“Reine Infinitesimalgeometrie,” Math. Z. **2** (1918), 384-411.

Pure infinitesimal geometry.

By

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§ 1. Introduction. On the relationship between geometry and physics.

The real world in which our consciousness is forced to reside *is not there*, all at a single moment, but *happens;* it elapses, being destroyed and born anew in each moment, a continuous one-dimensional sequence of states in *time*. The arena of this timelike happenstance is a three-dimensional Euclidian *space*. Its properties are examined by *geometry;* on the other hand, it is the problem of *physics* that real things exist in space to be regarded conceptually and to be founded on lasting laws, despite the ephemeral nature of phenomena. Physics is thus a science that has geometry at its foundations; however, the concepts by which it represents reality – matter, electricity, force, energy, electromagnetic field, gravitational field, etc. – belong to a completely different sphere from geometry.

This old insight regarding the relationship between form and content in reality, between geometry and physics, has been overturned by Einsteinian relativity theory [1]). The *special theory of relativity* leads to the knowledge that space and time are melded into an indissoluble unified entity that we will call the *world*; as a consequence of this theory, the world is a four-dimensional Euclidian manifold – Euclidian, with the modification that the quadratic form that is the basis for the world-metric is not positive-definite, but has an index of inertia equal to 1. The *general theory of relativity* says – entirely in the spirit of modern local action physics – that this is valid only infinitesimally, and takes the world metric to then be the general concept that was presented by Riemann in his Habilitation lecture in which he claimed that such a measure was based on a quadratic *differential* form. His principal innovation was the following insight: The metric is not a property of the world in itself; rather, spacetime as a phenomenon takes the form of a completely formless four-dimensional continuum, in the sense of Analysis Situs, but the metric expresses something real that exists in the world, that physical actions are exerted on matter through centrifugal and gravitational forces, and conversely, the state of the metric is naturally determined by the distribution and properties of matter. Since I wanted to liberate Riemannian geometry, which we will regard as “local geometry,” from one of its currently unresolved inconsistencies, a last global geometric element emerges that is suggested by its very Euclidian past itself, I arrived at a world metric from which not only gravitational, but also electromagnetic effects, emerge, which, as one may with good reason assume, thus account for all

[1]) I refer to the presentation in my book “Raum, Zeit, Materie,” Springer 1918 (denoted by RZM, in the sequel), and the literature that was cited in it.

physical phenomena [2]). In this theory, *all real events that occur in the world are manifestations of the world metric;* the physical concepts are nothing but geometric ones. The single difference between geometry and physics consists of the fact that geometry generally begins with a set of axioms that the metric concept essentially embodies [3]), but physics must arrive at these laws and pursue their consequences in order to distinguish the real world among all possible four-dimensional metric spaces [4]).

In this note, I would like to develop that *pure infinitesimal geometry* of which I am convinced the physical world is understood to be a special case. The construction of local geometry is properly performed in three steps. At the first step, one finds a *continuum*, in the sense of Analysis Situs, that is barren of all measurements – physically speaking, this is *the vacuum*. At the second step one finds the *affinely connected* continuum – which is what I call a manifold in which the concept of the infinitesimal parallel displacement of vectors has meaning; in physics, the affine connection appears in the form of the *gravitational field*. Finally, at the third step, one finds the *metric* continuum – physically, this is the "ether," whose states are manifested by the phenomena of matter and electricity.

§ 2. Topological space (vacuum).

As a result of the difficulty involved with grasping the intuitive character of continuous connections through a purely logical construction, a completely satisfactory analysis of the concept of an *n-dimensional manifold* is not possible at present [5]). The following shall suffice: An *n*-dimensional manifold may be described by *n* coordinates x_1, x_2, …, x_n, each of which take on a definite numerical value at each point of the manifold; different points correspond to different systems of values for the coordinates. If $\bar{x}_1, \bar{x}_2$, …, $\bar{x}_n$ is a second system of coordinates then there are specified relations:

$$x_i = f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n), \qquad (i = 1, 2, \ldots, n)$$

between the x and $\bar{x}$ coordinates of the same arbitrary point, in which the f_i are purely logico-arithmetically constructed functions; about them, we assume only that they are continuous and that they possess continuous derivatives:

$$\alpha_{ik} = \frac{\partial f_i}{\partial \bar{x}_k}$$

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[2]) A first communication on this matter appeared with the title of "Gravitation and Elektrizität," in Sitzungsber. d. K. Preuß. Akad. d. Wissenschaften 1918, pp. 465.

[3]) Traditional geometry immediately goes freely from this particular problem to a lesser one, in principle, by no longer making the space itself the object of one's investigation, but the special classes of possible structures in space that are suggested by the space metric.

[4]) I am sufficiently audacious as to believe that the totality of all physical phenomena may be derived from a single universal law of Nature of the utmost mathematical simplicity.

[5]) On this, cf., H. Weyl, Das Kontinuum (Leipzig 1918), in particular, pp. 77 et seq.

whose determinant does not vanish. The last condition is necessary and sufficient for affine geometry to be valid in the infinitesimal limit and for the coordinate differentials in both systems to be related by invertible linear relations:

$$dx_i = \sum_k \alpha_{ik} d\overline{x}_k \,. \tag{1}$$

We assume the existence and continuity of higher derivatives whenever it becomes necessary in the course of our investigation. Thus, in each case the concept of continuous and continuously differentiable functions of position, as well as 2, 3, … times continuous differentiability, has an invariant sense that is independent of the coordinates; the coordinates themselves are such functions. We shall call an *n*-dimensional manifold, about which we shall consider no other properties than the ones that are intrinsic to *n*-dimensional manifolds – to use physical terminology – an (*n*-dimensional) *vacuum*.

The relative coordinates dx_i of one of the infinitesimally close points $P' = (x_i + dx_i)$ to a point $P = (x_i)$ are the components of a *line element* at *P*, or an *infinitesimal displacement* $\overrightarrow{PP'}$ of *P*. When we transform to another coordinate system these components satisfy formulas (1), in which α_{ik} means the values of the appropriate derivatives at the point *P*. In general, any *n* given numbers ξ^i ($i = 1, 2, \ldots, n$) in a definite sequence at a point *P* – when one establishes a particular coordinate system for the neighborhood of *P* – characterize a *vector* (or a *displacement*) at *P*; the components ξ^i ($\overline{\xi}^i$, resp.) of the same vector in any two coordinate systems - the "unprimed" and the "primed" systems – are connected by the same linear transformation formulas (1):

$$\xi^i = \sum_k \alpha_{ik} \overline{\xi}^k \,.$$

One can add vectors at *P* and multiply them with numbers; they therefore define a "linear" or "affine" collection. There are *n* "unit vectors" $\mathfrak{e}_i$ at *P* that are associated with any coordinate system, namely, the ones that possess the components:

$$\begin{array}{c|ccccc} \mathfrak{e}_1 & 1, & 0, & 0, & \cdots, & 0 \\ \mathfrak{e}_2 & 0, & 1, & 0, & \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathfrak{e}_n & 0, & 0, & 0, & \cdots, & 1 \end{array}$$

in the chosen coordinate system.

Any two (linearly independent) line elements at *P* with the components dx_i (δx_i, resp.) span a (two-dimensional) surface element at *P* with the components:

$$dx_i\, \delta x_k - dx_k\, \delta x_i = \Delta x_{ik} \,;$$

any three (independent) line elements dx_i, δx_i, $\mathrm{d}x_i$ at *P* span a (three-dimensional) volume element with the components:

$$\begin{vmatrix} dx_i & dx_k & dx_l \\ \delta x_i & \delta x_k & \delta x_l \\ \mathrm{d}x_i & \mathrm{d}x_k & \mathrm{d}x_l \end{vmatrix} = \Delta x_{ik} \; ;$$

etc. A linear form at P that depends upon an arbitrary line, (surface, volume, …, resp.) element at P is called a *linear tensor* of rank 1 (2, 3, …,resp.). By the use of a chosen coordinate system the coefficients a of these linear forms:

$$\sum_k a_i\, dx_i \qquad \left(\frac{1}{2!}\sum_{ik} a_{ik}\,\Delta x_{ik}\,,\ \frac{1}{3!}\sum_{ikl} a_{ikl}\,\Delta x_{ikl}\,,\ \ldots,\ \text{resp.}\right)$$

can be normalized uniquely by the alternation requirement; it says that in the last-described case – viz., the index triple (ikl) – when the triple is subjected to an even permutation one obtains the same coefficient a_{ikl}, whereas the sign of the coefficient changes under an odd permutation. Hence:

$$a_{ikl} = a_{kli} = a_{lik} = -\,a_{kil} = -\,a_{lki} = -\,a_{ilk}\,.$$

The coefficients, thus normalized, will be referred to as the *components* of the tensor in question. By differentiation, to a scalar field f there corresponds a linear tensor field of rank 1 with the components:

$$f_i = \frac{\partial f}{\partial x_i}\,;$$

to a linear tensor field of rank 1 f_i there corresponds a second rank tensor field:

$$f_{ik} = \frac{\partial f_i}{\partial x_k} - \frac{\partial f_k}{\partial x_i}\,;$$

to a linear a second rank tensor field there corresponds a tensor field of rank 3:

$$f_{ikl} = \frac{\partial f_{ik}}{\partial x_l} + \frac{\partial f_{li}}{\partial x_k} + \frac{\partial f_{ik}}{\partial x_l}\,;$$

etc. These operations are independent of the coordinate system used [6]).

