"Équations de Madelung de l'électron magnétique non relativiste," Revue Scientifique **79** (1940), 209-215.

Madelung equations of the non-relativistic magnetic electron

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Introduction

The quantum theory of the electron that was given by Dirac is a theory of an imposed field, but not a theory of the field produced by the electron itself, a field that is, in particular, responsible for the absorption of radiation.

In an imposed field, the electron is described by four wave functions. These functions depend upon the space and time coordinates.

The set of wave functions defines a spinor, a geometric entity whose intervention is new in physics. This spinor is coupled to two scalars, two vectors, and a bivector by bilinear formulas. These various vectorial magnitudes collectively have a familiar physical significance.

One will note that this classification relating to the vectorial magnitudes that we just indicated and involving two scalars, two vectors, and a bivector, supposes a limitation on the possible changes of the reference system that is used. Indeed, we use the world rotation to the exclusion of reversals. If the reversals must be likewise used then the nomenclature of the vectorial magnitudes becomes a bit more complicated.

In an earlier paper ([1]), I established the partial differential equations that permit one to predict the evolution of the vectorial magnitudes as functions of time, equations that must contain only the vectorial magnitudes, and which must not make the wave functions appear explicitly. An analogous study was carried out some time ago by Madelung on the subject of particles whose mechanics is ruled by the Schrödinger equation, and in which he showed the existence of a hydrodynamical form for undulatory mechanics. This result was found to be true in the context of the Dirac electron. Moreover, on the one hand, the vectorial equations have permitted us to obtain the limited theory of the spin of the electron when Planck's constant goes to zero. The electron then has properties that one may arrive at classically as a gyroscope with just one moment of inertia that is endowed with magnetism. It leads one, on the other hand, to introduce naturally an angular parameter that is closely linked with the negative energy states. (There is good reason to return to the study of that parameter.)

The vectorial equations are rather easy to work with. Indeed, these equations, which are not linear, involve a large number of parameters. The examination of these equations shows, nevertheless, that their complexity may, in part, be due to an inadequate representation. It seems to me that by combining scalars, vectors, and bivectors in such a manner as to obtain new geometric parameters, one perhaps arrives at new parameters

([1]) Journal de Physique, 1940, pp. 18.

with a less uninviting aspect. However, a conducting wire fails to provide the necessary transformations. In that regard, it also seems advantageous to me to first study a particular case – viz., the case in which one neglects relativity. It is the results of that study that constitute the basis for this article. The reader will confirm that the introduction of a new velocity vector and the use of a pressure tensor render the formulas simpler to read and strongly emphasizes their hydrodynamical character.

The non-relativistic electron that we have occupied ourselves with is described, in spinorial notations, by the Pauli equations. In order to obtain the vectorial equations, a method thus consists in working with the Pauli equations as Madelung did with the Schrödinger equations. However, since the Pauli equations are a non-relativistic approximation to the Dirac equations, we arrive at the desired result with less work by looking for the non-relativistic approximation for the vectorial equations that are equivalent to the Dirac equations.

In the paper in *Journal de Physique* that we referred to, we have already written down the Madelung equations for a case that is even more particular: One supposes, as one does here, that relativity is negligible, and one further attributes a direction to the spin that is essentially variable.

Before going further, we must refute an objection to the principle that one might oppose the use made by us of a profusion of scalar or vectorial fields.

1. – Local magnitudes

In undulatory mechanics, the fundamental magnitudes are calculated upon starting with wave functions by integration in space. For example, take a particle whose properties are governed by an ordinary Schrödinger equation. The impulse of the particle along *Ox* is:

$$\int \Psi' \left(-ih \frac{\partial}{\partial x} \Psi \right) d\tau \,. \tag{1}$$

