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The Dirac-Madelung equations

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Summary. – Madelung has put the mechanics of Schrödinger into a form that approaches the mechanics of classical electrodynamics that relates to a continuous electric fluid, when applied to the electron. In the present article, we show that the relativistic mechanics of Dirac may be subjected to the same transformation. The tensorial form that is obtained here further presents a certain classical electrodynamical aspect. It highlights the geometric character of scalars, vectors, and tensors that are attached to the electron. These results may be of service in order to obtain the improvements that seem necessary in the present quantum theory of radiation.

1. Notations. – We utilize the geometric language and notations that are proper to the special relativity. An arbitrary point of the universe – or *spacetime* – is defined by its four (real) contravariant coordinates:

$$x_1, x_2, x_3 \qquad \text{(space)}, \qquad x_4 = ct \qquad \text{(time)}.$$

and the modulus-squared of the vector that joins the origin to that point is:

$$g^{ij}\, x_i\, x_j\,. \tag{1}$$

In practice, we choose ordinary rectangular axes. The preceding modulus-squared may then be written:

$$x_1^2 + x_2^2 + x_3^2 - x_1^2\,, \tag{2}$$

so the *timelike* vectors have a negative modulus-squared; this is the case for velocities. The spacelike vectors have a positive modulus-squared.

A great number of our formula that are established in rectangular axes are also valid in arbitrary oblique axes, provided that the change of coordinates results from a (linear) transformation of determinant equal to + 1.

Our notations demand a justification. Following the example of a good many physicists, it seems convenient to adopt lower indices for the contravariant components and upper indices for the covariant ones (†), quite simply because the contravariant components are more currently used than the other ones. On the other hand, the

(†) [D.H.D.: This is, of course, the opposite of the modern convention.]

modulus-squared (2) is often taken with a different sign; that convention seems preferable only for problems that relate to the mechanics of a material point.

The volume of spacetime that is spanned by four vectors a_i, b_i, c_i, d_i , namely:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix},$$

will be denoted by:

$$\alpha^{ijkl}\, a_i\, b_j\, c_k\, d_l\ .$$

The α^{ijkl} are covariant components of a tensor; their only possible values are – 1, 0, + 1. In particular, $\alpha^{1234} = + 1$.

In the geometry of spacetime, the vector product of two vectors a_i and b_j is an anti-symmetric tensor:

$$x_{ij} = a_i\, b_j - a_j\, b_i$$

that one calls a *bivector*. Among the non-zero components, for the applications, one envisions, in general, the six independent components:

$$x_{23}\ , \quad x_{31}\ , \quad x_{42}\ , \quad x_{14}\ , \quad x_{24}\ , \quad x_{34}\ .$$

We associate a bivector with an anti-symmetric tensor that is called an *associated bivector* and is defined by:

$$y_{ij} = \tfrac{1}{2}\, \alpha_{ijkl}\, x^{ij}.$$

In the fundamental rectangular system of coordinates, this definition is written more explicitly as:

$$\begin{aligned} y_{23} &= x_{14}\ , & y_{31} &= x_{24}\ , & y_{12} &= x_{34}\ , \\ y_{14} &= - x_{23}\ , & y_{24} &= - x_{31}\ , & y_{34} &= - x_{12}\ . \end{aligned}$$

We note that:

$$x_{ij} = -\tfrac{1}{2}\ \alpha_{ijkl}\, y^{kl}.$$

Finally, we utilize an abbreviated notation for partial derivatives, which is possible here because we have to imagine derivatives as being with respect to just the parameters x_1, x_2, x_3, x_4 . We simply set:

$$\partial^i = \frac{\partial}{\partial x_i}\ ;$$

in other words, the ∂^i are the contravariant symbolic components of the spacetime gradient operator.

2. Classical electrodynamical formulas. – The Maxwell-Lorentz theory of electric phenomena may not be applied today to the domain of microscopic physics. It is, nonetheless, at least an indispensable guide for the edification of more perfect descriptions of nature. We shall also transcribe some of its equations into our formalism. One must stop at the following problem, which was solved long ago, in principle, by reason of the indecision that has reigned up to now regarding the choice of notations.

One imagines a medium that is governed by the geometry of special relativity, in which one finds an electric current density world-vector j_i defined at each point; this vector is timelike. j_4 is the electric density of space in electrostatic units. In the case of an electron, it is a negative quantity. j_1, j_2, j_3 are the components of the spatial electric current density in electromagnetic units. Finally, there is a conservation law of electricity that may be written:

$$\partial^i j_i = 0. \tag{3}$$

The current is coupled with a potential world-vector that is given by the retarded potential formula:

$$V_i = \iiint \frac{1}{r} j_l \left(t - \frac{r}{c} \right) dx_1 dx_2 dx_3 , \tag{4}$$

which is valid only in the fundamental axes, and only when it is sufficiently specified. The derivative of the potential vector is the anti-symmetric tensor of the electromagnetic field:

$$F_{ij} = \partial_i V_j - \partial_j V_i , \tag{5}$$

a tensor that splits, in the ordinary language, into a magnetic field:

$$H_x = F_{23} , \qquad H_y = F_{31}, \qquad H_z = F_{12} , \tag{6}$$

and an electric field:

$$E_x = - F_{14} , \qquad E_y = - F_{24}, \qquad E_z = - F_{34} . \tag{7}$$

The magnetic field is expressed in electromagnetic units and the electric field, in electrostatic ones.