A linear tensor of rank 1 at P may be referred to as a *force* that acts on it. By choosing a particular coordinate system, such a tensor will therefore be characterized by n numbers ξ_i that transform contragediently to the displacement components under the transition to another coordinate system:

$$\bar{\xi}_i = \sum_k \alpha_{ki}\,\xi_k\,.$$

[6]) RZM, § 13.

If η^i are the components of an arbitrary displacement at P then $\sum_k \xi_i \eta^i$ is an invariant. We will generally understand a *tensor* at P to mean a linear form of one or more arbitrary displacements and forces at P. For example, if one is dealing with a linear form composed of three arbitrary displacements ξ, η, ζ and two arbitrary forces ρ, σ:

$$\sum a_{ikl}^{pq} \xi^i \eta^k \zeta^l \rho_p \sigma_q \,,$$

then we speak of a tensor of rank 5 that is covariant in the indices *ikl* and contravariant in the indices *pq* of the components of *a*. A displacement is itself a contravariant tensor of rank 1 and a force is a covariant tensor of rank 1. The fundamental operations of tensor algebra are [7]):

1. Addition of tensors and multiplication by a number;
2. Multiplication of tensors;
3. Contraction.

Tensor algebra may thus be established in the vacuum – it assumes no measurements – but, by contrast, only the "linear" tensors of tensor analysis can be defined.

A "*motion*" in our manifold is given when each value s of a real parameter is associated with a point in a continuous manner; by the use of a coordinate system x_i the motion is expressed by formulas $x_i = x_i(s)$, in which the x_i on the right are understood to symbolize functions. If we assume continuous differentiability then we obtain, independent of the coordinate system, a vector at each point $P = P(s)$ of the motion that has the components:

$$u^i = \frac{dx_i}{ds}\,,$$

namely, the *velocity*. Two motions that transform between themselves by means of a continuous monotone transformation of the parameter s describe the same *curve*.

§ 3. Affinely connected manifolds (world with a gravitational field).

I. Concept of an affine connection.

If P' is infinitely close to the fixed point P then P' is *affinely connected* with P when one establishes how each vector at P goes to a vector at P' by a parallel displacement of P to P'. It is self-explanatory that the parallel displacement of all the vectors at P to P' must therefore satisfy the following requirement:

A. *The transplantation of all vectors at P to the infinitely close point P' by parallel displacement yields a map of the vectors at P to the vectors at P'.*

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[7]) RZM, § 6.

If we use a coordinate system in which P has the coordinates x_i , P' has the coordinates $x_i + dx_i$, an arbitrary vector at P has the components ξ^i, and the vector at P' that it goes to under parallel displacement has components $\xi^i + d\xi^i$ then $d\xi^i$ must therefore depend upon the ξ^i linearly:

$$d\xi^i = -\sum_r d\gamma^i_r \xi^r .$$

the $d\gamma^i_r$ are infinitesimal quantities that depend only upon the point P and the displacement $\overrightarrow{PP'}$ whose components are dx_i , but not on the vector that ξ that is been subjected to the parallel displacement. From now on, we consider only affinely connected manifolds; in such a manifold each point P is affinely connected with all of its infinitesimally neighboring points. We must place yet another requirement upon the concept of parallel displacement, that of *commutativity:*

B. *If P_1, P_2 are two points that are infinitely close to P and if the infinitesimal vector $\overrightarrow{PP_1'}$ goes to $\overrightarrow{P_2P_{21}'}$ under a parallel displacement of P to P_2 , but $\overrightarrow{PP_2'}$ goes to $\overrightarrow{P_1P_{12}'}$ under a parallel displacement to P_1 then P_{12} and P_{21} must coincide.* (They define an infinitely small parallelogram.)

If we denote the components of $\overrightarrow{PP_1'}$ by dx_i and those of $\overrightarrow{PP_2'}$ by δx_i then this requirement obviously says that:

$$d\delta x_i = -\sum_r d\gamma^i_r \cdot \delta x_r \tag{2}$$

is a symmetric function of both line elements d and δ. As a result, $d\gamma^i_r$ must be a linear form in the differentials dx_i :

$$d\gamma^i_r = \sum_s \Gamma^i_{rs}\, dx_s ,$$

in which the coefficients Γ, which depend only upon P and are called the *"components of the affine connection,"* must satisfy the symmetry condition:

$$\Gamma^i_{sr} = \Gamma^i_{rs} .$$

Due to the manner by which we formulated requirement **B** in terms of infinitesimal quantities, it can be argued that it lacks a precise meaning. For that reason, we would like to establish explicitly by a rigorous proof that the symmetry condition of (2) is independent of any coordinate system. To this end, we consider a (twice continuously differentiable) scalar field f. From the formula for the total differential:

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i ,$$

we extract the fact that when ξ^i are the components of an arbitrary vector at *P* then:

$$df = \sum_i \frac{\partial f}{\partial x_i} \xi^i$$

is an invariant this is independent of any coordinate system. We define a change in it by means of a second infinitesimal displacement δ, under which the vector ξ will be displaced parallel to it from *P* to P_2, and we obtain:

$$\delta df = \sum_{ik} \frac{\partial^2 f}{\partial x_i \partial x_k} \xi^i \delta x_k - \sum_{ir} \frac{\partial f}{\partial x_i} \cdot \delta\gamma^i{}_r \xi^i .$$

If we again replace ξ^i in this equation with dx_i and switch d and δ then this yields the invariant:

$$\Delta f = (\delta d - d\delta) f = \sum_i \left\{ \frac{\partial}{\partial x_i} \sum_r (d\gamma^i{}_r \delta x_r - \delta\gamma^i{}_r dx_r) \right\}.$$

The relations:

$$\sum_r (\delta\gamma^i{}_r \, \delta x_r - \delta\gamma^i{}_r \, dx_r) = 0$$

yield the necessary and sufficient condition for this, that any scalar field must satisfy the equation $\Delta f = 0$.

In physical terminology, an affinely connected continuum refers to a universe that is ruled by a *gravitational field*. The quantities $\Gamma^i{}_{rs}$ are the components of the gravitational field. We will not need to give the formulas by which these components transform under a transition to another coordinate system here. Under linear transformations, the $\Gamma^i{}_{rs}$ behave like the components of a tensor that is covariant in *r* and *s* and contravariant in *i*, but they lose this character under nonlinear transformations. However, the variations $\delta\Gamma^i{}_{rs}$ that the quantities Γ experience when one varies the affine connection of the manifold arbitrarily are actually the components of a generally invariant tensor of the assumed character.

