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Intuitive aspects of the theory of relativity

I. Linear coordinates and g_{ik} coefficients in the special theory of relativity.

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In order to clarify the physically-intuitive meaning of general-relativistic coordinates and the individual g_{ik} coefficients in general relativity, it is preferable to introduce general linear coordinates in place of **Galilean** ones in special relativity. In the present treatise, that will be done in an elementary way, in which the intuitive difference between proper and improper relativistic coordinate will emerge clearly. In numbers **15**, **16**, **22**, and **23**, the properties of two interesting kinds of improper relativistic coordinates will be discussed.

A simple Gedanken experiment for the sequential measurement of the g_{ik} coefficients will be given. First, one must measure the $\sqrt{-g_{ik}}$ with a "rest" chronometer and ascertain the components a_α of the "asymmetry vector" from "light velocity measurements" in two opposite directions. One then measures the $\gamma_{\alpha\beta}$ coefficients of the spatial fundamental form by means yardsticks "at rest" and ultimately calculates the remaining g_{ik} coefficients, which are expressed simply in terms of the remaining quantities.

In that way, one can also do without the measurement of the speed of light completely when one establishes the asymmetry vector from "dynamical experiments" (numbers **10** and **26**).

1. – A large part of the difficulties that an intuitively-minded physicist will encounter in the study of relativity theory is probably based upon the fact that the discipline was developed and expanded mainly by mathematicians. Hence, the intuitive treatment of many details often suffered in favor of the abstract generality of the representation, and in particular, a thorough analysis of the connection between the symbols used and actual measurements has been lacking up to today. The elimination of that deficiency would demand a new textbook on relativity that would seem to deviate from the current ones. In this treatise, I will only discuss some questions that would be connected with such a thing.

2. – Ordinarily, the fundamental metric tensor g_{ik} is first introduced in general relativity. However, in order to recognize the physical meaning ([1]) of the individual g_{ik}

([1]) It is obvious that here we use the expression "physical meaning" in the older, well-established sense of "measurable," "observable," "intuitive," and not, like so many relativity theorists, in connection with the use of the expression "geometric meaning" in geometry; hence, "absolute" or "invariant" mean "independent of the choice of coordinates" (relative to a group of coordinate transformations that is often not given explicitly). In the latter sense, concepts like velocity, acceleration, energy, and the like would have no "physical meaning"!

coefficients, it is advisable to already introduce that tensor in special relativity. That will afford one an excellent opportunity to represent special relativity in general linear coordinates in such a way that one will avoid the "main pedagogical failure" of the present manner of representation in relativity, and at the same time, alter the meaning of the basic coordinates, in principle, when one goes from the special theory to the general one. In order to include this fundamental difference in the notation, as well, in what follows, we will denote the **Galilean** coordinates of special theory with large symbols and the general **Einsteinian** space-time coordinates (relativistic coordinates) with small ones.

From the physically-intuitive standpoint, we must distinguish three kinds of g_{ik} coefficients ([1]), according to whether both of the indices, one of them, or none of them has a temporal character, resp. The coefficients $g_{4\alpha}$ will pertain to our remarks especially, since they have no analogue in classical physics. However, we will see [and this seldom emerges as sufficiently clear (*) ([2])] that with the general ["non-orthochronous" ([3])] coordinates, the coefficients $g_{\alpha\beta}$ with two spatial indices also have a different physical meaning from the corresponding coefficients in geometry (surface theory), which we will denote by $\gamma_{\alpha\beta}$ in order to distinguish them ([4]).

Above all, we would like to remark here that the analogy between the $\gamma_{\alpha\beta}$ coefficients and the g_{ik} coefficients has only a formal nature, and it must be regarded as completely inappropriate from a pedagogical standpoint, as well as from an epistemological one, when one explains the geometrically-intuitive meaning of the $\gamma_{\alpha\beta}$ coefficients in surface theory, and then, with no further assumptions, when one goes over to the g_{ik} coefficients of **Einstein**'s theory (*). Indeed, the measurement of length in three-dimensional space and the measurement of the length of an interval in the four-dimensional space-time continuum are two entirely different processes. That difference emerges most clearly for the $g_{4\alpha}$ coefficients, whose physically-intuitive meaning has nothing to do with the cosines of angles.

3. In the interests of our plan of giving relativity theory the most intuitive representation possible, it is probably advisable to always use all geometric expressions with their original three-dimensional meanings and to provide the corresponding four-dimensional concepts with derived names. In what follows, we would always like to understand a "point" to mean a space-time point then, and we will always regard a vector

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([1]) The indices *i*, *k* shall assume the values 1, 2, 3, 4, while α, β will assume only the values 1, 2, 3.

([2]) Asterisks (with no explicit root symbol) shall direct the attention of the reader to points that often are represented unclearly or completely falsely (even by the most recognized theoreticians of relativity).