When the electric fluid is subject to the action of an electromagnetic field, which is either the electromagnetic field that it produces or an imposed field, there is good reason to consider a force density vector at each point of the universe:

$$f_i = F_{ik} j^k . \tag{8}$$

Introduce the vector u_i, with a modulus-squared that equals −1, and which is parallel to the current density vector, but in the opposite sense. We call this vector the *world-velocity*, regardless of whether its dimensions are those of velocity. We then set:

$$j_i = - e D u_i . \tag{9}$$

In this formula, e is the absolute value of charge of the electron, in electrostatic units. D is a scalar world-density that is always positive.

The transport of electricity is accompanied by a transport of matter. The simplest hypothesis consists of assuming that the representative vectors of these transports have a constant ratio. In other words, it consists of writing the matter current density vector as:

$$m_0 \, D u_i \, . \tag{10}$$

m_0 is the rest mass of the electron. Under the influence of resultant force density f_i , the motion of the electricity is then deduced from the relation:

$$m_0 \, c^2 \, D u_k \, \partial^k \, u_i = f_i \, . \tag{11}$$

Taking (5), (8), (9) into account, this relation may be further written:

$$u^k \left[\partial_i \left(m_0 c u_k - \frac{e}{c} V_k \right) - \partial_k \left(m_0 c u_i - \frac{e}{c} V_i \right) \right] = 0. \tag{12}$$

Introduce the impulsion-energy density vector:

$$D g_i = D \left(m_0 c u_i - \frac{e}{c} V_i \right). \tag{13}$$

A particular solution of (12) is obtained by assuming that g_i verifies the equation:

$$\partial_i \, g_j - \partial_j \, g_i = m_0 \, c \, (\partial_i \, g_j - \partial_j \, u_i) - \frac{e}{c} F_{ij} = 0. \tag{14}$$

One will note that the preceding summary ignores the existence of electric and magnetic polarization.

3. Dirac equations. – We must now direct our attention to a more correct theory of electric phenomena. The Dirac equations permit us to treat the evolution of an electron in an imposed electromagnetic field that is given by its potential world-vector V_j . This treatment conforms to the principle of special relativity. The electron is envisioned as a continuous fluid that is defined at point of space by four imaginary wave functions:

$$\Psi_\alpha , \quad \Psi_\beta , \quad \Psi_\gamma , \quad \Psi_\delta ,$$

which define, in an appropriate linear combination ([1]), not a vector, but a *spinor*.

The reader knows that one habitually considers the electron to be point-like. In this conception, the continuous fluid in question must have purely statistical significance. That way of looking at things renders the greatest service when one poses complex problems, such as the problem of the interaction of two electrons. However, it is

([1]) E. CARTAN, *Leçons sur la théorie des spineurs,* II, Paris, 1938.

absolutely superfluous for the intrinsic examination of the Dirac equations. It seems that we may better arrive at an understanding of them by not speaking of what they contain.

By starting with the wave functions, we may attach the following tensorial magnitudes ([2]) to each point of space:

Two invariants……………………………	Ω_1 and Ω_2
The current density vector……………….	j_i
The spin density vector…………………..	σ_i
The anti-symmetric magnetic and electric polarization tensor………………………..	μ_{ij}

We shall make the meaning of the tensor more precise. In a spatial volume element $d\tau$ there exists a small magnet whose electric moment has the components:

$$\mu_{23}\, d\tau, \qquad \mu_{31}\, d\tau, \qquad \mu_{12}\, d\tau,$$

and a small electric couple with components:

$$\mu_{14}\, d\tau, \qquad \mu_{24}\, d\tau, \qquad \mu_{34}\, d\tau .$$

We set:

$$\mu_x = \mu_{23}\,, \qquad \mu_y = \mu_{31}\,, \qquad \mu_z = \mu_{12}\,.$$

μ_z , μ_y , μ_x are the components of the magnetic polarization, or the intensity of magnetism. Likewise:

$$\mathfrak{M}_x = \mu_{14}\,, \qquad \mathfrak{M}_y = \mu_{24}\,, \qquad \mathfrak{M}_z = \mu_{34}\,,$$

are the components of the electric polarization.

We attach two current density vectors to the polarization, one of which is the electric current density of polarization:

$$\partial^j \mu_{ij}\,, \tag{14}$$

while the vector:

$$j_i - \partial^j \mu_{ij}$$

then represents the "true" electric current.