What we are to understand by the *parallel displacement of a force* at *P* to an infinitely close point P' is a result of the requirement that the invariant product of this force and an arbitrary vector at *P* remains invariant under parallel displacement. If ξ_i are the components of the force and η^i those of the displacement then from [8]):

$$d(\xi_i \, \eta^i) = (d\xi_i \cdot \eta^i) + \xi_r \, d\eta^r = (d\xi_i - d\gamma^r{}_i \, \xi_r) = 0$$

and we deduce the formula:

[8]) In the sequel, we employ the Einstein convention that one always sums over indices that appear twice in a term of a formula, because without it, it would be deemed necessary for us to place a summation sign in front of each one.

$$d\xi_i = \sum_r d\gamma^r_i\, \xi_r .$$

One can introduce a coordinate system x_i – which I call *geodetic* at P – at each location P in such a way that the components Γ^i_{rs} of the affine connection vanish at the location P in such a coordinate system. Next, if x_i are arbitrary coordinates that vanish at P and Γ^i_{rs} mean the components of the affine connection at the location P in this coordinate system then one obtains a geodetic $\bar{x}_i$ by the transformation:

$$x_i = \bar{x}_i - \tfrac{1}{2}\sum_{rs}\Gamma^i_{rs}\bar{x}_r\bar{x}_s . \tag{3}$$

Namely, if we consider the $\bar{x}_i$ to be the independent variables and their differentials $d\bar{x}_i$ to be constants then we have, in the Cauchy sense, at the location P ($\bar{x}_i = 0$):

$$dx_i = d\bar{x}_i , \qquad d^2x_i = -\Gamma^i_{rs}d\bar{x}_r d\bar{x}_s ;$$

hence:

$$d^2x_i + \Gamma^i_{rs}\, dx_r\, dx_s = 0 .$$

Due to its invariant nature, the latter equation reads like:

$$d^2\bar{x}_i + \bar{\Gamma}^i_{rs}d\bar{x}_r d\bar{x}_s = 0 .$$

However, for arbitrary constant $d\bar{x}_i$ it is satisfied only when all of the $\bar{\Gamma}^i_{rs}$ vanish. *The gravitational field can therefore always be made to vanish at a single point for a certain choice of coordinate system.* By the requirement of "geodesy" at P, coordinates in the neighborhood of P are determined up to third order when one is given any linear transformation; i.e., if $x_i, \bar{x}_i$ are two geodetic coordinate systems at P and the x_i, as well as the $\bar{x}_i$, vanish at P then by neglecting terms of third and higher order in $\bar{x}_i$ one has linear transformation formulas $x_i = \sum_k \alpha_{ik}\bar{x}_k$ with constant coefficients α_{ik} .

II. Tensor analysis. Straight line.

Tensor analysis may first be completely established in an affinely connected space. For example, if f_i^k are the components of a tensor field of rank 2 that is covariant in i and contravariant in k then we make use of an arbitrary displacement ξ and a force η, construct the invariant:

$$f_i^k\, \xi^i\, \eta_k$$

and its variation under an infinitely small shift d of the argument point P, under which ξ and η will be parallel displaced along with P. One has:

$$d(f_i^{\;k}\, \xi^i\, \eta_k) = \frac{\partial f_i^{\;k}}{\partial x_l}\, \xi^i\, \eta_k\, dx_l - f_r^{\;k}\, \eta_k\, d\gamma^r{}_i\, \xi^i + f_i^{\;r}\, \xi^i\, d\gamma^k{}_r\, \eta_k\,,$$

hence:

$$f_{il}^{\;k} = \frac{\partial f_i^{\;k}}{\partial x_l} - \Gamma_{il}^r f_r^{\;k} + \Gamma_{rl}^k f_i^{\;r}$$

are the components of a tensor field of rank 3 that is covariant in l and contravariant in k that arises from the given second rank tensor field in a manner that is independent of the coordinate system.

In an affinely connected space the concept of a *straight* or *geodetic line* takes on a precise meaning. A line comes about when one consistently displaces a vector parallel to itself in its own direction and follows the motion of the initial point of this vector; it can thus be characterized as the only curve that leaves its direction unchanged. If u^i are the components of such a vector then in the course of its motion the equations:

$$du^i + \Gamma^i{}_{\alpha\beta}\, u^\alpha\, dx_\beta = 0\,,$$
$$dx_1 : dx_2 : \ldots : dx_n = u_1 : u_2 : \ldots : u_n$$

are consistently valid. If we then represent the curve in terms of the parameter s then we can normalize it in such a way that one has:

$$\frac{dx_i}{ds} = u^i$$

identically in s , and the differential equations of the straight line then read:

$$w^i = \frac{d^2 x_i}{ds^2} + \Gamma^i_{\alpha\beta} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0\,.$$

For any arbitrary motion $x_i = x_i\,(s)$ the left-hand sides of these equations are the components of a vector that is invariantly linked with the motion at the point s, the *acceleration*. In fact, when ξ_i is an arbitrary force at some point that is parallel displaced by a transition to the point $s + ds$, one has:

$$\frac{d(u^i \xi_i)}{ds} = w^i \xi_i\,.$$

A motion whose acceleration vanishes identically is called a *translation*. A straight line – as one can also understand from our explanation above – is to be understood as the path of motion of a translation.

III. Curvature.

If P and Q are two points that are connected by a curve with a given initial vector then one can displace it from P to Q parallel to itself along the curve. The *vector translation* thus obtained is generally *non-integrable;* i.e., the vector that one obtains at Q is dependent upon the path of displacement along which the transition took place. Only in the special case where integrability exists is there any sense to speaking of the *same* vector at two different points P and Q; one would then understand such vectors to the ones that go to each other under parallel displacement. In that case, one calls the manifold *Euclidian.* One may introduce special "linear" coordinate systems in such a manifold that are distinguished by the fact that in such systems equal vectors at distinct points have equal components. Any two such linear coordinate systems are connected by linear transformation formulas. The components of the gravitational field vanish identically in a linear coordinate system.

In the infinitesimal parallelogram that was constructed above (§ 3, I., **B**.) we took an arbitrary vector with the components ξ^i at a point P, displaced it parallel to itself first to P_1 and then to P_{12}, and another time from P to P_2 and then to P_{21} . Since P_{12} coincides with P_{21} we can take the difference of these two vectors at that point and thus obtain a vector whose components are obviously:

$$\Delta\xi^i = \delta d\xi^i - d\delta\xi^i .$$

From:

$$d\xi^i = -d\gamma^i_k\xi^k = -\Gamma^i_{kl}\,dx_l\,\xi^k$$

it follows that:

$$\delta d\xi^i = -\frac{\partial\Gamma^i_{kl}}{\partial x_m}dx_l\delta x_m\xi^k - \Gamma^i_{kl}\delta dx_l\cdot\xi^k + d\gamma^i_k\delta\gamma^r_k\xi^k ,$$

and due to the symmetry of δdx_l :

$$\Delta\xi^i = \left\{\left(\frac{\partial\Gamma^i_{km}}{\partial x_l} - \frac{\partial\Gamma^i_{kl}}{\partial x_m}\right)dx_l\delta x_m + (d\gamma^i_r\delta\gamma^r_k - d\gamma^r_k\delta\gamma^i_r)\right\}\xi^k .$$

We thus obtain:

$$\Delta\xi^i = \Delta R^i_k\,\xi^k ,$$

in which ΔR^i_k is the linear form of the two shifts d and δ, which are independent of the displaced vector ξ, or furthermore, of the surface element that they span, which has the components:

$$\Delta x_{lm} = dx_l\,\delta x_m - dx_m\,\delta x_l ,$$

namely:

$$\Delta R^i_k = R^i_{klm}\,dx_l\,\delta x_m = \tfrac{1}{2}R^i_{klm}\,\Delta x_{lm} , \qquad (R^i_{kml} = -R^i_{klm}), \tag{4}$$

$$R^i_{klm} = \left(\frac{\partial\Gamma^i_{km}}{\partial x_l} - \frac{\partial\Gamma^i_{kl}}{\partial x_m}\right) + \left(\Gamma^i_{lr}\Gamma^r_{km} - \Gamma^i_{mr}\Gamma^r_{kl}\right). \tag{5}$$

If the η_i are components of an arbitrary force at P then $\eta_i \Delta x^i$ is an invariant. It follows that R^i_{klm} are the components of a tensor of rank 4 at P that is covariant in klm and contravariant in i, namely, the *curvature*. The vanishing of the curvature identically is the necessary and sufficient condition for the manifold to be Euclidian. Along with the "skew" symmetry described in (4) that the components of the curvature satisfy, they also satisfy the "cyclic" identity:

$$R^i_{klm} + R^i_{lmk} + R^i_{mkl} = 0 .$$

In essence, the curvature at a point P is a linear map, or transformation, $\Delta\mathbf{P}$ that associates each vector ξ with a vector $\Delta\xi$; this transformation itself depends linearly upon the surface element at P:

$$\Delta\mathbf{P} = \mathbf{P}_{ik}\, dx_i\, \delta x_k = \tfrac{1}{2}\mathbf{P}_{ik}\, \Delta x_{ik} \qquad (\mathbf{P}_{ki} = -\mathbf{P}_{ik}) .$$

The curvature is therefore best understood as a "linear transformation tensor of rank 2."