An integration by parts allows one to write this impulse in the form:

$$\frac{1}{2} ih \int \left(\Psi \frac{\partial}{\partial x} \Psi' - \Psi' \frac{\partial}{\partial x} \Psi \right) d\tau \,, \tag{2}$$

the integration being real this time. In order to make the language more precise, we call the impulsion (1) or (2) and *integral impulse*. One may regard the expression (2) as resulting from a calculation of the mean. It is then convenient to say that:

$$\frac{1}{2} ih \left(\Psi \frac{\partial}{\partial x} \Psi' - \Psi' \frac{\partial}{\partial x} \Psi \right)$$

represents the impulse density at *x*, *y*, *z*. However, since $\Psi\Psi'$ is the density (*tout court*), the expression:

$$\frac{1}{2}ih\left(\Psi\frac{\partial}{\partial x}\Psi' - \Psi'\frac{\partial}{\partial x}\Psi\right)/\Psi\Psi' \tag{3}$$

represents a magnitude that has the dimensions of an impulse. This impulse is what we call the *local impulse.*

This poses the question of knowing whether the use of such a terminology presents an incontestable necessity. Indeed, when one has defined a whole series of local magnitudes – impulse, energy, etc. – one perceives that the relations that exist between these magnitudes are different from the ones in classical physics that we are used to, while the classical relations are safeguarded on the subject of integral magnitudes, thanks to the method of operators. This fact compels one to adopt serious precautions in the use of local magnitudes, in spite of the fact that numerous physicists believe it preferable to not use them.

Nonetheless, the use of local magnitudes is imposed when one attempts to introduce a statistical interpretation into undulatory mechanics. This is why all of the authors assume that one argues on the basis of the density $\Psi\Psi'$, which is an essentially local quantity with a well-known statistical interpretation. As for the local impulse (3), it must, in the statistical language, be regarded, not as an impulse that is taken by the particle when it is found in the element *dt* – that impulse not being defined rigorously – but as the mean of all possible impulses when that circumstance is realized.

Since the local impulse is a mean magnitude, one must not expect to see it verify the classical relations. In that regard, the situation with the local magnitudes is the same in undulatory mechanics and the kinetics of gases. This is why the kinetic energy is not equal to the square of the impulse divided by twice the mass of the particle in either undulatory mechanics or the kinetics of gases. With that remark, the use of local magnitudes seems to be completely justified in undulatory mechanics, as it is in the kinetics of gases; i.e., hydrodynamics, in practice.

With that argument, we believe that there is no serious objection to the use of arguing with local magnitudes. In the sequel, we will make use of such magnitudes uniquely; e.g., density, velocity vectors, spin…

By definition, one of the major justifications for the use of local magnitudes resides in the elegance of the formulas that they verify. We hope that some of the formulas that we obtain later on tend to support that opinion.

We add that the combined use of local spin and wave functions in the problems that relate to the chemical valence removes their often paradoxical character and disposes of the reasoning that is carried out in terms of discontinuous functions of spin, as found in the method of Slater. These discontinuous functions are ingenious, but highly abstract. The merit of local magnitudes is, at least pedagogically, that of having a concrete sense.

2. – Summary of the previous results

In quantum mechanics, any particle is endowed with a point-like character and an undulatory character. Nevertheless, for the treatment of certain limit problems that relate to a unique particle, it is possible to abstract from the point-like character of the particle. It may then be regarded as smeared into a sort of statistical cloud. This cloud evolves in

the manner of a fluid in hydrodynamics, or, if one is dealing with a charged particle, in the manner of a charged continuous fluid, which brings in numerous methods of macroscopic electrodynamics. They are, at least, very reasonable for comparisons.

In particular, if one starts with the Dirac equation then the magnetic electron may be represented by a fluid of this type.

At each point of space, one defines a world-density D, a velocity vector u_i, whose modulus-squared is – 1, a vector s_i that is coupled to spin, whose modulus-squared is + 1, a bivector μ_{ij} of electric and magnetic polarization density, and an angle η. The vectors u_i and s_i are orthogonal. The electric current density is:

$$- e D u_i .$$

This current is the sum of a "true" electric current and a "free" current. The latter current results from the local temporal variations of the polarization. Let:

$$\partial^i$$

denote the operator:

$$\partial/\partial x_i .$$

The current that is coupled to the polarization is then written:

$$\partial^i \mu_{ij} .$$

The spin density is:

$$\tfrac{1}{2} h D s_i .$$

Finally, we refer to the aforementioned for the expression of the bivector μ_{ij} as a function of D, u_i, s_i, η.