Now, imagine the bivector:

$$v_{ij} = \frac{1}{2}\,\alpha_{ijkl}\,\mu^{kl}. \tag{15}$$

There is a magnetic current density of polarization ([3]) whose components may be written:

$$-\,\partial^j v_{ij}\,. \tag{16}$$

([2]) L. DE BROGLIE, *L'électron magnétique*, Paris, 1934.
([3]) E. HENRIOT, *Les couples de radiation et les moments électromagnétiques*, Paris, 1936.

The following table recalls how the sixteen physical magnitudes are deduced from the wave functions ([4]).

In order to obtain the physical magnitudes, multiplied by the constant that we mentioned, it suffices to take the indicated wave functions in the corresponding row, with its sign, and multiply each of them by the conjugate imaginary that is at the head of the column, and then add them. The electric density j_4 is necessarily negative.

	Ψ'_α	Ψ'_β	Ψ'_δ	Ψ'_γ
$\frac{1}{e}j_1$ ……….	Ψ_δ	Ψ_γ	Ψ_β	Ψ_α
$-i\,\frac{1}{e}j_2$ …….	Ψ_δ	$-\Psi_\gamma$	Ψ_β	$-\Psi_\alpha$
$\frac{1}{e}j_3$ ……….	Ψ_γ	$-\Psi_\delta$	Ψ_α	$-\Psi_\beta$
$\frac{1}{e}j_4$ ……….	$-\Psi_\alpha$	$-\Psi_\beta$	$-\Psi_\gamma$	$-\Psi_\delta$
$\frac{2}{h}\sigma_1$ ………	Ψ_β	Ψ_α	Ψ_δ	Ψ_γ
$\frac{2}{h}\sigma_2$ ………	Ψ_β	$-\Psi_\alpha$	Ψ_δ	$-\Psi_\gamma$
$\frac{2}{h}\sigma_3$ ………	Ψ_α	$-\Psi_\beta$	Ψ_γ	$-\Psi_\delta$
$\frac{2}{h}\sigma_4$ ………	$-\Psi_\gamma$	$-\Psi_\delta$	$-\Psi_\alpha$	$-\Psi_\gamma$
$\frac{2m_0c}{eh}\mu_{23}$ …	Ψ_β	Ψ_α	$-\Psi_\delta$	$-\Psi_\gamma$
$-i\,\frac{2m_0c}{eh}\mu_{31}$ …	Ψ_β	$-\Psi_\alpha$	$-\Psi_\delta$	$-\Psi_\gamma$
$\frac{2m_0c}{eh}\mu_{12}$ …	Ψ_α	$-\Psi_\beta$	$-\Psi_\gamma$	Ψ_δ
$-i\,\frac{2m_0c}{eh}\mu_{14}$ …	$-\Psi_\delta$	$-\Psi_\gamma$	Ψ_β	Ψ_α
$\frac{2m_0c}{eh}\mu_{24}$ …	Ψ_δ	$-\Psi_\gamma$	$-\Psi_\beta$	Ψ_α
$-i\,\frac{2m_0c}{eh}\mu_{34}$ …	$-\Psi_\gamma$	Ψ_δ	Ψ_α	$-\Psi_\beta$
Ω_1 ………….	Ψ_α	Ψ_β	$-\Psi_\gamma$	$-\Psi_\delta$
$-i\Omega_2$ ………….	Ψ_γ	Ψ_δ	$-\Psi_\alpha$	$-\Psi_\beta$

([4]) ψ' represents the imaginary conjugate of ψ, $2\pi h$ is the Planck constant.

The various components that we just envisioned are not independent. One can prove the following relations ([5]):

$$j_k \, \sigma_k = 0, \tag{17}$$

$$\Omega_1^2 + \Omega_2^2 = -\frac{1}{e^2} j_k j^k = \frac{4}{h^2} \sigma_k \sigma^k , \tag{18}$$

$$\mu_{ij} = \frac{1}{m_0 c} (\Omega_1^2 + \Omega_2^2)^{-1} [\Omega_1 \, \alpha_{ijkl} \, j^k \sigma^l - \Omega_2 \, (j_i \, \sigma_j - j_j \, \sigma_i)]. \tag{19}$$

One may have no other independent relation. It is advantageous to write these formulas by introducing the unitary vector u_i and a new unitary vector s_i whose modulus-squared is equal to + 1 and is parallel to the vector σ_i in the same sense. One then finds that:

$$u_k \, s^k = 0, \tag{20}$$

$$\sigma_i = \frac{1}{2} h \, D \, s_i , \tag{21}$$

$$\Omega_1^2 + \Omega_2^2 = D^2. \tag{22}$$

We may set:

$$\Omega_1 = D \cos \eta, \quad \Omega_2 = D \sin \eta. \tag{23}$$

The polarization density may now be written:

$$\mu_{ij} = \frac{eh}{2m_0 c} D \, [\sin \eta \, (u_i \, s_j - u_j \, s_i) - \cos \eta \cdot \alpha_{ijkl} \, u^k \, s^l]. \tag{24}$$