In order to rigorously prove the invariance of the curvature tensor beyond objections of the sort that might perhaps be raised for the infinitesimal changes described above, one employs a force field f_i, defines the change $d(f_i\, \xi^i)$ of the invariant product $f_i\, \xi^i$ in such a manner that under the infinitesimal shift d the vector ξ will be displaced parallel to itself. If one replaces the infinitesimal shift dx with an arbitrary vector ρ at P in the resulting expression then one obtains an invariant bilinear form in two arbitrary vectors ξ and ρ at P. With it, one defines the change that arises from a second infinitesimal shift δ and takes the vectors ξ, ρ along with it in a parallel fashion, and then replaces the second shift with a vector σ at P. One finds the form:

$$\delta d(f_i\, \xi^i) = \delta df_i \cdot \xi^i + df_i\, \delta\xi^i + \delta f_i\, d\xi^i + f_i\, \delta d\xi^i .$$

Due to the symmetry of δdf_i, switching d and δ and then subtracting yields the invariant:

$$\Delta(f_i\, \xi^i) = f_i\, \Delta\xi^i ,$$

and one thus achieves the desired proof.

§ 4. Metric manifold (the ether).

I. Concept of a metric manifold.

A manifold *carries a measure at the point P* when the lengths of line elements at P can be compared; we thus assume the validity of the Pythagorean-Euclidian law in the infinitesimal domain. Therefore, a number $\xi \cdot \eta$ shall correspond to any two vectors ξ, η at P, namely, the *scalar product*, which is a symmetric bilinear form in its dependence upon both of them; this bilinear form is clearly not absolutely determined, but only up to an arbitrary proportionality factor this different from 0. Therefore, it is not actually the form $\xi \cdot \eta$, but the equation $\xi \cdot \eta = 0$ that is given; two vectors that satisfy it will be called *perpendicular*. We assume that this equation is non-degenerate; i.e., that the only vector at P that is perpendicular to all of the vectors at P is the vector 0. However, we do not assume that the associated quadratic form $\xi \cdot \xi$ is positive-definite. If it has an index of inertia q and one has $n - q = p$ then we say briefly that the manifold is $(p + q)$-dimensional at the point in question; due to the arbitrariness of the proportionality factor, the two numbers p, q are defined only up to their ordering. We now assume that our manifold carries a measure at every point. In order to facilitate the goal of the analytical representation, we imagine: 1. a choice of coordinate system has been made and 2. a choice of arbitrary proportionality factor in the scalar product has been made at each location; one thus arrives at a "*reference system*" [9]) for the analytical representation. If the vector ξ at the point P with the coordinates x_i has the components ξ^i, and η has the components η^i then one will have:

$$(\xi \cdot \eta) = \sum_{ik} g_{ik}\, \xi^i\, \eta^k \qquad (g_{ki} = g_{ik}),$$

in which the coefficients g_{ik} are functions of the x_i . The g_{ik} shall not only be continuous, but twice continuously differentiable. Since they are continuous and their determinant g, by assumption, is nowhere vanishing the quadratic form $(\xi \cdot \xi)$ has the same index of inertia q at every location; we can therefore regard the manifold as $(p + q)$-dimensional in all of its aspects. If we keep the same coordinate system, but make a different choice of the undetermined proportionality factor then, instead of g_{ik} , we arrive at the new quantities:

$$g'_{ik} = \lambda \cdot g_{ik}$$

for the coefficients of the scalar product, where λ is a nowhere-vanishing continuous (and twice continuously differentiable) function of position.

As a result of the foregoing assumptions, the manifold is only endowed with an *angle measure;* the geometry that this alone will support is called *"conformal geometry."* As is well known, in the realm of two-dimensional manifolds ("Riemann spheres"), due to its importance in the theory of complex functions, it has attained a far-reaching level of development. If we make no further assumptions, then the individual points of the manifold remain completely isolated from each other in the metric context. A metric connection from point to point will then be first introduced in it when one proposes a *principle for comparing the length unit at a point P with the ones at infinitely close points.* Instead of this, Riemann made the very far-reaching assumption that the unit lengths of line elements could be compared with each other, not only at the same location, but also at any two finitely distant locations. *The possibility of such a "global geometrical" comparison can, however, not exist at all in a purely infinitesimal geometry.* The Riemannian assumption is also carried over into the Einsteinian world geometry of gravitation. Here, this inconsistency shall be removed.

Let P be a fixed point and let P_* be an infinitely close point that one arrives at by means of the displacement whose components are dx_i. We choose a particular coordinate

[9]) I thus distinguish between "coordinate system" and "reference system."

system. In terms of the length unit that was established at P (as well as the remaining points of the space), the square of the length of an arbitrary vector at P will be:

$$\sum_{ik} g_{ik}\, \xi^i\, \xi^k .$$

However, the square of the length of an arbitrary vector ξ_* at P_* will be, *when we transform the unit length that was chosen at P to P**, as we assumed would be possible, given by:

$$(1 + d\varphi)\sum_{ik}(g_{ik} + dg_{ik})\xi_*^i \xi_*^k ,$$

where $1 + d\varphi$ means a proportionality factor that deviates from 1 by an infinitely small quantity; $d\varphi$ must be a homogeneous function of the differentials dx_i of order 1. Namely, if we transplant the chosen length unit at the point P to a point along a curve that goes from P to the finitely distant point Q then, upon establishing the unit length at Q, we obtain the expression $g_{ik}\ \xi^i\ \xi^k$ for the square of the length of an arbitrary vector at Q, multiplied by a proportionality factor, that one obtains from the product of infinitely many factors of the form $1 + d\varphi$, which will take the form:

$$\Pi\ (1 + d\varphi) - \Pi\ e^{d\varphi} = e^{\Sigma d\varphi} = e^{\int_P^Q d\varphi}$$

under the transition from one point of the curve to the next. In order for the integral that appears in the exponent to be meaningful, $d\varphi$ must be a function of the differentials of the sort described above.

If one replace g_{ik} with $g'_{ik} = \lambda\, g_{ik}$ then, instead of $d\varphi$, another quantity $d\varphi'$ will appear. If λ is the value of this factor at the point P then one must have:

$$(1 + d\varphi')(\, g'_{ik} + a\ g_{ik}) = \lambda\,(1 + d\varphi)(g_{ik} + dg_{ik}),$$

which yields:

(6) $$d\varphi' = d\varphi - \frac{d\lambda}{\lambda} .$$

Of the next possible assumptions about $d\varphi$, that it is a linear differential form, the square root of a quadratic form, the cube root of a cubic form, etc., as we now see from (6), only the first one is meaningful. We have arrived at the following result:

The metric of a manifold is based on a quadratic differential form and a linear differential form:

(7) $$ds^2 = g_{ik}\, dx^i\, dx^k \qquad \textit{and} \qquad d\varphi = \varphi^i\, dx_i .$$

Conversely, if the metric is not, however, absolutely established by these forms, but by any pair of forms ds'^2, $d\varphi'$ *that originates in* (7) *by way of the equations:*

(8) $$ds'^2 = \lambda \cdot ds^2 , \qquad d\varphi' = d\varphi - \frac{d\lambda}{\lambda} ,$$

then these are equivalent to the former pair in the sense that both of them express the same metric. In this, λ is an arbitrary, nowhere-vanishing, continuous (more precisely: twice continuously differentiable) *function of position.* In all quantities or relations that analytically represent metric phenomena, the functions g_{ik}, φ_i must therefore be introduced in such a manner that one has invariance: 1. under arbitrary coordinate transformations ("coordinate invariance"), and 2. under the replacement of (7) with (8) ("scale invariance"). $\frac{d\lambda}{\lambda} = d \ln\lambda$ is a total differential. Thus, whereas an arbitrary proportionality factor for the quadratic form ds^2 remains at every location, there exists an indeterminacy in $d\varphi$ of an additive total differential.

We give a physical expression to a metric manifold by regarding it as a world full of *ether*. The particular metric that resides in the manifold represents a particular state of the ether-filled world. This state is therefore to be described, relative to a reference system, by being given the (arithmetic construction of the) functions g_{ik}, φ_i .