Let V_i be the potential world-vector of an imposed electromagnetic field. At each point, one defines an impulse-energy vector (a local magnitude):

$$g_i = - \frac{e}{c} V_i - mc \left[\cos\eta (D u_i + \frac{1}{e} \partial^i \mu_{ij}) - \frac{1}{e} \sin\eta \cdot \partial^i \nu_{ij} \right] / D .$$

in this formula, ν_{ij} is the bivector "associated to" μ_{ij} ([1]).

The partial differential equations are the following ones:

(4) $$\partial^i D u_i = 0,$$

(5) $$\partial^i D s_i + 2 \frac{mc}{h} S \sin \eta = 0,$$

([1]) At the risk of appearing to abuse the local magnitudes, in the first paper, g_i did not qualify as an impulse-energy. Only the product Dg_i qualifies as the "impulse-energy density." However, this is an unfortunate terminology (even if it is formally justified) because the product Dg_i does not enjoy any interesting property.

(6) $$mc\left[\sin\eta\left(Du_i+\frac{1}{e}\partial^i\mu_{ij}\right)+\frac{1}{e}\cos\eta\cdot\partial^i\nu_{ij}\right] = \tfrac{1}{2}h\,D\,s_i\,\partial^j g_i\,,$$

and:

$$\partial_i g_j-\partial^j g_i+\frac{1}{2}h\begin{vmatrix} u_1 & u_2 & u_3 & u_4\\ s_1 & s_2 & s_3 & s_4\\ \partial^i s_1 & \partial^i s_2 & \partial^i s_3 & \partial^i s_4\\ \partial^j s_1 & \partial^j s_2 & \partial^j s_3 & \partial^j s_4\end{vmatrix}-\frac{1}{2}h\begin{vmatrix} u_1 & u_2 & u_3 & u_4\\ s_1 & s_2 & s_3 & s_4\\ \partial^i u_1 & \partial^i u_2 & \partial^i u_3 & \partial^i u_4\\ \partial^j u_1 & \partial^j u_2 & \partial^j u_3 & \partial^j u_4\end{vmatrix}=0.$$

I finally emphasize that in the course of our old argument, no effort was made to attach the local impulse (or energy) to an impulse (or energy) integral. Since the integral magnitudes are fundamental magnitudes, as they are easily calculated by starting with the wave functions, a qualifier that is attributed to a local magnitude is justified only if an integration analogous to (2) allows one to pass to the integral magnitude that bears the same name. By referring to the vector g_i with the term "impulse-energy," we thereby make a hypothesis that might seem arbitrary. This hypothesis will be justified later on by the non-relativistic case.

3. – Non-relativistic approximation

We abandon the notations of tensor calculus. At the same time, we change some other notations, either in order to recall the usual notations from non-relativistic theory or for the sake of convenience.

We then set:

$$\rho = Du_4, \qquad \varepsilon = -e.$$

On the subject of the velocity vector, one must regard the component u_4 as close to 1. u_1, u_2, u_3 are then infinitely small. If c is the velocity of light then we set:

$$v_x = c\,u_1, \qquad v_y = c\,u_2, \qquad v_z = c\,u_3\,.$$

v_x, v_y, v_z are finite quantities. A value that is sufficiently close to u_4 is therefore:

$$u_4 = 1 + \frac{1}{2}c^{-2}(v_x^2+v_y^2+v_z^2)\,.$$

From the orthogonality of u_1 and s_1, it results that s_4 is infinitely small and that it may be written:

$$s_4 = (s_x\,v_x + s_y\,v_y + s_z\,v_z)\,/\,c \qquad \text{upon setting} \quad s_x = s_4\,,\ \ldots$$

while the other components of s_1 are linked by the relation:

$$s_x^2+s_y^2+s_z^2 = 1,$$

which is sufficiently exact The local magnetic moment is:

$$\frac{\varepsilon h}{2mc} s_x ,$$

while the electric moment is negligible.

