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Elementary electrodynamical laws

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Translated by D. H. Delphenich

1. – Foundations of the theory.

1. Foreword. – The recent theory of electrodynamics that is advocated by Maxwell, by its distinction between ether and matter in the interior of a body that is perceptible to the sense reverts to the opinions of the older schools to such a degree that the former distinction no longer exists. The *electrical particles* of the old theory are, in turn, justified. However, we have learned how to follow their interactions through the intermediary of the medium between them. Hence, the great problem was solved that was indeed formulated many times before Maxwell, although it resisted all attempts at solution. Moreover, Maxwell's contribution no longer seems to be an overthrow of the previous theories, as much as an advance in their natural order of development.

H. A. Lorentz was the first to successfully evaluate the distinction between ether and matter in **Maxwell**'s theory, and in that way he drew attention to the way that it approached the older theories from the outset, which then presents itself. In the belief that in the interests of science, not enough weight can be placed upon that, I would like to attempt to lay another brick towards the union of the old and new theories.

For the notations, I shall refer to my contribution to the *Festschrift für die Feier der Enthüllung des Gauss-Weber-Denkmals* in Göttingen, 1899 (¹). I shall also refer to it for a more detailed treatment.

2. – I shall use the terms *ether* and *matter* in a way that is entirely similar to that of **H. A.** Lorentz. Since I have already spoken of that on several occasions, fewer words will suffice here.

The optical behavior of streaming currents and similar phenomena show that the motion of matter that is perceptible to the senses is not itself carried along in the interior of light waves. As **H. A. Lorentz** showed, the aberration of light leads to the same conclusion. When we, with **Maxwell**, regard light as an electrodynamical process, we will be compelled to assume that there is a carrier of electrodynamical phenomena in the interior of matter that does not participate in the motions that are perceptible to the senses. *In order to clothe that fact in words conveniently, we*

^{(&}lt;sup>1</sup>) Published by **B. G. Teubner**, Leipzig.

shall distinguish between ether and matter. Speculating on their mutual relationship is not necessary for the immediate purpose of electrodynamics. Thus, we can postpone, for example, a discussion of whether we are dealing with different materials or the same material in different states. Matter and ether are only images that we see from our mortal standpoint in nature. The further advance in science of deciding what they correspond to in reality is yet to be achieved.

Whereas matter confronts us with a multitude of variations in its type, aggregation, and motion, for the representation of our experiences, the assumption will suffice that ether fills up all of the universe that is accessible to us with no noticeable gaps and no noticeable motions, and everywhere with the same extremely-simple properties. That is also true of the region that matter occupies, such that it seems to be completely saturated by the ether without displacing it noticeably.

A single constant will suffice to characterize the electrodynamical properties of the ether, namely, the speed of light *V* in the absence of matter. One must employ directed quantities, namely, vectors, in order to describe electrodynamical processes.

3. – We would like to start with *optics in free ether*. Very different physical meanings for vectors can come under consideration in that way. If *K* is one such vector then we can assume that the oscillation equation for one component K_v that is parallel to the arbitrary direction *v* is:

(1)
$$\frac{\partial^2 K_{\nu}}{\partial t^2} = V^2 \left(\frac{\partial^2 K_{\nu}}{\partial x^2} + \frac{\partial^2 K_{\nu}}{\partial y^2} + \frac{\partial^2 K_{\nu}}{\partial z^2} \right).$$

In that way, the following condition will express the transversality of the oscillations and the assumption that excitations are also possible:

(2)
$$\frac{\partial K_{\nu}}{\partial x} + \frac{\partial K_{\nu}}{\partial y} + \frac{\partial K_{\nu}}{\partial z} = 0$$

A certain second vector H can be associated with every chosen vector K that has a reciprocal relationship with it. We get H by the defining equations:

(3)
$$\begin{cases} \frac{\partial H_x}{\partial t} = V\left(\frac{\partial K_x}{\partial y} - \frac{\partial K_y}{\partial x}\right),\\ \frac{\partial H_y}{\partial t} = V\left(\frac{\partial K_x}{\partial z} - \frac{\partial K_z}{\partial x}\right),\\ \frac{\partial H_z}{\partial t} = V\left(\frac{\partial K_y}{\partial x} - \frac{\partial K_x}{\partial y}\right),\\ \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0. \end{cases}$$

Along with, (1) and (2), they give the analogue to (3):

(5)
$$\begin{cases} \frac{\partial K_x}{\partial t} = -V\left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right),\\ \frac{\partial K_y}{\partial t} = -V\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right),\\ \frac{\partial K_z}{\partial t} = -V\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right),\end{cases}$$

and as an analogue to (1):

(6)
$$\frac{\partial^2 H_{\nu}}{\partial t^2} = V^2 \left(\frac{\partial^2 H_{\nu}}{\partial x^2} + \frac{\partial^2 H_{\nu}}{\partial y^2} + \frac{\partial^2 H_{\nu}}{\partial z^2} \right).$$

Hertz derived the formal system (2), (3), (4), (5), as a replacement for (1) and (2) [(4) and (6), resp.] from **Maxwell**'s theory in 1884.

A third system comes under consideration in electrodynamics that is more advantageous than the second one in many cases and is more closely connected to **Maxwell**'s. The *vector potential* of one of the vectors *K* and *H* is employed in it. We would like to choose *H* and denote the potential by Γ , and then set:

(7)
$$H_x = -\left(\frac{\partial\Gamma_z}{\partial y} - \frac{\partial\Gamma_y}{\partial z}\right), \qquad H_y = -\left(\frac{\partial\Gamma_x}{\partial z} - \frac{\partial\Gamma_z}{\partial x}\right), \qquad H_z = -\left(\frac{\partial\Gamma_y}{\partial x} - \frac{\partial\Gamma_x}{\partial y}\right).$$

 Γ will still be undetermined by that. From the above, one must consider the value of:

$$\frac{\partial \Gamma_x}{\partial x} + \frac{\partial \Gamma_y}{\partial y} + \frac{\partial \Gamma_z}{\partial z}$$

to remain arbitrary. We shall reserve the right to make a suitable convention later.