From (6), it follows that the linear tensor of rank 2 with the components:

$$F_{ik} = \frac{\partial \varphi_i}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_i}$$

is uniquely determined by the metric on the manifold; I call it the *metric rotation.* It is, I believe, the same thing as what one calls the *electromagnetic field* in physics. It satisfies the "first system of Maxwell equations:"

$$\frac{\partial F_{kl}}{\partial x_l} + \frac{\partial F_{il}}{\partial x_k} + \frac{\partial F_{ik}}{\partial x_l} = 0 .$$

Its vanishing is the necessary and sufficient condition for the change in length unit to be integrable, and therefore any assumption that Riemann based metric geometry upon to be valid. We thus understand, as Einstein did in his world geometry by directing his mathematical hindsight to Riemann, that only the gravitational phenomena, but not the electromagnetic ones, could be accounted for.

II. Affine connection on a metric manifold.

In a metric space, in place of the requirement **A** for the concept of infinitesimal parallel displacement that was posed in § 3, I., one has the far-reaching requirement:

A*: *that the parallel displacement of all of the vectors at a point P to an infinitely close point P′ must be not only an affine, but also a congruent transplantation of this collection of vectors.*

By the use of the previous notations, this requirement yields the equation:

$$(1 + d\varphi)(g_{ik} + dg_{ik})(\xi^i + d\xi^i)(\xi^k + d\xi^k) = g_{ik}\,\xi^i\,\xi^k\,. \tag{9}$$

For any quantities a^i that carry an upper index (i), we define the "lowering" of this index by the equation:

$$a_i = \sum_k g_{ik}\,a^i$$

(and the inverse process of raising of an index by the inverse equations). For (9), we can write, in terms of these symbols:

$$(g_{ik}\,\xi^i\,\xi^k)\,d\varphi + \xi^i\,\xi^k\,dg_{ik} + 2\,\xi_i\,d\xi^i = 0\,.$$

The last term is:

$$= -\,2\,\xi_i\,\xi^k\,d\gamma^i{}_k = -\,2\,\xi^i\,\xi^k\,d\gamma_{ik} = -\,2\,\xi^i\,\xi^k\,(d\gamma_{ik} + d\gamma_{ki})\,;$$

one must therefore have:

$$d\gamma_{ik} + d\gamma_{ki} = dg_{ik} + g_{ik}\,d\varphi\,. \tag{10}$$

This equation may be solved for certain only when $d\varphi$ is a linear differential form; an assumption that we already insisted above was the only reasonable one. From (10), or:

$$\Gamma_{i,\,kr} + \Gamma_{k,\,ir} = \frac{\partial g_{ik}}{\partial x_r} + g_{ik}\,\varphi_r\,, \tag{10*}$$

it follows, by taking into account the symmetry property $\Gamma_{r,\,ik} = \Gamma_{r,\,ki}$ that:

$$\Gamma_{r,\,ik} = \tfrac{1}{2}\left(\frac{\partial g_{ir}}{\partial x_k} + \frac{\partial g_{kr}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_r}\right) + \tfrac{1}{2}(g_{ir}\,\varphi_k + g_{kr}\,\varphi_i - g_{ik}\,\varphi_r) \tag{11}$$

$$(\Gamma_{r,\,ik} = g_{rs}\,\Gamma^r{}_{ik}).$$

This shows that in a metric space the concept of infinitesimal parallel displacement of a vector is uniquely established by the given requirement [10]). I consider it to be the *fundamental fact of infinitesimal geometry* that when not only a metric, but also an affine connection, is given on a manifold *the principle of unit length displacement, with nothing else, leads to the displacement of directions,* or physically speaking, *the state of the ether determines the gravitational field.*

When the quadratic form $g_{ik}\,dx_i\,dx_k$ is indefinite, among the geodetic lines one can distinguish the *null lines*, along which this form vanishes. Since this depends only upon

[10]) On this, cf., Hessenberg, Vectorielle Begründung der Differentialgeometrie, Math. Ann Bd. 78 (1917), pp. 187-217, especially pp. 208.

the behavior of the g_{ik}, but not at all on the φ_i , these are therefore conformal geometric structures [11]).

We have placed certain axiomatic requirements on the concept of parallel displacement and showed that in a metric manifold they can be satisfied in one and only one way. It is, however, also possible to define these concepts explicitly in a simpler way. If P is a point of our metric manifold then we would like call a reference system *geodetic* at the point P when the φ_i vanish and the g_{ik} assume stationary values in such a system:

$$\varphi_i = 0 , \qquad \frac{\partial g_{ik}}{\partial x_r} = 0 .$$

D. *There is a geodetic reference system at each point P. If ξ is a given vector at P and P' is, however, an infinitely close point, then we understand the parallel displacement of ξ to the corresponding vector at P' to mean that vector at P' that possesses the same components as ξ in the geodetic reference system associated with P. This definition is independent of the choice of geodetic coordinate system.*

It is not difficult to prove the claim that was uttered in this statement independently of the line of reasoning that was followed here by direct computation, and in the same way, to show that the process of parallel displacement so defined will be described in an arbitrary coordinate system by the equations:

$$d\xi^r = -\Gamma^r_{ik}\, \xi^i\, dx_k , \tag{12}$$

with the coefficients Γ being taken from (11) [12]). Here, however, where the invariant meaning of equation (12) is already certain, we can conclude this in a simpler way. From (11), the Γ^r_{ik} vanish in a geodetic reference system, and equations (12) reduce to $d\xi^r = 0$. The concept of parallel displacement that we deduced from an axiomatic requirement therefore agrees with the one that was defined in **D**. It only remains for us to prove the existence of a geodetic reference system. To this end, we choose a geodetic coordinate system x_i at P that has the point P itself for its origin ($x_i = 0$). If the unit length at P and in its neighborhood is chosen arbitrarily and φ_i then means the values of these quantities at P then one needs only to carry out the transition from from (7) to (8) with:

$$\lambda = e^{\sum_i \varphi_i x_i}$$

in order to arrive at the fact that, along with the Γ^i_{rs} , also the φ_i , vanish at P. From this – see (10*) – the geodetic nature of the reference system thus obtained follows. The coordinates of a geodetic reference system at P are defined up to terms of third order in the immediate neighborhood of P when one is freely given a linear transformation, but the unit length is given up to terms of second order as long as the addition of a constant factor is given freely.

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[11]) With this remark, I would like to correct an oversight on page 183 of my book "Raum, Zeit, Materie."

[12]) One can thus follow the path that I took in RZM § 14.

III. Computationally convenient extension of the concept of a tensor.

The quantities that we introduced in § 2 as tensors are dimensionless; their components depend completely upon the choice of coordinate system, but not on the choice of unit length. In metric geometry, an extension of this concept proves to be preferable: by a tensor of weight *e*, we shall understand a linear form of one or more displacements and forces at a point that are independent of the coordinate system, but depend on the unit length in such a way that the form takes on the factor λ^e under the replacement of (7) with (8). The g_{ik} themselves are the components of a covariant tensor of rank 2 and weight 1. Incidentally, we regard this extended concept of a tensor only as an aid that we introduce merely for the sake of computational convenience; we ascribe an objective meaning only to the tensors of weight 0. Therefore, in the sequel whenever we speak of tensors with no additional mention of their weight, the concept is always to be understood in its original sense.

Any computational convenience resides in the following fact: If we perform the process of raising one or more indices in the components a_{ik} of a covariant tensor of weight *e* then we obtain the mixed components of a tensor of weight $e - 1$ in the case of a_i^k or a^i_k, and a contravariant tensor of weight $e - 2$ in the case of a^{ik}. We cannot decide, as would usually be the case, how to identify the resulting tensors with the original ones since, along with depending upon those tensors, they also depend upon the metric – the state of the world ether – and we will not consider this to be given a priori in the slightest, but leave open the possibility of subjecting it to arbitrary virtual variations.

