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FASCICLE LIII

The Monge problem

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THE

MONGE PROBLEM

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INTRODUCTION

The Monge problem in one independent variable, in the broad sense, consists of explicitly integrating a system of k ($k \leq n - 1$) Monge equations:

$$(\alpha) \qquad F_i\,(x_1, x_2, \ldots, x_{n+1}\,;\, dx_1, dx_2, \ldots, dx_{n+1}) = 0 \qquad (i = 1, 2, \ldots, k),$$

in which the F are homogeneous functions of the $dx_1, dx_2, \ldots, dx_{n+1}$.

By the term "explicitly integrating," we mean expressing the x variables as well-defined functions of one parameter, $n - k$ arbitrary functions of that parameter and their derivatives up to a certain order, and that those functions can also contain a finite number of arbitrary constants.

Monge solved that problem for the case $n = 2$, $k = 1$. Monge's result can be extended to certain equations or indeterminate systems of the form (α) in which $n > 2$.

In the case of $n > 2$, $k < n - 1$, one meets up with systems of Monge equations that have been the object of work by Serret, Darboux, Hadamard, Goursat, Cartan, and others. Beyond any doubt, it was Hilbert that established a fact that was foreseen by several geometers in relation to the impossibility of integrating explicitly in the general cases.

The Monge problem is linked with the problem of reducing a system of Pfaff equations to a canonical form.

If $k = n - 1$ then the general solution depends upon an arbitrary function of one argument, and the Monge problem is equivalent to the problem of explicitly integrating a Pfaff system of n equations in $n + 2$ variables of a well-defined system. That amounts to the problem of the equivalence of two systems of n total differential equations in $n + 2$ variables under of the group of point-like transformations in $n + 2$ variables. That was how Cartan could recognize whether a system of the form (α) was explicitly integrable in the case $k = n - 1$.

Vessiot found a theorem that was equivalent to that of Cartan under somewhat more general hypotheses by applying his new general theory of integration problems, which was based upon the consideration of sheaves of infinitesimal transformations. That theory, which correlates with Cartan's theory, opens up a vast horizon of research into the Monge problem. For the same problem with two unknown functions in several independent variables, one has some very essential results of E. Goursat.

CHAPTER I

THE FIRST-ORDER MONGE EQUATION.

1. The equation:

$$(1) \qquad f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) = 0.$$

Integral curves. Monge's method. – The problem of integrating equation (1) can be formulated as follows:

Determine the curves that are tangent to one of the generators of the cone (*T*):

$$(T) \qquad f\left(x, y, z, \frac{Y-y}{X-x}, \frac{Z-z}{X-x}\right) = 0$$

at each of their points when that cone has its summit at that point.

We first seek the condition that p and q must satisfy in order for the plane:

$$(2) \qquad Z - z = p\,(X - x) + q\,(Y - y)$$

to have two generators in common with the cone that coincide with a well-defined generator. If one sets $\frac{Y-y}{X-x} = t$ then one must express the idea that the equation:

$$(3) \qquad F\,(x, y, z, t, p + qt) = 0$$

has a double root at t. Hence, the desired condition will be the result of the elimination of t from equation (3) and its derivative with respect to t. One will obviously arrive at the same condition if one eliminates y', x' from the equations:

$$f\,(x, y, z, y', z') = 0, \qquad z' = p + qy', \qquad f_{y'} + q\,f_z = 0.$$

Let:

$$(4) \qquad F\,(x, y, z, p, q) = 0$$

be the result of the elimination. If (*S*) is an integral surface of equation (4), when one considers it to be a partial differential equation, then one will have that at each point *M* of the surface (*S*), the cone (*T*) will touch the tangent plane to the surface along a generator. Therefore: *Each Monge equation* (1) *corresponds to an equation* (4) *that is the tangential equation to* (1). Equation (4) is also called the *adjoint* equation to (1).

Conversely, let a nonlinear partial differential equation have the form (4); it couples the angular coefficients p, q of the tangent plane to an integral surface that passes through a given point in space. The position of that plane will then depend upon just one arbitrary parameter, and one can deduce from this that this enveloping plane is, in general, a cone (*T*) that has the point *M* for its summit. The equation of that cone is the result of the elimination of p, q from (2), (4), and the equation:

$$(Y - y)\frac{\partial F}{\partial p} - (X - x)\,\frac{\partial F}{\partial q} = 0.$$

It then results that: *A partial differential equation* (4) *corresponds to a Monge equation* (1) *that one finds by elimination of p, q from equation* (4), *and the equations:*

$$dz = p\,dx + q\,dy, \qquad \frac{\partial F}{\partial p}dy - \frac{\partial F}{\partial q}dx = 0.$$

The cone (T), which is the envelope of the ∞^1 planes that are represented by equation (2) when the coefficients p, q verify (4), is called the *elementary cone* that is associated with the point (x, y, z). Upon recalling that a contact element whose elements (x, y, z, p, q) satisfy (4) is called an *integral contact element*, one can say that the elementary cone that is associated with the point (x, y, z) is the envelope of integral contact elements that belongs to that point. Let a surface (S) be an integral of equation (4), let (x, y, z) be a point of that surface, and let (T) be the cone that corresponds to it. As one knows, just one characteristic of (S) will pass through the point (x, y, z) that has a generator of (T), which has that point for its summit, as its tangent.

We seek a *non-characteristic* curve that is situated on (S) and tangent at each of its points to the characteristic (S) that passes through that point. One sees that, in general, such a curve will exist on (S), since one can consider the surface (S) as being generated by a family of characteristics, each of which meets the characteristic that is infinitely close to it, and therefore those characteristics will have an envelope. At each point, that curve will admit the generator of (T) that relates to that point as its tangent. Conversely, let (Γ) be a curve that satisfies (1) without being a characteristic of (4). The locus of characteristic curves that are tangent to (Γ) will then be an integral surface of (4), and the curve (Γ) will be the envelope of characteristics.

