"Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung," Göttinger Nachrichten (1921), pp. 99-112.

On infinitesimal geometry: relationship with projective and conformal concepts †.

(From a letter to F. Klein)

By

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I. The construction of pure infinitesimal geometry, as I have described most rigorously in chaps. 3 and 4 of my book "Space, Time, and Matter," may be naturally carried out in three steps, which are described by the catchphrases *continuous connection, affine connection, metric* ([1]). *Projective* and *conformal* geometry originate by abstraction from affine (metric, resp.) geometry. It is characteristic of the *conformal character* of a metric space that each point is associated with an infinitesimal cone of null directions:

(1) $$g_{ik}\,dx_i\,dx_k = 0\,.$$

If one changes the metric of the space in such a way that this cone remains unchanged at each point then the conformal character is preserved; such a change can generally be chosen in such a way that the fundamental quadratic form $g_{ik}\,dx_i\,dx_k$ is changed linearly, but otherwise arbitrarily. – It is characteristic of the *projective character* of an affinely connected space that there is a parallel displacement that acts on an arbitrary *direction* at an arbitrary point P, when P itself is infinitesimally displaced in this direction. If one changes the affine connection in such a way that this parallel displacement takes directions to themselves – or, what amounts to the same thing, *geodetic lines* to themselves – then the projective properties of the manifold are not altered. If Γ^r_{ik} are the components of the affine connection and $[\Gamma^r_{ik}]$ those of the change in it then the condition for the projective character to be unaffected during the alteration is that for arbitrary quantities ξ^i one must have that:

(2) $$[\Gamma^r_{ik}]\,\xi^i\,\xi^k \text{ is proportional to } \xi^r.$$

A simple algebraic consideration shows that this is the case when and only when:

(3p) $$[\Gamma^r_{ik}] \text{ has the form } \delta^r_i\psi_k + \delta^r_k\psi_i$$

† Translated by D.H. Delphenich.

[1] In § 18 of chap. 4, I have formulated the space problem, which seems to be the actual basis for this construction: meanwhile, I have arrived at its solution in the sense that was presumed therein. [Added in correction, April 1921.]

(the ψ_i are thus arbitrary). One thus equates it with the formula for the change in the affine connection, which yields, when the metric of a metric space is changed while preserving its conformal character:

$$[\Gamma^r_{ik}] = \tfrac{1}{2}(\delta^r_i \varphi_k + \delta^r_k \varphi_i - g_{ik}\varphi^r) . \tag{3k}$$

In relativity theory, the projective and conformal characters have an immediate intuitive meaning. The former, the persistence of the world-direction of a moving particle, which singles out a certain "natural" motion when it is released from a particular point, is a unification of inertia and gravitation that Einstein posed in place of either notion, for which, however, no suggestive name has emerged, as of yet. The infinitesimal cone (1), however, describes the difference between past and future in the neighborhood of a world-point; the conformal character is the cause-and-effect structure of the universe, through which one may determine which world-points can possibly be causally connected to each other. Therefore, the following theorem expresses something that is also physically meaningful:

Theorem 1. The projective and conformal character of a metric space determine that metric uniquely.

Therefore, if there exist two metrics in the same space for which the fundamental quadratic forms agree, whereas the coefficients of both fundamental linear forms differ by φ_i then the difference between then corresponds to affine connections that satisfy equation (3k). If the projective character is preserved under the transition from one metric to another one then (2) must be true, and here that gives:

$$(g_{ik}\,\xi^i\,\xi^k)\,\varphi^r \text{ is proportional to } \xi^r .$$

One need only choose two different directions at a point for which $g_{ik}\xi^i\,\xi^k$ does not vanish in order to conclude from this that $\varphi_i = 0$. It follows from this theorem that the world-metric can be established only by the observation of the "natural" motions of material particles and their effects, in particular, the radiation of light; measuring sticks and clocks are not necessary for this.