([3]) Since the orthogonality of a time-like and a space-like direction in the **Minkowski** diagram has nothing to do with orthogonality in space, intuitively, I propose to call coordinates in which the time axis is orthogonal in the **Minkowski** sense to the spatial axes – so, ones for which $g_{4\alpha} = 0$ – "orthochronous" coordinates. As we will see below, the orthochronicity of a coordinate system is a property of the "mutual calibration" of coordinate chronoscopes (number **18**).

([4]) We then set $s^2 = \sum_{i,k} g_{ik}\, x_i\, x_k$ for the fundamental metric form in the four-dimensional space-time continuum – i.e., the square of the "interval" – while we set $\sigma^2 = \sum_{\alpha,\beta} \gamma_{\alpha\beta}\, x_\alpha\, x_\beta$ for the square of the spatial distance.

quantity in the sense of classical physics, etc. We can call the analogous four-dimensional structures space-time points (event-points), four-vectors, hypersurfaces, etc.

Obviously, we will also use the word "geometry" in the sense of "practical geometry," or even better, "physical geometry." We accordingly understand "point" to be the limit of a body that becomes ever smaller, and not an abstract "thing," as in "mathematical geometry," etc.

I.

4. – We now place ourselves upon special relativity as a base, and we shall assume that its usual general form of representation is known ([1]). We will call the coordinates that will be used in it **Galilean**, and we shall denote them with large symbols X, Y, Z, T. Those coordinates (or more precisely, their differences) have an immediate physical meaning: They can be considered to be the results of well-defined measurements with normal-unit yardsticks and chronometers ([2]).

We shall now choose four arbitrary, real, single-valued, continuously-differentiable functions of the **Galilean** coordinates and set:

$$x_i = f_i\,(X, Y, Z, T). \tag{1}$$

The single restriction that we shall provisionally impose upon the functions f_i consists of the demand that equations (1) shall be uniquely soluble in the entire domain under consideration. We will generally call the numbers x_1, x_2, x_3, x_4 ([3]) *Einsteinian coordinates*. They are quadruples of numbers that are that are associated with event-points in a manner that is one-to-one (in a certain space-time domain), continuous, and "smooth" ([*]), but otherwise arbitrary. Other than that, they have no immediate physical meaning. They can be first introduced into physical equations in connection with the coefficients of the fundamental metric tensor.

5. – Here, we have defined the **Einsteinian** coordinates by starting with **Galilean** ones. We shall not go into the question here of whether they can be "given" in a more general way, but only remark that the "given" of the coordinates in the universe that surrounds us will, in the final analysis, always lead back (directly or indirectly) to

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([1]) Mind you, we will not be dealing here with a new foundation for relativity theory in general linear coordinates without the use of **Galilean** coordinates then, even if such a foundation might be possible, and would perhaps even be desirable as a better explanation for some of the main questions.

([2]) Since the word "clock" is used in various senses in relativity theory, I propose to call devices that do nothing but show the passage of time "chronoscopes," while the ones that are intended for the "measurement" of time will be referred to as "chronometers." A proper relativistic coordinate system will then be described intuitively as a "reference observer (Ger. *Bezugsmolluske* = "reference mollusk") with embedded chronoscopes." Whether or not the coordinate-clocks of the linear systems "tick uniformly" and differ from the chronometers of **Galilean** systems only by the "rate of ticking" (and their mutual calibration), we will call them chronoscopes here, since the consideration of linear coordinates is typically regarded as the first step towards a better understanding of the general-relativistic coordinates.

([3]) We will also often set $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ct$. Cf., footnote 2, pp. 5.

"individual instructions ([1])" that pertain to certain material objects. However, those individual instructions have no fundamental connection with the individual instructions of a normal-unit yardstick or a normal clock (see below, number **9**).

6. – The general **Einsteinian** coordinates have so little intuitive meaning that we have no right to distinguish one of those coordinates as a time coordinate and to regard the other ones as space coordinates. However, they will seldom be used with that degree of generality.

If it is possible to associate a "reference observer with embedded chronoscopes" with one of the coordinates x_i then we would like to call the coordinates *reference observer coordinates;* we will choose the coordinate that is shown by the chronoscopes to be the x_4 coordinate. It is only when the "mutual calibration" of the chronoscopes is subject to certain further restricting conditions that we can call the x_4 coordinate a *proper relativistic coordinate* ([*]). We will discuss the difference between these three kinds of coordinates in more detail in Part II in terms of simple examples, in order to formulate them in general after that in Part III.

7. – All three of the coordinate types that we just enumerated can also appear in special relativity as *linear coordinates* ([2]). For our purposes, it will suffice to restrict our consideration to those coordinates. From now on, we will then assume that the functions in (1) are linear. As is known, the motion of a freely-moving mass point and any light signal in a vacuum are expressed by means of linear equations on the basis of such coordinates. We will not go further here into the question of the extent to which, or in what sense, that property is characteristic of those coordinates. Intuitively, one can refer to those coordinates as spatially-affine "unaccelerated" relativistic coordinates.