The Ansatz (7) fulfills (4), and due to (3), it will imply that:

$$-\frac{\partial^2 \Gamma_x}{\partial y \partial t} + \frac{\partial^2 \Gamma_y}{\partial z \partial t} = V \left(\frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z} \right),$$

along with two similar equations. The entire system shows that *V K* can differ from the vector $(-\partial\Gamma_x / \partial t, -\partial\Gamma_y / \partial t, -\partial\Gamma_z / \partial t)$ only by a vector part that possesses a scalar potential. If we denote that by Φ then we must set:

(8)
$$K_{\nu} = -\frac{\partial \Phi}{\partial \nu} - \frac{1}{V} \frac{\partial \Gamma_{\nu}}{\partial t},$$

in which ν means an arbitrary direction. Now the system (3) is also fulfilled with that. What remains are (2) and (5), and (2) yields:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{V} \frac{\partial}{\partial t} \left(\frac{\partial \Gamma_x}{\partial x} + \frac{\partial \Gamma_y}{\partial y} + \frac{\partial \Gamma_z}{\partial z} \right) = 0.$$

It follows from (5) that for an arbitrary direction v:

$$\frac{\partial^2 \Gamma_{\nu}}{\partial t^2} = V^2 \left(\frac{\partial^2 \Gamma_{\nu}}{\partial x^2} + \frac{\partial^2 \Gamma_{\nu}}{\partial y^2} + \frac{\partial^2 \Gamma_{\nu}}{\partial z^2} \right) - V \frac{\partial}{\partial \nu} \left[\frac{\partial \Phi}{\partial t} + V \left(\frac{\partial \Gamma_x}{\partial x} + \frac{\partial \Gamma_y}{\partial y} + \frac{\partial \Gamma_z}{\partial z} \right) \right].$$

In regard to the indeterminacy in Γ , we now add the convention that:

(9)
$$\frac{\partial \Phi}{\partial t} + V \left(\frac{\partial \Gamma_x}{\partial x} + \frac{\partial \Gamma_y}{\partial y} + \frac{\partial \Gamma_z}{\partial z} \right) = 0.$$

The replacement for (2) and (5) then follows that:

(10)
$$\frac{\partial^2 \Phi}{\partial t^2} = V^2 \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right),$$

(11)
$$\frac{\partial^2 \Gamma_{\nu}}{\partial t^2} = V^2 \left(\frac{\partial^2 \Gamma_{\nu}}{\partial x^2} + \frac{\partial^2 \Gamma_{\nu}}{\partial y^2} + \frac{\partial^2 \Gamma_{\nu}}{\partial z^2} \right).$$

(9), (10), (11), in conjunction with (7) and (8), represent the promised *system of Maxwell equations*. As we know it is not symmetric. That initially seems to be a drawback (which can be easily remedied, moreover), but in reality it is not, because due to the place of the theory of optics in the theory of electrodynamics, we will find ourselves in a position to accommodate precisely the asymmetry between electric and magnetic phenomena that exists according to experiments.

Maxwell did not employ the simplified relation (9). Namely, to him, Γ was not merely an auxiliary mathematical quantity, as it is for us, but a function of the state with special meaning, and therefore he had to leave the value of:

$$\frac{\partial \Gamma_x}{\partial x} + \frac{\partial \Gamma_y}{\partial y} + \frac{\partial \Gamma_z}{\partial z}$$

undetermined, although "not related to any physical phenomenon." In an interesting paper that was cited in Section 12 of **Levi-Civita** (1897) ([†]), (9) was given as a consequence of hypothetical assumptions on Φ and Γ .

4. Electric charge. – From **Maxwell**'s theory, the electric and magnetic excitation in the free ether can be represented by one vector from the pair of K and H of optics. Since we must consider the asymmetry that was just touched upon, the *electric force* will be denoted by K and the *magnetic force* by H in our case.

For a space in which equations (2) and (4) are fulfilled everywhere, one has the following rules for any closed surface:

$$\int_{0}^{0} d\sigma K_{\nu} = 0, \qquad \int_{0}^{0} d\sigma H_{\nu} = 0,$$

in which $d\sigma$ means a surface element and ν is its normal. The index ⁰ is intended to recall that one is dealing with a closed surface. Indeed, if the surface itself lies in free ether, but it encloses matter, then (2) and (4) will be useless, and from (3) and (5), one can only conclude that the surface integral is independent of time. (2) and (4) likewise imply that all surfaces that enclose the same matter must also have the same values for their integrals. Experience teaches that non-zero values can appear for only the electric excitation, but not for the magnetic excitation.

We accordingly set:

$$\int_{0}^{0} d\sigma K_{\nu} = 4\pi e , \qquad \int_{0}^{0} d\sigma H_{\nu} = 0 .$$

The quantity *e*, which should be referred to the outward-pointing normal, depends upon the only the matter that is enclosed, but not upon the special form of the enclosing surface. It is called the *total amount of electricity that is contained in matter*.

If two bodies come into temporal contact then experiments show that they will often exhibit different charges from before. From the theorems that we just derived, the sum of the changes must survive the contact, so one body must have taken on just as much charge as the other one lost. *The law of conservation of electricity* then seems to be a consequence of (5).

From electrolysis, one can conclude that the electric charge goes back to the molecular structure of matter, since the individual atoms or groups of atoms can assume only a well-defined positive or negative charge or a whole-number multiple of it.

5. What does a change in charge mean now? – Until recently, that question raised special difficulties in electrodynamics. In 1895, **H. A. Lorentz** wrote (¹):

"Therefore, if the assumption of that transition or exchange of ion charges (which is, of course, a very mysterious process) is the requisite extension of any

^{(&}lt;sup>†</sup>) Translator: The citation to this paper in in a footnote on pp. 11.