IV. Curvature in metric spaces.

If ξ^i, η^i are two arbitrary displacements at the point *P*, but f_i are the components of a force field, then it follows that:

$$f_i\,\eta^i = f^i\,\eta_i\,;$$
$$\Delta(f_i\,\eta^i) = f_i\,\Delta\eta^i = \Delta(f^i\,\eta_i) = f^i\,\Delta\eta_i\,;$$

hence:

(13) $$\xi_i\,\Delta\eta^i = \xi^i\,\Delta\eta_i\,.$$

On the other hand, when the vectors are, as always, parallel displaced by virtual displacements one has:

$$d(\xi^i\eta_i) + (\xi^i\eta_i)\,d\varphi = 0\,,$$
$$\delta d(\xi^i\eta_i) + \delta(\xi^i\eta_i)\,d\varphi + (\xi^i\eta_i)\,\delta d\varphi = 0\,.$$

The middle term in the latter equation is:

$$= -(\xi^i\eta_i)\,\delta\varphi\,d\varphi\,,$$

and the first one is:

$$= \eta_i\,\delta d\xi^i + \delta\eta_i\,d\xi^i + d\eta_i\,\delta\xi^i + \xi^i\,\delta d\eta_i\,.$$

If one exchanges *d* and δ and subtracts then this yields:

$$(\eta_i\,\Delta\xi^i + \xi^i\,\Delta\eta_i) + (\xi^i\eta_i)\,\Delta\varphi = 0\,,$$

or, on account of (13):

$$(\eta_i\,\Delta\xi^i + \xi_i\,\Delta\eta^i) + (\xi^i\,\eta_i)\,\Delta\varphi = 0\,.$$

Thus, if we set:

(14) $$\Delta\xi^i = \bar{\Delta}\xi^i - \tfrac{1}{2}\,\xi^i\,\Delta\varphi$$

then we have decomposed $\Delta\xi^i$ into components that are perpendicular to ξ^i and components that are parallel to ξ^i. One has:

$$\Delta\varphi = \tfrac{1}{2} F_{ik}\,\Delta x_{ik}\,,$$

and we write:

$$\bar{\Delta}\xi^i = \Delta\bar{R}^i_k\,\xi^k\,, \qquad \Delta\bar{R}^i_k = \tfrac{1}{2}\bar{R}^i_{klm}\Delta x_{lm}\,.$$

One then has:

(15) $$R^i_{klm} = \bar{R}^i_{klm} - \tfrac{1}{2}\delta^i_k F_{lm}\,, \qquad \delta^i_k = \begin{cases} 1 & (i = k) \\ 0 & (i \neq k). \end{cases}$$

If we lower the index i then the quantities are skew-symmetric, not only in l and m, but also i and k. In the decomposition (15), we refer to the first summand as the direction curvature and the second one as the length curvature. Length curvature = metric rotation. By the nature of the corresponding decomposition (14) of $\Delta\xi^i$, a theorem follows that justifies our terminology: The tensor $\bar{R}$ of direction curvature vanishes when and only when the parallel displacement of a vector subjected to a change of direction is integrable; the tensor F of length curvature vanishes when and only when the likewise altered length is integrable.

Here, we give the explicit expression for the direction curvature. We introduce, as usual, the Christoffel three-index symbols and the Riemannian curvature components by the equations:

$$\begin{bmatrix} ik \\ r \end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{ir}}{\partial x_k} + \frac{\partial g_{kr}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_r}\right), \qquad \begin{bmatrix} ik \\ r \end{bmatrix} = \sum_s g_{rs} \begin{Bmatrix} ik \\ s \end{Bmatrix},$$

$$G^i_{klm} = \frac{\partial}{\partial x_l}\begin{Bmatrix} km \\ i \end{Bmatrix} - \frac{\partial}{\partial x_m}\begin{Bmatrix} kl \\ i \end{Bmatrix} + \begin{Bmatrix} lr \\ i \end{Bmatrix}\begin{Bmatrix} km \\ r \end{Bmatrix} - \begin{Bmatrix} mr \\ i \end{Bmatrix}\begin{Bmatrix} kl \\ r \end{Bmatrix},$$

and further set, for an arbitrary quadratic system of numbers a_{ik}:

$$(g_{il}\,a_{km} + g_{km}\,a_{il} - g_{im}\,a_{kl} - g_{kl}\,a_{im}) = \bar{a}_{iklm}$$

and define:

$$\frac{\partial\varphi_i}{\partial x_k} - \begin{Bmatrix} ik \\ r \end{Bmatrix}\varphi_r = \Phi_{ik}\,,$$

$$\varphi_i\,\varphi_k - \tfrac{1}{2} g_{ik}\,(\varphi_r\,\varphi^r) = \varphi_{ik}\,,$$

which makes:

$$\bar{R}_{iklm} = G_{iklm} - \bar{\Phi}_{iklm} + \tfrac{1}{2}\tilde{\varphi}_{iklm} .$$

One observes here that the individual terms on the right-hand side have no intrinsic significance: they clearly possess "coordinate" invariance, but not "scale" invariance. For the contracted tensors:

$$\bar{R}^i_{kim} = \bar{R}_{km} , \qquad G^i_{kim} = G_{km}$$

one has:

$$\bar{R}_{ik} = G_{ik} - \frac{n-2}{2}(\Phi_{ik} - \tfrac{1}{2}\varphi_{ik}) - \tfrac{1}{2} g_{ik} (\Phi - \tfrac{1}{2}\varphi),$$

where:

$$\Phi = \Phi^i_i = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\varphi^i)}{\partial x_i} , \qquad \varphi = \varphi^i_i = -\frac{n-2}{2} (\varphi_i\, \varphi^i) .$$

When we set:

$$\bar{R}^i_i = \bar{R} = R, \qquad G^i_i = G,$$

another contraction yields:

$$R = G - (n-1)\left\{\Phi + \frac{n-2}{4}(\varphi_i\varphi^i)\right\} .$$

One can derive a tensor from the directional curvature that depends only upon the g_{ik} in the following manner:

$$*R_{iklm} = (n-2)\,\bar{R}_{iklm} - (g_{il}\bar{R}_{km} + g_{km}\bar{R}_{il} - g_{im}\bar{R}_{kl} - g_{kl}\bar{R}_{im}) + \frac{1}{n-1}(g_{il}g_{km} - g_{im}g_{kl})\bar{R} .$$

These numbers $*R_{iklm}$ are analogous to the $*G_{iklm}$ that one defines by means of G_{iklm}; thus, if one raises the index i again then $*G^i_{klm} = *R^i_{klm}$ are the components of an invariant tensor of conformal geometry. This tensor always vanishes for $n = 2$ and $n = 3$, and first plays a role for $n \geq 4$. Its vanishing is a necessary (but not sufficient) condition for the manifold to be mapped to a Euclidian one in a manner that preserves angles.

§ 5. Scalar and tensor densities.

I. In topological space.

If $\int \mathfrak{W}\, dx$ – I briefly write dx for the integration element $dx_1\, dx_2\, \ldots\, dx_n$ – is an integral invariant then $\mathfrak{W}$ is a quantity that depends upon the coordinate system in such a way that under a transition to another coordinate system it is multiplied by the absolute value of the functional determinant. If we regard this integral as a measure on an integration domain that is filled with quantum matter then $\mathfrak{W}$ is its density. For that reason, a quantity of the sort described may be referred to as a *scalar density*. This is an important concept that stands on a par with that of "scalar" and does not reduce to it in the slightest [13]). Analogously, we will call a linear form in one or more displacements and forces that depends upon the coordinate system in such a way that it gets multiplied by the absolute value of the functional determinant under a coordinate transition a *tensor density*. We are justified in thinking of tensors as *intensities* and tensor densities as *quantities.* The covariant and contravariant expressions will be employed for tensors. The general concept of tensor density belongs to pure topology. However, in this type of geometry the basis for the analysis of tensor densities may be constructed only to an analogous degree compared to the analysis of tensors.

In § 2, we called a tensor linear when it is covariant and its components satisfy the requirement that they be alternating. We shall call a tensor density linear when it is contravariant and possesses alternating components. A linear tensor density of rank 1 can be regarded as a "current strength." If $\mathfrak{w}^i$ is such a tensor density then:

$$\frac{\partial \mathfrak{w}^i}{\partial x_i} = \mathfrak{w} \tag{16}$$

is a scalar density that is coupled with it; if $\mathfrak{w}^{ik}$ is a linear density of rank 2 then:

$$\frac{\partial \mathfrak{w}^{ik}}{\partial x_k} = \mathfrak{w}^i \tag{17}$$

is a linear tensor density of rank 1, etc. One proves (16) in a well-known manner by showing that the left-hand side represents the source strength that is associated with the current strength. From this, one obtains (17) with the aid of a force field $f_i = \frac{\partial f}{\partial x_i}$ that arises from a potential f, and defines the divergence of $\mathfrak{w}^{ik} f_i$:

$$\frac{\partial(\mathfrak{w}^{ik} f_i)}{\partial x_k} = \frac{\partial \mathfrak{w}^{ik}}{\partial x_k} \cdot f_i ,$$

etc.