II. We now direct our attention to the curvature tensor:

$$F^\alpha_{ikl} = \left(\frac{\partial F^\alpha_{il}}{\partial x_k} - \frac{\partial F^\alpha_{ik}}{\partial x_l}\right) + (F^\alpha_{kr}F^r_{il} - F^\alpha_{lr}F^r_{ik})$$

and its contraction $F^\alpha_{i\alpha k} = F_{ik}$. In a metric space one can further construct the tensor $g_{ik}\,F$ by yet another contraction $F = F^i_i$. How can we alter the curvature tensor when we alter the affine connection (metric, resp.) without affecting the projective (conformal, resp.) character of the manifold in so doing? In the first case, a brief calculation based on (3p) yields the following result: if one sets:

(4p) $$\Psi_{ik} = \left(\frac{\partial \psi_i}{\partial x_k} - \Gamma^r_{ik}\psi_r\right) - \psi_i\,\psi_k$$

and, for an arbitrary system of numbers u_{ik}, defines:

(5p) $$\hat{\Psi}^{\alpha}_{ikl} = \delta^{\alpha}_i (u_{kl} - u_{lk}) + (\delta^{\alpha}_k u_{il} - \delta^{\alpha}_l u_{ik})$$

then the change in the curvature $[F^{\alpha}_{ikl}]$ is determined from:

$$[F^{\alpha}_{ikl}] + \hat{\Psi}^{\alpha}_{ikl} = 0 .$$

By contraction, it then follows that:

$$[F_{ik}] + (n\,\Psi_{ik} - \Psi_{ik}) = 0 .$$

If one then defines a tensor G_{ik} by the equation:

(6p) $$n\,G_{ik} - G_{ki} = F_{ik} ,$$

then the tensor of rank four:

(7p) $$F^{\alpha}_{ikl} - \hat{G}^{\alpha}_{ikl} = \text{proj.}\, F^{\alpha}_{ikl}$$

experiences no change under our process: it depends only upon the projective character of the manifold and we therefore refer to it as the *projective curvature*. When we ignore the trivial case of $n = 1$, equation (6p) may always be solved and gives:

$$(n-1)\,(n+1)\,G_{ik} = n\,F_{ik} + F_{ki} .$$

For $n = 2$ the projective curvature is identically null, so it is not until $n = 3$ that it plays a role.

I have already discussed the *conformal curvature* ([1]). One must construct:

(4k) $$\Phi_{ik} = \left(\frac{\partial \varphi_i}{\partial x_k} - \Gamma^r_{ik}\varphi_r\right) - \tfrac{1}{2}\,\varphi_i\,\varphi_k + \tfrac{1}{4}\,g_{ik}\,(\varphi_r\,\varphi^r)$$

and for a system of numbers u_{ik} one generally sets:

(5k) $$\hat{\Phi}_{iklm} = \tfrac{1}{2}(g_{il}\,u_{km} + g_{km}\,u_{il} - g_{im}\,u_{kl} - g_{kl}\,u_{im}), \quad u^i_i = u ;$$

then it follows from (3k) that:

$$[F^{\alpha}_{ikl}] + \hat{\Phi}^{\alpha}_{ikl} = 0 .$$

[1] Math. Zeit. Bd. **2** (1918), pp. 404.

(The upper index α precedes the lower ones.) Contraction gives:

$$[F_{ik}] + \{(n-2)\,\Phi_{ik} + g_{ik}\,\Phi\} = 0 .$$

If one thus defines a tensor H_{ik} by the equation:

(6k) $$(n-2)\,H_{ik} + g_{ik}\,H = 2\,F_{ik} ,$$

then the conformal curvature is:

(7k) $$\text{conf.}\,F^{\alpha}_{ikl} = F^{\alpha}_{ikl} - \tilde{H}^{\alpha}_{ikl} .$$

For $n > 2$, (6k) can be solved:

$$(n-1)\,H = F, \qquad (n-1)\,(n-2)\,H_{ik} = 2\,(n-1)\,F_{ik} - g_{ik}\,F .$$

However, the conformal curvature also vanishes for $n = 3$, so it is not until $n = 4$ that it plays a role.

Theorem 2. Along with the (affine) total curvature F^{α}_{ikl}, *there is also a projective curvature and a conformal curvature that may be determined from the total curvature by means of equations* (5p), (6p), (7p) ((5k), (6k), (7k), resp.). *The total curvature plays a role from* $n = 2$ *on up, but the projective curvature first plays a role for* $n = 3$ *and the conformal curvature, for* $n = 4$.

III. I will use the word "flat" in the Euclidian sense. An affinely connected space is flat the components of the affine connection vanish identically for a certain choice of coordinate system. A metric space is flat when and only when the coefficients of the fundamental quadratic form are constant and those of the fundamental linear form vanish for a certain choice of coordinate system and gauge. Only the second part of this claim requires a proof. If the components of the affine connection vanish then it follows from the equations that link the two fundamental metric forms $g_{ik}\,dx_i\,dx_k$, $\varphi_i\,dx_i$ with the affine connection that:

(8) $$\frac{\partial g_{ik}}{\partial x_r} + g_{ik}\,\varphi_r = 0 .$$

From this, one likewise obtains:

$$\frac{\partial \varphi_r}{\partial x_s} - \frac{\partial \varphi_s}{\partial x_r} = 0 .$$

From this, one can choose a gauge in such a way that $d\varphi = \varphi_i\,dx_i$ vanishes, and it then follows from (8): $g_{ik} = \text{const.}$

Theorem 3. The vanishing of the curvature is not only a necessary, but also a sufficient, condition for a manifold to be flat.