^{(&}lt;sup>1</sup>) Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern, pp. 6 and 7. Leiden 1895.

theory that assumes a migration of electricity by ions then a sustained electric current can also never consist of a convection *alone*... **Giese** is of the opinion that an actual convection cannot come into play at all in metals. However, since it does not seem possible that "jumps" of charges are included in the theory, I hope that I might be excused for ignoring such processes completely, for my own part, and imagining that a current in a metal wire is simply a motion of charged particles."

I had to take a similar approach in my own theoretical work in electrodynamics. The only alternative that seemed satisfactory to me was one that **Helmholtz**, among others, referred to in 1881 in a talk at a conference that was held in honor of **Faraday**: We must regard electricity as having exactly the same constitution as matter, that is, we must also ascribe well-defined unvarying atoms to it.

Of course, we cannot regard electrical atoms as *imponderable*, in the sense of the older intuitions, because as a result of the electrodynamical processes that are connected with the motion of the ether, it will have a *kinetic energy*, so a *mass*, in the sense of mechanics. If we then contemplate the part that electrical particles play in the structure of perceptible matter then they will appear to be just such a thing. "That offers the enticing prospect of unifying matter and electricity into a higher viewpoint." When I wrote that in an outline of a theory of electrodynamics in 1894, the purely-hypothetical character of the statement clearly emerged. It was only in the Spring of 1896 that I was able to give the widely-separated numbers 10^{-7} and 1 as the limits on the atomic weight of the special electrical atoms that are exchanged in the transfer of molecular charges. However, since that time, I have quickly become more certain of that fact. **Zeeman**'s discovery came about in the same year, and its explanation by **H. A. Lorentz** that suggested that one might assume that the atomic weight of the special electrical atoms was about 1/1000. In that Winter, my own investigations with cathode rays led me to the conclusion that they consisted of special electrical atoms and that their atomic weight was between 1/2000 and 1/4000 (¹).

It is well-known that a great number of works would appear later that established that result and extended it in many directions. The numerical values for both phenomena were determined more precisely and approached each other.

6. Electron theory of electrodynamics. – If we combine everything together then we can assert the following with great confidence:

The charge of any material particle is peculiar to it for all time, so it never changes.

In order to make the actual sense of that hypothesis more precise, one must recall that we recognize the "amount of electricity" as a measure of the electrodynamical interconnection with the ether. It then emerges immediately that nothing further can be said besides the fact that the electrodynamical interconnection with the ether with any material particle (to the extent that it can be measured by the "amount of electricity") is something that characterizes it for all time.

An electrical conduction current is always to be regarded as a flow of material particles then. That has been known for a long time for electrolytic conduction. What is new is that we must also

^{(&}lt;sup>1</sup>) E. Wiechert, Phys.-ökonom. Ges. zu Königsberg i Pr. 38 (1897), 7 January.

assume something similar for metallic conduction. If there are only special negative electrical atoms, and not also positive atoms of a similar type, then the motion would result in only the negative direction of the electrical current.

As **H. A. Lorentz** first showed, it is possible *to regard the electrodynamical processes as nothing but consequences of the motions of electrical particles.* It might be likely that the actual phenomena are not exhausted in that way, but in any event, we tentatively have the right to make that simplifying assumption in the implementation of our theory.

With that, we have returned to the basic picture of the old theory. The only difference is that we no longer regard the electrical fluid as imponderable, but as material. **Stoney** (1874) gave the name of *electron* to the smallest molecular charge. Since we shall place it at the center of our theory, we can also call it the *electron theory* of electrodynamics.

7. Field equations. – For the free ether, we assume (Section **3**):

$$\begin{split} \frac{\partial K_x}{\partial t} &= -V\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right), & \qquad \frac{\partial H_x}{\partial t} &= -V\left(\frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z}\right), \\ \frac{\partial K_y}{\partial t} &= -V\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right), & \qquad \frac{\partial H_y}{\partial t} &= -V\left(\frac{\partial K_x}{\partial z} - \frac{\partial K_z}{\partial x}\right), \\ \frac{\partial K_z}{\partial t} &= -V\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right), & \qquad \frac{\partial H_z}{\partial t} &= -V\left(\frac{\partial K_y}{\partial x} - \frac{\partial K_x}{\partial y}\right), \\ \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} + \frac{\partial K_z}{\partial z} &= 0, & \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} &= 0. \end{split}$$

Those equations do not change for material systems, because we must take the electrons into account. However, it is not necessary to make further hypotheses, due to our assumption that the ether also penetrates matter and has the same properties, which will take the system that was written down to:

(12)
$$\begin{cases} \frac{\partial K_x}{\partial t} = -V\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - 4\pi \gamma_x, \\ \frac{\partial K_y}{\partial t} = -V\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_x}{\partial x}\right) - 4\pi \gamma_y, \\ \frac{\partial K_z}{\partial t} = -V\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) - 4\pi \gamma_z, \end{cases}$$

(13)
$$\begin{cases} \frac{\partial H_x}{\partial t} = -V\left(\frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z}\right),\\ \frac{\partial H_y}{\partial t} = -V\left(\frac{\partial K_x}{\partial z} - \frac{\partial K_x}{\partial x}\right),\\ \frac{\partial H_z}{\partial t} = -V\left(\frac{\partial K_y}{\partial x} - \frac{\partial K_x}{\partial y}\right),\end{cases}$$

(14)
$$\frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} + \frac{\partial K_z}{\partial z} = 4\pi \chi,$$

(15)
$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0,$$

in which γ denotes the electrical current, and χ denotes the electrical density, and the following consequence will emerge from the foregoing equations:

(16)
$$-\frac{\partial \chi}{\partial t} = \frac{\partial \gamma_x}{\partial x} + \frac{\partial \gamma_y}{\partial y} + \frac{\partial \gamma_z}{\partial z}.$$

That system combines all of the experimental laws regarding the electric and magnetic field excitations. K, H, γ , χ represent mean values inside of matter, which would correspond to the assumption of a molecular constitution.