Consequently, there exists just one curve on an integral surface that satisfies (1) without being characteristic, and it is *the envelope of its characteristics.* By analogy with the case of developable surfaces, one calls it the *edge of regression*. Lie gave such curves the name of *integral curves.* One then calls any curve that satisfies (1) without being characteristic an *integral curve.*

If V (x, y, z, a, b) = 0 is a complete integral of (4) then an arbitrary integral surface will be defined by the characteristics:

$$V = 0, \qquad \frac{\Delta V}{\Delta a} = 0 \qquad [b = \varphi(a)],$$

and the envelope of the characteristics will be defined by the equations:

$$(5) \qquad V = 0, \qquad \frac{\Delta V}{\Delta a} = 0, \qquad \frac{\Delta^2 V}{\Delta a^2} = 0,$$

so all of the integral curves can be represented by those equations (5), which permits one to calculate x, y, z as functions of the parameter a, while φ is an arbitrary function of a.

If an integral curve is tangent to an integral surface of (4) then the contact will be of order at least two. That property of integral curves, as well as a certain number of other ones, was pointed out by Sophus Lie.

As an example, let the equation $dx^2 + dy^2 = k^2\, dz^2$. The adjoint equation is $k^2\,(p^2 + q^2) = 1$, and formulas (5) will give the general solution, in which:

$$V = (1 - a^2)\, x + K\,(1 + a^2)\, z + 2ay + 4\, f\,(a) = 0,$$

and $f\,(a)$ is an arbitrary function of a.

Euler was the first to find the explicit integral to the equation:

$$dx^2 + dy^2 = dz^2.$$

2. Solving the equation:

$$dx^2 + dy^2 + dz^2 = ds^2. \tag{6}$$

Serret's formulas. – The integration of equation (6) was first performed by Serret by means of a geometric interpretation of (6) in rectangular coordinates. He sought to express x¸ y, z, and s as functions of one parameter θ for an arbitrary curve. Following the ideas of Monge, he sought to envision any curve as an edge of regression of a developable surface. That surface, which is the geometric locus of tangents to the curve, is represented by the equations:

$$\Phi = z - px - qy + u = 0, \qquad \delta\Phi = du - x\,dp - y\,dy = 0,$$

in which p, q, u are considered to be functions of one parameter q. The edge of regression will then be represented by the system of equations:

$$\Phi = 0, \qquad \delta\Phi = 0, \qquad \delta^2\Phi = 0.$$

One will easily deduce the expressions for dx, dy, dz as functions of p, dp, $d^{\,2}p$, $d^{\,3}p$, q, dq, $d^{\,2}q$, $d^{\,3}q$, u, du, $d^{\,2}u$, $d^{\,3}u$ from this by differentiation, while taking the equations themselves into account, and here one takes:

$$ds = \sum A_i\, d^{\,i}u \qquad (i = 1, 2, 3),$$

in which the A_i do not contain u. One can express s as a function of ψ, p, q, and its successive derivatives Upon integrating each term of ds by parts and setting $u = \dfrac{\psi'}{A_3''' - A_2'' + A_1'}$. The Ψ, p, q are considered to be functions of one independent variable. We remark that the independent variable θ has remained indeterminate, up to now. Serret chose it in such a manner that one can take simpler formulas, and he then expressed x, y, z, s as functions of one parameter θ and two arbitrary functions $\psi\,(\theta)$ and $\varphi\,(\theta)$ and the successive derivatives of those two functions [**36**].

Some new formulas for the solution of equation (3) were given by Darboux by using a method that we shall now discuss.

3. Darboux's method for the equation:

(7) $$f(dx_1, dx_2, \ldots, dx_n) = 0.$$

– First consider the Serret equation (6), when it is written:

(6′) $$dx_1^2 + dx_2^2 + dx_3^2 = dx_4^2,$$

and then set:

$$dx_i - a_i\,dx_4 = 0$$

and

(8) $$x_i - a_i\,x_4 = b_i \qquad (i = 1, 2, 3).$$

One will have:

(9) $$\sum a_i^2 = 1$$

and

(10) $$\frac{db_i}{da_i} = -x_4,$$

and the problem of integrating the equation will come down to the following one:

Determine the most general expressions a_i, b_i that satisfy the equations:

(11) $$\frac{db_1}{da_1} = \frac{db_2}{da_2} = \frac{db_3}{da_3}.$$

We see that one has six functions to determine. Since there are three relations between them, there will then be three arbitrary functions. We take two of the a_i to be such arbitrary functions – for example, a_1, a_2, and another U, whose choice will lead to the theory of contact. Indeed, we remark that the relations (11) express the idea that the two curves (A) that are described by the point (a_1, a_2, a_3) and (B) that are described by the point (b_1, b_2, b_3) must have their tangent planes, and consequently their osculating planes, parallel to each other, which will show that (B) is the edge of regression of a developable surface whose tangent planes are parallel to the osculating planes of (A), hence, it follows that if one sets:

$$\begin{vmatrix} X & Y & Z \\ a_1' & a_2' & a_3' \\ a_1'' & a_2'' & a_3'' \end{vmatrix} - U = \Phi,$$

in which U is an arbitrary function of t, and the values of X, Y, Z that are deduced from the equations $\Phi = 0$, $d\Phi / dt = 0$, $d^2 \Phi / dt^2 = 0$ will be precisely those of b_1, b_2, b_3, which, by virtue of formulas (8), (10), will give one the x_1, x_2, x_3, x_4 as functions of t.

Darboux extended his method for integrating equation (7) by setting:

$$dx_i - a_i\, dx_n = 0 \qquad (i = 1, 2, \ldots, n - 1)$$

and

(12) $$x_i - a_i\, x_n = b_i\,,$$

so

(13) $$\frac{db_i}{da_i} = -x_n\,.$$

The problem then comes down to the following one: Determine the most general expressions for a_i, b_i as functions of a certain parameter that satisfy the equations:

(14) $$\frac{db_1}{da_1} = \frac{db_2}{da_2} = \ldots = \frac{db_{n-1}}{da_{n-1}}\,,$$

(15) $$f(a_1, a_2, \ldots, a_{n-1}, 1) = 0.$$

The number of functions a_i, b_i is $2(n - 1)$, and the number of relations (14) and (15) is $n - 1$. We choose the $n - 1$ arbitrary functions to be $n - 2$ of the a_i and one function U, which will lead us along a path that is analogous to the preceding one. Indeed, we set:

$$\begin{vmatrix} b_1 & b_2 & \cdots & b_{n-1} \\ a_1' & a_2' & \cdots & a_{n-1}' \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{(n-2)} & a_2^{(n-2)} & \cdots & a_{n-1}^{(n-2)} \end{vmatrix} = U.$$