Equation (5) entails that sin η is infinitely small. We occupy ourselves with just the ordinary electron – i.e., a negative electron. η is then close to π. Set:

$$\eta = \pi + \theta / c;$$

one gets:

$$\sin \eta = -\theta / c, \qquad \cos \eta = -1 + \theta^2 / 2c^2.$$

Equation (5) then becomes:

$$\rho\, \theta = \frac{h}{2m}\left(\frac{\partial}{\partial x}\rho s_x + \frac{\partial}{\partial y}\rho s_y + \frac{\partial}{\partial z}\rho s_z \right). \tag{8}$$

The conservation equation (4) takes the habitual form:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z = 0. \tag{9}$$

The impulse is:

$$\mathbf{g} = m\mathbf{v} - \frac{1}{2}\frac{h}{\rho}\,\mathrm{rot}\,\rho\mathbf{s} + \frac{1}{c}\mathbf{A}; \tag{10}$$

in this expression, **A** represents the ordinary vector potential in three dimensions. The energy is ([1]):

$$W = cg_4 - mc^2 = \frac{1}{2}m(v^2 - \theta^2) - \frac{1}{2}h\mathbf{s} \times \mathrm{grad}\,\theta + \frac{1}{2}h\,\frac{1}{\rho}\,\mathrm{div}\, r(\mathbf{v} \wedge \mathbf{s}) + s\,\mathbf{V}; \tag{11}$$

in it, V represents the scalar potential. The electric polarization (in the non-relativistic approximation, it reduces to an ordinary vector) is:

$$\frac{1}{2} sh\,\mathrm{rot}\,\rho\,\mathbf{s}.$$

Equation (6) is written:

$$\left(\frac{\partial}{\partial t} + v_x + \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) = \mathbf{s} \wedge (\mathrm{grad}\,\theta - \mathrm{rot}\,\mathbf{v}). \tag{12}$$

Finally, of the two determinants that figure in (7), the second one is negligible and the first one reduces, with mixed notations, to:

([1]) The sign $\times$ represents the scalar product, while the $\wedge$ sign represents the vectorial product.

$$- \begin{vmatrix} s_x & s_y & s_z \\ \partial^i s_x & \partial^i s_y & \partial^i s_z \\ \partial^j s_x & \partial^j s_y & \partial^j s_z \end{vmatrix} .$$

If one sets:

$$s_x = \cos\varphi \sin\alpha, \qquad s_y = \sin\varphi \sin\alpha, \qquad s_z = \cos\alpha$$

then this determinant is, moreover:

$$\partial^i(\cos\alpha\ \partial^j\varphi) - \partial^j(\cos\alpha\ \partial^i\varphi).$$

4. – True velocity

Our equations take on a better form by eliminating the parameter θ and introducing a new velocity. Indeed, the velocity of the components v_x , v_y , v_z is a complex magnitude that is coupled to the flow that results from the two electric fluids of different types. The true fluid velocity is:

$$\mathbf{w} = \mathbf{v} - \frac{1}{2}\frac{h}{m}\frac{1}{\rho}\,\mathrm{rot}\,\rho\,\mathbf{s}.$$

We eliminate **v** with the aid of that equation. It then becomes:

$$\mathbf{g} = m\mathbf{w} + \frac{e}{c}\mathbf{A}, \tag{13}$$

an expression that shows that the free fluid plays no actual role in the impulse.

Equation (7) gives:

$$\mathrm{rot}(m\mathbf{w} + \frac{e}{c}\mathbf{A} + h\cos\alpha\cdot\mathrm{grad}\,\varphi) = 0,$$

or, if **H** is the magnetic field:

$$m\ \mathrm{rot}\ \mathbf{w} + \frac{e}{c}\mathbf{H} + \frac{1}{2}h\ \mathrm{grad}\cos\alpha \wedge \mathrm{grad}\,\varphi = 0. \tag{14}$$

5. – Evolution of spin

We transcribe the evolution of spin (12) by eliminating θ and **v**. A rot **w** appears in the right-hand side, and there is some advantage to making it disappear by using (14). Having done all calculations, it becomes:

(15)
$$m\left(\frac{\partial}{\partial t}+w_x\frac{\partial}{\partial x}+w_y\frac{\partial}{\partial y}+w_z\frac{\partial}{\partial z}\right)\mathbf{s}$$
$$=\mathbf{s}\wedge\left\{\frac{\varepsilon}{c}\mathbf{H}+\frac{1}{2}h\left[\Delta\mathbf{s}+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\rho}{\partial x}\frac{\partial}{\partial z}\right)\mathbf{s}\right]\right\}.$$