Hence, instead of (1), we can now write ([3]):

$$\left.\begin{aligned} X &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4, \\ Y &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + A_{24}x_4, \\ Z &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + A_{34}x_4, \\ cT &= A_{41}x_1 + A_{42}x_2 + A_{43}x_3 + A_{44}x_4. \end{aligned}\right\} \tag{L}$$

To simplify, we have already assumed that the functions are not only linear, but also homogeneous, which can always be arranged by a trivial shift of the coordinate origin. We shall temporarily assume that the constants A_{ik} define a non-vanishing determinant.

([1]) Cf., **H. Weyl**, *Raum, Zeit, Materie*, 5th ed., pp. 8.
([2]) Cf., **H. Weyl**, *loc. cit.*, 4th ed., pp. 160.
([3]) The notations are chosen such that the A_{ik} can be considered to be dimensionless constant. Cf., footnote 2, pp. 5.

II.

8. – In order for the physically-intuitive viewpoint to emerge better, we would first like to discuss some special cases of the transformation (L) in as elementary a way as possible, and only then go on to a more general formulation.

We then start from an arbitrarily-chosen, but fixed, **Galilean** system, which we would like to call G, and introduce new linear coordinate systems by means of transformation equations of type (L). We will refer to the new coordinate systems with the same symbols that we use for the transformation equations that lead to them.

9. We can already make the same remarks from our viewpoint in regard to the simplest of all transformation equations:

$$X = Ax\,. \tag{L_z}$$

At first, it shall be expressly shown that this equation [as well as equations (L), in general] should be regarded as the transformation of the coordinates *with no change in the physical unit of length.* Hence, only a new notation for the points on the x-axis will be introduced by that equation. The physical unit of length is an "individually designated" body ([1]); e.g., the normal meter of platinum-iridium that is preserved in Sèvres, which has largely nothing to do with the enumeration of the points along the x-axis ([2]). It first becomes meaningful for the measurement of lengths and the determination of the $\gamma_{\alpha\beta}$ and g_{ik} coefficients.

Nonetheless, coordinate transformations are often referred to as changes in the unit of length (along the x-axis) by mathematicians. In a similar sense, one often speaks of different length units in different directions. Should that manner of speaking be maintained, one must distinguish between "physical" and "mathematical" units of length. The mathematical unit of length would then be nothing but what the physicist intuitively calls the "unit scale along an axis." The length measurement would be regarded as a comparison of the physical unit of length with the mathematical one. If the physicist speaks roughly of a unit of length then we must, in any event, assume that he means a physical unit of length in the sense that was given above.

10. – As is known, it is practical to give the result of a measurement of length along the x-axis in the form:

$$\sigma^2 = \gamma_{11}\, x^2. \tag{2}$$

([1]) Cf., number **5**.

([2]) For that reason, it would be most logical to consider the x_i as dimensionless quantities. However, as is known, dimensions are only agreements. They should be defined as practically as possible. That requirement probably best corresponds to endowing all four coordinates x_i with the dimension of length. The equation $x_4 = ct$ then keeps its usual sense, and for the concrete interpretation of special gravitational fields, such as, e.g., static or stationary fields, one can, with no further assumptions, identify x_1, x_2, x_3 with certain lengths and x_4 / c with certain time durations. In the final analysis, our way of establishing dimensions rests upon the agreement that under any change in the unit of length, the enumeration of the space-time points should also suffer a corresponding change.

One can briefly explain the physically-geometric meaning of the γ_{11} coefficients as follows: $\sqrt{\gamma_{11}}$ is the length (in cm) of a scale unit ($\Delta x = 1$) along the x-axis. In that, we understand length to mean the result of a measurement that is performed with a normal yardstick.

In surface theory, as well as in "true" geometry (in which only "spatial" coordinates come under consideration), above all, the state of motion of the yardstick is not mentioned at all, so one can probably think of the measurement of length in physical geometry as nothing but something that is performed by means of yardsticks "at rest." First of all, "at rest" must mean "at rest relative to a **Galilean** system." Special complications will arise for an "accelerated" reference system (i.e., in the absence of gravitational fields) that shall be discussed in the second article. However, in the second place – as we shall soon see – even for linear systems, the $\gamma_{\alpha\beta}$ coefficients that are measured by means of yardsticks that are "relatively at rest" (i.e., "co-moving") are generally not equal to the corresponding $g_{\alpha\beta}$ coefficients (*).

11. – With those remarks, there is no longer anything that prevents us from introducing the differential form:

$$d\sigma^2 = \sum_{\alpha,\beta} \gamma_{\alpha\beta}\, dx_\alpha\, dx_\beta\,, \tag{3}$$

which represents the square of the spatial separation, or explaining the geometric meaning of the individual $\gamma_{\alpha\beta}$ coefficients. As is known, the $\gamma_{\alpha\alpha}$ correspond to the γ_{11} in (2); for $\alpha \neq \beta$, one has $\gamma_{\alpha\beta} = \sqrt{\gamma_{\alpha\alpha}}\sqrt{\gamma_{\beta\beta}}\cos(\alpha, \beta)$, where (α, β) means the angle between the x_α-axis and the x_β-axis. However, to once more stress this fact, only the $\gamma_{\alpha\beta}$ coefficients actually affect the usual general manner of representation for the geometrically-intuitive meaning of the "g_{ik} coefficients," and thus, neither the $g_{\alpha\beta}$ for non-orthochronous coordinates nor the $g_{4\alpha}$ and g_{44} coefficients (cf., nos. **2** and **10**).

