (*after an arbitrary choice of unit of measurement* was made in any coordinate system) determined a matrix of numbers a_{ik} that defined the coefficients of an invariant bilinear form of two arbitrary infinitesimal displacements:

(11) $$a_{ik}\, dx_i\, \delta x_k\ .$$

Here, we speak of a tensor when the components a_{ik} are determined uniquely when one bases them upon a coordinate system and *after making a well-defined choice of the proportionality factor that is contained in the* g_{ik}, and indeed, in such a way that the form (11) remains invariant under coordinate transformations when one replaces the g_{ik} with $\lambda\, g_{ik}$, but the a_{ik} go to $\lambda^e\, a_{ik}$. We then say that the tensor has *weight e*, or also, when we ascribe the dimension "length = l" to the line element ds, that it has dimension l^{2e}. Absolutely invariant tensors only have weight 0; the field tensor with the components F_{ik} is of that type. From (10), it satisfies the first system of MAXWELL's equations:

$$\frac{\partial F_{kl}}{\partial x_i} + \frac{\partial F_{li}}{\partial x_k} + \frac{\partial F_{ik}}{\partial x_l} = 0.$$

Once one has established the concept of parallel displacement, geometry and tensor calculus can be developed with no effort.

a) *Geodetic lines.* If a point P and a vector at it are given then the geodetic line that goes through P in the direction of that vector will arise in such a way that one consistently displaces the vector parallel to itself in its own direction. With the use of a suitable parameter τ, the differential equation for the geodetic line will read:

$$\frac{d^2 x_i}{d\tau^2} + \Gamma^i_{rs} \frac{dx_r}{d\tau} \frac{dx_s}{d\tau} = 0.$$

(Naturally, it cannot be characterized as the line of shortest length here, since the concept of the length of the curve has no sense.)

b) *Tensor calculus.* In order to, e.g., derive a tensor field of rank 2 from a covariant tensor field of rank 1 and weight 0 with components f_i by differentiation, we take an arbitrary vector ξ^i at the point P as an aid, define the invariant $f_i\,\xi^i$ and its infinitely-small change when one goes from the point P with the coordinates x_i to the neighboring point P' with the coordinates $x_i + dx_i$, while displacing the vector ξ parallel to itself under that transition. One will get:

$$\frac{\partial f_i}{\partial x_k} \xi^i dx_k + f_r\, d\xi^r = \left(\frac{\partial f_i}{\partial x_k} - \Gamma^r_{ik} f_r \right) \xi^i dx_k$$

for that change. The quantities in parentheses on the right-hand side are then the components of a tensor field of rank 2 and weight 0 that is defined by the field f in a completely invariant way.

c) *Curvature.* In order to construct the analogue of the RIEMANNian curvature tensor, one recalls the infinitely-small parallelogram figure that was used above and consists of the points P, P_1, P_2, and $P_{12} = P_{21}$. If one displaces a vector $\mathfrak{x} = (\xi^i)$ at P parallel to itself to P_1 and from there, to P_{12}, and another time, to P_2, and from there to P_{21} then since P_{12} and P_{21} coincide, it will make sense to define the difference $\Delta\mathfrak{x}$ of the two vectors that one obtains at that point. One will get:

$$\Delta\xi^i = R^i_j\, \xi^j \tag{12}$$

for its components, in which R^i_j are independent of the displaced vector $\mathfrak{x}$, but linearly-dependent upon the surface element that is spanned by the two displacements $\overrightarrow{PP_1} = (dx_i)$, $\overrightarrow{PP_2} = (\delta x_i)$:

$$R^i_j = R^i_{jkl}\, dx_k\, dx_l = \tfrac{1}{2} R^i_{jkl}\, \Delta x_{kl}\,.$$

The curvature components R^i_{jkl}, which depend upon only the location of *P*, have the two symmetry properties:

1. Their signs change when one switches the last two indices *k* and *l*.

2. If one performs the three cyclic permutations of *jkl* and adds the associated components then that will yield 0.

If we lower the index *i* then will get R_{ijkl}, which are the components of a covariant tensor of rank 4 and weight 0. Even without calculation, one can reach a simple conclusion that *R* splits into two summands in a natural way:

$$R^i_{jkl} = P^i_{jkl} - \tfrac{1}{2}\,\delta^i_j\,F_{kl} \qquad \delta^i_k = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{13}$$

of which, the first one P_{ijkl} is skew-symmetric in not only the indices *kl*, but also in *i* and *j*. While the equations $F_{ik} = 0$ characterize our space as one with no electromagnetic field, – i.e., as one in which the problem of translation of length is integrable – as would emerge from (13), $P^i_{jkl} = 0$ are the invariant conditions for no gravitational field to exist in it; i.e., for the problem of the translation of direction to be integrable. Only Euclidian space is devoid of both electricity and gravitation.