We further note the useful formula:

$$\nu_{ij} = \frac{eh}{2m_0 c} D \, [\cos \eta \, (u_i \, s_j - u_j \, s_i) + \sin \eta \cdot \alpha_{ijkl} \, u^k \, s^l]. \tag{25}$$

Now write the Dirac partial differential equations, which regulate the evolution of the wave functions, and, in turn, the physical magnitudes:

$$\left.\begin{aligned}
(-i\partial^4 + X_4 + K)\Psi_\alpha + (i\partial^1 + X_1 - \partial^2 + iX_2)\Psi_\delta + (i\partial^3 + X_3)\Psi_\gamma = 0,\\
(-i\partial^4 + X_4 + K)\Psi_\beta + (i\partial^1 + X_1 + \partial^2 - iX_2)\Psi_\delta + (i\partial^3 + X_3)\Psi_\delta = 0,\\
(-i\partial^4 + X_4 - K)\Psi_\gamma + (i\partial^1 + X_1 - \partial^2 + iX_2)\Psi_\delta + (i\partial^3 + X_3)\Psi_\alpha = 0,\\
(-i\partial^4 + X_4 - K)\Psi_\delta + (i\partial^1 + X_1 + \partial^2 - iX_2)\Psi_\delta + (i\partial^3 + X_3)\Psi_\beta = 0.
\end{aligned}\right\} \tag{26}$$

In order to suggest any series of intermediate calculations, we have set:

$$K = m_0 \, c : h.$$
$$X_i = V_i \cdot e : eh.$$

([5]) L. DE BROGLIE, *loc. cit.*, and J. YVON, *C. R. Acad. Sc.* **205** (1937), 1367.

4. Elimination of the wave functions. We now propose to eliminate the wave functions in such a manner as to establish the theory of the relations in which the current, spin, the two invariants, and the tensorial magnitudes that they are attached to figure explicitly. We thus set about to do, *a propos* of the Dirac electron, what Madelung ([6]) did *a propos* of the non-relativistic quantum electron.

We first recall two differential relations that are easily established by linearly combining the Dirac equations with the aid of the conjugate imaginaries of the wave functions. They are the equation of conservation of electricity:

$$\partial^i j_i = 0 \qquad \text{or} \qquad \partial^i Du_i = 0 \tag{27}$$

and the equation that shows that, in general, spin density vector is not conservative:

$$\partial^i Ds_i + \frac{2m_0c}{h} D \sin \eta = 0. \tag{28}$$

The wave functions may always be put into the form:

$$\Psi_\omega = F_\omega e^{i\varphi}, \tag{29}$$

where ω is one of the indices $\alpha, \beta, \gamma, \delta$, and φ is a real quantity. We choose φ in such a manner that F_δ, for example, is a real quantity. Upon linearly combining the formulas that issue from Table I, one finds that:

$$4\,\Psi_\delta \Psi_\delta = Du_4 - Ds_3 - \Omega_1 + \frac{2m_0c}{h}\mu_{12}\,. \tag{30}$$

We may then set, with an unvarying sign in a domain where F_δ is non-zero:

$$2F_\delta = \pm\left(Du_4 - Ds_3 - \Omega_1 + \frac{2m_0c}{h}\mu_{12}\right)^{1/2}. \tag{31}$$

We further set:

$$\left.\begin{aligned}
&a_1 = -Du_1 + \frac{2m_0c}{eh}\mu_{24,} && a_2 = -Du_2 - \frac{2m_0c}{eh}\mu_{14,}\\
&a_3 = D(u_3 - s_1), && a_4 = \Omega_2 + \frac{2m_0c}{eh}\mu_{34,}\\
&a_5 = Ds_1 - \frac{2m_0c}{eh}\mu_{23,} && a_6 = Ds_2 - \frac{2m_0c}{eh}\mu_{31,}\\
&a_7 = Ds_1 - Ds_3 - \Omega_1 + \frac{2m_0c}{eh}\mu_{12.}
\end{aligned}\right\} \tag{32}$$

([6]) F. MADELUNG, *Zeit. für Phys.* **40** (1926), 322.

the indices that are affected with the letter *a* have no natural tensorial significance. The parameters *a* are intermediate tools for our calculation. They permit us to eliminate the *F* with the aid of the relations:

$$\left.\begin{aligned} &4F_\delta F_\alpha = a_1 + ia_2, \; 4F_\delta F_\beta = a_3 + ia_4, \\ &4F_\delta F_\gamma = a_5 + ia_6 \quad\;\; 4F_\delta^2 = a_7. \end{aligned}\right\} \tag{33}$$