12. – Similar arguments relate to the transformation:

$$T = Bt. \tag{L_t}$$

For a clock, it corresponds to a new calibration that changes the rate of ticking by a constant ratio. However, if we imagine two neighboring clocks – e.g., a chronometer in the system G and a neighboring L-chronoscope – then B will give the ratio of the rates at which those two clocks tick. The time T that is shown by the chronometer will also be called "proper time" τ for that chronometer, as well as for the "comoving" chronoscope. If we set $-B^2 = g_{44}$ then we can then write:

$$ds^2 = -c^2\, dT^2 = -c^2\, dt^2 = -g_{44}\,(c\, dt)^2 = -g_{44}\,(dx^4)^2. \tag{4}$$

That formula can be regarded as a special case of the general formula for ds^2, as long as the relativistic coordinate systems to which the two clocks belong are mutually at rest.

Namely, the equations $dx = 0 = dy = dz$ will then follow from the equations $dX = 0 = dY = dZ$, and conversely.

We then see that we are dealing with just simple ratios here, since a comoving chronometer will always measure the g_{44} coefficient (independently of whether the coordinates are orthochronous or non-orthochronous), and indeed $\sqrt{-g_{44}}$ gives *the time that is shown by a comoving chronometer (in sec) that corresponds to the time duration* ($\Delta t = 1$) *of a tick of the system chronoscope.* One can also say that this root gives the ratio by which the coordinate chronoscope ticks slower than the comoving normal chronometer ([1]).

13. – We now go on to a discussion of linear transformations that act upon a spatial coordinate = e.g., x – and a temporal coordinate t at the same time. Speaking intuitively, we then consider a "line world"; i.e., the "events along the x-axis" or "the static existence" in a two-dimensional **Minkowski** (xt)-diagram (whose use we would like to avoid here as much as possible, however).

We would like to put the general linear homogeneous transformation that leads us from G to the coordinate system x, t in the following form:

$$\left.\begin{aligned} X &= A(x - wt), \\ T &= B(t - ax). \end{aligned}\right\} \qquad (L_2)$$

It is specialized only insofar as the non-vanishing of the determinant AB $(1 - aw)$ implies the non-vanishing of A_{11} and A_{44} . We will see directly that one must restrict the absolute values of the coefficients w and a as above if one would like to interpret x as the spatial coordinate and t as the time coordinate ([2]).

14. First, we consider the following especially simple special case of (L_2):

$$X = x - wt, \qquad T = t \quad (w > 0). \qquad (L_2')$$

That has the form of the well-known **Galilei** transformation of classical kinematics, although the interpretation that we shall now give it is completely different. Namely, we consider (L_2') to be the defining equations of new coordinates, without postulating the fundamental equivalence of the new reference system. In classical kinematics, one regards (L_2') as:

([1]) Since only linear relativistic coordinates will be considered in this article, comoving chronometers will also be moving "without acceleration" or "falling together." The difficulties that occur for time and length measurements in gravitational fields shall be discussed in particular in the second article.

([2]) On the grounds of convenience, one always takes $B > 0$, and usually $A > 0$, as well. We shall also do that, and in addition, we will consider w and a to be positive quantities in any event for concrete examples.

1. A reference system that moves with the constant velocity w in the direction of the X-axis relative to the **Galilean** system G.

2. A **Galilean** system.

Here, in the special theory of relativity, 2 is false, as we said. However, 1 can only be asserted when:

$$| w | < c, \tag{5}$$

since no material point can move faster than or equally fast as light relative to a **Galilean** system.

15. – If one has:

$$w > c \tag{6}$$

then one can still always employ the number-pair x, t in order to characterize the individual event-points in our line world, but the event x = const. can no longer correspond to successive states of the same mass point. There can then be no reference observer in the **Einsteinian** sense. However, if one would like to further establish that the x-numbers can be regarded as characteristic of the "spatial points" then one would come to some entirely remarkable results. Bodies at rest in L_2' cannot exist at all, and the light signals would only propagate in one direction, but with two different velocities $w + c$ and $w - c$, along the x-axis.

We then arrive at the simplest example of an "improper relativistic coordinate system without a reference observer" whose use (like that of all improper relativistic coordinate systems, moreover) is possible, in principle, but quite impractical. Both axes would be space-like in a **Minkowski** diagram.

16. – To complete the picture, we would like to briefly outline how things work in the (3+1)-dimensional case. We consider a linear reference system that arises from G by the transformation (L_2') when one adds the equations $Y = y$, $Z = z$. We assume that a flash of light is generated at any point P in the system at any time $t = T$. After a certain time Δt, one will find all of the emitted light signals on the surface of an ellipsoid (which is just a sphere in our simple special case). However, when $w > c$, we will find the emission point outside of that ellipsoid! It will be found inside of it only for proper coordinates when $w < c$, and at the center of the ellipsoid, only for orthochronous coordinates, and thus, only for $w = 0$ for the example that is considered now.