I will briefly present the proof of this long-known theorem once more since it is not only fundamental in what follows, but it is also typical of the integrability considerations that will be further examined. The assumption has the consequence that a vector can be parallel displaced in a manner that is entirely independent of the path, i.e., that the equations:

(9) $$\Xi^i_k = \frac{\partial \xi^i}{\partial x_r} - \Gamma^i_{kr}\xi^r = 0$$

posses a solution ξ^i that agrees with an arbitrary previously-given initial value ξ^i_0 at the origin. If we generally set:

$$u^i_{k;l} - u^i_{l;k} = \left(\frac{\partial u^i_k}{\partial x_l} - \frac{\partial u^i_l}{\partial x_k}\right) + (\Gamma^i_{lr}u^r_k - \Gamma^i_{kr}u^r_l),$$

$$u_{i,k/l} - u_{i,l/k} = \left(\frac{\partial u_{ik}}{\partial x_l} - \frac{\partial u_{il}}{\partial x_k}\right) + (\Gamma^r_{ik}u_{rl} - \Gamma^r_{il}u_{rk}),$$

for a tensor field u^i_k (u_{ik}, resp.), then for vanishing curvature one will have:

(10) $$\Xi^i_{k/l} - \Xi^i_{l/k} = 0,$$

which might also be true for the functions ξ^i. One can now satisfy equations (9) by way of the existence theorem for ordinary differential equations in all circumstances, in such a way that (9) is true identically in $x_1, x_2, \ldots, x_k$, as long as one sets the remaining variables = 0. On the basis of the identities (10), one then easily shows that they are then already satisfied identically in all variables. In order to arrive at the "linear" coordinate system *yi*, one must now similarly treat the equations:

$$\frac{\partial x_i}{\partial y_k} = \xi^i_{(k)}(x_1, x_2, \cdots, x_n),$$

whose right-hand side $\xi^i_{(1)}, \xi^i_{(2)}, \ldots, \xi^i_{(n)}$ consists of the those solutions of (9) that agree with the initial values:

1, 0, 0, …, 0 ;
0, 1, 0, …, 0 ;
. . . . . .
0, 0, 0, …., 1

resp.

It is clear when one should regard a manifold as *flat* in the *projective* sense, as well as in the *conformal* sense. A necessary condition to which this character is bound is the vanishing of the projective (conformal, resp.) curvature. It must therefore be true that in the case of a given tensor G_{ik} of that type one has:

(I p) $$F^{\alpha}_{ikl} = \hat{G}^{\alpha}_{ikl}, \qquad \text{viz.,} = \delta^{\alpha}_{i}(G_{kl} - G_{lk}) + (\delta^{\alpha}_{k} G_{il} - \delta^{\alpha}_{l} G_{ik})$$

and in the other case a tensor H_{ik} must satisfy:

(I k) $$F^{\alpha}_{ikl} = H^{\alpha}_{ikl}, \qquad \text{viz.,} = \tfrac{1}{2}(\delta^{\alpha}_{k} H_{il} - \delta^{\alpha}_{l} H_{ik}) + \tfrac{1}{2}(H^{\alpha}_{k} g_{il} - H^{\alpha}_{l} g_{ik}).$$

The aforementioned change in the affine connection (the fundamental linear metric form, resp.), which converts the manifold into a flat one, is then determined from the equations:

(II p) $$\Psi_{ik} = G_{ik},$$
or:
(II k) $$\Phi_{ik} = H_{ik},$$

resp. A calculation that is most comfortably carried out in a geodetic coordinate system (in which all Γ vanish at the point in question) immediately yields:

(11) $$\Psi_{i,k/l} - \Psi_{i,l/k} + (\hat{\Psi}^{\alpha}_{ikl} - F^{\alpha}_{ikl})\,\varphi_{\alpha} = 0,$$

hence, due to (II p) and (I p):
$$\Psi_{i,k/l} - \Psi_{i,l/k} = 0.$$

As a result, G_{ik} must also satisfy the same conditions:

(III p) $$G_{i,k/l} - G_{i,l/k} = 0.$$

In the conformal case, one finds by precisely the same argument that:

(III k) $$H_{i,k/l} - H_{i,l/k} = 0.$$

Conditions (I p), (III p) are, however, not only necessary, but also sufficient, for a manifold to be projectively flat. Under these conditions, equations (II p) then have a solution [for which the ψ_i assume arbitrary given initial values, as well] ([1]). In fact, when the integrability condition (III p) is satisfied, equation (11), which is true for arbitrary ψ_i, teaches us that the difference $D_{ik} = \Psi_{ik} - G_{ik}$ satisfies the identity:

$$D_{i,k/l} - D_{i,l/k} + \hat{D}^{\alpha}_{ikl}\psi_{\alpha} = 0,$$
or:
(12) $$^{*}D_{i,k/l} - {}^{*}D_{i,l/k} = 0,$$

in which the left-hand side is analogous to $D_{i,k/l} - D_{i,l/k}$, but refers to the expression constructed from the altered affine connection. Next, one can satisfy (II p) such that this

[1] This arbitrariness corresponds to the freedom to map the flat space onto itself by an arbitrary collineation; in the conformal case, instead of collineations, one uses the spherical affinities (Liouville's theorem).

equation is true identically in $x_1, x_2, \ldots, x_k$ when one sets the remaining variables equal to zero. The identities (12) then teach us that they are true without restriction. Hence, the total curvature of the altered affine connection is zero, and, from theorem 3, the manifold is flat. One also clearly recognizes the step that one must take in order to determine those homogeneous variables for which the equations for any geodetic are linear. – The conformal problem is completely analogous.

IV. Herr Schouten ([1]) has made the noteworthy discovery that for $n > 3$ the integrability condition (III k) is a result of (II k). For me,this investigation of Herr Schouten was, the the first time that someone seriously examined the projective and conformal standpoints that I only touched upon in my earlier representation of infinitesimal geometry. One has the following analogue of Schouten's theorem: For $n > 2$, the integrability condition (III p) is a consequence of (I p). We thus arrive at this result:

Theorem 4. Among the affinely connected manifolds, the projectively flat ones are characterized: in the case of $n = 2$, *by the validity of equations* (III p); *in the case* $n \geq 3$, *by the vanishing of the projective curvature. A necessary and sufficient condition for a metric space to be conformally mapped to a flat space is: for* $n = 3$, *equations* (III k) *must be satisfied* (Cotton); *when* $n \geq 4$, *the conformal curvature must vanish* (Schouten).

I will give the calculations that lead up to this theorem shortly, in which, for the sake of convenience, I employ a geodetic coordinate system. Since G_{ik} essentially agrees with F_{ik}, we next compute:

$$\frac{\partial F_{ik}}{\partial x_l} - \frac{\partial F_{il}}{\partial x_k}.$$

Since F_{ik} is, up to an expression that involves the components of the affine connection *quadratically:*

$$= \frac{\partial \Gamma^{\alpha}_{ik}}{\partial x_\alpha} - \frac{\partial \gamma_i}{\partial x_k}, \quad \gamma_i = \Gamma^{\alpha}_{i\alpha},$$

one has:

$$\frac{\partial F_{ik}}{\partial x_l} - \frac{\partial F_{il}}{\partial x_k} = \frac{\partial^2 \Gamma^{\alpha}_{ik}}{\partial x_l \partial x_\alpha} - \frac{\partial^2 \Gamma^{\alpha}_{il}}{\partial x_k \partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \Gamma^{\alpha}_{ik}}{\partial x_l} - \frac{\partial \Gamma^{\alpha}_{il}}{\partial x_k}\right) = -\frac{\partial F^{\alpha}_{ikl}}{\partial x_\alpha}. \tag{13}$$

In this, I then replace F^{α}_{ikl} in the right-hand side with the expression (I p) and write, to abbreviate:

$$G_{ik} - G_{ki} = \gamma_{ik}$$

– hence, from (6 p):

$$(n - 1)\gamma_{ik} = F_{ik} - F_{ki} = \frac{\partial \gamma_k}{\partial x_i} - \frac{\partial \gamma_i}{\partial x_k} \tag{14}$$

– which yields:

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[1] To appear in Math. Zeit.

