# A remark about Einstein's new formulation of the principle of general relativity 

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It will be shown that Einstein's theory of teleparallelism permits a general relativistically-invariant formulation of Dirac's theory of the electron.
§ 1. - In a note that appeared recently (*), A. Einstein gave a method for simultaneously describing the electromagnetic field and the gravitational field. In order to characterize the metric, he did not employ the fundamental metric tensor $g_{\mu \nu}$, but introduced an infinitesimal Lorentzian ( ${ }^{* *}$ ) coordinate system at each point. He denoted the covariant components of the four axes $(a=0,1,2,3)$ of that coordinate system in the basic Gaussian system by $h_{\mu a}$, and the contravariant ones by $h_{a}^{\mu}{ }^{* * *}$. The contravariant Gaussian components $v^{\mu}$ of a vector with the components $v_{a}$ in a Lorentzian system are then:

$$
\begin{equation*}
v^{\mu}=h_{a}^{\mu} v_{a}, \tag{1}
\end{equation*}
$$

so the length of the vector $v$ is:

$$
\begin{equation*}
g_{\mu \nu} v^{\mu} v^{v}=g_{\mu \nu} h_{a}^{\mu} h_{b}^{v} v_{a} v_{b}=L_{a b} v_{a} v_{b} . \tag{2}
\end{equation*}
$$

In this, one has, by definition:

$$
L_{a b}=\left\{\begin{array}{rll}
0 & \text { for } & a \neq b, \\
1 & { }^{\prime} & a=b=0, \\
-1 & " & a=b \neq 0 .
\end{array}\right.
$$

[^0]One will then have for the matrices $\underline{\underline{g}}=\left(g_{\mu \nu}\right), \bar{h}=\left(h_{a}^{\mu}\right), L=\left(L_{a b}\right)$ that:

$$
\begin{equation*}
\bar{h}_{\underline{\underline{g}}}^{\underline{\underline{g}}} \bar{h}=L \tag{3}
\end{equation*}
$$

The covariant components of $v_{a}$ are:

$$
\begin{equation*}
v_{\mu}=h_{\mu a} v_{a}=g_{\mu \nu} v^{v}=g_{\mu \nu} h_{a}^{v} v_{a}, \tag{4}
\end{equation*}
$$

and one further has $\underline{h}=\left(h_{\mu a}\right)$ :

$$
\begin{equation*}
\underline{h}=\underline{\underline{g}} \bar{h}, \quad \bar{h}^{\prime} \underline{h}=L, \quad \overline{h^{\prime}}=L h^{-1} . \tag{5}
\end{equation*}
$$

The transpose of the last equation is:

$$
\begin{equation*}
\underline{h}^{\prime}=\bar{h}^{\prime} \underline{\underline{g}}, \quad \underline{h^{\prime}} \bar{h}=L, \quad \bar{h}=\underline{h}^{\prime-1} L \tag{6}
\end{equation*}
$$

Since $L=L^{-1}$, we will then get:

$$
\begin{equation*}
\bar{h}_{\underline{\prime}}^{\underline{g}} \bar{h}=L \underline{h}^{-1} g h^{\prime-1} L=L, \quad \underline{\underline{g}}=\underline{h} L h^{\prime} . \tag{7}
\end{equation*}
$$

Likewise, since $\overline{\bar{g}}=\left(g^{\mu \nu}\right)$ :

$$
\begin{equation*}
g=g^{-1}=\bar{h} L \bar{h}^{\prime} \tag{8}
\end{equation*}
$$

It is clear that the metric form $g_{\mu \nu}$ does not determine the components $h_{\mu a}$ of the infinitesimal Lorentzian coordinate system completely, since an arbitrary Lorentz transformation will still remain free at each point of the world. That manifests itself in the fact that one can also write $\underline{h} T$ for $\underline{h}$ in all of the equations up to now (if $T L T^{\prime}=L$, i.e., $T$ is indeed different from world-point to world-point, but it is a Lorentz transformation everywhere; one will then also have $T^{\prime} L T=L$ ), so $g$ will remain unchanged by that. Nevertheless, A. Einstein further assumed that $h_{\mu a}$ would have meaning, up to a single space-dependent Lorentz transformation, when they determine not only the metric form, but also the electromagnetic potentials $\Phi_{\mu}\left({ }^{*}\right)$ :

$$
\begin{equation*}
\Phi_{\mu}=C \Lambda_{\mu \alpha}^{a}, \quad \quad \Lambda_{\mu \alpha}^{\beta}=\frac{1}{2}\left(\Delta_{\mu \alpha}^{\beta}-\Delta_{\alpha \mu}^{\beta}\right), \quad \quad \Delta_{\mu \alpha}^{\beta}=h_{a}^{\beta} \frac{\partial h_{\mu b}}{\partial x^{a}} L_{a b} . \tag{9}
\end{equation*}
$$

Moreover, Einstein could derive, on the one hand, the equations of the older theory of gravitation, and on the other hand, Maxwell's equations for the vacuum, from some action principles in the first approximation (i.e., when $\underline{h}$ is approximately an identity matrix).