The simplest invariant of a linear map such as (12), which associates every vector $\mathfrak{x}$ with a vector $\Delta\mathfrak{x}$, is its "trace":

$$\frac{1}{n} R^j_i .$$

Here, from (13), it will take the form:

$$-\tfrac{1}{2} F_{ik}\, dx_i\, \delta x_k ,$$

which we have encountered already. The simplest invariant of a tensor like $-\tfrac{1}{2}F_{ik}$ is the square of its magnitude:

$$L = \tfrac{1}{4} F_{ik}\, F^{ik} .$$

Since the tensor *F* has weight 0, *L* is obviously an invariant of weight – 2. If *g* is minus the determinant of g_{ik} then:

$$d\omega = \sqrt{g}\; dx_0\, dx_1\, dx_2\, dx_3 = \sqrt{g}\; dx$$

will be the volume of an infinitely-small volume element, and it is known that MAXWELL's theory will arise from the electrical quantity of action that is equal to the integral $\int L\, d\omega$ of that simplest invariant when it is taken over an arbitrary world-domain, and indeed, in the sense that for arbitrary variations of the g_{ik} and ϕ_i that vanish on the boundary of the world-domain, one will have:

$$\delta\int L\,d\omega = \int (S^i\,\delta\phi_i + T^{ik}\delta g_{ik})\,d\omega,$$

in which:

$$S^i = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\,F^{ik})}{\partial x_k}$$

gives the left-hand side of the inhomogeneous MAXWELL equations (on whose right-hand side, one finds the components of the four-current), and the T_{ik} define the energy-impulse tensor of the electromagnetic field. Since L is an invariant of weight – 2, but the volume element is one of weight n / 2 in n-dimensional geometry, the integral $\int L\,d\omega$ will make sense only when the dimension is $n = 4$. *The possibility of arriving at Maxwell's theory, with our interpretation, is then linked with dimension* 4. However, in the four-dimensional world, the electromagnetic quantity of action will be a pure number. Therefore, no matter how large the quantity of action 1 turns out to be in the traditional mass units of the CGS system, admittedly, it can be first ascertained only when a calculation has been performed on an observation of the physical problem to be tested – e.g., the electron – on the basis of our theory.

Going over to the geometry of physics, from the model of MIE's theory ([1]), we must assume that all of the legitimacy of nature rests upon on well-defined integral invariant, namely, the *quantity of action:*

$$\int W\,d\omega = \int \mathfrak{W}\,dx \qquad\qquad (\mathfrak{W} = W\sqrt{g}\,),$$

in such a way that *the real world is distinguished from all possible four-dimensional metric space by the fact that any quantity of action that is contained in any world-domain will assume an extremal value* for any variations of the potentials g_{ik}, ϕ_i that vanish on the boundary of the world-domain in question. W, the world-density of action, must be an invariant of weight – 2. *The quantity of action is a pure number in any case.* Hence, our theory gives an account, from the outset, of the atomistic structure of the world, which takes on a fundamental meaning in the current way of looking at things: namely, the quantum of action. The simplest and most natural Ansatz that we can make for W reads:

$$W = R^i_{jkl}\,R_i^{jkl} = |\,R\,|^2. \tag{14}$$

From (13), that will also yield:

$$W = |\,P\,|^2 + 4L.$$

(At worst, the factor 4 that the second [electrical] term L includes when it is added to the first one can still be doubtful here.) However, even without specializing the quantity of action further, we can draw some general conclusions from the action principle. Namely, we will show: *In the same way* that the investigations of HILBERT, LORENTZ, EINSTEIN, KLEIN, and the author ([2]) connected the *four conservation laws of matter*

([1]) Ann. Phys. (Leipzig) **37**, **39**, **40** (1912-13).
Cf., also WEYL, *Raum, Zeit, Materie*, Berlin, 1918, § 25.