 γ is the sum of a series of different physical processes: viz., *convection*, *conduction current*, *dielectric polarization*, and *magnetization*.

Hertz (and **Heaviside**) appealed to the system of equations (12) to (16). In **Maxwell**'s manner of representation, using the notations of Section **3**, we will get:

(17)
$$H_{x} = -\left(\frac{\partial\Gamma_{z}}{\partial y} - \frac{\partial\Gamma_{y}}{\partial z}\right), \quad H_{y} = -\left(\frac{\partial\Gamma_{x}}{\partial z} - \frac{\partial\Gamma_{z}}{\partial x}\right), \quad H_{z} = -\left(\frac{\partial\Gamma_{y}}{\partial x} - \frac{\partial\Gamma_{x}}{\partial y}\right),$$

(18)
$$K_{\nu} = -\frac{\partial \Phi}{\partial \nu} - \frac{1}{V} \frac{\partial \Gamma_{\nu}}{\partial t},$$

(19)
$$\frac{\partial^2 \Phi}{\partial t^2} = V^2 \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right) + 4\pi V^2 \chi ,$$

(20)
$$\frac{\partial^2 \Gamma_{\nu}}{\partial t^2} = V^2 \left(\frac{\partial^2 \Gamma_{\nu}}{\partial x^2} + \frac{\partial^2 \Gamma_{\nu}}{\partial y^2} + \frac{\partial^2 \Gamma_{\nu}}{\partial z^2} \right) + 4\pi V \gamma_{\nu},$$

(21)
$$-\frac{\partial \chi}{\partial t} = \frac{\partial \gamma_x}{\partial x} + \frac{\partial \gamma_y}{\partial y} + \frac{\partial \gamma_z}{\partial z}$$

The last equation formulates the law of conservation of electricity. The relation:

(22)
$$\frac{\partial \Phi}{\partial t} + V \left(\frac{\partial \Gamma_x}{\partial x} + \frac{\partial \Gamma_y}{\partial y} + \frac{\partial \Gamma_z}{\partial z} \right) = 0$$

appears as a consequence of (19), (20), (20), (21) here.

The system (17) to (21) is completely equivalent to the system (12) to (15), so that fundamental system can be taken to be that of the field excitations.

8. Effect of the ether on matter. – Up to now, we have observed only the excitation of the ether, such that the foundations of the theory of electrodynamics are still not complete. What is lacking is to establish the effect of the ether on matter. **H. A. Lorentz** was the first to show that the following two hypotheses will suffice for that:

An electrical particle of charge e experiences a mechanical force || K of intensity e K due to the electric excitation of the ether independently of its motion.

An electrical particle of charge e that moves with a velocity v experiences a mechanical force $\perp v$ and $\perp H$ of intensity e v H sin (v, H) / V due to the magnetic excitation of the ether.

9. Concluding remarks. – The circle of fundamental hypotheses for a theory of electrodynamics is now complete. Given the wealth of phenomena that they embrace, there are very few of them, and all of them are either closely linked with experiments or chosen from the simplest possible ones: viz., the law of light motion in free ether, **Maxwell**'s assumptions, the fact that magnetic and electric excitations come into play in that way, the assumption of a carrier of those excitations that is omnipresent, everywhere at rest, and has the same behavior everywhere, which we call the "ether," the idea that the interactions between ether and matter are coupled to just electrical particles and their motion, and finally, the two laws of the previous section.

Up to now, electrodynamical energy was still not considered. That was done intentionally in order to show that it does not need to be considered when one is establishing the basic concepts. However, when one applies the principle of energy, that will imply that the energy must be ascribed to the electrodynamically-excited ether, and that one must satisfy the requirements of the principles most simply by means of **Maxwell**'s energy formula and **Poynting**'s concept of the energy current.

II. – Elementary laws.

10. Formulation of the problem. – It is characteristic of the theory being developed that it assumes a propagation of electrodynamical excitations with the speed of light in the free ether. That suggests that it must be possible to represent the respective excitation at any location as a consequence of processes that take place at every location in space and go back so far in time that they correspond to that speed of propagation. Furthermore, since we assume that all excitations of the ether have their origin in electrical particles, we will suspect that it must also be possible, in the sense of the old theory, to relate the definitive processes to those electrical particles alone.

Similar problems in the theory of elasticity, as well as optics, have been treated many times before, and we will easily reach our goal when we make use of the methods that were developed in those theories. In what follows, I would like to go down the path along which **Beltrami** formulated **Huygens**'s principle analytically.

11. Elementary laws of space elements. – It would not be expedient to appeal to the equations for *K* and *H* directly, because the separation of both vectors leads to inconvenient auxiliary conditions. In such cases, one cares to introduce suitable auxiliary quantities. In our case, that was already done with Γ and Φ , so with no further assumptions, we can exploit the system:

(19)
$$\frac{\partial^2 \Phi}{\partial t^2} = V^2 \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right) + 4\pi V^2 \chi ,$$

(20)
$$\frac{\partial^2 \Gamma_{\nu}}{\partial t^2} = V^2 \left(\frac{\partial^2 \Gamma_{\nu}}{\partial x^2} + \frac{\partial^2 \Gamma_{\nu}}{\partial y^2} + \frac{\partial^2 \Gamma_{\nu}}{\partial z^2} \right) + 4\pi V^2 \frac{\gamma_{\nu}}{V}$$

for the determination of Φ and Γ .