One can write:

$$U = \sum \lambda_i\, b_i\,,$$

in which the λ_i are coupled by the relations that were encountered in contact theory:

$$\sum \lambda_i\, a_i^{(k)} = 0,$$

in which:

$$a_i^{(k)} = \frac{d^{(k)} a_i}{dt^k} \qquad (i = 1, 2, \ldots, n - 1;\; k = 1, 2, \ldots, n - 2).$$

One easily sees that the values of b_i are determined by the equations:

$$U = \sum \lambda_i\, b_i\,, \quad \frac{dU}{dt} = \sum b_i \frac{d\lambda_i}{dt}\,, \qquad \ldots, \qquad \frac{d^{n-2}U}{dt^{n-2}} = \sum b_i \frac{d^{n-2}\lambda_i}{dt^{n-2}}\,,$$

hence, by virtue of the relations (12), (13), (15), one can determine the $x_1, x_2, \ldots, x_n$ as functions of the $a_1, a_2, \ldots, a_{n-2}, U$, and its successive derivatives, where U is an arbitrary function [**14**].

It is obvious that several geometric questions will find their solution in the method of G. Darboux.

4. A particular class of equations. Method of J. Hadamard. – While studying a problem in physics, Hadamard was led to search for the general solution to a system of $n - 1$ differential equations of the form:

$$\sum_k F_{ik}(y_k) = 0 \qquad (k = 1, 2, \ldots, n;\ i = 1, 2, \ldots, n-1),$$

in which we let F denote a differential operation of the form:

$$A_0\, D^n + A_1\, D^{n-1} + \ldots + A_n\,,$$

in which D is the symbol of a derivation, and the A are arbitrary functions of the independent variable. We first remark that one can always reduce the given system to a system of the form:

$$\sum a_{ik} y'_k + \sum b_{ik} y_k = 0 \qquad (k = 1, 2, \ldots, m\,;\ i = 1, 2, \ldots, m-1),$$

in which the a_{ik}, b_{ik} are functions of the independent variable, by introducing auxiliary unknowns.

Each left-hand side can be considered to be the sum of two terms, one of which contains the derivatives, and the other of which contains only variables. Since the number of equations is $m - 1$, we will have $m - 1$ terms that contain m derivatives. It will then suffice to replace each of those terms with a derivative by a convenient change of variables in order to obtain a new system in $m - 1$ derivatives. That is always possible in the case that we are addressing, since one has:

$$\sum a_{ik} y'_k = \left(\sum a_{ik} y_k\right)' - \sum a'_{ik} y_k \qquad (k = 1, 2, \ldots, m),$$

and if one sets $\sum a_{ik} y_k = z_i$ then one will take:

$$z'_i + \sum_{\rho=1}^{m-1} \gamma_{i\rho} z_\rho + \delta_{im}\, y_m = 0,$$

in which the $\gamma_{i\rho}$, δ_{im} denote constants or functions of t. We have supposed that the $\sum a_{ik} y_k$ are independent of each other in such a way that one can consider the z_i to be independent. Upon eliminating the y_m, we will get $m - 2$ equations in $m - 1$ variables of the same form as the given system. One then proceeds with it in the same manner. Upon doing that, one will arrive at just one equation of the form:

$$u_1' + b_1 u_1 + b_2 u_2 = 0,$$

which will define the function u_2 after one chooses u_1 arbitrarily. If one then repeats the preceding series of calculations that were performed then one will arrive at expressions for $y_1, y_2, \ldots, y_m$ with the aid of an arbitrary function and its successive derivatives up to order $m - 1$.

5. The Monge equation:

(16) $$f\left(x_1, x_2, \ldots, x_{n+1}; \frac{dx_2}{dx_1}, \frac{dx_3}{dx_1}, \ldots, \frac{dx_{n+1}}{dx_1}\right) = 0.$$

That equation admits an infinitude of solutions that depend upon $n - 1$ arbitrary functions, because one can take:

$$x_h = f_h(x_1) \qquad (h = 2, 3, \ldots, n)$$

arbitrarily, and what will remain is one equation that determines x_{n+1} as a function of x_1. Here, I call any curve that satisfies equation (16) an *integral curve.* If one introduces the variables:

$$x_j' = \frac{dx_j}{dx_1} \qquad (j = 2, 3, \ldots, n + 1)$$

then the equation will be equivalent to the system:

$$f(x_1, x_2, \ldots, x_{n+1} ;\ x_2',\ x_3', \ldots,\ x_{n+1}') = 0,$$

$$\frac{dx_1}{1} = \frac{dx_2}{x_2'} = \ldots = \frac{dx_{n+1}}{x_{n+1}'}.$$

I keep the variables x_λ ($\lambda = 1, 2, \ldots, n + 1$) and make a change of variables x_j' by taking the following types of transformations:

$$x_{n+1}' = p_1 + \sum p_h x_h', \qquad \frac{\partial f}{\partial x_h'} + p_h \frac{\partial f}{\partial x_{n+1}'} = 0,$$

so the new system will be:

(17) $$F(x_1, x_2, \ldots, x_{n+1} ; p_1, p_2, \ldots, p_n) = 0,$$

(18) $$\frac{dx_i}{P_i} = \frac{dx_{n+1}}{\sum p_i P_i} \qquad (i = 1, 2, \ldots, n),$$

in which F is the transform of the function f, if we suppose that we are dealing with the general case. Hence, each equation (16) will correspond to an equation of the form (17) that one calls the *adjoint equation* to (16). If I consider $x_1, x_2, \ldots, x_n$ to be n independent variables, x_{n+1}, to be a function of those n variables, and $p_1, p_2, \ldots, p_n$ to be partial derivatives then I will have a partial differential equation for the characteristic curves, to which one must add to the equations (18), the equations:

$$\frac{dx_1}{P_1} = \frac{-dp_i}{X_i + p_i X_{n+1}}.$$

Conversely, each partial differential equation corresponds to an equation (16) that one obtains by eliminating $p_1, p_2, \ldots, p_n$ from (17) and (18). Here, we can also consider the corresponding *elementary cone.*

6. Necessary and sufficient conditions that the x **of any integral curve must satisfy.** – Let $V(x_1, x_2, \ldots, x_{n+1}; a_1, \ldots, a_n) = 0$ be the complete integral of equation (17); we have proved that one can replace equation (16) by the system:

(19) $$V = 0, \qquad \sum_\lambda \frac{\partial V}{\partial x_\lambda} dx_\lambda = 0, \qquad \sum_\lambda \frac{\partial}{\partial x_\lambda}\left(\frac{\partial V / \partial a_1}{\partial V / \partial a_n}\right) dx_\lambda = 0 \qquad (\lambda = 1, 2, \ldots, n+1).$$

For the proof of that [**50**], we have replaced the variables a_i with the p_i that are defined by the relations:

(20) $$\frac{\partial V}{\partial x_i} + p_i \frac{\partial V}{\partial x_{n+1}} = 0,$$

and by applying the properties of determinants, we deduced a relation of the form:

$$\sum_\lambda \frac{\partial}{\partial x_\lambda}\left(\frac{\partial V / \partial a_i}{\partial V / \partial a_k}\right) dx_\lambda = 0 \qquad (i, k = 1, 2, \ldots, n),$$

and thus, the proposition.

Our proposition can result more directly by a process that was pointed out to me by Engel, which proceeds as follows:

I once more change the variables x_λ, p_i into x_λ, a_i by taking a type of transformation of the form (20). The variables x_λ, a_i are coupled by the relation $V = 0$, so one can consider them to be coordinates in the equation $F(x_i, p_i) = 0$.

Because of the equation $V = 0$, the conditions for the elements of equation (17) to be united will become:

$$\sum \frac{\partial V}{\partial x_\lambda} dx_\lambda + \sum \frac{\partial V}{\partial a_i} da_i = 0,$$

$$\sum \frac{\partial V}{\partial x_\lambda} dx_\lambda = 0,$$

or furthermore:

$$(21) \qquad \begin{cases} \sum \dfrac{\partial V}{\partial x_\lambda} dx_\lambda + \sum \dfrac{\partial V}{\partial a_i} da_i = 0, \\ \sum \dfrac{\partial V}{\partial a_i} da_i = 0. \end{cases}$$

We now remark that if one sets:

$$\omega = \sum \frac{\partial V}{\partial a_i} da_i$$

then one will have, upon letting ω' denote the bilinear covariant:

$$\omega' = \sum \delta\left(\frac{\partial V}{\partial a_i}\right) da_i - \sum d\left(\frac{\partial V}{\partial a_i}\right) \delta a_i,$$

or furthermore:

$$\omega' = \sum \sum \frac{\partial}{\partial x_\lambda}\left(\frac{\partial V}{\partial a_i}\right) \delta x_\lambda \, da_i - \sum \frac{\partial}{\partial x_\lambda}\left(\frac{\partial V}{\partial a_i}\right) dx_\lambda \, \delta a_i .$$

Now, in order to get the differential equations of the characteristic system (20), one must only add the equation $\omega' = 0$ to equations (21) and consider the δx_λ, δa_i to be arbitrary quantities that are subject to only the conditions:

$$\sum \frac{\partial V}{\partial x_\lambda} \delta x_\lambda = 0, \qquad \sum \frac{\partial V}{\partial a_i} \delta a_i = 0.$$

One can then take the equations:

$$\sum \frac{\partial}{\partial x_\lambda}\left(\frac{\partial V / \partial a_i}{\partial V / \partial a_n}\right) dx_\lambda = 0$$

for the dx_λ, which will lead to our proposition.

The elimination of the a_i from equations (19) provides equation (16). One then obtains the necessary and sufficient conditions that the x of any integral curve must be subject to. Those conditions can be put into the form:

$$(22) \qquad V = 0, \qquad \sum \frac{\partial V}{\partial a_i} da_i = 0, \qquad \sum \frac{\partial}{\partial x_\lambda}\left(\frac{\partial V / \partial a_i}{\partial V / \partial a_n}\right) dx_\lambda = 0.$$

7. Various applications. – It is easy to see that one can deduce the following equation from equations (22):

$$\frac{\Delta^2 V}{\Delta a^2} = \sum_i \sum_k \frac{\partial^2 V}{\partial a_i\, \partial a_k} a_i' a_k' + \sum \frac{\partial V}{\partial a_i} a_i'' = 0 \qquad (i, k = 1, 2, \ldots, n),$$

in which the a are considered to be functions of one independent variable. Hence, the three conditions are:

$$(23) \qquad V = 0, \qquad \frac{\Delta V}{\Delta a} = 0, \qquad \frac{\Delta^2 V}{\Delta a^2} = 0.$$

Note that here one must suppose that the a_i are not constant; i.e., that equations (23) belong to any integral curve, but *they are not characteristic.* Hence, if one calls any curve that satisfies equation (16), *but is not characteristic*, an *integral curve* then one will have that the x_λ verify equations (23) and the differential equations:

$$\sum \frac{\partial}{\partial x_\lambda}\left(\frac{\partial V / \partial a_i}{\partial V / \partial a_n}\right) dx_\lambda = 0 \qquad (\lambda = 1, 2, \ldots, n + 1),$$

and therefore we will have equations (23) for the general solution that gives the x for any integral curve.

If one sets $\frac{\partial V}{\partial a_i} : \frac{\partial V}{\partial a_n} = - b_i$ then one can take the following equations for equations (22):

$$(24) \qquad \begin{cases} V = 0, & \sum_i \frac{\partial V}{\partial a_i} a_i' = 0, \\ \frac{\partial V}{\partial a_i} + b_i \frac{\partial V}{\partial a_n} = 0, & b_i + \sum_k \frac{\partial}{\partial a_k}\left(\frac{\partial V / \partial a_i}{\partial V / \partial a_n}\right) a_k' = 0. \end{cases}$$

Botasso [**3**] appealed to those equations in order to establish some theorems that gave necessary and sufficient conditions for a simply-infinite sequence Σ of characteristics of (17) to admit an envelope outside of the singular integral.

Note that one can deduce different families of integral curves from equations (19), (22), or (24) if one subjects the arbitrary functions a to conveniently-chosen relations.

CHAPTER II

HIGHER-ORDER MONGE EQUATIONS. MONGE SYSTEMS.