[^1]§ 2. - The foregoing remark had the aim of applying the theory thus-presented to the Dirac equations ( ${ }^{*}$ ) of the rotating electron.

From the physical standpoint, the introduction of a local Lorentz coordinate system is not at all reasonable in Dirac's theory. The probability for an electron to move in, say, the $X$ or $-X$ direction will be independent of the probability of it moving in the $Y$ or $-Y$ direction only when those directions are perpendicular to each other. One will soon be led to a preference for introducing a local coordinate system by this and similar considerations. Now, our convention is that the relative orientation of those local coordinate systems to each other is influenced by the electromagnetic field.

The equation of motion of the electron is then:

$$
\begin{equation*}
\sum_{k, a, b} \frac{1}{2}\left[\gamma_{a} h_{b}^{k} L_{a b}\left(p_{k}+e \Phi_{k}\right)+\left(p_{k}+e \Phi_{k}\right) \gamma_{a} h_{b}^{k} L_{a b}\right] \psi=m \psi, \tag{10}
\end{equation*}
$$

in which the expression on the left must be the symmetrized Dirac expression, instead of the simple one, since $p_{k}$ and $h_{b}^{k}$ do not commute. If we introduce the covariant derivative $\frac{h}{2 \pi i} \nabla_{k}$ in place of $p_{k}$ then when we set the determinant $\left|h_{k a}\right|=H=\sqrt{-\left|g_{\mu \nu}\right|}$ and perform all differentiations, we will get:

$$
\begin{equation*}
\sum_{k, a, b}\left[\gamma_{a} h_{b}^{k} L_{a b}\left(\frac{h}{2 \pi i} \frac{\partial}{\partial x^{k}}+e \Phi_{k}\right)+\frac{h}{2 \pi i} \frac{1}{2 H} \gamma_{a} \frac{\partial H h_{b}^{k}}{\partial x^{k}} L_{a b}\right] \psi=m \psi, \tag{10}
\end{equation*}
$$

in which the equations:

$$
\begin{equation*}
\frac{1}{2}\left(\gamma_{a} \gamma_{a^{\prime}}+\gamma_{a} \gamma_{a}\right)=L_{a a^{\prime}} \tag{12}
\end{equation*}
$$

are valid for the $\gamma$, as in (10). The matrices $\gamma_{1}, \gamma_{2}, \gamma_{3}$ are skew-Hermitian then, while the matrix $\gamma_{0}$ is symmetric. One easily convinces oneself that (10) or (11) is "rotationallyinvariant" when introduces $\sum_{c} h_{c}^{k} T_{c b}$ in place of $h_{b}^{k}$ and observes that $T L T^{\prime}=L=T^{\prime} L T$. With that, the four components of $\psi$ transform amongst each other in precisely the same way that the four components of the ordinary Dirac $\psi$ transform under spatial rotations. Their invariance under general transformations of the Gaussian coordinate system will follow when one assumes that $\psi$ is an invariant. $\frac{\partial \psi}{\partial x^{k}}$ will be a vector then, and $h_{b}^{k} \frac{\partial \psi}{\partial x^{k}}$ will once more be a scalar. Likewise, $\frac{1}{H} \frac{\partial H h_{b}^{k}}{\partial x^{k}}$ is a scalar, namely, the divergence of $h_{b}$.

[^2]It will already follow from these properties of $\psi$, as well as what was done at the end of § 1, that $\left[\psi\left(x_{0}, x_{1}, x_{2}, x_{3}\right), \psi\left(x_{0}, x_{1}, x_{2}, x_{3}\right)\right]$ means the probability that the electron is at the point $x_{1}, x_{2}, x_{3}$ in volume 1 at the time $x_{0}$. If we construct a surface of area $1 / \Delta x_{0}$ that is perpendicular to the $x_{a}$-axis at the point $x_{1}, x_{2}, x_{3}$ then:

$$
\left(\psi\left(x_{0}, x_{1}, x_{2}, x_{3}\right), \gamma_{0} \gamma_{a} \psi\left(x_{0}, x_{1}, x_{2}, x_{3}\right)\right)
$$

will be the probability that the electron enters that surface at a time between $x_{0}$ and $x_{0}+$ $\Delta x_{0}$ from the large values of $x_{a}$ minus the probability that it enters from small values of $x_{a}$ . Since the unit operator $\left(1+\gamma_{0} \gamma_{a}\right) / 2$ belongs to the double eigenvalue +1 of the matrix $\gamma_{0} \gamma_{a},\left(\psi, \frac{1+\gamma_{0} \gamma_{a}}{2} \psi\right)$ will be the probability that this surface will be entered by the electron from the side of large $x_{a}$ and $\left(\psi, \frac{1-\gamma_{0} \gamma_{a}}{2} \psi\right)$ will be the probability that it is entered from the side of small $x_{a}$. All of the statements that are true in ordinary Dirac theory are valid in local coordinate systems.