Beltrami (¹) employed the following mathematical lemma: If *U* is a function of the coordinates *x*, *y*, *z*, and *r* then one will have:

$$4\pi U_0 = \int d\sigma \left[\frac{\partial}{\partial r} \left(\frac{U}{r} \right) \cos(n, r) - \frac{1}{r} \frac{\partial U}{\partial n} \right] + \int \frac{d\omega}{r} \left[\frac{\partial^2 U}{\partial r^2} - \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) \right]$$

for an arbitrary point (0), in which r is understood to mean the distance from (0), the first integral refers to an arbitrary surface that surrounds the point (0), and the second one refers to the space that it encloses. n means the inward-pointing normal. In the differentiation with respect to r, one must regard x, y, z as constants, while in the differentiations with respect to n, x, y, z, one must

^{(&}lt;sup>1</sup>) **E. Beltrami**, Rend. Accad. Lincei (5) **4** (1895), pp. 51. **W. Voigt** gave a German presentation in *Compendium der theoretische Physik*, v. 2, Leipzig, 1896, pp. 776.

regard r as constant. In our case, we imagine that the surface is pushed out to infinity and assume that the associated integral can then be set equal to zero. What will then remain is:

$$4\pi U_0 = \int \frac{d\omega}{r} \left[\frac{\partial^2 U}{\partial r^2} - \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) \right].$$

We set $r = (t_0 - t)$ V in that, where t_0 is regarded as a constant and t is regarded as variable, such that U will go to a function of x, y, z, and t, and we obtain:

$$4\pi (U_0)_{t=t_0} = \int \frac{d\omega}{r} \left[\frac{1}{V^2} \frac{\partial^2 U}{\partial r^2} - \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) \right]_{t=t_0 - r/V}.$$

If we apply that theorem to Φ and Γ_{ν} then, with the use of the differential equations (19) and (20), that will immediately give:

(23)
$$\Phi_{t=t_0} = \int \frac{d\omega}{r} \chi_{t=t_0-r/V} ,$$

(24)
$$(\Gamma_{\nu})_{t=t_0} = \int \frac{d\omega}{r} \left(\frac{\gamma_{\nu}}{V}\right)_{t=t_0-r/V} ,$$

with which, the following statement can be made: One will get the value of Φ and Γ_v for any location (0) and any time t_0 by summing the components:

$$\frac{d\omega}{r}\chi$$
 and $\frac{d\omega}{r}\frac{\gamma_{\nu}}{V}$

over all volume elements $d\omega$. In that, r means the distance from the volume element to (0) and the values of $\chi(\gamma_v, \text{resp.})$ are chosen to be the ones that go back to a distant time when an excitation that is emitted with the speed of light would have arrived at (0) at precisely time t_0 . The potential components of the individual volume elements then seem to spread out with the speed of light.

The laws (23) and (24), along with formulas (17), (18), for the determination of K and H, and the law of conservation of electricity (21), give us a new *representation of the field equations that* replaces local action with forces-at-a-distance according to the model of the old theories.

12. Historical remarks. – In 1858, Riemann $(^1)$ already sought to exploit formula (19) in electrodynamics, along with the law (23) that it is linked with, which was derived here with the use of a lemma by **Beltrami**. However, since he considered only the electric force (but not the

^{(&}lt;sup>1</sup>) **B. Riemann**, Pogg. Ann. **131** (1867), pp. 237.

magnetic one), his process remained fruitless. (23) and (24), or corresponding theorems, were later exploited by **Poincaré** (¹) (1891), and to the greatest extent by **H. A. Lorentz** (²) (1892 and 1895). In 1897, **Levi-Civita** (³) showed that one would arrive at the **Hertz-Heaviside** formulas when formulas similar to (23) and (24) were assumed in **Helmholtz**'s theory.

13. Elementary laws of electrons. – Only one last step remains for us: *We must resolve the electrodynamical effect of matter into the contributions from the individual electrons using the process of W. Weber.* With that, we then come to the actual subject of the present article.

At first, we can suspect that, in connection with (23) and (24) for an isolated *electron of charge* $i = \int d\omega \chi$ and velocity v, we can simply set:

$$\Phi_{t=t_0} = \frac{t}{r_{t=t_0-r/V}}, \qquad (\Gamma_v)_{t=t_0} = t \left(\frac{1}{r} \frac{v_v}{V}\right)_{t=t_0-r/V},$$

and in fact, that was assumed for Φ in the time of **Riemann**. However, that path leads to contradictions with the fundamental assumptions of our theory, such as for example, in the treatment of any of the problems in Part III will show directly, so it is impassable. That is based in the fact that it is not permissible to pass to the limit of a point-like body, even before formulas (23) and (24) are applied. Indeed, those formulas are valid for *spatially*-distributed electricity, so that would demand that the passage to the limit would have to be first carried out after they have been applied. That amounts to the same thing as saying that the formulas (23) and (24) can be applied to only infinitely-small, but not point-like, bodies.

In that way, it shall be assumed that the body that we would like to call an "electron" is symmetric in all directions and exhibits no rotations. In the other cases, mean values would have to be taken.

Let (1) be the position of the center of the body at the earlier time t_1 , from which an excitation was emitted with the speed of light *V* and came to precisely the point (0) at time t_0 . If the distance (0)-(1) is then denoted by r_1 then that will make:

$$t_1 = t_0 - \frac{r_1}{V} \,.$$

Due to the assumed infinitely-small extent of the electron, in the application of (23):

$$\Phi_{t=t_0}=\int\!\frac{d\omega}{r}\,\chi_{t=t_0-r/V}\,,$$

^{(&}lt;sup>1</sup>) **H. Poincaré**, C. R. Acad. Sci. Paris **113** (1891), pp. 515.

^{(&}lt;sup>2</sup>) **H. A. Lorentz**, *La théorie éléctromagnétique de Maxwell*, etc., Leiden, 1892, and also Arch. Néerl. **25** (1892), pp. 363. *Versuch einer Theorie*, etc., Leiden, 1895.