We will soon see that one comes to more remarkable results for other types of improper relativistic coordinates when one would like to establish the interpretation of x_1, x_2, x_3 as spatial coordinate and t as the time coordinate (see below, number **22**).

17. – We now return to the proper relativistic coordinates L_2' and assume that the condition (5) is valid. If we define the "velocity" in the system L_2' to be the derivative of the *x*-coordinate with respect to the *t*-coordinate ([1]) then its absolute value will be greater or smaller than the corresponding velocity in the Galilean system *G*. The (scalar) "speeds of light" in both directions ([2]), viz., c_+ and c_- , will be correspondingly equal to $c + w$ and $c - w$, resp.

We can write the square of the interval in L_2' as follows:

$$\left.\begin{aligned} s^2 = X^2 - c^2T^2 &= x^2 - 2\frac{w}{c}x(ct) - \left(1-\frac{w^2}{c^2}\right)(ct)^2 \\ &= g_{11}x^2 + 2g_{41}x(ct) + g_{44}(ct)^2. \end{aligned}\right\} \tag{7}$$

That gives us:

$$g_{41} = \frac{c_- - c_+}{2c}, \tag{8}$$

which is an equation that clarifies the "physical sense" of that coefficient in the case that was just considered [cf., number **24**, equation (19)].

One can say the following about the physical meaning of the other two coefficients $g_{11} = 1$ and $g_{44} = -\left(1-\frac{w^2}{c^2}\right)$: Although $t = T$, a chronometer that is at rest in L_2' does not show the "time" *t*. Since it moves in the "synchronous" Galilean system *G*, in which $T = t$, it will move slower with the ratio of $\sqrt{1-\frac{w^2}{c^2}}$: 1. From the general rule (number **12**), however, it measures the coefficient g_{44} , and $\sqrt{-g_{44}} = \sqrt{1-\frac{w^2}{c^2}}$.

By contrast, a normal unit yardstick that is at rest in L_2' does not directly measure the corresponding $\sqrt{g_{11}}$, but only $\sqrt{\gamma_{11}}$. In our case, we see that best when we imagine that this yardstick moves relative to *G* with the velocity *w*. Therefore, due to the Lorentz contraction, the contraction of its ends will correspondingly be $\Delta X = \sqrt{1-\frac{w^2}{c^2}}$. However, since $\Delta x = \Delta X$, that yardstick, which rests along the *x*-axis, will cover less than a unit interval, and indeed $\sqrt{\gamma_{11}} = 1\Big/\sqrt{1-\frac{w^2}{c^2}}$, while $\sqrt{g_{11}} = 1$ ([3]). One clarifies the study of this important fact most easily by the difference between g_{11} and γ_{11} in non-orthogonal coordinates in connection with the transformation (L_2^0) (see number **21**).

([1]) The word "velocity" is used with various meanings ([*]) in the theory of relativity.
([2]) Cf., number **24**, last paragraph.
([3]) The given value of $\sqrt{\gamma_{11}}$ follows from (7), (27), and (32) by calculation.

Here, we can point out that one can also measure $\sqrt{g_{11}}$ directly with a normal unit yardstick, but we must give it a velocity w in the direction of the x-axis in order to do that. That yardstick is then at rest in G, so its endpoints will correspondingly give ΔX = 1, and since the times that are given in G and L_2' agree, we can write $s^2 = (\Delta X)^2 = (\Delta x)^2$, and it will follow from this that $g_{11} = 1$, as it should be, from (7).

18. – We direct our attention to another simple special case of (L_2), and indeed to the transformation:

$$X = x, \quad T = t - ax \qquad (a > 0) , \tag{L_2^0}$$

which is especially interesting, insofar as it first played a role with the appearance of the theory of relativity, although it had also been used in classical physics, in principle. One can characterize that transformation in a physically-intuitive way as the "mutual calibration" of the coordinate chronoscopes without changing their state of motion and its x-direction. We leave one of those chronoscopes – e.g., the one that is found at $X = x = 0$ – at rest, while we shift another one – $X = x = 1$ – while all of the other ones are proportional to their distance from the first chronoscope. One can then call a the *coefficient of the mutual calibration of the coordinate chronoscopes*, or also (see number **20**) the *asymmetry coefficients* of the relativistic coordinate system considered.

It is interesting that all velocities (with the exception of the velocity 0) experience a change by the mutual calibration of the coordinate chronoscopes, even though neither the "unit of length along the x-axis" nor the ticking rate of the coordinate chronoscopes will change, and the two coordinate systems G and L_2^0 will be found in a state of relative rest. In this, we understand the word "velocity" to be the derivative of the x-coordinate with respect to the t-coordinate, as in number **17** above.