II. In affinely connected and in metric spaces.

In an affinely connected manifold one can not only define the divergence of linear tensor densities, but also arbitrary ones. We shall consider a vector field ξ^i at a point P to be stationary when the vectors ξ in the neighboring points P' to P go over to the vector ξ at P under parallel displacement, i.e., when there are total differential equations:

[13]) The comparison between scalars and scalar densities corresponds completely to that of functions and Abelian integrals in the theory of algebraic functions.

$$d\xi^i + \Gamma^i_{rs}\,\xi^r\,dx_s = 0 \qquad \left(\text{or } \frac{\partial \xi^i}{\partial x_s} + \Gamma^i_{rs}\xi^r = 0\right)$$

at P. Obviously, there is a vector field that is stationary at P that is associated with any arbitrary given vector ξ at the point P. One can define an analogous concept for force fields. If one now defines, e.g., the divergence of a mixed tensor density $\mathfrak{w}_i^k$ of rank 2 then one makes use of a vector field ξ that is stationary at P and constructs the divergence of the tensor density $\xi^i\,\mathfrak{w}_i^k$:

$$\frac{\partial(\xi^i\,\mathfrak{w}_i^k)}{\partial x_k} = \frac{\partial \xi^r}{\partial x_k}\mathfrak{w}_r^k + \xi^i\,\frac{\partial \mathfrak{w}_r^k}{\partial x_k} = \xi^i\left(-\Gamma^r_{ik}\,\mathfrak{w}_r^k + \frac{\partial \mathfrak{w}_r^k}{\partial x_k}\right).$$

This quantity is a scalar density, and therefore:

$$\frac{\partial \mathfrak{w}_i^k}{\partial x_k} - \Gamma^r_{is}\,\mathfrak{w}_r^s$$

is a tensor density of rank 1 that arises from $\mathfrak{w}_i^k$ in a manner that is independent of the coordinate system.

However, one can not only construct a tensor density that has a rank that is less by one from such a tensor density by taking its *divergence*, but also construct another tensor density that has a rank that is higher by one from it by *differentiation*. Next, if $\mathfrak{s}$ means a scalar density, which we can regard as the density of a substance that fills the manifold, and if $dV = dx_1\,dx_2\,\dots\,dx_n$ is an infinitely small volume element then $\mathfrak{s}\,dV$ is the quantum of the substance that fills this element. We now subject dV to the infinitesimal displacement δ (with the components δx_i); by this, we understand a process by which the individual points of dV experience infinitesimal displacements that take them to new points by parallel displacement. The difference between the matter quanta that fill up dV and those that fill the displacement of dV to the surrounding neighborhood amounts to:

$$(\delta\mathfrak{s} - \mathfrak{s}\;\Gamma^r_{ir}\,dx_i)\,dV = (\delta\mathfrak{s} - \mathfrak{s}\;\gamma^r_r\;)\,dV\,.$$

One thus has that:

$$\frac{\partial\,\mathfrak{s}}{\partial x_i} - \Gamma^r_{ir}\,\mathfrak{s} \tag{18}$$

are the components of a covariant tensor density of rank 1 that arises from the scalar density $\mathfrak{s}$ in a manner that is independent of the coordinate system. Its vanishing at a location shows that the substance itself is uniformly distributed. Moreover, (18) can also be derived in a more computational fashion as follows: One makes use of a vector field ξ^i that is stationary at P and takes the density of the current strength $\mathfrak{s}\xi^i$:

$$\frac{\partial(\mathfrak{s}\xi^i)}{\partial x_i} = \frac{\partial\mathfrak{s}}{\partial x_i}\xi^i + \mathfrak{s}\frac{\partial\xi^i}{\partial x_i} = \left(\frac{\partial\mathfrak{s}}{\partial x_i} - \Gamma^r_{ir}\,\mathfrak{s}\right)\xi^i .$$

In order to facilitate the transition from the differentiation of scalar tensor densities to arbitrary ones – e.g., mixed tensor densities $\mathfrak{w}_i^k$ of rank 2 – one takes, in a now familiar sort of way, a vector ξ^i that is stationary at P and a stationary force field η_i , and differentiates the scalar density $\mathfrak{w}_i^k\,\xi^i\,\eta_k$. Contracting the tensor density that results from differentiation over the differentiation index and a contravariant one gives one the divergence.

The analysis of tensor densities is therefore already accomplished in affine geometry. What *metric* geometry now provides is merely the following *method of generating* tensor densities: one multiplies an arbitrary tensor of weight $-\frac{n}{2}$ by $\sqrt{g}$, where g is the determinant of g_{ik} . – Example: The real world is a (3 + 1)-dimensional manifold; g is, however, negative, and we use the positive $-g$ in place of it. From the covariant metric rotation tensor F_{ik}, which has weight 0, we obtain the contravariant F^{ik} of weight – 2, and from it, upon multiplication by $\sqrt{-g}$, we obtain:

$$\sqrt{-g}\,F^{ik} = \mathfrak{F}^{ik} .$$

These are therefore the components of a certain linear tensor density of rank 2 that is invariant of the state of the ether; we will refer to it as the *metric rotation density (electromagnetic field density).*

$$\frac{\partial\mathfrak{F}^{ik}}{\partial x_k} = \mathfrak{s}^i \tag{19}$$

is thus a current strength (linear density of rank 1). In (19), we have the second system of Maxwell equations before us, which admittedly first takes on a definite meaning when the "*electrical current*" $\mathfrak{s}^i$ is expressed in yet another way in terms of the state of the ether. In any event, from our interpretation of the electromagnetic field, it can, however, give anything like an electromagnetic field density and an electrical current only in a four-dimensional world. The integral of

$$\mathfrak{S} = \tfrac{1}{4}F_{ik}\,\mathfrak{F}^{ik},$$

which can be taken over any world domain, appears in physics as the *quantity of electromagnetic action* that is contained in this domain. Its meaning is based on the fact that the infinitely small change that it experiences under an infinitesimal variation δg_{ik} , $\delta\varphi_i$ of the state of the ether that vanishes on the boundary of the domain is:

$$= \int (\mathfrak{s}^i\delta\varphi_i + \tfrac{1}{2}\mathfrak{S}^{ik}\delta g_{ik})\,dx \qquad (\mathfrak{S}^{ki} = \mathfrak{S}^{ik}),$$

in which $\mathfrak{s}^i$ are the components of the current strength that are defined by (19), and the mixed tensor density of rank 2 with the components:

$$\mathfrak{S}_i^k = \mathfrak{S}\delta_i^k - F_{ir}\mathfrak{F}^{kr}$$

represents the *energy-momentum tensor density* of the electromagnetic field. *The existence of all of these quantities is completely linked with the dimension number 4. In the first place, the interpretation of physical phenomena that is advocated here gives us reasonable grounds for recognizing that the world is four-dimensional*.

$$\Delta\varphi = F_{ik}\, dx_i\, \delta x_k$$

is the "trace" of any transformation:

$$\Delta\mathfrak{P} = \mathfrak{P}_{ik}\, dx_i\, \delta x_k$$

that the curvature defines. From the form of $\mathfrak{S}$, we can define the transformation:

$$\tfrac{1}{4}\sqrt{-g}\,\mathfrak{P}_{ik}\,\mathfrak{P}^{ik}$$

(in which multiplication means concatenation). The trace of $\mathfrak{M}$ itself is a scalar density that is uniform near $\mathfrak{S}$.