([2]) HILBERT, "Die Grundlagen der Physik," I. Comm. Gött. Nachr., 20 Nov. 1915.
H. A. LORENTZ, in four articles in the Versl. K. Ak. van Wetensch., Amsterdam, 1915-16.

(viz., the energy-impulse tensor) with the *invariance of the quantity of action under coordinate transformations*, which contained four arbitrary functions, *the new "yardstick invariance* [viz., the transition from (8) to (9)] *that appears here is linked with the law of conservation of electricity*, which brings in a fifth arbitrary function. The type and manner by which the latter is joined with the energy-impulse principle seems to be one of the strongest general arguments in favor of the theory that is proposed here, to the extent that one can speak of its confirmation at all in purely speculative terms.

For an arbitrary variation that vanishes on the boundary of the world-domain in question, we set:

$$\delta\int \mathfrak{W}\,dx = \int (\mathfrak{W}^{ik}\,\delta g_{ik} + \mathfrak{w}^i\,\delta\phi_i)\,dx \qquad (\mathfrak{W}^{ki} = \mathfrak{W}^{ik}). \tag{15}$$

The laws of nature will then read:

$$\mathfrak{W}^{ik} = 0, \qquad \mathfrak{w}^i = 0. \tag{16}$$

We can refer to the first one as the law of the gravitational field, while the second one is that of the electromagnetic field. The quantities W^i_k, w^i that are introduced by way of:

$$\mathfrak{W}^i{}_k = \sqrt{g}\,W^i_k, \qquad \mathfrak{w}^i = \sqrt{g}\;w^i$$

are the mixed (contravariant, resp.) components of a tensor of rank 2 (1, resp.) of weight – 2. Five extra equations are included in the system of equations (16), according to the invariance properties. That is expressed in the five following invariant identities that exist between their left-hand sides:

$$\frac{\partial \mathfrak{w}^i}{\partial x_i} \equiv \mathfrak{W}^i{}_i, \tag{17}$$

$$\frac{\partial \mathfrak{W}^i{}_k}{\partial x_i} - \Gamma^s_{kr}\mathfrak{W}^r{}_s \equiv \tfrac{1}{2}F_{ik}\,\mathfrak{w}^i. \tag{18}$$

The first one results from yardstick-invariance. Namely, if we assume an infinitely-small function of position $d\rho$ for $\ln\lambda$ in the transition from (8) to (9) then we will get the variations:

$$\delta g_{ik} = g_{ik}\,\delta\rho, \quad \delta\phi_i = \frac{\partial(\delta\rho)}{\partial x_i}.$$

(15) must vanish for them. When one exploits the invariance of the quantity of action under coordinate transformations by an infinitely-small deformation of the world-continuum ([1]), one will get the identities:

A. EINSTEIN, Berl. Ber. 1916. pp. 1111-1116.
F. KLEIN, Gött. Nachr. 25 January 1918.
H. WEYL, Ann. Phys. (Leipzig) **54** (1917), 121-125.

([1]) WEYL, Ann. Phys. (Leipzig) **54** (1917), 121-125. F. KLEIN, Gött. Nachr. Session on 25 Jan. 1918.

$$\left(\frac{\partial \mathfrak{W}^i{}_k}{\partial x_i} - \frac{1}{2}\frac{\partial g_{rs}}{\partial x_k}\mathfrak{W}^{rs}\right) + \frac{1}{2}\left(\frac{\partial \mathfrak{w}^i}{\partial x_i}\cdot \phi_k - F_{ik}\,\mathfrak{w}^i\right) \equiv 0,$$

which will be converted into (18) when is replaced with $g_{rs}\,\mathfrak{W}^{rs}$ according to $\partial\mathfrak{w}^i / \partial x_i$ using (17). From the laws of gravitation alone, one already gets that:

$$\frac{\partial \mathfrak{w}^i}{\partial x_i} = 0, \tag{19}$$

and from the laws of the electromagnetic field alone, one gets that one must have:

$$\frac{\partial \mathfrak{W}^i{}_k}{\partial x_i} - \Gamma^s_{kr}\mathfrak{W}^r{}_s = 0. \tag{20}$$