Here, we see a fictitious magnetic field of quantum origin appear that gets added to the imposed magnetic field. If the latter does not exist then one may say that the spin is subject to a classical evolution along the stream lines (this amounts to the true current). This is what one comes to when the electron is distributed on a plane wave. Its position is completely indeterminate. **s** is then just a function of time. Its precessional motion is given by:

$$m\frac{d}{dt}\mathbf{s}=\frac{e}{c}\mathbf{s}\wedge\mathbf{H}.$$

The corresponding phenomenon might not be observed with electrons, because one does not know how to polarize an electronic sheaf. By contrast, the observation has been used with neutrons ([1]). In principle, the phenomenon is the same for electrons and neutrons, provided that the particles propagate parallel to the magnetic field in such a manner that the latter does not act on the charge of the electron.

6. – Energy

One applies the same method to equation (11). It becomes:

(16)
$$W=\frac{1}{2}mw^2+eV-\frac{h^2}{8m}\left[4\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}+(\sin\alpha\,\mathrm{grad}\varphi)^2+(\mathrm{grad}\,\alpha)^2\right]$$
$$-\mathbf{s}\times\left\{\frac{\varepsilon}{2mc}\mathbf{H}+\frac{h^2}{4m}\left[\Delta\mathbf{s}+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial}{\partial x}+\frac{\partial\rho}{\partial x}\frac{\partial}{\partial z}\right)\mathbf{s}\right]\right\}.$$

In this energy, one can distinguish the classical kinetic term, a term that represents the energy of a magnet in a magnetic field, the magnetic field in question being the one that figures already in the evolution of spin, and finally a group of terms that may qualify as the potential energy. It involves the imposed potential energy and a term that one may consider to be a potential energy of quantum origin.

In some cases, it might be useful to replace:

$$(\sin\alpha\,\mathrm{grad}\,\varphi)^2+(\mathrm{grad}\,\alpha)^2$$

([1]) F. Bloch, *Phys. Rev.* **50** (1936), 259; **51** (1937), 994.

by a more obviously invariant expression, after a change of coordinate axes. This expression is:

$$\left(\frac{\partial}{\partial x}s_x\right)^2+\left(\frac{\partial}{\partial y}s_x\right)^2+\left(\frac{\partial}{\partial z}s_x\right)^2+\left(\frac{\partial}{\partial x}s_y\right)^2+\left(\frac{\partial}{\partial y}s_y\right)^2+\left(\frac{\partial}{\partial z}s_y\right)^2+\left(\frac{\partial}{\partial x}s_z\right)^2+\left(\frac{\partial}{\partial y}s_z\right)^2+\left(\frac{\partial}{\partial z}s_z\right)^2.$$

7. – Function *S*

Equation (14) expresses the idea that a certain vector has zero rotation. We introduce a conveniently-chosen function *S* that permits us to write:

$$m\mathbf{v}+\frac{\varepsilon}{c}\mathbf{A}+\frac{1}{2}h\cos\alpha\cdot\operatorname{grad}\varphi=\operatorname{grad}S. \tag{17}$$

Equation (14) expresses only a part of what is contained in (7). In order to take it into account completely, we must now add the relation:

$$W-\frac{1}{2}h\cos\alpha\frac{\partial}{\partial t}\varphi=-\frac{\partial S}{\partial t} \tag{18}$$

to (17).

Upon eliminating **w** between (17) and (18), after replacing *W* with its expression (16) in (18), one obtains an equation that governs the evolution of *S*. The function *S* that is used here plays a role that is analogous to the function with the same name that figures in the Madelung equation.

We complete our equations with a conservation equation that we may write:

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}\rho w_x+\frac{\partial}{\partial y}\rho w_y+\frac{\partial}{\partial z}\rho w_z=0. \tag{19}$$

All of the theory of the non-relativistic, quantum, magnetic electron is contained in the latter equation and equations (15), (16), (17), (18).

