"Zur Erweiterung der Wellenmechanik," Zeit. Phys. **83** (1933), 338-440.

On the extension of wave mechanics

By **Raschko Zaycoff** in Sofia

Translated by D. H. Delphenich

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Increasing the number of wave functions ψ will account for the difference between positive and negative electrons ([1]).

§ 1. – Let:

$$\lambda_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \lambda_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \lambda_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1}$$

be the two-rowed **Pauli** matrices. If we set:

$$\mathfrak{P} \equiv (\lambda_1, \lambda_2, \lambda_3, i\varepsilon) \tag{2}$$

then, as is known:

$$\psi^* \mathfrak{P}\, \psi \tag{2'}$$

will represent the radiation vector. However, it is the four-rowed **Dirac** matrices:

$$\left.\begin{array}{l} \sigma_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_3 \end{bmatrix}, \quad \delta = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}, \\ \rho_1 = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} 0 & -i\varepsilon \\ i\varepsilon & 0 \end{bmatrix}, \quad \rho_3 = \begin{bmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix} \end{array}\right\} \tag{3}$$

that are definitive for the electron, and with their help, we can construct the nine matrices:

$$\sigma_a\, \rho_b \qquad (a, b = 1, 2, 3)\,. \tag{4}$$

([1]) According to **Rutherford**, there also exists a positive electron with the same mass as the usual (negative) electron.

In total, there are then sixteen distinct matrices present. In fact, if we set:

$$\mathfrak{J} \equiv (\sigma_1\,\rho_1\,,\ \sigma_2\,\rho_1\,,\ \sigma_3\,\rho_1\,,\ i\ \delta) \tag{4}$$

then:

$$\psi^*\,\mathfrak{J}\ \psi \tag{5}$$

will represent the four-current. If we further write:

$$\mathfrak{S} \equiv (\sigma_1\,,\ \sigma_2\,,\ \sigma_3\,,\ i\ \rho_1) \tag{6}$$

then:

$$\psi^*\,\mathfrak{S}\ \psi \tag{6′}$$

can be interpreted as the four-spin. We also set:

$$\mathfrak{M} \equiv (\sigma_1\,\rho_3\,,\ \sigma_2\,\rho_3\,,\ \sigma_3\,\rho_3\,,\ -i\ \sigma_1\,\rho_2\,,\ -i\ \sigma_2\,\rho_2\,,\ -i\ \sigma_3\,\rho_2) \tag{7}$$

and remark that:

$$\psi^*\,\mathfrak{M}\ \psi \tag{7′}$$

represents the tensor of the magnetization and polarization. Finally:

$$\psi^*\,\rho_3\,\psi \tag{8}$$

is the proper mass density and:

$$\psi^*\,\rho_2\,\psi \tag{9}$$

is the proper charge density.

If we set:

$$\psi_1 = \psi_3\,, \qquad \psi_2 = \psi_4 \tag{10}$$

then the quantities (7′), (8), (9) will vanish, and we will get:

$$\mathfrak{J} = \mathfrak{S} = 2 \cdot \mathfrak{P}\,. \tag{11}$$

Dirac’s theory, which probably subsumes **Pauli**’s theory, only makes it possible to describe negative electrons, but yields no explanation for the positive proton. In order to obtain a consistent theory of matter, we must increase the number of wave functions.

§ 2. – We now construct the eight-rowed matrices:

$$\left.\begin{array}{llll}
\alpha_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}, & \alpha_2 = \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}, & \alpha_3 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}, & 1 = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, \\
\beta_1 = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_1 \end{bmatrix}, & \beta_2 = \begin{bmatrix} \rho_2 & 0 \\ 0 & \rho_2 \end{bmatrix}, & \beta_3 = \begin{bmatrix} \rho_3 & 0 \\ 0 & \rho_3 \end{bmatrix}, & \\
\gamma_1 = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, & \gamma_2 = \begin{bmatrix} 0 & -i\,\delta \\ -i\,\delta & 0 \end{bmatrix}, & \gamma_3 = \begin{bmatrix} \delta & 0 \\ 0 & -\delta \end{bmatrix}. &
\end{array}\right\} \tag{12}$$