8. The equation:

(25) $$f(x, y, z, y', z', y'', z'') = 0.$$

Ed. Goursat's theory. – Let an equation of the form:

(26) $$V(x_1, x_2, x_3 ; a_1, a_2, a_3) = 0$$

be given; append the equations:

(27) $$\sum \frac{\partial V}{\partial x_i} dx_i = 0 \qquad (i, k = 1, 2, 3),$$

(28) $$\sum \frac{\partial^2 V}{\partial x_i\, \partial x_k} dx_i\, dx_k + \sum \frac{\partial V}{\partial x_i} d^2 x_i = 0$$

to it. One will then deduce that:

(29) $$\sum \frac{\partial V}{\partial a_i} da_i = 0, \qquad \sum \frac{\partial^2 V}{\partial x_i\, \partial a_k} dx_i\, da_k = 0.$$

Moreover, let an equation:

(30) $$\psi(a_1, a_2, a_3 ; da_1, da_2, da_3) = 0$$

be given that is homogeneous in the da. Upon eliminating the da_1, da_2, da_3 from equations (29), (30), one will arrive at an equation of the form:

(31) $$F(a_1, a_2, a_3 ; x_1, x_2, x_3 ; dx_1, dx_2, dx_3) = 0$$

that is homogeneous in the dx. If one now eliminates the a from equations (26), (27), (28), (31) then one can take an equation of the form:

(32) $$A + \sum B_i\, d^2 x_i = 0,$$

in which the A, B_i do not contain $d^2 x_i$. Suppose that if one considers x_3 to be a function of the x_1, x_2 then equation (26) will be a complete integral of a linear system in involution of the second-order partial differential equation that represents a family of surfaces (Σ) that depend upon three parameters a_1, a_2, a_3 . In addition, let equation (30) define the relation that the a_1, a_2, a_3 must satisfy in order for the envelope E of the surface (Σ) to likewise be an integral of the system in involution. If one has taken the a_1, a_2, a_3 to be functions of one variable parameter a that satisfy the relation (30) then the characteristics of the moving surface (Σ) will have an envelope (A) that we, with Goursat, call the *edge of regression* of the integral surface (E). All of the curves (A) satisfy the same second-order Monge equation.

Indeed: Consider x_1, x_2, x_3 to be functions of one independent variable that define the edge of regression (A) and remark, with Goursat, that the surface (Σ) has second-order contact with (A) at the point where the characteristic that is situated in (Σ) touches that envelope. One will then have the x_i as coordinates of the curve (A) that satisfy equations (26), (27), (28), (29), (30), and consequently equation (31), so a Monge equation of the form (32) will result. If one sets $x_1 = x$, $y_1 = y$, $z_1 = z$ and considers x to be an independent variable then equation (32) will take the form:

$$z'' = M(x, y, z, y', z')\, y'' + N(x, y, z, y', z'). \tag{32′}$$

One also sees that integrating equation (32′) comes down to integrating (30), which is a first-order Monge equation.

Goursat likewise showed how, if one is given a linear system in involution:

$$\begin{cases} r + \lambda s + \mu = 0, \\ s + \lambda t + \nu = 0 \end{cases} \tag{33}$$

one can then obtain the corresponding equation directly without knowing the complete integral, and that also suggests that if an equation of the form (32′) is given then one can know whether it corresponds to a system in involution by algebraic operations and differentiation. Finally, one can construct that system. Therefore, one difference between the first-order Monge equation and the second-order one is obvious: In general, any equation (1) will correspond to a first-order partial differential equation, while an equation (25) will not, in general, correspond to a system in involution, and that will be true even when equation (25) is linear in y'' and z'' [**28**].

Beudon has used procedures that relate to the second-order Monge equation in order to express x, y, $\mathcal{J}$ as functions of one argument with no quadrature sign by setting:

$$\mathcal{J} = \int \{M(x, y, y')y'' - N(x, y, y')\}dx.$$

In order to do that, he sought to determine a function $a(x, y, y')$ in such a manner that the Monge equation:

$$M(x, y, y')\, y'' - N(x, y, y') = \frac{\partial a}{\partial x} + \frac{\partial a}{\partial y} y' + \frac{\partial a}{\partial y'} y'' + z''$$

will result from a system in involution of the form (33); it comes down to determining a from a second-order partial differential equation [**2**]. Those questions have been studied by E. Cartan by means of the theory of bilinear covariants, which we shall address later on.

9. The equation:

(34) $$f(x_1, x_2, x_3, x_4; dx_1, \ldots, dx_4; d^2x_1, \ldots, d^2x_4) = 0.$$

One can also make a first-order partial differential equation in three independent variables correspond to a second-order equation by starting with a complete integral of the latter equation, and in that very particular case, the solution to the Monge equation will be given by very simple formulas that can be considered to be an extension of the Monge formulas.

Indeed, let:

(35) $$F(x_1, x_2, x_3, x_4; p_1, p_2, p_3) = 0$$

be a first-order partial differential equation, and let:

(36) $$V(x_1, x_2, x_3, x_4; a_1, a_2, a_3) = 0$$

be a complete integral of that equation. If one forms the following relations:

(37) $$dV = 0, \qquad d^2 V = 0$$

then one can infer the equations:

(38) $$\Delta V = 0,$$

(39) $$\sum\sum \frac{\partial^2 V}{\partial a_i \, \partial x_\lambda} da_i \, dx_\lambda = 0,$$

in which Δ denotes the total differential with respect to the *a*. Add to these, the equation:

(40) $$\sum\sum \frac{\partial^2 V}{\partial a_i \, \partial x_k} da_i \, dx_k = 0 \qquad (i, k = 1, 2, 3).$$

If one eliminates the da_i from equations (38), (39), and (40) then one will get an equation that contains the x, dx, and a; one eliminates the a from them and (36), (37). In general, one will then arrive at an equation of the form (34) that is linear in the $d^2 x$. One further sees that one can deduce:

(41) $$\sum \frac{\partial V}{\partial a_i} \, d^2 a_i = 0$$

from equations (36), (39), (40), and consequently the Monge equation that corresponds to equation (35) in the manner that was cited above will have the solution that is given by equations (36), (38), (40), (41) for a solution.

10. Monge systems of $n - 1$ equations in $n + 1$ variables. Goursat's method. – Goursat gave a very elegant method for the integration of a Monge system [**24**].