^{(&}lt;sup>3</sup>) **T. Levi-Civita**, Nuovo Cimento (4) **6** (1897), pp. 93.

the only times *t* and distances *r* that will come under consideration are the ones that lie infinitely close to t_1 and r_1 , resp. The intersecting spherical surfaces can be regarded as planes at the scale of the electron. $r - r_1$ is its distance from (1). In the integration, each plane is associated with a certain section through the electron. We ask how its distance *R* from the center is connected with $r - r_1$. If *v* is the velocity of the electron then its center will lie at a distance of $(t - t_1) v \cos(v, r)$ from the plane through (1) at time *t*. It will follow immediately from this that:

$$R = r - r_1 - (t - t_1) v \cos(v, r)$$
,

and therefore, since $r = (t_0 - t) V$, $r_1 = (t_0 - t_1) V$:

$$R = (r - r_1) \left[1 + \frac{v}{V} \cos(v, r) \right] \,.$$

In the integration that defines Φ , and for every section r = const., the value of χ is chosen to be the one that belongs to R, so the integration can be carried out as if the electron were at rest with its center at (1), assuming that we imagine that its dimensions are varied in the ratio:

$$|R|: |r-r_1| = \left|1 + \frac{v}{V}\cos(v, r)\right|: 1$$

without changing the χ -value that is parallel to r_1 . The symbol || should suggest that the absolute value is taken. The variation of the denominator r does not come under consideration for infinitely-small dimensions, so we will then get:

$$\Phi_{t=t_0} = \left(\frac{1}{r\left|1+\frac{v}{V}\cos\left(v,r\right)\right|}\int d\omega \chi\right)_{t=t_0-r/V}$$
$$= t \left(\frac{1}{r\left|1+\frac{v}{V}\cos\left(v,r\right)\right|}\right)_{t=t_0-r/V}.$$

Things take a similar form for Γ_{ν} , in which we set *i* v_{ν} in place of $d\omega \chi_{\nu}$, and we will then get the following pair of equations as the *elementary law for an isolated electron:*

$$\Phi_{t=t_0} = \iota \left(\frac{1}{r \left| 1 + \frac{v}{V} \cos\left(v, r\right) \right|} \right)_{t=t_0 - r/V},$$

(25)

$$(\Gamma_{v})_{t=t_{0}} = t \left(\frac{\frac{v_{v}/V}{r \left| 1 + \frac{v}{V} \cos(v, r) \right|} \right)_{t=t_{0} - r/V}$$

In the definition of $\cos(v, r)$, *r* is taken to be the direction that leads from (0) to the electron, so $v \cos(v, r)$ means the component of *v* that points away from (0).

Just as was expected, the condition $t = t_0 - (r/V)$ means that in the determination of Φ (Γ_v , resp.) the previous location of the electron was chosen from which an excitation that spread out with the speed of light *V* would arrive at the point in question at the time t_0 . If there are possibly more such places then Φ (Γ_v , resp.) will be set equal to the sum of the individual contributions.

As long as *v* is smaller than the speed of light *V*, $1 + v \cos(v, r) / V \cosh(v)$ only be positive. Negative values are also possible when v > V. In such a case, the condition that the absolute value shall be true will be applicable: One then substitutes $-[1 + v \cos(v, r) / V]$.

If arbitrarily-many electrons are present in the field then one must add up the contributions to Φ and Γ_v that are determined by (25) and (26). With the addition of:

$$K_{\nu} = -\frac{\partial \Phi}{\partial \nu} - \frac{1}{V} \frac{\partial \Gamma_{\nu}}{\partial t} \qquad \text{and} \qquad H_{\nu} = -\left(\frac{\partial \Gamma_{\mu}}{\partial \lambda} - \frac{\partial \Gamma_{\lambda}}{\partial \mu}\right),$$

in which λ , μ , ν means an arbitrary cyclic permutation of x, y, z, x, ..., we will get a representation of the field excitation that hearkens back to the spirit of W. Weber's thoughts on isolated electrical particles.

It is a characteristic feature of the theory that we regard electrons as point-like. In cases where that is not allowed, we must resolve the electron into volume elements and replace t with $\chi d\omega$.

III. – Some applications of the elementary laws of electrons.

14. Linear stationary or semi-stationary currents. – In order to show the usefulness of the elementary law (25), (26) of electrons, some applications of it shall now be given. We first turn to the classical problem of the older theories that is suggested by linear currents.

Let the current be stationary and let its conductor be *at rest*. In order to define Φ and Γ for the point (0), our problem is to sum the contributions that the elementary law gives over all electrons. We imagine that we construct two spherical surfaces *r* and *r* – *dr* around (0) that cut out the line

element $d\lambda$ from the current conductor, and in order to apply the elementary law, we delimit the time-element between:

$$t = t_0 - \frac{r}{V}$$
 and $t + dt = t_0 - \frac{r - dr}{V}$

which has the length:

$$dt = \frac{dr}{V}$$

Which electrons come under consideration for dt? Since we cannot assume everywhere-equal velocity, we shall next consider those groups of them whose velocities are parallel to $d\lambda$ and lie between v and v + dv. Let $d\chi$ be the linear density of their electricity, so $d\chi d\lambda$ is the amount of electricity that they imply for $d\lambda$. An electron that is found on the spherical surface r at time t and at the endpoint dc is shifted to:

$$v dt = \frac{v}{V} dr = \frac{v}{V} \cos(v, r) dl$$

at time t + dt. In the summation, not only will the electrons along a segment $d\lambda$ belong to dt (dr and $d\lambda$, resp.), but also all of the ones along a segment:

$$d\lambda + v dt = d\lambda \left[1 + \frac{v}{V} \cos(v, r) \right],$$

and $d\lambda$ accordingly gives the contribution to Φ :

$$d\Phi = \int d\chi \frac{d\lambda \left[1 + \frac{v}{V} \cos(v, r)\right]}{r \left[1 + \frac{v}{V} \cos(v, r)\right]} = \frac{d\lambda \int d\chi}{r} = \frac{de}{r},$$

if *de* means the amount of electricity that is found along the respective $d\lambda$. Thus, the formula of electrostatics for Φ :

$$\Phi = \int \frac{de}{r}$$

is also true for stationary linear currents.