They will then produce the 54 matrices:

$$\alpha_a\,\beta_b\,, \;\; \alpha_a\,\gamma_b\,, \;\; \beta_a\,\gamma_b\,, \;\; \alpha_a\,\beta_b\,\gamma_c \qquad (a, b, c = 1, 2, 3)\,. \tag{13}$$

We will then have 64 distinct matrices in total. We now set:

$$\mathfrak{J}^- \equiv (\alpha_1\,\beta_1\,, \alpha_2\,\beta_1\,, \alpha_3\,\beta_1\,, i\;1)\,. \tag{14}$$

The quantity:

$$\psi^*\,\mathfrak{J}^-\,\psi \tag{14′}$$

represents the four-current of the (–)-electrons. We also write:

$$\mathfrak{J}^+ = \mathfrak{J}^- \cdot \gamma_1 \tag{15}$$

and can regard the quantity:

$$\psi^*\,\mathfrak{J}^+\,\psi \tag{15′}$$

as the four-current of the (+)-electrons. If we further set:

$$\psi_1 = \psi_5\,, \qquad \psi_2 = \psi_6\,, \qquad \psi_3 = \psi_7\,, \qquad \psi_4 = \psi_8 \tag{16}$$

then it will follow that:

$$\mathfrak{J}^- = \mathfrak{J}^+ = 2\;\mathfrak{J}\,, \tag{17}$$

in which $\mathfrak{J}$ has the meaning that it gets from formula (5). When we set:

$$\left.\begin{array}{c}
\mathfrak{S}^- \equiv (\alpha_1, \alpha_2, \alpha_3, i\,\beta_1), \quad \mathfrak{S}^- \equiv \mathfrak{S}^- \cdot \gamma_1, \\
\mathfrak{M}^- \equiv (\alpha_1\,\beta_3, \alpha_2\,\beta_3, \alpha_3\,\beta_3, -i\,\alpha_1\,\beta_2, -i\,\alpha_2\,\beta_2, -i\,\alpha_3\,\beta_2), \\
\mathfrak{M}^- \equiv \mathfrak{M}^- \cdot \gamma_1,
\end{array}\right\} \tag{18}$$

we will have the following four-vectors and tensors:

$$\psi^* \mathfrak{S}^- \psi\,, \qquad \psi^* \mathfrak{S}^+ \psi\,, \qquad \psi^* \mathfrak{M}^- \psi\,, \qquad \psi^* \mathfrak{M}^+ \psi\,, \tag{18′}$$

as well as the invariants:

$$\psi^* \beta_3 \psi\,, \qquad \psi^* \beta_2 \psi\,, \qquad \psi^* \beta_3 \gamma_1 \psi\,, \qquad \psi^* \beta_2 \gamma_1 \psi\,. \tag{19}$$

§ 3. – We now construct the four-vectors:

$$\left.\begin{array}{ll} \psi^* \mathfrak{J}^- \gamma_2 \psi, & \psi^* \mathfrak{J}^- \gamma_3 \psi, \\ \psi^* \mathfrak{S}^- \gamma_2 \psi, & \psi^* \mathfrak{S}^- \gamma_3 \psi, \end{array}\right\} \tag{20}$$

the tensors:

$$\psi^* \mathfrak{M}^- \gamma_2 \psi\,, \qquad \psi^* \mathfrak{M}^- \gamma_3 \psi\,, \tag{21}$$

and the invariants:

$$\psi^* \beta_3 \gamma_2 \psi\,, \qquad \psi^* \beta_2 \gamma_2 \psi\,, \qquad \psi^* \beta_3 \gamma_3 \psi\,, \qquad \psi^* \beta_2 \gamma_3 \psi\,. \tag{22}$$

For (16), all of the quantities (20), (21), (22) will vanish. However, then cannot, unfortunately, be interpreted physically on the basis of our knowledge up to now. They certainly owe their existence to the difference between both types of electrons. It is possible that they are connected with the presence of magnetic currents and charges.