$$\frac{\partial F_{ikl}^{\alpha}}{\partial x_\alpha} = \frac{\partial \gamma_{kl}}{\partial x_i} + \left(\frac{\partial G_{il}}{\partial x_k} - \frac{\partial G_{ik}}{\partial x_l} \right);$$

however, in the left-hand side, I use (6 p) to replace F_{ik} with:

$$n\, G_{ik} - G_{ki} = (n-1)\, G_{ik} + \gamma_{ik}\,.$$

The result is:

$$(n-1)\left(\frac{\partial G_{ik}}{\partial x_l} - \frac{\partial G_{il}}{\partial x_k} \right) + \left(\frac{\partial \gamma_{ik}}{\partial x_l} - \frac{\partial \gamma_{il}}{\partial x_k} \right) = \left(\frac{\partial G_{il}}{\partial x_l} - \frac{\partial G_{il}}{\partial x_k} \right) - \frac{\partial \gamma_{kl}}{\partial x_i},$$

or:

(15 p) $$(n-2)\left(\frac{\partial G_{ik}}{\partial x_l} - \frac{\partial G_{il}}{\partial x_k} \right) + \left(\frac{\partial \gamma_{ik}}{\partial x_l} + \frac{\partial \gamma_{kl}}{\partial x_i} + \frac{\partial \gamma_{il}}{\partial x_k} \right) = 0\,.$$

However, the second bracket = 0 here. Expression (14) is then valid in an arbitrary coordinate system when one neglects those terms that involve the components of the affine connection *quadratically;* hence, by differentiation in a geodetic coordinate system, one has:

$$(n-1)\,\frac{\partial \gamma_{ik}}{\partial x_l} = \frac{\partial}{\partial x_l}\left(\frac{\partial \gamma_k}{\partial x_i} - \frac{\partial \gamma_i}{\partial x_k} \right),$$

and therefore:

(16 p) $$(n-1)\left(\frac{\partial \gamma_{ik}}{\partial x_l} + \frac{\partial \gamma_{kl}}{\partial x_i} + \frac{\partial \gamma_{il}}{\partial x_k} \right) = 0\,.$$

One thus obtains the desired result for $n > 2$.

I shall omit the analogous computations in the conformal case. By the corresponding substitutions, one then finds, instead of (15 p):

(15 k) $$(n-3)\left(\frac{\partial H_{ik}}{\partial x_l} - \frac{\partial H_{il}}{\partial x_k} \right) + \left\{ g_{ik}\left(\frac{\partial H}{\partial x_l} - \frac{\partial H_l^{\alpha}}{\partial x_\alpha} \right) - g_{il}\left(\frac{\partial H}{\partial x_k} - \frac{\partial H_k^{\alpha}}{\partial x_\alpha} \right) \right\} = 0\,.$$

We may apply the "conservation law" that is well-known in gravitation theory:

$$\frac{\partial F}{\partial x_i} = 2\,\frac{\partial F_i^{\alpha}}{\partial x_\alpha},$$

since, from (I k), our space Riemannian; From (I k), it then follows that the "dilational curvature" ([1]) $F_{\alpha kl}^{\alpha}$ vanishes. By the way, this formula also follows from (13), likewise by

[1] *Raum, Zeit, Materie,* chap. 4, pp. 114. $F_{\alpha ik}^{\alpha}\, dx_i\, \delta x_k$ is the relative increment that the volume of an infinitely small "compass" experiences when it goes around the surface element that is spanned by the line elements dx, δx.

contraction, under the considering the fact that here, (I k) implies that $F_{\alpha ikl}$ is not only skew-symmetric in kl, but also in αi. This gives:

$$(n-1)\frac{\partial H}{\partial x_i} = \frac{\partial F}{\partial x_i} = 2\frac{\partial F_i^\alpha}{\partial x_\alpha} = (n-2)\frac{\partial H_i^\alpha}{\partial x_\alpha} + \frac{\partial H}{\partial x_i},$$

hence:

(16 k) $$(n-2)\left(\frac{\partial H}{\partial x_i} - \frac{\partial H_i^\alpha}{\partial x_\alpha}\right) = 0 .$$

We have thus reached out goal.