It is easy to show that the *a* verify the relations:

$$\left.\begin{aligned} &a_7(\partial^1\varphi - X_1) + a_5(\partial^3\varphi - X_3) - a_7(\partial^4\varphi - X_4) \\ &\quad = Ka_4 - \partial^3 a_6 + \partial^2 a_2 + \frac{1}{2a_7}(-a_7\partial^2 + a_6\partial^3 - a_2\partial^4)a_5, \\ &a_7(\partial^2\varphi - X_2) + a_6(\partial^3\varphi - X_3) - a_2(\partial^4\varphi - X_4) \\ &\quad = Ka_2 + \partial^3 a_5 - \partial^4 a_1 + \frac{1}{2a_7}(a_7\partial^1 - a_5\partial^3 + a_4\partial^4)a_7, \\ &a_6(\partial^1\varphi - X_1) - a_5(\partial^2\varphi - X_2) - a_4(\partial^4\varphi - X_4) \\ &\quad = Ka_4 + \partial^4 a_5 - \partial^4 a_3 + \frac{1}{2a_7}(-a_5\partial^1 - a_6\partial^2 - a_7\partial^3 + a_3\partial^4)a_7, \\ &a_2(\partial^1\varphi - X_1) - a_1(\partial^2\varphi - X_2) - a_4(\partial^3\varphi - X_3) \\ &\quad = \partial^1 a_1 + \partial^2 a_2 - \partial^3 a_3 + \frac{1}{2a_7}(-a_1\partial^1 - a_2\partial^2 + a_3\partial^3 - a_3\partial^4)a_7 \end{aligned}\right\} \tag{34}$$

and

$$\left.\begin{aligned} &a_4(\partial^1\varphi - X_1) + a_3(\partial^2\varphi - X_2) + a_2(\partial^3\varphi - X_3) - a_6(\partial^4\varphi + X_4) \\ &\quad = -Ka_6 + \partial^1 a_3 - \partial^2 a_4 + \partial^3 a_1 + \partial^4 a_5 \\ &\qquad + \frac{1}{2a_7}(-a_3\partial^1 + a_4\partial^2 - a_1\partial^3 + a_4\partial^5)a_7, \\ &a_3(\partial^1\varphi - X_1) - a_4(\partial^2\varphi - X_2) + a_1(\partial^3\varphi - X_3) - a_5(\partial^4\varphi + X_4) \\ &\quad = -Ka_5 - \partial^1 a_4 - \partial^2 a_3 - \partial^3 a_2 + \partial^4 a_6 \\ &\qquad + \frac{1}{2a_7}(a_4\partial^1 + a_5\partial^2 + a_2\partial^3 - a_3\partial^2 - a_6\partial^4)a_7, \\ &a_1(\partial^1\varphi - X_1) + a_2(\partial^2\varphi - X_2) - a_3(\partial^3\varphi - X_3) - a_7(\partial^4\varphi + X_4) \\ &\quad = -Ka_7 - \partial^1 a_2 + \partial^2 a_1 + \partial^3 a_4 \\ &\qquad + \frac{1}{2a_7}(a_2\partial^1 - a_1\partial^2 - a_4\partial^3)a_7, \\ &a_5(\partial^1\varphi - X_1) + a_6(\partial^2\varphi - X_2) - a_7(\partial^3\varphi - X_3) - a_3(\partial^4\varphi + X_4) \\ &\quad = -Ka_7 - \partial^1 a_6 + \partial^2 a_5 + \partial^4 a_4 \\ &\qquad + \frac{1}{2a_7}(a_6\partial^1 - a_5\partial^2 - a_4\partial^4)a_7. \end{aligned}\right\} \tag{35}$$

Consider the four expressions:

$$\partial^1\varphi - X_1, \qquad \partial^2\varphi - X_2, \qquad \partial^3\varphi - X_3, \qquad \partial^4\varphi - X_4$$

to be unknowns. Equations (34) have a determinant with respect to these unknowns that must be simply – $(2\Omega_2\ a_7)^2$. The solution is not a hopeless enterprise; of the four expressions that we obtain, we retain only the last two:

$$\left.\begin{aligned} 2\Omega_2(\partial_3\varphi - X_3) &= -\frac{2m_0c}{eh}\partial^i v_{3i} + \frac{1}{a_7}(a_5\partial_3 a_1 + a_6\partial_3 a_2 + a_7\partial_3 Du_3 + Ds_4\partial_3 a_7), \\ 2\Omega_2(\partial_4\varphi - X_4) &= -\frac{2m_0c}{eh}\partial^i v_{4i} + \frac{1}{a_7}(a_5\partial_4 a_4 + a_6\partial_4 a_2 + a_7\partial_4 Du_3 + Ds_4\partial_4 a_7). \end{aligned}\right\} \qquad (36)$$

Some linear combinations that are obtained by trail and error when one now starts with the eight equations (34) and (35) finally give us, at the same time, four relations that are summarized by:

$$\begin{aligned} &2\Omega_1(\partial^i\varphi - X_i) \\ &\quad = \frac{2m_0c}{eh}(eDu_i + \partial^j\mu_{ij}) \\ &\qquad + \frac{1}{2a_7}(a_2\,\partial_i\,a_4 - a_4\,\partial_i\,a_2 + a_4\,\partial_i\,a_3 - a_3\,\partial_i\,a_4 + a_6\,\partial_i\,a_5 - a_5\,\partial_i\,a_6). \end{aligned} \qquad (37)$$

In the domain where we placed ourselves, the Dirac equations are equivalent to the set of equations (27), (28), (36), and (37).