If both (+) and (–)-electrons are present at the same time then the unified wave equations will read:

$$\frac{h}{2\pi i} v_\rho \left(\frac{\partial \psi}{\partial x^\rho} + \frac{2\pi i e}{hc} \varphi_\rho \gamma_3 \psi \right) + m\,c\,v_5\,\psi = 0\,, \tag{23}$$

in which:

$$v_\rho \equiv (\alpha_1 \beta_1,\ \alpha_2 \beta_1,\ \alpha_3 \beta_1,\ i\,1)\,, \qquad v_5 = \beta_3\,. \tag{24}$$

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On the extension of wave mechanics (Second communication)

By **Raschko Zaycoff** in Sofia

Translated by D. H. Delphenich

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A reinterpretation of the tensors that arise from our eight-component wave mechanics ([1]) is proposed that will yield a unified representation of the various electrical particles. In conjunction with a previous paper ([2]), a covariant form of the wave equations will then be given.

§ 1. – Now that it has been established experimentally (**Anderson**, *et al*) that there exist electrons whose ratio of rest mass to charge proves to have the opposite sign to the usual electrons, it is clear that **Dirac**'s wave equation, which yields the aforementioned difference, in addition to the spin, describes the correct distribution of electrically-charged matter. We would now like to show that the eight-component extension of the equations produces the distribution of protons, along with that of electrons, since they can be written in a microscopic form. We would now like to set:

$$\nu_1 = \alpha_1 \beta_1 , \qquad \nu_2 = \alpha_2 \beta_1 , \qquad \nu_3 = \alpha_3 \beta_1 , \qquad \nu_4 = i\, 1, \qquad \nu_5 = \beta_3 . \tag{1}$$

We can then interpret the quantity:

$$-e\psi^* \gamma_3 \nu_{k'} \psi \qquad\qquad (k' = 1, 2, 3, 4) \tag{2}$$

as the four-current of the total electricity and the quantity:

$$-e\psi^* \gamma_3 \nu_5 \psi \tag{3}$$

as the proper electrical density. We now consider the fourth component of (2), which will read:

$$ie(-\psi_1^* \psi_1 - \psi_2^* \psi_2 - \psi_3^* \psi_3 - \psi_4^* \psi_4 + \psi_5^* \psi_5 + \psi_6^* \psi_6 + \psi_7^* \psi_7 + \psi_8^* \psi_8) \tag{4}$$

([1]) **R. Zaycoff**, Zeit. Phys. **83** (1933), 338; cf., formulas (1), (3), (12), (13), (23), (24).
([2]) **R. Zaycoff**, Zeit. Phys. **82** (1933), 267. Cited as I in what follows.

when it is written out.

It represents the electric charge density, multiplied by i ([1]) and can be either negative or positive. The distribution of electrons, as well as protons, is then given by it. In a similar way, we can convince ourselves that (2) and (3) are applicable to both type of electricity.

By means of a "gauge transformation" of the electromagnetic four-potential φ_μ :

$$\varphi'_\mu = \varphi_\mu - \lambda_{,\mu} , \tag{I, (27)}$$

the wave functions ψ can be transformed as follows:

$$\psi'_s = \psi_s \cdot e^{ia\lambda} , \qquad \psi'_t = \psi_t \cdot e^{-ia\lambda} \qquad (s = 1, 2, 3, 4,\ t = 5, 6, 7, 8), \tag{5}$$

in which a represents a positive constant that is proportional to e. The quantities (2), (3) are thus gauge invariant.

We further set:

$$s_1 = \alpha_1 , \qquad s_2 = \alpha_2 , \qquad s_3 = \alpha_3 , \qquad s_4 = i\,\beta_1 \tag{6}$$

and interpret the quantities:

$$\frac{h}{4\pi} \cdot \psi^* s_{k'} \psi \qquad (k' = 1, 2, 3, 4) \tag{7}$$

as the components of the four-spin of electricity. They are gauge invariant. The real parts of the quantities:

$$\left.\begin{aligned} p_\rho &= \frac{h}{2\pi i} \psi^* \left(\psi_{,\rho} + \frac{2\pi i e}{hc} \gamma_3 \varphi_\rho \psi \right) \qquad (\rho = 1, 2, 3) \\ p &= \frac{hc}{2\pi i} \sum_{\rho=1}^{3} \psi^* v_\rho \left(\psi_{,\rho} + \frac{2\pi i e}{hc} \gamma_3 \varphi_\rho \psi \right), \end{aligned}\right\} \tag{8}$$

in which v_1, v_2, v_3 mean the first three matrices, are gauge invariant and represent the mechanical impulse (mechanical energy, resp.) of the electrons and protons. Finally:

$$b_0 \psi^* v_5 \psi , \tag{9}$$

in which b_0 means a quantity ([2]) with the dimension of mass, can be regarded to be a proper mass density, which is likewise gauge invariant.

([1]) The fourth coordinate x^4 was taken to be imaginary.
([2]) Which will be explained later.

§ 2. – If we set:

$$\begin{aligned} &\mu_{23} = \alpha_1\,\beta_1\,, && \mu_{31} = \alpha_2\,\beta_3\,, && \mu_{12} = \alpha_3\,\beta_3\,, \\ &\mu_{41} = -\,i\,\alpha_1\,\beta_2\,, && \mu_{42} = -\,i\,\alpha_2\,\beta_2\,, && \mu_{43} = -\,i\,\alpha_3\,\beta_2 \end{aligned} \tag{10}$$

then the quantities [1]:

$$\psi^* \mu_{k'l'}\,\psi \qquad (k' \neq l') \tag{11}$$

play the role of a mechanical dipole moment tensor. Likewise:

$$\psi^* \nu_{k'}\,\psi \tag{12}$$

can be interpreted as the mechanical four-current. The quantities (11) and (12) are gauge invariant. However, they always occur in physical laws multiplied by a very small factor, which explains why they are neglected.

The following quantities:

$$\psi^* \gamma_3\, s_{k'}\,\psi\,, \qquad \psi^* \gamma_3\, \mu_{k'l'}\,\psi\,, \qquad \psi^* \beta_2\,\psi\,, \qquad \psi^* \gamma_3\,\beta_2\,\psi \tag{13}$$

are also gauge invariant, but they do not appear anywhere.

Now, the quantities:

$$\left.\begin{aligned} &\psi^*\gamma_1\,\nu_{k'}\,\psi, \quad \psi^*\gamma_1\,\nu_5\,\psi, \quad \psi^*\gamma_2\,\nu_{k'}\,\psi, \quad \psi^*\gamma_2\,\nu_5\,\psi, \quad \psi^*\gamma_1\,\beta_2\,\psi, \quad \psi^*\gamma_2\,\beta_2\,\psi, \\ &\psi^*\gamma_1\,s_{k'}\,\psi, \quad \psi^*\gamma_1\,\mu_{k'l'}\,\psi, \quad \psi^*\gamma_2\,s_{k'}\,\psi, \quad \psi^*\gamma_2\,\mu_{k'l'}\,\psi \end{aligned}\right\} \tag{14}$$

are no longer gauge invariant, and therefore be dropped from consideration [2]. When one neglects gravitation, the unified wave equations will now read [3]:

$$\sum_{\rho=1}^{4} \nu_\rho \left(\psi_{,\rho} + \frac{2\pi\, i\, e}{h\, c}\, \gamma_3\, \varphi_\rho\, \psi \right) + \frac{2\pi\, i\, e}{h}\, \nu_5 (b_0 - b\,\gamma_3\,\varphi)\psi = 0\,. \tag{15}$$

In fact, such a variability in the rest mass exists as a result of the packing effect between electrons and protons. In all situations where only electrons are present, one can set:

$$\psi_5 = \psi_6 = \psi_7 = \psi_8 = 0\,, \qquad b_0 - b\,\gamma_3\,\varphi = m^-, \tag{16}$$

and in all situations where there are only protons:

([1]) In general, the primed Latin symbols shall run from 1 to 4 and the unprimed ones from 1 to 5.
([2]) Our previous interpretation did not account for gauge invariance.
([3]) b_0 and b are certain constants, and φ is a function.

$$\psi_1 = \psi_2 = \psi_3 = \psi_4 = 0\,, \qquad b_0 - b\,\gamma_3\,\varphi = m^+. \tag{17}$$

Equations (15) then have a microscopic character.