V. Having considered the flat space, we will now treat the next simplest case, the "sphere." If $E(x) = \sum_{i=1}^{n} \pm x_i^2$ is the unit quadratic form of inertial index q (the first $n-q$ signs are +, the last q are $-$), λ is any number, then I call n-dimensional manifold that is represented through the equation:

$$x_0^2 + \lambda\, E(x) = 1$$

in an $(n + 1)$-dimensional flat space with the fundamental metric form:

$$\frac{dx_0^2}{\lambda} + E(dx)$$

a *sphere*, regardless of whether λ is positive or negative, and whatever value the inertial index possesses. If we express x_0 in the fundamental metric form in terms of $x_1, x_2, \ldots, x_n$ then one understands a sphere (of inertial index q and curvature λ) to mean an n-dimensional metric manifold for which one has that the fundamental quadratic form:

$$= E(dx) + \frac{\lambda E^2(x, dx)}{1-\lambda E(x)}$$

for any choice of coordinate system, but the linear form vanishes; the coordinates vary in a region in which $1 - \lambda\, E(x) > 0$. One sees that a distinction between the cases $\lambda = 0$ and $\lambda \neq 0$ – which I consider to be completely unjustified in the context of an arbitrary real number λ – is not required here. For the sphere, one has:

(17) $$F_{ikl}^{\alpha} = \lambda(\delta_k^\alpha g_{il} - \delta_l^\alpha g_{ik}) .$$

A metric space R that satisfies such an equation shall be referred to as a space with *scalar curvature*. The curvature λ itself is a scalar of gauge weight -1; the requirement that λ = const. therefore has no meaning independently of the gauge. When $\varphi_i\, dx_i$ is the fundamental linear form of R, the invariant gradient of λ is, moreover, given by:

(18)
$$\frac{\partial \lambda}{\partial x_i} - \lambda\, \varphi_i\,.$$

This is identically 0 for the sphere. – We pose the question: How can one characterize the sphere in an invariant way amongst all other metric spaces (of equal inertial index q) in the contexts of metric, affine, projective, and conformal geometry?

Projectively and *conformally*, spheres are identical with flat manifolds. By means of:

(19 p)
$$x_i = \frac{y_i}{\sqrt{1+\lambda E(y)}}\,, \qquad (i = 1, 2, \ldots, n)$$

or:

(19 k)
$$x_i = \frac{y_i}{\sqrt{1-\lambda E(y)}}\,, \qquad (i = 1, 2, \ldots, n)$$

resp., the sphere gets projectively (conformally, resp.) mapped onto the flat space with the fundamental metric form $E(dx)$. Here, everything is therefore carried out by means of theorem 4. The *affine* question reverts to the metric one due to:

Theorem 5: A metric manifold that can be affinely mapped onto a sphere also agrees with it in the metric context.

Since the manifold must have the same affine connection – hence, the same curvature F_{ikl}^{α} – as the sphere, its dilatational curvature must satisfy $F_{\alpha ik}^{\alpha} = 0$. It is therefore a Riemannian space, and we can take its fundamental linear form to be = 0 from now on. Furthermore, I must now distinguish between the cases $\lambda = 0$ and $\lambda \neq 0$, resp. In the former case, one gets immediately that $\overset{*}{g}_{ik}$ = const. (the starred quantities refer to the manifold in question and the unstarred ones to the sphere), and one may, in general, assume that the affine map has already been arranged to make $\overset{*}{g}_{ik}\, dx_i\, dx_k = E(dx)$, by an additional linear transformation, if needed. In the latter case, however, the affine map is also metric-preserving with no further assumptions. One thus concludes: By assumption, one has:

(20)
$$\overset{*}{g}_{ik,\alpha} \equiv \frac{\partial \overset{*}{g}_{ik}}{\partial x_\alpha} - \Gamma_{\alpha k}^{r}\,\overset{*}{g}_{ir} - \Gamma_{\alpha i}^{r}\,\overset{*}{g}_{kr} = 0\,.$$

If one then constructs that tensor that has the following expression in a geodetic coordinate system:

$$\frac{\partial \overset{*}{g}_{ik,\alpha}}{\partial x_\beta} - \frac{\partial \overset{*}{g}_{ik,\beta}}{\partial x_\alpha}\,,$$

then, as one observes in a geodetic coordinate system with no further assumptions, (20) gives us the following relationship:

$$g^*_{ir} F^r_{k\alpha\beta} + g^*_{kr} F^r_{i\alpha\beta} = 0 \ .$$

We substitute (17) into this and drop the factor $\lambda \neq 0$:

$$(g^*_{i\alpha} g_{k\beta} + g^*_{k\alpha} g_{i\beta}) - (g^*_{i\beta} g_{k\alpha} + g^*_{k\beta} g_{i\alpha}) = 0 \ .$$

From this, one painlessly concludes that g^*_{ik} is proportional to g_{ik} , and, from (20), one sees that the proportionality factor must be constant.