In MAXWELL's theory, $\mathfrak{w}^i$ has the form:

$$\mathfrak{w}^i \equiv \frac{\partial(\sqrt{g}\,F^{ik})}{\partial x_i} - \mathfrak{s}^i \qquad (\mathfrak{s}^i = \sqrt{g}\,s^i),$$

in which s^i means the four-current. Since the first part satisfies equation (19) identically here, that will yield the law of conservation of electricity:

$$\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\,s^i)}{\partial x_i} = 0.$$

Similarly, in EINSTEIN's theory, $\mathfrak{W}^i{}_k$ consists of two terms, the first of which satisfies equation (20) identically, and the second of which is the mixed components $T^{\,i}_{\,k}$ of the energy-impulse tensor, multiplied by $\sqrt{g}$. Hence, equations (20) lead to the four conservation laws of matter. An entirely analogous situation will be true in our theory when we choose the Ansatz (14) for the quantity of action. The five conservation principles are "eliminated" by the field laws; i.e., they follow from them in two ways, and therefore exhibit the fact that five of them are superfluous.

For example, for the Ansatz (14), MAXWELL's equation read:

$$\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\,F^{ik})}{\partial x_i} = s^i,$$

(21) and the current

$$s^i \text{ is } = \frac{1}{4}\left(R\phi_i + \frac{\partial R}{\partial x_i}\right).$$

R refers to the invariant of weight – 1 that arises from R^i_{jkl} when one first contracts i, k, and the j and l. If R^* means the RIEMANN curvature invariant that is constructed from only the g^{ik} then calculation will give:

$$R = R^* - \frac{3}{\sqrt{g}} \frac{\partial(\sqrt{g}\,\phi^i)}{\partial x_i} + \frac{3}{2}(\phi_i\,\phi^i)\,.$$

In the static case, where all spatial components of the electromagnetic potential vanish, and all quantities are independent of time x_0, from (21), one must have:

$$R = R^* + \tfrac{3}{2}\phi_0\,\phi^0 = \text{const.}$$

However, one can also arrive at R = const. = ± 1 in full generality in a world-domain in which $R \neq 0$ by suitably fixing the arbitrary unit of length, except that one must expect that there are surfaces with $R = 0$ for time-varying states, which would obviously play a certain singular role. One should not use R as the action density (R^* appears, as such, in EINSTEIN's theory of gravitation), since it does not have weight – 2. That has the consequence that our theory will probably lead to MAXWELL's electromagnetic equations, but not to EINSTEIN's gravitational equations; fourth-order differential equations enter in their place. However, it is, in fact, also very unlikely that EINSTEIN's gravitational equations are strictly incorrect, and above all, due to the fact that the gravitational constant that occurs in them falls outside the scope of the usual constants of nature, such that the gravitational radius of the charge and mass of an electron, for instance, has a completely different order of magnitude (namely, 10^{20} [10^{40}, resp.] times smaller) than the radius of the electron itself ([1]).

My only intent here was to develop briefly the general foundations of theory. Naturally, the problem arises of inferring its physical consequences, based upon the special Ansatz (14), and to compare it with experiments, and in particular, to examine whether the existence of electrons and the peculiarities of the up-to-now unexplained processes in the atom can be derived from it. From the mathematical standpoint, that problem is exceptionally complicated, since obtaining approximate solutions by restricting to the linear terms is excluded from it. Since neglecting terms of higher order in the interior of the electron is certainly not permitted, the linear equations that arise by such an oversight can essentially possess only the solution 0. I shall reserve the task of returning to all of these things more thoroughly to another occasion.

Addendum. EINSTEIN has remarked about the foregoing work:

"If light rays are the single means of ascertaining the metric behavior in the neighborhood of a world-point empirically then a factor will generally remain undetermined in the distance ds (as well as in the g_{ik}). However, that indeterminacy will not be present when one appeals to results of measurements that are obtained from

([1]) Cf., WEYL, "Zur Gravitationstheorie," Ann. Phys. (Leipzig) **54** (1917), pp. 133.