19. – We now assume that:

$$|\, a \,| < \frac{1}{c}, \tag{9}$$

in which L_2^0 is associated with a proper relativistic coordinate system. The relation:

$$v = \frac{V}{1 \pm aV} \tag{10}$$

exists between the absolute value of the velocity V in the Galilean system G and that of v in L_2^0, where the plus or minus sign in the denominator is taken according to whether the velocity points in the direction of the positive or negative x-axis, resp. In particular, the speeds of light in both directions are:

$$c_+ = \frac{c}{1+ac}, \qquad c_- = \frac{c}{1-ac}. \tag{11}$$

In this, we see the reason that we had to set $|a|c < 1$ above. The complications that arise when one drops that inequality will be discussed in number **22**.

20. – From (11), we calculate:

$$a = \frac{c_- - c_+}{2c_-c_+}. \tag{12}$$

For that reason, as we remarked before, we can call a the *asymmetry coefficient* of the system L_2^0, and all the more so, since a similar formula:

$$a = \frac{v_- - v_+}{2v_-v_+} \tag{13}$$

will be true for any two "dynamically equivalent" velocities v_+ and v_- that correspond to two equal and opposite velocities in the comoving Galilean system G, as one easily calculates from (10). If we then separate – e.g., two identical mass-points that are at rest in L_2^0 – by means of a symmetric agency that acts between them – e.g., a homogeneously-compressed spring that is cut in the middle – and measure its (scalar) velocity v_+ and v_- in this reference system then we will have performed a "dynamical experiment" for the measurement of the asymmetry coefficient a.

If one mutually adjusts the coordinate chronoscopes in such a way that light moves with the same velocity in both x directions then that will also imply the same velocities in both directions for all "dynamical symmetry experiments." One can introduce that fact as one of the basic postulates of the theory of relativity (cf., footnote 1, pp. 3).

21. – We can summarize the physically-intuitive interpretation for the formula for the square of the interval in L_2^0:

$$s^2 = (1 - c^2 a^2)\, x^2 + 2ca \cdot x\,(ct) - (ct)^2 \tag{14}$$

as follows: From the general rule of number **12**, a normal clock that is at rest in L_2^0 measures $\sqrt{-g_{44}}$, but on the other hand, it also shows the "time coordinate," since in the comoving Galilean system G, in which $\Delta T = \Delta t$ (for $X = x =$ const.), is likewise at rest; hence, $g_{44} = -1$, as it should be, from (14).

A normal unit yardstick that is at rest in a linear reference system always has a "rest length" of one, regardless of the calibration of the coordinate chronoscope; the separation σ of its two endpoints is equal to one. However, the interval between the two event-points that correspond to the same time coordinate t is only equal to one when those events are simultaneous in the Galilean system in which the yardstick is at rest. Otherwise, a correction that corresponds to the term $-c^2 (\Delta T)^2$ would have to be subtracted from the expression for the square of the interval in **Galilean** coordinates. One can clarify the state of affairs in the two coordinate systems L_2^0 and G, which both have the same reference observer, in an elementary intuitive way. If we lay a yardstick at rest along the x-axis then it will cover a piece $\Delta X = 1 = \Delta x$ of that axis, and therefore $\gamma_{11} = 1$; hence, the g_{11} in (14) will be different from one, on the grounds that were given before ([1]).

From (12), the coefficient g_{44}, which is equal to ca in our case, is proportional to the difference of the speeds of light in both directions, as in the case of L_2'. The general formulas will be given in number **24**.

22. – If we now assume that:

$$a > \frac{1}{c} \tag{15}$$

then we will get the simplest example of an "improper relativistic coordinate system with a reference observer." Such a thing certainly exists in our case, since it is identical with the reference observer of the Galilean system G that is relatively at rest, except that the coordinate chronoscopes in L_2^0 are mutually calibrated so clumsily that one would arrive at entirely remarkable results when one would like to interpret the given of those chronometers as "time" in that reference system.

A certain velocity in G would then correspond to an infinite "velocity" in L_2^0, and (which is probably even harder to imagine) many velocities in G would correspond to opposite "velocities" in L_2^0, even though the two reference systems G and L_2^0 are at rest with respect to each other! In particular, the "velocities" of both light signals that emanate from a point would prove to be positive. However, one must not say that both signals propagate in the same direction L_2^0 (since they indeed move next to each other in the comoving system G), but rather, that one of them moves "into the past." The noteworthy sign of its "velocity" does not originate in the inversion of the dx in the numerator, but in the dt in the denominator. That will perhaps become clearer when one arranges a chronometer in such a way that it moves with a speed that is larger than $1 / a$ in the negative X-direction in G. Even though all of the coordinate chronoscopes in L_2^0 "run forwards," the evolution of this chronometer, when evaluated by the L_2^0 chronoscopes, will run backwards!