In order to define Γ , we must imagine that the electrons with velocities that lie between v and v + dv contribute $di = v d\chi$ to the current strength, which contributes:

$$\frac{d\chi d\lambda \left[1 + \frac{v}{V}\cos(v, r)\right]}{v \left[1 + \frac{v}{V}\cos(v, r)\right]} \frac{v \cos(v, v)}{V} = \frac{di d\lambda \cos(\lambda, r)}{V r}$$

to Γ_v .

That will give the contribution:

$$d\Gamma_{v} = \frac{i}{V} \cdot \frac{d\lambda \cos\left(\lambda, r\right)}{r}$$

for $d\lambda$, and the known formula for the total system of currents:

$$\Gamma_{\nu} = \int \frac{i}{V} \cdot \frac{d\lambda \cos{(\lambda, r)}}{r},$$

which gives the distribution of magnetic force, and in conjunction with the second law of Section **8** on the mechanical effect of the magnetically-excited ether, it will also represent the ponderomotive forces between current systems that correspond to experiments.

If the current varies very slowly, so it is *semi-stationary*, then our formulas for Φ and Γ will still be valid approximately. By means of:

$$K_{\nu} = -\frac{\partial \Phi}{\partial \nu} - \frac{1}{V} \frac{\partial \Gamma_{\nu}}{\partial t},$$

we will then get the *induced electromotive force* from the second term on the right. If we integrate over a closed ring then that will immediately give **Neumann**'s formula, which is a sign that our calculations also lead to correct results here. For the induction in moving bodies, according to the second law in Section **8**, there will be a component that is due to the motion in the magnetic field, which likewise corresponds to experiment.

15. Elementary law for volume elements. – For material systems of currents, the elementary law of electrons must lead back to the same starting point, namely, the elementary law for volume elements that was given Section **11**. One can easily verify that this is actually the case when one arranges the electrons in the summation of their contributions to Φ and Γ_v for the spatial point (0) and time t_0 according to the distances r and time t that occur in the calculation, similar to what was done in the previous section.

Once more:

$$t=t_0-rac{r}{V},$$
 $t+dt=t_0-rac{r-dr}{V},$ $dt=rac{dr}{V}$

might belong together. Let v_r be the velocity component that points forward at (0). We next direct our investigation to the electrons for which v_r lies between v_r and $v_r + dv_r$; let the spatial density

of its electricity be $d\chi$. The layer that is found at the distance *r* at time *t* has attained a distance at time t + dt that is greater by:

$$v_r dt = \frac{v_r}{V} dr = \frac{v}{V} \cos(v, r) dr.$$

For *dr*, *dt*, an electron layer of density:

$$dr + \frac{v}{V}\cos(v, r) dr = \left[1 + \frac{v}{V}\cos(v, r)\right] dr$$

will then enter into the calculation. Its contribution to Φ is:

$$d\Phi = \int d\sigma dr \frac{d\chi \left[1 + \frac{v}{V} \cos(v, r)\right]}{r \left[1 + \frac{v}{V} \cos(v, r)\right]} = \int d\sigma dr \frac{d\chi}{r},$$

when $d\sigma$ denotes a surface element. If we integrate over χ and r then it will follow that:

$$\Phi_{t=t_0} = \int \frac{d\omega}{r} \chi_{t=t_0-r/V}$$
 ,

that is, the previous formula. Similarly, that will give a contribution of:

$$d\Gamma_{v} = \int d\sigma dr \frac{d\chi v_{v} \left[1 + \frac{v}{V}\cos(v, r)\right]}{rV \left[1 + \frac{v}{V}\cos(v, r)\right]} = \int d\sigma dr \frac{d\chi v_{v}}{r} \frac{V}{V}$$

to Γ_{v} .

Now let $d\chi v_{\nu}$ be the contribution from the selected group of electrons to $d\gamma_{\nu}$. If we make use of that and integrate over γ and *r* then the law that we are still lacking will follow:

$$(\Gamma_{\nu})_{t=t_0} = \int \frac{d\omega}{r} \left(\frac{v_{\nu}}{V}\right)_{t=t_0-r/V}.$$

16. An isolated electron in uniform rectilinear motion. (¹). – Let *v* be the velocity. We refer everything to a coordinate system whose *z*-axis is || v, and whose origin lies at the location of the electron at the time t_0 for which we seek the distribution of Φ and Γ . That will then give the distribution of electrodynamical excitations relative to the electron.

A very simple calculation will show that for the space point (0), the position (1) of the electron that must be considered is the one for which *r* has the value:

$$\left[\frac{v}{V}z + \sqrt{(x^2 + y^2)\left(1 - \frac{v^2}{V^2}\right) + z^2}\right] \left(1 - \frac{v^2}{V^2}\right)^{-1},$$
$$r\left[1 + \frac{v}{V}\cos(v, r)\right]$$

will have the value:

and

$$\sqrt{(x^2+y^2)\left(1-\frac{v^2}{V^2}\right)+z^2}$$
.