Some very long calculations then permit us to show that:

$$\begin{aligned} &(a_5\,\partial\,a_4 + a_6\,\partial\,a_2 + a_7\,\partial\,Du_3 + Ds_4\,\partial\,a_7) \\ &\quad = + a_7\,Ds^i\,\partial u_i \cdot \cos\eta \\ &\qquad + D^2\sin\eta \times \{[u_2\cos\eta + (u_2\,s_3 - u_3\,s_2)]\,\partial u_1 \\ &\qquad + [-u_1\cos\eta + (u_3\,s_1 - u_1\,s_3)]\,\partial u_2 \\ &\qquad + [u_1\sin\eta + (u_1\,s_2 - u_2\,s_1)]\,\partial u_3 - u_3\sin\eta\,\partial u_1 \\ &\qquad + [-s_2\sin\eta - (u_2\,s_4 - u_4\,s_2)]\,\partial s_1 \\ &\qquad + [s_2\cos\eta + (u_1\,s_4 - u_4\,s_1)]\,\partial s_2 - s_4\sin\eta\,\partial s_3 \\ &\qquad + [s_3\cos\eta - (u_1\,s_2 - u_2\,s_1)]\,\partial s_4 + (-u_3 + s_4)\,\partial\eta \\ &\qquad + \left(\sin\eta - \frac{2m_0c}{eh}\frac{\mu_{34}}{D}\right)\cdot s^i\,\partial u_i\}, \end{aligned} \qquad (38)$$

and that:

$$\begin{aligned} &(a_2\,\partial\,a_1 - a_1\,\partial\,a_2 + a_4\,\partial\,a_3 - a_3\,\partial\,a_4 + a_6\,\partial\,a_5 - a_5\,\partial\,a_6) \\ &\qquad = -2a_7\,D \cdot s^i\,\partial u_i \cdot \sin\eta + 2\,D^2\cos\eta\,\{\ldots\}. \end{aligned} \qquad (39)$$

The indefinite bracket that figures in (39) is the same as the one that figures in (38). We may now eliminate $\partial^3\varphi - X_3$ from the first equation of (36) and the third equation of (37), and operate in an analogous manner on $\partial_4\varphi$. The two relations obtained are written in a general manner as:

$$m_0 c\left[\sin\eta\left(Du_i + \frac{1}{e}\partial^i\mu_{ij}\right) + \frac{1}{e}\cos\eta\cdot\partial^i\nu_{ij}\right] = \frac{h}{2} D\, s^j \partial_i u_j\,. \tag{40}$$

The first relation obtained corresponds to the index 3, while the second one, to the index $i = 4$. In reality, the relations that correspond to the index $i = 1$ and the index $i = 2$ are likewise verified. Indeed, these four relations are not independent. Taking into account the properties of the vectors u_i and s_i (viz., orthogonality, modulus-squared equal to + 1 or – 1) and equations (27) and (28), of these four relations, only two of them are independent, which may be chosen arbitrarily; their verification entails that of the other two.

We have thus replaced relations (36) with relations (40) in our system of equations. We shall now direct our attention to equations (37). We may eliminate φ, which does not have a tensorial character, by forming the six expressions:

$$\partial_i(\partial_j\varphi) - \partial_j(\partial_i\varphi).$$

The calculation is prohibitive in practice due to the complexity of the expression (39) that figures in the right-hand side. One begins by using the relativistic invariance of the Dirac theory and by imagining that one uses particular axes for which the orthogonal vectors u_i and s_i are each reduced to having just one component at a given point of spacetime in the neighborhood that we are considering. It is then easy to return to arbitrary orthogonal axes. We thus find that:

$$\partial_i(\partial_j\varphi) - \partial_j(\partial_i\varphi)$$
$$= \partial_i g_j - \partial_j g_i + \frac{h}{2}\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ s_1 & s_2 & s_3 & s_4 \\ \partial_i s_1 & \partial_i s_2 & \partial_i s_3 & \partial_i s_4 \\ \partial_j s_1 & \partial_j s_2 & \partial_j s_3 & \partial_j s_4 \end{vmatrix} - \frac{h}{2}\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ s_1 & s_2 & s_3 & s_4 \\ \partial_i u_1 & \partial_i u_2 & \partial_i u_3 & \partial_i u_4 \\ \partial_j u_1 & \partial_j u_2 & \partial_j u_3 & \partial_j u_4 \end{vmatrix} = 0. \tag{41}$$