(infinitely-small) rigid bodies (yardsticks) and clocks in the definition of *ds*. A timelike *ds* can then be measured directly by a unit clock whose world-line includes *ds*."

"Such a definition of the elementary distance *ds* would then be only illusory if the concepts of "unit yardstick" and "unit clock" were based upon a largely false assumption. That would be the case if the length of a unit yardstick (the period of a unit clock, resp.) depended upon its history. If that were true in nature then there could be no chemical elements with spectral lines of definite frequencies, but rather the relative frequencies of two (spatially-close) atoms of the same kind would generally be different. Since that is not the case, it seems to me that the basic hypothesis of the theory is sadly untenable, although its boldness and profundity must fill any reader with wonder."

Response from the author. I would like to thank EINSTEIN for the fact that he has given me the opportunity to confront directly the objection that he has raised. If fact, I do not believe that he is correct. According to the special theory of relativity, a rigid yardstick will always have the same rest length when it comes to rest in a suitable reference space, and a properly-functioning clock will always possess the same period under those circumstances, when it is measured in proper time (MICHELSON experiment, DOPPLER effect). However, one cannot by any means speak of a clock measuring proper time $\int ds$ for an arbitrary turbulent motion (as long as, say, one goes through nothing but equilibrium states in the thermodynamics of a gas that is heated arbitrarily rapidly and non-uniformly). Above all, that is not the case when the clock (e.g., an atom) is subjected to effect of a strongly-varying electromagnetic field. In the general theory of relativity, one can then say at most this much: *A clock in a static gravitational field will measure the integral* $\int ds$ *in the absence of an electromagnetic field.* How a clock will behave for an arbitrary motion under the combined effect of an arbitrary electromagnetic and gravitational field can first be learned by developing a theory of dynamics that is based upon physical laws. Due to this problematic behavior of yardsticks and clocks, in my book *Raum, Zeit, Materie*, I have founded the basic measurement of the g_{ik} upon only the observation of the arrival of light signals. (cf., pp. 181, *et seq.*) In that way (in the event that EINSTEIN's theory is correct), not only the ratios of those quantities can, in fact, be determined, but also (with a choice of a fixed unit of measurement) their absolute values. KRETCHMANN arrived at the same ideas, independently of me ["Über die physikalischen Sinne der Relativitätspostulate, Ann. Phys. (Leipzig) **53** (1917), 755.].

In the theory that was developed here, outside the interior of the atom, for a suitable choice of coordinates and the undetermined proportionality factor, the quadratic form ds^2 reads the same as in the special theory of relativity, to a good approximation, and to the same degree of approximation, the linear form is equal to 0. In the case of the absence of an electromagnetic field (linear form is rigorously = 0), ds^2 is even determined with complete exactitude by the requirement that is expressed in the parentheses (up to a *constant* proportionality factor that, in fact, also remains arbitrary for EINSTEIN; the same thing will occur when only one electromagnetic field is present). The most plausible assumption that one can make about a clock that is at rest in a static field is that the integral measures *ds*, thus-normalized. In my theory, as well as in EINSTEIN's, the problem remains of deriving that fact [1] from an explicitly-developed theory of dynamics. However, in any case, an oscillating structure of well-defined constitution that is continually at rest in a certain static field will behave in a uniquely-determined way. (The influence of any possible turbulent history will rapidly subside) I do not believe that my theory will contradict anything in this experiment (which is confirmed by the existence of chemical elements for the atoms). One should observe that the mathematically-ideal process of vector displacement that the mathematical construction of geometry is based upon has nothing to do with the real-world process of the motion of a clock whose evolution is determined by the laws of nature.

The geometry that was developed here is the true local geometry, and that must be emphasized from the mathematical standpoint. It would be remarkable if a partial and inconsistent local geometry with a hidden electromagnetic field were true in nature in its place. However, I might naturally be on the wrong track with my entire way of looking at things. Here, we are actually dealing in pure speculation; a comparison with experiment is a self-explanatory demand. However, in order to do that, the consequences of the theory must be inferred. I am hoping for assistance with that difficult problem.

([1]) Whose experimental verification is still partially incomplete (e.g., redshift of the spectral lines in the vicinity of large masses).