8. – Euler equation

In hydrodynamics, the evolution of the velocity is given by the Euler equations. When the motion of a fluid is defined without vortices, the Euler equations come down to one equation that defines the evolution of the velocity potential. This potential is analogous to our function *S*. Conversely, in our problem, we may imagine defining equations that are analogous to the Euler equations. They will be more general than our equations (17) and (18), but they might serve to help us prove certain theorems.

In order to form these equations, one reverts, for the moment, to relativistic notations. One considers w_x, w_y, w_z to be the spatial components of a world-vector w_i . One multiplies equation (7) by w_i, term-by-term, then one reverts to the old notation and the

non-relativistic approximation. Thanks to (15), one then eliminates the derivatives with respect to time of **s**, φ, and α that appeared in the course of the calculation. One finally gets:

$$(20)\qquad m\left(\frac{\partial}{\partial t}+w_x\frac{\partial}{\partial x}+w_y\frac{\partial}{\partial y}+w_z\frac{\partial}{\partial z}\right)w_x = \varepsilon\left[E_x+\frac{1}{c}(w_yH_z-w_zH_y)\right]$$
$$+\frac{h^2}{8m}\frac{\partial}{\partial x}\left[4\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}+(\sin\alpha\,\mathrm{grad}\,\varphi)^2+(\mathrm{grad}\,\alpha)^2\right]$$
$$+\mathbf{s}\times\frac{\partial}{\partial x}\left\{\frac{\varepsilon h}{2m_0c}\mathbf{H}+\frac{h^2}{4m}\left[\Delta\mathbf{s}+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial}{\partial y}+\frac{\partial\rho}{\partial z}\frac{\partial}{\partial z}\right)\mathbf{s}\right]\right\},$$

and two other analogous equations in w_x and w_y, respectively. In the last row, one sees that a force with a classical aspect is exerted in the electron magnet when the total magnetic field (i.e., imposed plus quantum) is non-uniform.

9. – Conservation equations

In undulatory mechanics, the conservation theorems – for example, the conservation of energy – may be established according to a method that was put to work *à propos* of Ehrenfest's theorem. Since it is the simplest method, one may also develop the magnitude in question with the aid of proper orthonormal functions. If the magnitude is independent of time then the time does not figure in the development. Here, that naturally amounts to an integral magnitude.

A third method consists of attaching the integral magnitude – namely, the impulse **G** – to a local magnitude – namely, the impulse **g** – by a relation of the type:

$$\mathbf{G} = \iiint \rho\mathbf{g}\,dx\,dy\,dz\,.$$

As a result, it suffices to show that the local magnitude verifies a differential equation of the appropriate form. For our example, that equation will have the following form:

$$\frac{\partial}{\partial t}\rho\mathbf{g}+\frac{\partial}{\partial x}\mathbf{A}+\frac{\partial}{\partial y}\mathbf{B}+\frac{\partial}{\partial z}\mathbf{C} = 0.$$

A, **B**, **C** are the magnitudes whose precise expression has no influence on the result ([1]).

10. – Conservation of impulse

Starting with equations (19) and (20), one forms:

([1]) That method was employed by Schrödinger *à propos* of the Gordon equation (Annalen der Physik, 82, 1927).

$$\frac{\partial}{\partial t}\rho(mw_x+\frac{e}{c}A_x)\,.$$

In order to give the result a simple form, we introduce, as in the hydrodynamics of an anisotropic fluid, a pressure tensor that is symmetric and has nine components. These components are:

(21)
$$P_{xx}=\frac{h^2}{2m}\left[\left(\frac{\partial}{\partial x}\sqrt{\rho}\right)^2+\sqrt{\rho}\,\frac{\partial^2}{\partial x^2}\sqrt{\rho}-\frac{1}{2}\rho\mathbf{s}\times\frac{\partial^2}{\partial x^2}\mathbf{s}\right],$$

$$P_{xy}=\frac{h^2}{2m}\left[\left(\frac{\partial}{\partial x}\sqrt{\rho}\right)\left(\frac{\partial}{\partial y}\sqrt{\rho}\right)-\sqrt{\rho}\,\frac{\partial^2}{\partial x\,\partial y}\sqrt{\rho}-\frac{1}{2}\rho\mathbf{s}\times\frac{\partial^2}{\partial x\,\partial y}\mathbf{s}\right],$$