We have introduced a new parameter g_i that is defined by:

$$Dg_i = \frac{e}{c}DV_i - m_0 c\left[\cos\eta\left(Du_i + \frac{1}{e}\partial^j\mu_{ij}\right) - \frac{1}{e}\sin\eta\,\partial^j\nu_{ij}\right]. \tag{43}$$

The vector Dg_i seems to generalize the energy-impulsion density vector of the classical theory. What we called the transport of matter in paragraph 2 takes on a very complicated expression. One part of the matter transport is coupled to the true electric current, as in paragraph 2, on the condition that one first multiply the corresponding current density by – cos η. Likewise, a matter transport is coupled with the magnetic current, always with the same proportionality relation, on the condition that one first multiply the density of that current by + sin η.

Equations (41), in which the g_i are unknowns, are integrable, which results from the manner itself by which they were obtained, but this result may also be established directly by profiting from the properties of the vectors u_i and s_i that figure in the two determinants.

In formulas (40) and (42), we specify the polarization densities as functions of the vectors u_i and s_i ; taking into account of (27) and (28), one comes to the equivalent relations:

$$\alpha_{ijkl}\, u^k s^l \partial^j \eta + s^j \partial_j u_i - u^j \partial_j s_i = s^j \partial_j u_i\,, \tag{43}$$

$$-g_i = \frac{e}{c} V_i + m_0 c \cos \eta \cdot u_i + \frac{1}{2} h \left[(u_i s_j - u_j s_i) \partial^j \eta - \frac{1}{D} \alpha_{ijkl} \partial^j D u^k s^l \right], \tag{44}$$

respectively. Any solution of the Dirac equation corresponds to a solution of the tensorial equation. The converse is not true. The discussion of these equations therefore necessitates a study that we shall not commence in this article. Nevertheless, here are several remarks:

Of course, *a propos* of tensorial equations, we shall first consider only the solutions that are uniform (except, perhaps, as far as the parameter η is concerned), continuous, and differentiable. D then necessarily has the same sign everywhere. It is, by convention, the positive sign.

With the Dirac equations, u_4 is everywhere positive. With the tensorial equations, it not, *a priori*, impossible that u_4 must change sign at some point or another; we leave these solutions aside. However, there certainly exist solutions for which u_4 is everywhere negative. Let a certain solution be:

$$u_i\,, \qquad s_i\,, \qquad \eta, \qquad D, \qquad u_4 > 0, \tag{45}$$

which corresponds to a certain imposed potential V_i that changes from V_i into $-V_i$. We immediately obtain a solution of the new problem with:

$$-u_i\,, \quad s_i\,, \qquad \eta, \qquad D, \qquad u_4 < 0, \tag{46}$$

if (45) represents a negative electron and (46) represents a positive electron. We immediately obtain another solution of the new problem:

$$u_i\,, \quad -s_i\,, \quad \eta + \pi, \quad D, \qquad u_4 > 0. \tag{46}$$

We are permitted to abstract from the formula:

$$j_i = -\, e\, D\, u_i$$

the fact that the electron (47) is a positive electron, as the following paragraph shows clearly.

We now limit ourselves to solutions for which u_4 is positive. They are solutions of the Dirac system only if the parameter a_7 defined by (32) is positive. We quickly point out that this parameter is surely positive if cos η is negative, but if cos η is positive, an appropriate choice of axes might make a_7 negative (u_4 always remains positive after the change), at least in certain domains. As one sees, the case in which cos η is positive is attached to the positive electron as it is interpreted by (47). It thus seems that the various tensorial equations are quite profound, as far as the two types of electricity are concerned. It is, moreover, impossible to expand upon this subject in the absence of the theory of radiation.

5. Non-quantum approximation. – Our tensorial equations easily permit us to find out what happens to the electron when the Planck constant goes to zero. Equation (28) shows us that in the limit:

$$\sin \eta = 0,$$

and as a result:

$$\cos \eta = \pm 1.$$

Equations (44) and (41) become:

$$g_i = -\frac{e}{c} - m_0 c\, u_i \cdot \cos \eta, \tag{45}$$

$$\partial_i g_j - \partial_j g_i = 0. \tag{46}$$

The motion of the electricity is the one that we encountered in classical electrodynamics when the field that is defined by V is composed of only the imposed field, and the world-rotation of the energy-impulsion is zero. We take into account solutions such that u_4 is positive. The case cos $\eta = -1$ corresponds to the ordinary electron, while the case cos $\eta = +1$ corresponds to the positive electron. Equation (11) is then valid.