We can make the methods of Monge and Darboux even more profound by Goursat's method, and one we will see how we can extend Monge's results. Let:

(42) $$f_i(x_1, \ldots, x_{n+1}; dx_1, \ldots, dx_{n+1}) = 0 \qquad (i = 1, 2, \ldots, n-1)$$

be a system of $n - 1$ Monge equations, so the cone (T) that corresponds to the summit $M(x_1, \ldots, x_{n+1})$ will be represented by the equations:

(43) $$f_i(x_1, \ldots, x_{n+1}; X_1 - x_1, \ldots, X_{n+1} - x_{n+1}) = 0.$$

Let:

(44) $$X_{n+1} - x_{n+1} - \sum_i p_k(X_k - x_k) = 0 \qquad (k = 1, 2, \ldots, n)$$

be the plane (P), so equations (43), (44) will determine the generators of the cone (T) that are situated on the plane (P). If one sets:

$$\frac{X_2 - x_2}{X_1 - x_1} = a$$

then equations (43) will define the ratios:

$$\frac{X_\rho - x_\rho}{X_1 - x_1} \qquad (r = 3, 4, \ldots, n + 1)$$

as functions of a, and equation (44) will take the form:

(45) $$U(a) = \varphi_{n+1}(a) - p_1 - p_2\, a - \sum p_\mu\, \varphi_\mu(a) = 0 \qquad (\mu = 3, 4, \ldots, n).$$

We now seek to determine the coefficients p in such a fashion that the plane (P) will have n generators in common with the cone (T) that coincide with a well-defined generator, in which case, with Goursat, we say that the plane *osculates the cone* (T).

One will get necessary and sufficient conditions for the plane (P) to osculate the cone (T) in the form of $n - 1$ equations of the form:

(46) $$F_i(x_1, \ldots, x_{n+1}; p_1, \ldots, p_n) = 0.$$

They result from eliminating the a from the equations:

$$U(a) = 0, \qquad U'(a) = 0, \qquad \ldots, \qquad U^{(n-1)}(a) = 0,$$

which express the idea that equation (45) possesses a multiple root of order n. Equations (46) are called the *tangential* equations to the cone (T) whose summit is at $M(x_1, x_1, \ldots, x_{n+1})$.

If one considers $x_1, \ldots, x_n$ to be n independent variables and x_{n+1} to be a function of those n variables with $p_k = \partial x_{n+1} / \partial x_k$ then equations (46) will define a system of partial differential equations that we, with Goursat, call the *associated system* to the system (42).

Therefore, any Monge system (42) corresponds to an associated system of the form (46). Suppose that the system is in involution, and let:

$$V(x_1, x_2, \ldots, x_{n+1}; a, b) = 0$$

be a complete integral of that system. If $b = \varphi(a)$, where the function $\varphi(a)$ is an arbitrary function of a, and if:

$$\frac{\Delta V}{\Delta a} = \frac{\partial V}{\partial a} + \frac{\partial V}{\partial b}\varphi'(a), \ldots$$

then the formulas:

$$(47) \qquad V = 0, \qquad \frac{\Delta V}{\Delta a} = 0, \qquad \frac{\Delta^2 V}{\Delta a^2} = 0, \qquad \ldots, \qquad \frac{\Delta^n V}{\Delta a^n} = 0$$

will define the general integral of the Monge system (42).

11. Application of Goursat's method. – That method can be applied whenever the associated system is in involution.

Let a Monge system (a) be given in which $i < n - 1$. Goursat's method applies if one can adjoin to that system $n - i - 1$ new equations of the same form in such a fashion that the associated system of the system thus-formed is in involution. With Goursat, consider the Serret equation that was treated by Darboux. Suppose that one has i equations of the form (7).

Add to them the $n - i - 1$ equations:

$$(48) \qquad \frac{dx_\rho}{dx_1} = \psi_\rho\left(\frac{dx_2}{dx_1}\right) \qquad (r = 1, 2, \ldots, n - 1),$$

in which the ψ_ρ are arbitrary. Equations (a), (48) define a system of $n - 1$ Monge equations in $n + 1$ variables whose associated system is in involution, and Goursat's method will be applicable.

In that way, one will find that the general solution of the Serret equation (6′) is given by the formulas:

$$V = 0, \qquad \frac{\Delta V}{\Delta a} = 0, \qquad \frac{\Delta^2 V}{\Delta a^2} = 0, \qquad \frac{\Delta^3 V}{\Delta a^3} = 0,$$

in which:

$$V = x_4 - \sum p_k x_k - \psi(a) \qquad (k = 1, 2, 3),$$

where $\psi(a)$ is an arbitrary function of a, and p_k are functions of a that are defined by the equations:

$$p_1 + p_2\, a + p_3\, \varphi(a) = U(a), \qquad p_1 + p_3\, \varphi'(a) = U'(a), \qquad p_3\, \varphi''(a) = U''(a),$$

with

$$U(a) = \sqrt{1 + a^2 + \varphi^2(a)}\,,$$

and $\varphi(a)$ denote an arbitrary function of a.

We sought [**46**] to apply Goursat's method to the equations of the form:

$$f\left(x_1, \frac{dx_2}{dx_1}, \frac{dx_3}{dx_1}, \cdots, \frac{dx_n}{dx_1}\right) = \frac{dx_{n+1}}{dx_1}\,. \tag{49}$$

Append $n - 2$ relations of the form:

$$\frac{dx_h}{dx_1} = \varphi_h\left(x_1, \frac{dx_2}{dx_1}\right) \qquad (h = 3, 4, \ldots, n)$$

to (49). One sees that the associated system:

$$F_i(x, p) = 0$$

of the Monge system that is composed of equations (46) will take a form such that:

$$\frac{\partial F_k}{\partial p_1} = 0, \qquad \frac{\partial F_k}{\partial x_h} = 0, \qquad \frac{\partial F_1}{\partial p_1} = 0$$

$$(h = 2, 3, \ldots, n, n + 1;\ k = 2, 3, \ldots, n - 1)$$

and one concludes from this that in order for the associated system to be in involution, it is necessary and sufficient that one must have:

$$\frac{\partial F_k}{\partial p_1} = 0,$$

identically, which will happen perforce if the equation has the form:

$$f_1(x_1) + f_2\left(\frac{dx_2}{dx_1}, \frac{dx_3}{dx_1}, \cdots, \frac{dx_n}{dx_1}\right) = \frac{dx_{n+1}}{dx_1}\,,$$

so the adjoint equations will have the form:

$$\frac{dx_\rho}{dx_1} = \varphi_\rho\left(\frac{dx_2}{dx_1}\right) \qquad (\rho = 3, 4, \ldots, n).$$

It would be interesting to look for the Monge systems for which Goursat's method applies. One question that would emerge from such a search is the following one: In which cases will eliminating a from n equations of the form:

$$\sigma_i\,(x_1, x_2, \ldots, x_{n+1}, p_1, p_2, \ldots, p_n, a) = 0$$

give a system in involution?