III. The quantity of action and its variation.

We now return to pure mathematics. If $\mathfrak{W}$ is any scalar density that is uniquely defined by the state of the ether (independently of the coordinate system) then we shall (from the example of Maxwellian theory) refer to the integral invariant $\int \mathfrak{W}\, dx$ as the *quantity of action* that is contained in the domain of integration. Under an arbitrary variation of the state of the ether of the type that was described, we set:

$$(20) \qquad \delta\int \mathfrak{W}\, dx = \int (\mathfrak{w}^i\, \delta\varphi_i + \mathfrak{W}^{ik}\, \delta g_{ik})\, dx \qquad (\mathfrak{W}^{ki} = \mathfrak{W}^{ik}).$$

The $\mathfrak{w}^i$ are the components of a contravariant tensor density of rank 1 and the $\mathfrak{W}_i^k$ are those of a mixed tensor density of rank 2. *There exist n + 1 identities between these "Lagrangian derivatives" of the action function* $\mathfrak{W}$ *that arise from the invariance of the quantity of action.* First, one must have invariance when one replaces g_{ik} with λg_{ik} and, at the same time, φ with $\varphi_i - \frac{1}{\lambda}\frac{\partial\lambda}{\partial x_i}$; in this, we take λ to be a quantity $1 + \delta\lambda$ that deviates from 1 by an infinitely small amount then (20) must vanish for:

$$\delta g_{ik} = \delta g_{ik}\ \delta\lambda\,, \qquad \delta\varphi_i = -\frac{\partial(\delta\lambda)}{\partial x_i}\,.$$

This yields the first $n + 1$ identities:

$$\boxed{\frac{\partial \mathfrak{w}^i}{\partial x^i} + \tfrac{1}{2}\mathfrak{W}_i^i = 0.} \tag{21}$$

Secondly, we employ the invariance of the quantity of action under coordinate transformations by an infinitesimal deformation of the ether [14]). We displace the point $P = (x_i)$ of the ether to the point $\bar{P} = (\bar{x}_i)$. However, the displacement $P\bar{P}$ must vanish on the boundary of the region in question in such a way that this displaced region is still filled with the same quantum of ether. In a second coordinate system, we ascribe the coordinates x_i to point $\bar{P}$. If we displace the ether without changing its state then the metric at the point will be defined by:

$$g_{ik}\,(x)\,dx_i\,dx_k \qquad \text{and} \qquad \varphi_i\,(x)\,dx_i$$

after displacement in these coordinates, or, when we transform back to the old coordinates, by:

$$\bar{g}_{ik}(\bar{x})d\bar{x}_i d\bar{x}_k \quad \text{and} \quad \bar{\varphi}_i(\bar{x})d\bar{x}_i\,;$$

hence, at the point P by:

$$\bar{g}_{ik}(x)dx_i dx_k \quad \text{and} \quad \bar{\varphi}_i(x)dx_i\,.$$

For the state of the ether thus obtained, the quantity of action, due to its invariance, must possess the same values as it originally did. If this deformation is infinitesimal $\bar{x}_i = x_i + \delta x_i$ then this yields:

$$\delta g_{ik} = \bar{g}_{ik}(x) - g_{ik}\,(x) \quad = -\left\{ g_{ir}\,\frac{\partial(\delta x_r)}{\partial x_k} + g_{kr}\,\frac{\partial(\delta x_r)}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_r}\,\delta x_r \right\}$$

$$\delta\varphi_i = \bar{\varphi}_i(x) - \varphi_i\,(x) \quad = -\left\{ \varphi_r\,\frac{\partial(\delta x_r)}{\partial x_k} + \frac{\partial \varphi_i}{\partial x_r}\,\delta x_r \right\}.$$

(20) must vanish for these variations. If one ignores the derivatives of the components δx_i of the shift by partial integration then one obtains the equations:

$$\left\{ \frac{\partial \mathfrak{W}_i^k}{\partial x_k} - \frac{1}{2}\frac{\partial g_{rs}}{\partial x_i}\mathfrak{W}^{rs} \right\} + \left\{ \frac{\partial(\mathfrak{w}^k \varphi_i)}{\partial x_k} - \mathfrak{w}_k\,\frac{\partial \varphi_k}{\partial x_i} \right\} = 0\,.$$

If we use (21) then we find that the second of the two terms in curly brackets is:

[14]) Weyl, Ann. d. Physik Bd 54 (1917), pp. 117 (§ 2); F. Klein, Nachr. d. K. Gesellsch. d. Wissensch. zu Göttingen, math.-physik. Kl. Sitzung v. 25 Jan. 1918.

$$- g_{rs}\, \varphi_i \cdot \mathfrak{W}^{rs} + F_{ik}\, \mathfrak{w}^k .$$

Now, one has:

$$\frac{1}{2}\left(\frac{\partial g_{rs}}{\partial x_l} + g_{rs}\varphi_i\right)\mathfrak{W}^{rs} = \tfrac{1}{2}(\Gamma_{r,\,is} + \Gamma_{s,\,ir})\,\mathfrak{W}^{rs} ,$$

due to the symmetry of $\mathfrak{W}^{rs}$:

$$= \Gamma_{r,\,is}\,\mathfrak{W}^{rs} = \Gamma^r_{is}\,\mathfrak{W}^s_r .$$

Thus, the equations take the final form, in which their invariant character is evident:

(22)
$$\boxed{\left(\frac{\partial \mathfrak{W}^k_i}{\partial x_k} - \Gamma^r_{is}\mathfrak{W}^s_r\right) + F_{ik}\mathfrak{w}^k = 0.}$$

IV. Transition to physics.

In a metric manifold whose ether is found in a state of extremal action, such that in any region of the world that is subjected to arbitrary infinitesimal variations of φ_i and g_{ik} that vanish on the boundary one has:

(23)
$$\delta\int \mathfrak{W}\, dx = 0,$$

one has the Lagrangian equations:

(24)
$$\mathfrak{w}^i = 0 , \qquad \mathfrak{W}^k_i = 0 .$$

In physics, the first equation is referred to as the *law of electromagnetism* and the second one as the *law of gravitation.* As in mechanics, physics also states a Hamiltonian principle [15]): *The real world is such that its ether is found in a state of extremal action.* We know the laws of Nature, which are summarized by Hamilton's principle (23), that govern it when we know how the action density $\mathfrak{W}$ depends upon the state of the ether. Equations (24) are independent of each other, but five (n = 4) identities (21), (22) exist between them. In fact, the quantities g_{ik}, φ_i can be determined by the law (24) only to the extent that one can freely transform from a reference system to any another arbitrary system; however, such a transition depends upon five arbitrary functions. The vanishing of the divergence $\dfrac{\partial \mathfrak{w}^i}{\partial x_i}$, which is defined by the left-hand side of the electromagnetic equation, is therefore a consequence of the law of gravitation, and conversely, the vanishing of the divergence:

$$\frac{\partial \mathfrak{W}^k_i}{\partial x_k} - \Gamma^r_{is}\mathfrak{W}^s_r$$

[15]) On this, cf., G. Mie, Annalen der Physik, Bd. 37, 39, 40 (1912/13), or the representation of Mie's theory in RZM § 25; D. Hilbert, Die Grundlagen der Physik (1. Mitteilung), Nachr. d. K. Gesellsch. d. Wissensch. zu Göttingen, Sitzung of 20 Nov. 1915.

is a result of the law of electromagnetism. These five identities are closely connected with the so-called *conservation laws*, namely, the (one-component) law of the conservation of electricity and the (four-component) law of the energy-momentum principle. They teach us: the conservation law (upon whose validity mechanics rests) follows in two ways from the electromagnetic equations, as well as the gravitational equations; one would thus like to refer to it as the simultaneous validity of both groups of laws.

The only Ansatz for the action density in a (3 + 1)-dimensional world that one must reasonably consider is the following one:

$$\mathfrak{W} = \mathfrak{M} + \alpha\mathfrak{S},$$

in which α is a numerical constant, and the meaning of $\mathfrak{M}$ and $\mathfrak{S}$ is to be taken from part II of this section. Depending on the scope of that arena, one sees which of them is allowed by our theory of the laws of the world. In fact, as a first approximation, by restricting to the linear terms, Hamilton's principle gives the Maxwellian law of the electromagnetic field and the Newtonian law of gravitation. Thus, since the quantity of action is a pure number, there arises the possibility of a *quantum of action*, whose existence is regarded by the contemporary physics as the fundamental atomic structure of the cosmos.

Here, we shall not go any further into the physical implications of the theory, which only treats the systematic development of pure infinitesimal geometry and its associated analysis of tensors and tensor densities. Once more, we emphasize the points at which it departs from the usual theory. They are: the step-wise construction in the three levels of topology, affine geometry, and metric geometry, the liberation of the latter from one of the global-geometric inconsistencies that has stuck to it since its Riemannian conception, and the extension of the theory of tensors (intensities) to its opposite, the theory of tensor densities (or quantities).

(Received on 8 June 1918.)