§ 3. – Formula I, (17) can be replaced with the more general one:

$$h_{m\mu\,/\,\rho} = h_m^\sigma\, f_{\sigma\rho\mu} + n_m\, f_{\rho\mu}\,, \tag{18}$$

in which the quantity $f_{\rho\sigma\mu}$ is antisymmetric in all indices. Along with the formula:

$$f_{\mu\nu} = \varphi_{\nu,\,\mu} - \varphi_{\mu,\,\nu}\,, \tag{19}$$

the formula:

$$f_{\lambda\mu\nu} = \varepsilon_{\lambda\mu\nu\rho}\, g^{\rho\sigma} \varphi_{,\sigma} \tag{20}$$

also appears here, in which:

$$\varepsilon_{\lambda\mu\nu\rho} = \pm i \sqrt{g}\,, \tag{21}$$

according to whether the indices λ, μ, ν, ρ represents an even (odd, resp.) permutation of the number sequence 1 2 3 4, and φ represents a scalar.

The right-hand side of the expression for $\gamma_{km\rho}$ in the Appendix to I also has the quantity:

$$h_k^\sigma\, h_m^\kappa\, f_{\rho\sigma\kappa} \tag{22}$$

added to it. Instead of I, (26), we have:

$$G = R - f_{\kappa\rho}\, f^{\kappa\rho} - f_{\kappa\rho\sigma}\, f^{\kappa\rho\sigma}\,. \tag{23}$$

Moreover, the four-rowed matrices α_k that apply to all formulas I, (30), …, (69) must be replaced with the eight-rowed ones ν_k . In formulas (45), (48), (67), (70), the constant ε is replaced with the matrix:

$$-\,a\,\gamma_3\,.$$

We will then have, e.g., instead of I, (45):

$$\left.\begin{aligned} \psi_{/\rho} &= \psi_{,\rho} + (i\,a\,\varphi_\rho\,\gamma_3 - \tfrac{1}{4}\gamma_{kr\rho}\,\nu_k^+\,\nu_r)\psi\,, \\ \psi_{/\rho}^* &= \psi_{,\rho}^* - \psi^*(i\,a\,\varphi_\rho\,\gamma_3 - \tfrac{1}{4}\gamma_{kr\rho}\,\nu_k\,\nu_r^+)\,. \end{aligned}\right\} \tag{24}$$

§ 4. – In place of I, (54), we will have:

$$M = \frac{1}{i} h_r^\rho \, \psi^* \nu_r \, \psi_{/\rho} + n_r \, \psi^* \nu_r (b_0 - b \, \gamma_3 \, \varphi) \psi \,, \tag{25}$$

in which the $\psi_{/\rho}$ are determined from (24), and φ is the function in (20).

The covariant unified wave equations now read:

$$\nu_r \{ h_r^\rho \, \psi_{/\rho} + i \, n_r (b_0 - b \, \gamma_3 \, \varphi) \psi \} = 0 \,. \tag{26}$$

Correspondingly, the terms:

$$N \, \delta \varphi, \;\; L \, \delta \varphi$$

will enter into the integral on the right-hand sides of I, (58), (59). It then follows from this that along with the field equations I, (60), (61), (62), one also has the further field equation:

$$F \equiv N - L = 0, \tag{27}$$

from which the distribution function φ can be determined.

We then have 41 field equations, between which 15 identities exist. However, that will describe all of physical reality, i.e., the gravitational field, the electromagnetic field, the mass distribution, and the matter field of the electrons and protons.

Finally, instead of I, (77), we have the following equations:

$$(M_{r\alpha} \, h_r^\rho)_{/\rho} + f_{\alpha\rho} \, M^\rho + \varphi_{,\alpha} \cdot N = 0 \,. \tag{28}$$

It shows that in addition to the **Lorentz** ponderomotive force:

$$f_{\alpha\rho} \, M^\rho \,,$$

the **Poincaré** pressure force:

$$\varphi_{,\,\alpha} \, N$$

is also present.