23. – We shall once more consider the corresponding (3+1)-dimensional part of an "improper relativistic coordinate system," as in number **16**, but this time "with a

(1) For $\Delta t = 0$, we get from (L_2^0) that $\Delta X = \Delta x$, $\Delta T = -a\,\Delta x$, so $s^2 = \Delta X^2 - c^2\,\Delta T^2 = (1 - c^2 a^2)\,\Delta x^2$, and from that, $g_{11} = 1 - c^2 a^2$.

reference observer," and repeat the Gedanken experiment that was described there. This time, the light signals that are sent from a point P at the "time" t to the "time" $t + \Delta t$ will define a hyperboloid of two sheets (with a focus at P) ([1]). The ellipsoid in no. **16** can be a hyperboloid here, since the "speed of light" is infinitely large in some directions. One also suggests that only one sheet of the hyperboloid (viz., the one that is closer to the point P) corresponds to events that are later than the light flash at P, and the other one, to events that are earlier. In order to consider all light signals that are emitted from P at "time" t, one must consider yet another "time point" with a smaller t. The two hyperboloids together would correspond to one wave front that is emitted from P and another one that contracts to it.

The difficulties in the description of the "propagation" of light in L_2^0 for $ac > 1$ are obviously connected with the fact that the t-coordinate parts company with the "main property" of time that for two causally-connected events, the effect will always correspond to a larger value of that coordinate than the cause.

24. – We would like to add some words about the general (L_2)-transformation, since from now on, that will be only one step towards the general (3+1)-dimensional case. We would now like to exhibit matter in such a way that an arbitrary, proper, linear, relativistic coordinate system L_2 is given in our line-world, by which, we can go to ∞^1 Galilean systems by means of linear transformations of type (L_2). Two of them are coupled to L_2 in an especially simple way, and they are, in fact:

1. The "comoving Galilean system" or "proper system of the system L_2" (which we would like to call G_2) that we obtain when we set w equal to zero in (L_2).

2. The "synchronous Galilean system" – viz., G' – that we likewise get by setting a equal to zero.

The values of the three remaining coefficients in (L_2) are then coupled uniquely with the three g_{ik} coefficients. We shall now give formulas that can be seen to be direct generalizations of the ones that were discussed already. One will preserve their validity in the general case, as well (number **26**), by replacing the x-axis with the x_α-axis.

For the square of the interval, we get:

$$s^2 = (\Delta X)^2 - c^2 (\Delta T)^2 = g_{11} (\Delta x)^2 + 2\, g_{41}\, \Delta x\, \Delta ct + g_{44}\, (\Delta ct)^2, \tag{16}$$

with

$$g_{11} = A^2 - B^2 (ca)^2, \qquad g_{41} = -A^2 \frac{w}{c} + B^2\, ca, \qquad -g_{44} = B^2 - A^2 \frac{w^2}{c^2}.$$

We calculate from this that for $a = 0$:

([1]) It is interesting to exhibit the state of affairs in a (2+1)-dimensional **Minkowski** diagram, or in the (1+1)-dimensional case, to represent it in rectangular X, T coordinates, or even better, in rectangular x, t coordinates.

$$w = \frac{c_+ - c_-}{2}, \tag{17}$$

and for $w = 0$:

$$a = \frac{c_- - c_+}{2c_-\, c_+} = \frac{v_- - v_+}{2v_-\, v_+} = \frac{1}{c}\frac{g_{41}}{-g_{44}}, \tag{18}$$

in which v_+ and v_- mean a pair of "dynamically-equivalent" velocities, as in number **20**.

When we set $s = 0$ in (16) and divide by $(\Delta t)^2$, we will get a quadratic equation for the speed of light in L_2. The two roots of that equation will have opposite signs in a proper relativistic coordinate system; we set them equal to c_+ and c_-. The positive quantities c_+ and c_- then refer to the scalar speeds of light in the two directions of the x-axis. We get:

$$g_{41} = g_{44}\, c\frac{c_+ - c_-}{2c_+ c_-} = g_{44}\, c\frac{v_+ - v_-}{2v_+ v_-} = g_{11}\frac{c_- - c_+}{2c}. \tag{19}$$

III.

25. – After that elementary discussion, we can now briefly address the general (3+1)-dimensional case. Equations (L) mediate the transition from the starting Galilean system G to the linear system L. In order for a reference observer to exist for it, the points that are at rest in L must have velocities in G that are smaller than c. We then obtain the "condition for the reference observer" from (L) directly:

$$A_{14}^2 + A_{24}^2 + A_{34}^2 < A_{44}^2, \tag{20}$$

or, by substituting this in the expression for s^2:

$$g_{44} < 0. \tag{21}$$

However, in order for L to be a proper relativistic coordinate system, its chronoscopes must be correspondingly "mutually calibrated" (*), and in fact, in such a way that no point that moves in G with a speed that is below or above that of light can pass the L-chronoscopes at constant t. It must then follow from $\Delta t = 0$ that:

$$\left(\frac{\Delta X}{\Delta T}\right)^2 + \left(\frac{\Delta Y}{\Delta T}\right)^2 + \left(\frac{\Delta Z}{\Delta T}\right)^2 < c^2, \tag{22}$$

which says nothing but the fact that the form s^2 must be positive-definite for $\Delta t = 0$. As is known, the conditions for this are:

$$g_{11} > 0, \qquad \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} > 0, \qquad |\, g_{\alpha\beta} \,| > 0. \tag{23}$$

The inequalities (21) and (23) together then express the condition for the *L*-coordinates to be proper relativistic coordinates, and at the same time, they also guarantee that s^2 will be a nondegenerate quadratic differential form with an index of inertia of 3. It is known that the general condition for this, independently of the type of coordinate system that was used as a basis, is that only one change of sign occurs in the sequence:

$$1, \quad g_{11}, \quad \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}, \quad |g_{\alpha\beta}|, \quad |g_{ik}|. \tag{24}$$

It is easy to see that it follows from (21) and (23) that $|g_{ik}| < 0$, but not conversely, so the first four quantities in (24) will be positive for, e.g., four space-like axes (i.e., improper relativistic coordinates with no reference observer), and the change of sign must result at the last step, but then $g_{44} > 1$.