Therefore, we have made [**49**] some remarks relating to that question of the application of Goursat's method. That provided us with an opportunity to recover the preceding results for (49) as particular cases of more general results.

Gross [**29**] has also studied some cases in which one finds solutions to certain indeterminate differential systems without any quadrature.

CHAPTER III

IMPOSSIBILITY OF EXPLICIT INTEGRATION IN THE GENERAL CASE.

12. Impossibility of extending the Monge method. – Consider the Monge equation in four variables:

$$f(x_1, x_2, x_3, x_4 ; dx_1, dx_2, dx_3, dx_4) = 0,$$

and let $V(x_1, x_2, x_3, x_4 ; a_1, a_2, a_3)$ be the complete integral of the adjoint equation (Chap. I). One might be tempted to believe that the equations:

$$(50) \qquad V = 0, \qquad \frac{\Delta V}{\Delta a} = 0, \qquad \frac{\Delta^2 V}{\Delta a^2} = 0, \qquad \frac{\Delta^3 V}{\Delta a^3} = 0$$

will provide the general solution in a manner that is analogous to the case of three variables.

We have remarked [**44**] that such a general solution does not exist, in general.

For example, take equation (6′). Since $V = 0$, one will then have the equation:

$$x_4 - a_1 x_1 - a_2 x_2 - b\, x_3 - a_3 = 0, \qquad b^2 = 1 - a_1^2 - a_2^2 .$$

One cannot say that the x_k that are inferred from (50) provide the solution to equation (6′), since the a_i are *arbitrary* functions of the independent variables, and consequently, they are *mutually independent.*

In regard to that, we have proved that in order for (50) to give a solution, it is necessary that a_1, a_2 are not independent, but coupled by the relation:

$$a_1'^2 + a_2'^2 = (a_1 a_2' - a_2 a_1')^2 ,$$

and we have likewise given [**45**] a much more general theorem that says the following: It is, in general, impossible to deduce the equation $\Delta^3 V / \Delta a^3 = 0$ from equations (19) or (22). Hence, one is led to demand to known whether a function:

$$V(x_1, x_2, \ldots, x_{n+1} ; a_1, a_2, \ldots, a_{n+1})$$

does or does not exist such that (19), (22) can be put into the form of $n + 1$ other equations, four of which are the following ones:

$$V_1 = 0, \qquad \frac{\Delta V_1}{\Delta a} = 0, \qquad \frac{\Delta^2 V_1}{\Delta a^2} = 0, \qquad \frac{\Delta^3 V_1}{\Delta a^3} = 0.$$

For example, recall equation (6′). Goursat found a function:

$$V_1 = x_4 - \sum p_i\, x_i - b,$$

[in which the p_i are well-defined functions of one parameter a and an arbitrary function $\varphi(a)$, and b is a second arbitrary function $\psi(a)$] such that the equations:

$$V_1 = 0, \qquad \frac{\Delta V_1}{\Delta a} = 0, \qquad \frac{\Delta^2 V_1}{\Delta a^2} = 0, \qquad \frac{\Delta^3 V_1}{\Delta a^3} = 0$$

give the general solution of equation (6′), and more generally, Goursat's method shows how one can extend Monge's method.

13. Hilbert's theorem. Generalizations. – In an article [**31**] that was published in 1912, Hilbert proved a theorem that asserted the impossibility of expressing the general solution to the equation:

$$\frac{dz}{dx} = \left(\frac{d^2 y}{dx^2}\right)^2 \tag{51}$$

by the formulas:

$$\left\{\begin{aligned} x &= \varphi(t, w, w_1, w_2, \cdots, w_r), \\ y &= \psi(t, w, w_1, \cdots, w_r), \\ z &= \chi(t, w, \cdots, w_r), \end{aligned}\right. \tag{52}$$

in which φ, ψ, χ denote well-defined functions of their arguments, t is a parameter, w is an arbitrary function of t, and $w_1, \ldots, w_r$ are the successive derivatives of w.

In order to prove that, Hilbert started with the identity that equation (51) will lead to when there exists a solution of the form (52). After making some remarks about the form of that identity, one will first deduce from that neither side of the identity in question contains w_{r+2}, w_{r+1}. One then supposes that the first of equations (52) has been solved for w_r and that its value has been introduced into the other two equations. If one then takes:

$$\psi = f\,(t, w, w_1, \ldots, w_{r-1}, x),$$
$$\chi = g\,(t, w, w_1, \ldots, w_{r-1}, x),$$

then upon appealing to certain identities that one will easily find, one can conclude that:

$$f_{w_{r-1}} = 0, \qquad f_{w_{r-2}} = 0, \qquad \ldots, \qquad f_i = 0;$$

i.e., that f can contain only x. One also deduces, upon taking equation (51) into account, that $g_x = f_{xx}^2$ and then $g = X + W$, in which X is a function of only x, and W is a function of $t, w, w_1, \ldots, w_{r-1}$. One finally sees that W is a constant; i.e., g is a function of only x. Now, we suppose that $r \geq 1$ in the solution (52), and the impossibility of integrating (51) explicitly is established.

Hilbert's analysis can be extended to any equation that gives dz / dx as a function of x, y, z, dy / dx, d^2y / dx^2 by means of an expression that is not homographic to d^2y / dx^2.

We have generalized [**49**] Hilbert's theorem by using the same mode of proof and have asserted the impossibility of explicitly integrating other Monge equations.

14. Various remarks. –

I. In the paper that was cited above, before studying equation (51), Hilbert considered the first-order Monge equation (1). One can find an equation V such that the equations:

$$\sum \frac{\partial V}{\partial x} dx = 0, \tag{53}$$

$$\sum \frac{\partial^2 V}{\partial a \, \partial x} dx = 0 \tag{54}$$

result from the elimination of a from equation (1).