$$P_{xz}=\ldots$$

$$\ldots$$

The desired equation is then obtained in the form:

$$\begin{aligned}&\frac{\partial}{\partial t}\rho(mw_x+\frac{e}{c}A_x)+\frac{\partial}{\partial x}\left[w_x\rho(mw_x+\frac{e}{c}A_x)+P_{xx}\right]\\&\quad+\frac{\partial}{\partial y}\left[w_y\rho(mw_x+\frac{e}{c}A_x)+P_{xy}\right]+\frac{\partial}{\partial z}\left[w_z\rho(mw_x+\frac{e}{c}A_x)+P_{xz}\right]\\&\qquad=\varepsilon\rho\left[-\frac{\partial}{\partial x}V_x+\frac{1}{c}(w_x\frac{\partial}{\partial x}A_x+w_y\frac{\partial}{\partial y}A_y+w_z\frac{\partial}{\partial z}A_z)\right]+\frac{\varepsilon h}{2mc^2}\rho\mathbf{s}\times\frac{\partial\mathbf{H}}{\partial x}\,.\end{aligned}$$

It shows that there is conservation of the component of the impulse along *Ox* when the potentials are independent of the variable *x*. Moreover, it immediately leads to the Ehrenfest theorem relative to the non-relativistic quantum electron.

11. – Conservation of spin

Under a simple transformation, equations (15) and (19) lead us to write that:

$$\frac{\partial}{\partial t}\rho\mathbf{s}+\frac{\partial}{\partial x}(w_x\rho\mathbf{s}-\rho\mathbf{s}\wedge\frac{\partial}{\partial x}\mathbf{s})+\frac{\partial}{\partial y}(w_y\rho\mathbf{s}-\rho\mathbf{s}\wedge\frac{\partial}{\partial y}\mathbf{s})+\frac{\partial}{\partial z}(w_z\rho\mathbf{s}-\rho\mathbf{s}\wedge\frac{\partial}{\partial z}\mathbf{s})\;=\mathbf{s}\wedge\frac{\varepsilon}{c}\,\mathbf{H}_x\,.$$

This relation shows that the (integral) component of the spin along *Oz* – for example – is invariable when the magnetic field **H** is parallel to *Oz*.

12. – Conservation of energy

Form:

$$\frac{\partial}{\partial t}\rho W + \frac{\partial}{\partial x} w_x \rho W + \frac{\partial}{\partial y} w_y \rho W + \frac{\partial}{\partial z} w_z \rho W .$$

The result simply expresses that if we set, thus completing formulas (21):

$$P_{xt} = -\frac{h^2}{2m}\left[\left(\frac{\partial}{\partial x}\sqrt{\rho}\right)\left(\frac{\partial}{\partial t}\sqrt{\rho}\right) - \sqrt{\rho}\,\frac{\partial^2}{\partial x^2}\sqrt{\rho} - \frac{1}{2}\rho \mathbf{s}\times\frac{\partial^2}{\partial x\,\partial t}\mathbf{s}\right] = P_{tx},$$

$$P_{xt} = \ldots,$$

$$P_{xt} = \ldots$$

then we indeed obtain:

$$\frac{\partial}{\partial t}\rho W + \frac{\partial}{\partial x}(w_x \rho W + P_{tx}) + \frac{\partial}{\partial y}(w_y \rho W + P_{ty}) + \frac{\partial}{\partial z}(w_z \rho W + P_{tz})$$

$$= \varepsilon\rho\left[\frac{\partial}{\partial t}V - \frac{1}{c}(w_x \frac{\partial}{\partial t}A_x + w_y \frac{\partial}{\partial t}A_y + w_z \frac{\partial}{\partial t}A_z)\right] - \frac{\varepsilon h}{2mc}\mathbf{s}\times\frac{\partial \mathbf{H}}{\partial t} .$$

This equation entails that the integral:

$$\int \rho\, W\, d\tau$$

must be independent of time when the potentials do not depend upon it either. It is therefore necessary to identify the energy with the integral that is defined by starting with the wave functions, which is invariable under the same conditions. This is why the term "local energy" is justified *à propos* of *W*. An analogous conclusion is valid for the impulse and for the spin.