Furthermore, the components of the four-current in the Gaussian coordinate system:

$$
\begin{equation*}
J^{k}=\left(\psi, h_{b}^{k} \gamma_{0} \gamma_{a} L_{a b} \psi\right) \tag{13}
\end{equation*}
$$

The divergence of the current is calculated thus:
$\operatorname{Div} J^{k}=\frac{1}{H} \frac{\partial}{\partial x^{k}}\left(\psi, H h_{b}^{k} \gamma_{0} \gamma_{a} L_{a b} \psi\right)$

$$
\begin{align*}
& =\left(\frac{\partial \psi}{\partial x^{k}}, \gamma_{0} \gamma_{a} h_{b}^{k} L_{a b} \psi\right)+\left(\psi, \gamma_{0} \gamma_{a} \frac{1}{H} \frac{\partial H h_{b}^{k}}{\partial x^{k}} L_{a b} \psi\right)+\left(\psi, \gamma_{0} \gamma_{a} h_{b}^{k} L_{a b} \frac{\partial \psi}{\partial x^{k}}\right) \\
& =2 \text { Real part of }\left[\left(\psi, \gamma_{0} \gamma_{a} h_{b}^{k} L_{a b} \frac{\partial \psi}{\partial x^{k}}+\frac{1}{2} \gamma_{0} \gamma_{a} \frac{1}{H} \frac{\partial H h_{b}^{k}}{\partial x^{k}} L_{a b} \psi\right)\right], \tag{14}
\end{align*}
$$

since $\gamma_{0} \gamma_{a}$ are Hermitian for all $a$ and all $h_{b}^{k}$ are pure real. If we multiply (11) by $\gamma_{0} \frac{2 \pi i}{h}$ and substitute that in (14) then we will get:

$$
\begin{equation*}
\text { Div } J^{k}=2 \text { Real part of }\left[\left(\psi, \gamma_{0} \frac{2 \pi i}{h} m \psi-\gamma_{0} \gamma_{a} \frac{2 \pi i e}{h} h_{b}^{k} L_{a b} \Phi_{k} \psi\right)\right] . \tag{15}
\end{equation*}
$$

However, the expression on the right (15) will vanish, since $\gamma_{0} \frac{2 \pi i}{h} m$, as well as $\gamma_{0} \gamma_{a} \frac{2 \pi i e}{h}$ are skew-symmetric, so their bilinear form (15) will be pure-imaginary then. One will then have:

$$
\begin{equation*}
\operatorname{Div} J^{k}=0 . \tag{16}
\end{equation*}
$$

If we had not symmetrized (10), corresponding to the non-commutation of $p$ and the $h_{a}^{k}$, then it would not be possible to define a divergence-free current.

I would not like to go into the physical consequences of (11) at this point. All that I wanted to show was that the Dirac theory of the spinning electron can be generalized in a simple and natural way with the help of Einstein's new theory.

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[^0]:    (*) A. Einstein, Berl. Ber. (1928).
    ${ }^{(* *)}$ Einstein initially employed an infinitesimal Euclidian coordinate system. However, in order to free us to free ourselves from the relativistic $i$, it will be convenient to employ a Lorentzian system directly. We can then set the distance between two world-points equal to $d s^{2}=d t^{2}-d x^{2}-d y^{2}-d z^{2}=$ $d x_{0}^{2}-d x_{1}^{2}-d x_{2}^{2}-d x_{3}^{2}$ if we assume that the speed of light $c=1$. The non-quantum mechanical equations can be written out as real, as long as no electromagnetic potentials are in them.
    (**) The indices $a, b$ then refer to the axes of the Lorentzian coordinate system (as in Einstein), while all others refer to the axes of the Gaussian coordinate system.

[^1]:    ( ${ }^{*}$ ) $\Delta_{\mu \alpha}^{\beta}$ is not a tensor. The quantity $C$ is a (very large) constant that is essentially the reciprocal of the gravitational constant.

[^2]:    (") Mathematically, the difficulty is based upon the fact that the two-dimensional representation of the Lorentz group cannot be extended to any representation in affine space. That difficulty came to light in the paper of H. Tetrode, $\mathbf{5 0}$ (1928), 336, due to the fact that a unique assignment of the coefficients in his equation did not seem possible, but one would have to further draw upon Einstein's new theory. H. Tetrode had already clearly recognized that this difficulty exists.