The limiting system of equations also comprises the equation for conservation of electricity. By contrast, the spin and the polarization tensor are annulled. Nevertheless, equation (43), which governs the evolution of the unitary vector s_i, which is parallel to the spin density vector, preserves its meaning. It is written:

$$u^j \partial_j s_i = -s^j (\partial_i u_j - \partial_j u_i). \tag{47}$$

Once equations (45) and (46) have been solved, one may, on the one hand, solve the conservation equation that gives D, and, on the other hand, equation (47). It gives the evolution of the spin, which becomes infinitely small along a current trajectory. The spin plays only a passive role that is, nevertheless, interesting to examine. Equation (13) permits us to eliminate:

$$\partial_i u_j - \partial_j u_i$$

from equation (47), which is now written:

$$u^j\,\partial_j\,s_i = \mp\frac{e}{m_0c^2}F_{ij}\,s^j. \tag{48}$$

At the same time, we note the present expression for the polarization density:

$$\mu_{ij} = \mp\frac{eh}{2m_0c}\,\alpha_{ijkl}\,u^k\,s^l. \tag{49}$$

This evolution of the spin is identical to the one that Kramers ([7]) imagined. In the elementary theory of magnets, a magnet that is placed in a magnetic field is subject to a couple. The simplest possible relativistic generalization is given by the mechanical angular momentum density vector:

$$\mu_{ik}F_j^k - \mu_{jk}F_i^k\,, \tag{50}$$

or, from (49):

$$\pm\frac{eh}{2m_0c}D\ \alpha_{ijkl}\,(s_m\,F^{mk}\,u^l - u_m\,F^{mk}\,s_l).$$

Taking into account the general properties of u_i, s_i in (46), the relations (48) are equivalent to the relation:

$$\frac{1}{2}\,hcu_m\,\partial^m(\alpha_{ijkl}\,u^k\,s^l) = \pm\frac{eh}{2m_0c}\,\alpha_{ijkl}\,(s_m\,F^{mk}\,u^l - u_m\,F^{mk}\,s_l), \tag{51}$$

which shows the identity of the Kramers formulas and our own. In fact, Kramers argued on the basis of a point-like model, but it will be easy to convert our approximation, which is a "geometric optics" approximation to the point-like model. The tensor:

$$\frac{1}{2}\,hc\ \alpha_{ijkl}\,u^k\,s^l$$

plays the same role, *a propos* the action of the couples, as the vector:

$$m_0c\ u_i$$

does, *a propos* the action of forces.

It will be, without a doubt, interesting to recall this classical treatment of spin and replace the imposed field or external field by the total field and attempt to formulate the conservation theorems by following the method that was advocated by Henriot ([8]).

([7]) H. A. KRAMERS, *Physica* **1** (1934), 825.
([8]) E. HENRIOT, *loc. cit.*

6. Non-relativistic approximation. – One develops this approximation upon regarding $1/c$, u_1, u_2, u_3 as infinitely small; meanwhile, $\frac{e}{c}V$ remains finite. We suppose, moreover, that s_i has a direction that is basically variable – for example, that of *Oz*. s_3 is then close to + 1, s_1, s_2, s_4 ,are of first order, while sin η is of second order. One may regard cos η as constant, and if we limit ourselves to the case of the ordinary electron, as equal to + 1. The magnetic field is not arbitrary. It must be parallel to *Oz*. A function *S* permits one to eliminate the velocities u_1, u_2, u_3 by means of the relations:

$$m_0c\,u_1 = \frac{\partial S}{\partial x_1} + \frac{e}{c}V_1,$$
$$m_0c\,u_2 = \frac{\partial S}{\partial x_2} + \frac{e}{c}V_2,$$
$$m_0c\,u_3 = \frac{\partial S}{\partial x_3} + \frac{e}{c}V_3.$$

That function must verify the relation:

$$\frac{\partial S}{\partial t} - eV_4 + \frac{1}{2m_0}\left[\left(\frac{\partial S}{\partial x_1} + \frac{e}{c}V_1\right)^2 + \left(\frac{\partial S}{\partial x_2} + \frac{e}{c}V_2\right)^2 + \left(\frac{\partial S}{\partial x_3} + \frac{e}{c}V_3\right)^2\right] + \frac{eh}{2m_0c}H_3 - \frac{h^2}{2m_0}\frac{\Delta\sqrt{Du_4}}{\sqrt{Du_4}}$$
$$= 0,$$

to which, one must add the equation for the conservation of electricity. We have therefore obtained a Schrödinger equation of the problem in the form that was given by Madelung ([9]). The present article then consists essentially in a generalization of the ideas of Madelung.

7. Conclusion. – We think that we have clarified the significance of the tensorial magnitudes that relate to the magnetic electron, as well as the relations that exist between them. These relations show that the magnetic electron may be treated as an electrically-charged and polarized continuous fluid and that it then seems to present a fundamentally classical character, at least when one considers only a imposed electromagnetic field.

In this restricted domain, one may pretend that the undulatory mechanics of the electron is nothing but a consequence of its magnetism. Indeed, the undulatory properties seem to be closely coupled with the existence of a polarization tensor. Nevertheless, one must not forget that our tensorial system is not strictly equivalent to that of Dirac.

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([9]) F. MADELUNG, *loc. cit.*