We will then get:

$$\Phi = \frac{t}{\sqrt{(x^2 + y^2)\left(1 - \frac{v^2}{V^2}\right) + z^2}} ,$$

$$\Gamma_v = \frac{v_v}{V} \frac{t}{\sqrt{(x^2 + y^2)\left(1 - \frac{v^2}{V^2}\right) + z^2}} .$$

When that is combined with:

$$K_{\nu} = -\frac{\partial \Phi}{\partial \nu} - \frac{1}{V} \frac{\partial \Gamma_{\nu}}{\partial t}, \qquad \qquad H_{\nu} = -\left(\frac{\partial \Gamma_{\mu}}{\partial \lambda} - \frac{\partial \Gamma_{\lambda}}{\partial \mu}\right), \qquad \qquad \lambda, \mu, \nu = \dots, x, y, z, x, y, \dots,$$

that will, in fact, imply the known distribution of field excitations when one observes that due to the fact that we have assumed a moving coordinate system, we have to replace:

$$\frac{\partial \Gamma_{\nu}}{\partial t}$$
 with $-V \frac{\partial \Gamma_{\nu}}{\partial z}$.

^{(&}lt;sup>1</sup>) The same problem for a superluminal velocity was likewise treated in the Jubilee volume for **H. A. Lorentz** (pp. 652) and by my distinguished Göttingen colleague **Th. Des Coudres** (Zusatz 7, III, 1901) along a path that corresponded completely to the one that was followed here.

17. Oscillating electron. – In conclusion, we might consider the case that is of interest in optics, in which an electron performs sinusoidal oscillations. Since oscillations of the most general type can be resolved into linear oscillations in that theory, we can then confine ourselves to investigating *linear oscillations*.

The origin of the coordinate system might be laid at the center of oscillation with the *z*-axis parallel to the line of oscillation. We can then write:

$$\zeta = Z \sin 2\pi \, \frac{t}{T} \, ,$$

in which ζ is the respective *z*-coordinate of the electron, *Z* is the greatest deflection, and *T* is the period. The emitted light has the wavelength $\lambda = VT$.

Let ρ be the distance from the point (0) to the coordinate origin so to the center of oscillation. We restrict ourselves to the case in which Z proves to be infinitely-small compared to λ and ρ . We must then replace:

$$\frac{1}{r\left[1+\frac{v}{V}\cos(v,r)\right]} \quad \text{with} \quad \frac{1}{\rho}\left(1+\frac{\zeta z}{\rho^2}+\frac{v z}{V \rho}\right)$$

and

$$t = t_0 - \frac{r}{V}$$
 with $t = t_0 - \frac{\rho}{V} = t = t_0 - R\frac{\rho}{\lambda}$

in the right-hand sides of formulas (25), (26), and when we write t in place of t_0 in the final formula, we will get:

$$\Phi = \frac{\iota}{\rho} + \frac{\iota Z}{\rho} \frac{z}{\rho} \left[\frac{1}{\rho} \sin 2\pi \left(\frac{t}{T} - \frac{\rho}{\lambda} \right) + \frac{2\pi}{\lambda} \cos 2\pi \left(\frac{t}{T} - \frac{\rho}{\lambda} \right) \right],$$

$$\Gamma_x = 0, \qquad \Gamma_y = 0, \qquad \Gamma_z = \frac{\iota Z}{\rho} \frac{2\pi}{\lambda} \cos 2\pi \left(\frac{t}{T} - \frac{\rho}{\lambda} \right).$$

Those formulas yield a well-known case of the radiation from a luminous point.

Poynting's law will give information about the radiated energy most simply. If we apply it to a very large sphere then it will follow that:

$$-\frac{dE}{dt}=\frac{e^2Z^2}{3}\left(\frac{2\pi}{\lambda}\right)^4 V\,,$$

in which -dE denotes the energy loss from the oscillating system during dt.

That is connected with an interesting consequence in regard to the *damping of the oscillation of an electron* that results from the action of a central force that is proportional to the distance. In order for the theorems that were derived to be approximately valid, we must assume that the

damping is only very small. If *m* means the effective mass and $k \zeta$ means the restoring force then, except for the negligible influence of damping, one must set:

$$m\frac{d^2\zeta}{dt^2}=-k\,\zeta\,,$$

from which it follows that:

$$\frac{k}{m} = \left(\frac{2\pi}{T}\right)^2,$$

and that the energy of oscillation is:

$$E = \frac{1}{2}k Z^{2} = \frac{1}{2} \left(\frac{2\pi}{T}\right)^{2} Z^{2} m.$$

In conjunction with our formula for -dE/dt, that will give the value:

$$\tau = 2\frac{E}{\frac{dE}{dt}} = \frac{3}{4\pi^2} \cdot \frac{m}{\iota} \cdot \frac{\lambda^2 V}{\iota}$$

for the *relaxation time*, that is, the time it takes for the amplitude to drop down to 1/2.828... times its original value, and the value:

$$w = \tau V = \frac{3}{4\pi^2} \cdot \frac{m}{\iota} \cdot \frac{\lambda^2 V^2}{\iota}$$

for the length of the path that the light travels during the relaxation time.

We would like to apply that formula for the case of emission of light with one spectral line. We set λ equal to around 1/20000. ι/m might be assumed to have the value $4 \cdot 10^{17}$, which roughly corresponds to the Zeeman phenomenon and cathode rays. ι is known only imprecisely. According to whether one sets the number of molecules in a cubic centimeter of gas at 0° C and normal pressure equal to:

$$N = 10^{19}$$
 or $N = 10^{20}$

(which might characterize the limits that one defines), that will give:

$$\iota = 13 \cdot 10^{-19}$$
 or $\iota = 1.3 \cdot 10^{-10}$,

respectively, and we will get:

$$w = 3 \text{ m}$$
 or $w = 30 \text{ m}$

resp.

We can compare that with our observations of interference for large path differences. The give about 1/2 meter for the greatest path difference for which interference can be recognized.

We must conclude from that, in addition to the decline in the oscillations that results from the emission of light, other perturbing causes will be in effect that can prove to be stronger. If we set:

w > 0.5 m,

which would correspond to observations, then it will follow that:

 $\iota < 80 \cdot 10^{-10}, \qquad N > \frac{1}{6} 10^{19}.$

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