As one sees, whether the relationships in the cases $\lambda = 0$ and $\lambda \neq 0$ are completely different, nevertheless, the distinctions are also overcome here; however, this would necessitate a closer look into continuum analysis, which Brouwer and myself would like to pose in place of the presently untenable Atomism. Thus, I shall not bother you for now, until such time as the demands of continuum analysis can be proved directly from this theorem in a very beautiful and intuitive way.

Finally: How are the spheres to be invariantly recognized in the *metric* context? The answer that we give for this is: as spaces with scalar curvature in which the invariant gradient of the curvature scalar vanishes, as well. As for this, we see (the theorem of F. Schur([1])) that for $n > 2$ this obviates the second condition. Proof: the dilatational curvature $F^\alpha_{\alpha ik}$ of a space with a scalar curvature vanishes; it is therefore necessarily Riemannian, and we can assume that the fundamental linear form is = 0 once and for all. The second requirement then says that for this normal gauge one has λ = const. Our claim then immediately reduces to: The sphere is the only Riemannian space with constant scalar curvature. This is a theorem that is well-known and was already discussed by Riemann. Here, it thus yields to the following thorough proof. A Riemannian space R of constant scalar curvature satisfies the conditions (I p), (III p) that were given in section III, and indeed one has for such a space that $G_{ik} = \lambda\, g_{ik}$. It follows that it can be projectively mapped onto the flat space with the fundamental metric form $E(dx)$. We do this, by a linear transformation, in such a way that $g_{ik}\, dx_i\, dx_k$ agrees with $E(dx)$ at the origin, in order to integrate equations (II p) with the initial values $\psi_i = 0$, and finally, to make the map itself represent the proof of theorem 3. If x_i are the linear coordinates thus obtained (at which point, they will be denoted by y_i) then the components of the affine connection of R have the form:

(21) $$\Gamma^r_{ik} = \delta^r_i \psi_k + \delta^r_k \psi_i \ ,$$

i.e., one has:

(22) $$\frac{\partial g_{ik}}{\partial x_r} \ (= \Gamma_{i,kr} + \Gamma_{k,\,ir}) = g_{ir}\,\psi_k + g_{kr}\,\psi_i + 2\, g_{ik}\,\psi_r \ .$$

However, the condition on the scalar curvature that $G_{ik} = \lambda\, g_{ik}$ reads like:

[1] Math. Ann. Bd. **27** (1886), pp. 563.

(22′) $$\frac{\partial \psi_i}{\partial x_k} - \psi_i\,\psi_k + \lambda\, g_{ik} = 0\,.$$

At the origin, one has:

(23) $$g_{ik}\,dx_i\,dx_k = E(dx), \qquad \psi_i\,dx_i = 0\,.$$

For a given constant λ the differential equations (22), (22′) obviously have only one solution g_{ik}, ψ_i for these initial values, and it is, as we know, the sphere of curvature λ.

A space with scalar curvature always satisfies condition (I p) with $G_{ik} = \lambda\, g_{ik}$. From the calculations that were carried out in **IV.** it follows that for $n > 2$ we have equation (III p), which takes the form here:

(24) $$g_{ik}\frac{\partial \lambda}{\partial x_l} - g_{il}\frac{\partial \lambda}{\partial x_k} = 0\,; \quad \frac{\partial \lambda}{\partial x_i} = 0\,.$$

Theorem 6. For n > 2, *any metric space with scalar curvature is a sphere; in the case n* = 2, *one must add the requirement that the invariant gradient of the scalar curvature vanishes.*

VI. As appropriate as this method of proof is to the subject, it produces the following remarkable fact ([1]):

Theorem 7. The only projectively flat metric space is – when the dimension is greater than 2 *– the sphere.*

The formulation that this theorem implies has no analogue in the conformal case. The determination of all *metric* spaces that are *conformally flat* is achieved quite simply from the assumption that one uses the unit quadratic form $E(dx)$ and an arbitrary linear form for the fundamental linear form; in an entirely corresponding way, one obtains all *affinely connected* manifolds that are *projectively flat* by means of an arbitrary linear form $\psi_i\,dx_i$ by starting with (21). Here, however, we are asking which *metric* spaces can be projectively mapped onto a flat one. Our claim shows us that equation (I p) for a metric space of dimension more than two can be true only if $G_{ik} = \lambda\, g_{ik}$. This allows us to formulate the fact that *the Cayley metric is the only metric* (in our sense) *that can be installed in the* (more than two-dimensional) *space of ordinary projective geometry.*

The determination of all projectively flat metric spaces obviously produces the solution to the following differential equation:

(25) $$\frac{\partial g_{ik}}{\partial x_r} + g_{ik}\,\varphi_r = g_{ir}\,\psi_k + g_{kr}\,\psi_i + 2g_{ik}\,\psi_r$$

[1] For Riemannian spaces, this theorem was already proved by *Beltrami* (for $n = 2$) in another and essentially complicated way. Ann. di Matem. **7**, pp. 203; see also, Ann. di Matem., ser. II, v. **2**, pp. 232; by *Lipschitz* in the treatise in Crelles Journal, Bd. **72**, pp. 1, et seq., and by *F. Schur*, Math. Ann., Bd. **27**, pp. 537-567.

for the unkown functions φ_i, ψ_i, g_{ik} ($= g_{ki}$). If we set $2\psi_i - \varphi_i = f_i$, introduce the general notation:

$$\frac{\partial \varphi_i}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_i} = \varphi_{i/k} - \varphi_{k/i}$$

and understand, as before, that Ψ_{ik} means the expression $\frac{\partial \psi_i}{\partial x_k} - \psi_i \, \psi_k$, then the integrability relation:

$$\frac{\partial}{\partial x_s}\left(\frac{\partial g_{ik}}{\partial x_r}\right) - \frac{\partial}{\partial x_r}\left(\frac{\partial g_{ik}}{\partial x_s}\right) = 0 ,$$

when applied to equation (25), gives the following relation:

(26) $$(g_{ir}\,\Psi_{ks} + g_{kr}\,\Psi_{is}) - (g_{ir}\,\Psi_{ks} + g_{kr}\,\Psi_{is}) + g_{ik}\,(f_{r/s} - f_{s/r}) = 0 .$$

We may assume that for the case in question:

$$g_{ii} = e_i = \pm 1, \qquad g_{ik} = 0 \; (i \neq k);$$

one then obtains from (26) that can take $i = k = r$ (since $g_{ii} \neq 0$), and likewise:

$$\Psi_{ik} + \tfrac{1}{2}(f_{i/k} - f_{k/i}) = \lambda_i \, g_{ik} .$$

If one chooses $r = I$, $s = k$ for this then (since $g_{ii}\, g_{kk} - g_{ik}^2 \neq 0$) one has $\lambda_i = \lambda_k$, hence:

(27) $$\Psi_{ik} = -\lambda \, g_{ik} - \tfrac{1}{2}(f_{i/k} - f_{k/i}) .$$

If one substitutes this in (26) then the first term $\lambda\, g_{ik}$ gives no contribution at all, and what remains is an equation that, from (26), tells us that one can replace Ψ_{ik} with $-\tfrac{1}{2}(f_{i/k} - f_{k/i})$. It is satisfied identically for $n = 2$; however, in general, one obtains, by multiplication by g_{ik} and summing over i and k:

(28) $$(n - 2)\,(f_{i/k} - f_{k/i}) = 0 ;$$

such that in all cases (26) is equivalent to both equations (27), (28). When $n > 2$, it follows simply that:

(29) $$\Psi_{ik} = -\lambda \, g_{ik} ,$$

(30) $$f_{i/k} - f_{k/i} = 0 .$$

Our goal is thus attained; (29) then states that the tensor G_{ik} is proportional to g_{ik} . The knowledge that we obtained from our earlier results that for this reason a Riemannian space is necessarily produced and by the use of the normal gauge one will have $\lambda = \text{const.}$ can be easily stated here by the following reasoning: from (29), one concludes, however, that:

$$\psi_{i/k} - \psi_{k/i} = \Psi_{ik} - \Psi_{ki} = 0 ,$$

and from (30) that $\varphi_{i/k} - \varphi_{k/i} = 0$. Thus, one may assume that $\varphi_i = 0$. If one uses equation (29), which will, for the moment, be denoted by D_{ik}, to construct the integrability relation:

$$\frac{\partial D_{ik}}{\partial x_l} - \frac{\partial D_{il}}{\partial x_k} = 0 ,$$

and substitutes in it the expressions that one gets from equations (25), (29) for the derivatives of g_{ik} and ψ_i then one arrives at (24). At the origin, we may assume that the initial values (23) are valid. What now emerges is the same integration problem as in the conclusion of the previous section, whose only solution is the sphere of curvature λ .

Theorem 7 is also valid in the case of n = 2 when we restrict ourselves to Riemannian spaces. On the contrary, one may introduce other metrics than the Cayley metric into the ordinary two-dimensional projective plane when one allows the non-integrability of the dilatation displacement.

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