13. – The Pauli equations

Our point of departure has forced us to treat the problem posed without specifying the corresponding wave functions, and the equations that they verify. It is, nonetheless, necessary to formulate our results in a way that is possibly more applicable to other problems, to recall the properties of the functions, and the form of these equations.

The approximation of the Dirac equation, whose consequences we just developed, is called the *Pauli approximation*. It involves two wave functions $\Psi_1(x, y, z, t)$, $\Psi_2(x, y, z, t)$. Starting with these functions, one calculates the following local magnitudes:

1. The density ρ:

$$\rho_1 = \Psi_1 \Psi'_1 + \Psi_2 \Psi'_2 .$$

2. The spin:

$$\rho s_x = \Psi_1 \Psi'_2 + \Psi_2 \Psi'_1 ,$$
$$\rho s_y = -(\Psi_1 \Psi'_2 - \Psi_2 \Psi'_1) ,$$
$$\rho s_z = \Psi_1 \Psi'_1 - \Psi_2 \Psi'_2 .$$

3. The velocity along *Ox*; for example:

$$m \rho w_x = \frac{1}{2} ih \left(\Psi_1 \frac{\partial}{\partial x} \Psi'_1 - \Psi'_1 \frac{\partial}{\partial x} \Psi_1 + \Psi_2 \frac{\partial}{\partial x} \Psi'_2 - \Psi'_2 \frac{\partial}{\partial x} \Psi_2 \right) - \frac{\varepsilon}{\rho} \rho A_x .$$

4. The energy:

$$m \rho W = -\frac{1}{2} ih \left(\Psi_1 \frac{\partial}{\partial t} \Psi'_1 - \Psi'_1 \frac{\partial}{\partial t} \Psi_1 + \Psi_2 \frac{\partial}{\partial t} \Psi'_2 - \Psi'_2 \frac{\partial}{\partial t} \Psi_2 \right).$$

The evolution of the two wave functions as functions of time is given by the Pauli system of equations, which we give the following precise form, in order for the equivalence with what was said in the preceding paragraphs to be perfect:

$$\left[\frac{h^2}{2m} \Delta + ih \frac{\partial}{\partial t} - \varepsilon V - i \frac{\varepsilon h}{mc} \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) - \frac{\varepsilon^2}{2mc^2} \mathbf{A}^2 \right] \Psi_1$$
$$+ \frac{\varepsilon h}{2mc} [(H_x - iH_y) \Psi_2 + H_z \Psi_1] = 0,$$

$$\left[\frac{h^2}{2m} \Delta + ih \frac{\partial}{\partial t} - \varepsilon V - i \frac{\varepsilon h}{mc} \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) - \frac{\varepsilon^2}{2mc^2} \mathbf{A}^2 \right] \Psi_2$$
$$+ \frac{\varepsilon h}{2mc} [(H_x + iH_y) \Psi_1 - H_z \Psi_2] = 0.$$

(Here, one has assumed that div **A** = 0.) Starting with the preceding equations, it is possible to verify some of the less complicated equations in the preceding paragraphs: One substitutes the wave functions for the local magnitudes, and then one confirms that the relations obtained are verified identically, thanks to the Pauli relations. Therefore, our work is found to be confirmed. Nevertheless, it has not been possible to verify – for example – relation (16) in this way, since the calculations are complicated.

Conclusion

We have obtained the Madelung equations that apply to the non-relativistic, spinning, quantum electron. These equations take an interesting hydrodynamical form when one

eliminates the total electric current and substitutes the true electric current. In addition, it is convenient to introduce a tensorial pressure that is derived from the variations from point to point of the density and spin. These results encourage us to attempt to modify the Madelung equations for the Dirac electron along the same lines.

In passing, we have shown, with the aid of the conservation theorems, that it is justified to call the vector **g** the impulse and the scalar *W* the energy. The path that we followed has not been without its detours, but the direct identification of these magnitudes by substituting the wave functions for vectors has led to some more tractable calculations.

Finally, we now hope to have convinced the reader that the local magnitudes are not contrary to the spirit of undulatory mechanics and that their methodical usage is not dangerous.

(Manuscript received 8 April 1941)