26. – If we then assume that *L* is a proper linear relativistic coordinate system without observing how it would come about – as in number **24** (cf., number **24**). In order to briefly recapitulate the physical meaning of the g_{ik} coefficients in this coordinate system, we write the expression for s^2 in the following forms that emerge from each other by elementary algebraic conversions. In that, we distinguish between covariant and contravariant tensor components, and make use of **Einstein**'s summation rule (α, β from 1 to 3, *i*, *k* from 1 to 4):

$$s^2 = g_{ik}\,x^i\,x^k = g_{\alpha\beta}\,x^\alpha\,x^\beta + 2g_{4\alpha}\,x^\alpha\,(ct) + g_{44}\,(ct)^2, \tag{25}$$

$$= \gamma_{\alpha\beta}\,x^\alpha\,x^\beta + g_{44}\,(t - a_\alpha\,x^\alpha)^2 = \sigma^2 + g_{44}\,(t - a_\alpha\,x^\alpha)^2, \tag{25^0}$$

$$= g_{\alpha\beta}\,(x^\alpha - w^\alpha t)\,(x^\beta - w^\beta t) + \left(g_{44} - g_{\alpha\beta}\frac{w^\alpha w^\beta}{c^2}\right)(ct)^2, \tag{25^1}$$

in which:

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - c^2\,a_\alpha\,a_\beta, \tag{26}$$

$$a_\alpha = \frac{1}{c}\frac{g_{4\alpha}}{-g_{44}}, \tag{27}$$

$$w^\alpha = -\,c\,g^{\alpha\beta}\,g_{4\beta}. \tag{28}$$

Equations (25^0) and (25^1) can be interpreted as the way that one can go from *L* to a "comoving Galilean system" G^0 and a "synchronous Galilean system" G' (cf., number **24**) by means of linear coordinate transformations ([1]).

We can now summarize the Gedanken experiments that will lead to the measurement of the individual g_{ik} coefficients as follows: We first place a chronometer at rest in *L*; the time duration (in sec) of a tick of the coordinate chronoscope ($\Delta t = 1$) that one reads off of that chronometer will give us $\sqrt{-g_{44}}$ (cf., numbers **12**, **17**, and **21**). We then measure the speeds of light in the six directions of the positive and negative axes and obtain the

([1]) It is plausible that these two systems are defined up to an orthogonal space transformation.

components of the "asymmetry vector" from them by means of the formula (cf., numbers **20** and **24**):

$$a_\alpha = \frac{c_{\alpha-} - c_{\alpha+}}{2c_{\alpha-}c_{\alpha+}}. \tag{29}$$

We can also perform three "dynamical experiments" instead of them (numbers **20** and **24**) and use the formula:

$$a_\alpha = \frac{v_{\alpha-} - v_{\alpha+}}{2v_{\alpha-}v_{\alpha+}}. \tag{30}$$

We obtain the $g_{4\alpha}$ coefficients from g_{44} and a_α by using the formula [cf., (26) and numbers **21** and **24**]:

$$g_{4\alpha} = - g_{44}\, c\, a_\alpha. \tag{31}$$

We then measure the $\gamma_{\alpha\beta}$ coefficients of the fundamental spatial form σ^2 by means of yardsticks at rest in the known geometric way and obtain the $g_{\alpha\beta}$ coefficients from them using the formula [cf., (26), as well as numbers **21** and **24**]:

$$g_{\alpha\beta} = \gamma_{\alpha\beta} + g_{44}\, c^2\, a_\alpha a_\beta. \tag{32}$$

We can also measure the $g_{\alpha\beta}$ coefficients directly with yardsticks, but we must use yardsticks that move with the velocity w^α relative to L, instead of ones at rest (cf., number **17**, last section).

27. – Allow me to conclude our study with a quotation from **Felix Klein**'s wonderful lectures on the development of mathematics in the Nineteenth Century ([1]). As **Klein** said, one can "make the four-dimensional way of thinking true deductively from the outset" without having to "discuss the existing experiments and gradually reinterpret their originally three-dimensional conception," but immediately after that, to allude to the analogy with the Copernican and geocentric world-models, he added: "Admittedly, there still remains the half of astronomy that satisfies the Copernican picture only *in abstracto* and does not endeavor to go into the rigorous details of thinking through the geocentric observations precisely from that standpoint."

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([1]) Bd. II, pp. 75.