If one sets:

$$V(x, y, z, a) = b, \tag{55}$$

$$\frac{\partial V}{\partial a} = \gamma \tag{56}$$

then one will get equations (53), (550, (56), which give a, b, γ as functions of x, y, z, dy / dx, dz / dx. One also sees that:

$$\frac{db}{da} = \gamma, \tag{57}$$

and obtains a transformation of (1) into the special form (57).

Conversely, (55), (56), and the equation $\frac{\partial^2 V}{\partial a^2} = \frac{d\gamma}{da}$ that one deduces from (54), (56) give x, y, z as functions of a, b, γ, $d\gamma / da$.

More generally, we have considered [**49**] a system of $n - 1$ Monge equations (a) for which we suppose the existence of a function $V(x_1, x_2, \ldots, x_{n+1} ; a_1)$ such that the equations of the system result from the elimination of a_1 from:

$$(58)\qquad \begin{cases} \sum \dfrac{\partial V}{\partial x_1} dx_1 = 0, \\ \sum \dfrac{\partial^2 V}{\partial a_1\, \partial x_1} dx_1 = 0, \\ \dots\dots\dots\dots\dots\dots, \\ \sum \dfrac{\partial^n V}{\partial a_1^{n-1}\, \partial x_1} dx_1 = 0. \end{cases}$$

If one sets:

$$(59)\qquad V = a_2\,, \qquad \frac{\partial^\lambda V}{\partial a_1^\lambda} = a_{2+\lambda} \qquad (i = 1, 2, \dots, n + 1; \lambda = 1, 2, \dots, n - 1)$$

then one will have:

$$(60)\qquad \frac{da_\rho}{da_1} = a_{\rho+1} \qquad (\rho = 2, 3, \dots, n).$$

Equations (59) and the equation $\dfrac{\partial^n V}{\partial a_1^n} = \dfrac{da_{n+1}}{da_1}$ will then determine the x_i as functions of the a_i , da_{n+1} / da_1 .

Conversely, the values of a_1, a_2, …, a_{n+1} as functions of x, dx are inferred from equations (59) and:

$$\sum \frac{\partial V}{\partial x_i} dx_i = 0$$

verify equations (60) if one takes equations (α) into account.

II. In a general fashion, Hilbert attached the problem of explicitly integrating a system of indeterminate differential equation to a much more general problem that amounts to recognizing whether one can establish a one-to-one correspondence between the solutions to one given differential system and the solutions of another one. Cartan replaced that statement with another one that was much more precise by defining the equivalence of two differential systems, based upon the notion of the prolongation of the system [**10**].

CHAPTER IV

EQUIVALENCE OF THE MONGE PROBLEM AND THE INTEGRATION OF A PFAFF SYSTEM.

Suppose that the system (α) has been solved for:

$$\frac{dx_{n+\lambda-\rho}}{dx_1} \qquad (\rho = 1, 2, \ldots, k).$$

Set:

$$\frac{dx_\lambda}{dx_1} = u_\lambda \qquad (\lambda = 2, 3, \ldots, n + 1 - k).$$

If one considers the u_λ to be new variables then one is reduced to a Pfaff system of $2n + 1 - k$ variables:

$$\text{(61)} \qquad \begin{cases} \dfrac{dx_{n+2-\rho}}{dx_1} = f_\rho(x_1, x_2, \ldots, x_{n+1}; u_2, u_3, \ldots, u_{n+1-k}), \\ dx_\lambda - u_\lambda dx_1 = 0. \end{cases}$$

The two systems (α) and (61) are equivalent. One then sees that a Monge system of k equations in $n + 1$ variables can be replaced with a Pfaff system in which the number of equations is the number of variables, increased by $n - k$ units.

Consider the particular case $k = n - 1$, so the system (α) will be a system of $n - 1$ Monge equations in $n + 1$ variables, and the system (61) will be a system of n Pfaff equation in $n + 2$ variables.

15. Review of some results of the theories of E. Cartan. – Let S be a Pfaff system:

$$\text{(62)} \qquad \omega_1 = \sum_k X_{ik} dx_k = 0 \qquad [k = 1, 2, \ldots, n;\ i = 1, 2, \ldots, r\ (r < n - 1)].$$

The system S can be written in an infinitude of ways by replacing the variables x_k by a new arbitrary system of variables that are functions that are distinct from the latter ones. It is essential to know its *class* – i.e., the minimum number of variables that can enter into the equations of the system S under a change of variables. Let γ be the class of that system.

When the system S has been put into a form in which only γ variables and their differentials appear, we say that it has been converted into *reduced* form. We determine the class γ and put the system S into a reduced form by appealing to *characteristic elements*, which we shall now define.

One knows that a linear element (dx_k) is an *integral linear element* if the $dx_1, \ldots, dx_n$ verify equations (62). Two integral linear elements (dx_k) and (δx_k) are *in involution* if they verify the equations $\omega_i' = 0$.

An integral linear element is *characteristic* if it is in involution with all of the other integral linear elements that issue from the same point. In order to form the equations that define the characteristic elements, suppose – to fix ideas – that we have solved the system S for $dx_1, dx_2, \ldots, dx_r$, and then substituted the expressions for $\delta x_1, \delta x_2, \ldots, \delta x_r$ in $\omega_1', \omega_2', \ldots, \omega_i'$ that are inferred from the equations $\omega_i(\delta) = 0$. We then express the idea that after the aforementioned substitution, the $\omega_i' = 0$ will be independent of the δx_{r+1}, $\delta x_{r+2}, \ldots, \delta x_n$; i.e., we equate the coefficients of $\delta x_{r+1}, \ldots, \delta x_n$ to zero. One then takes certain equations:

$$\pi_1(d) = 0, \qquad \ldots, \qquad \pi_\mu(d) = 0,$$

which define the characteristic system of S, along with $\omega_i(d) = 0$.

Denote it by S_1. One proves that no matter what the system S, the characteristic system S_1 is completely integrable. In order for any integral linear element to be a characteristic element, it is necessary and sufficient that S should be completely integrable. One calls any integral of S_1 a *characteristic variable* and any multiplicity whose linear elements are all characteristic a *characteristic multiplicity*.

The number of linear equations that is independent of S_1 is called the *order* of S_1. One proves that the class of S is equal to the order of S_1 and that if one makes a change of variables in S by taking distinct characteristic variables for the independent variables then one will have converted S into a reduced form. Suppose that one has obtained p integrals $f_1, f_2, \ldots, f_p$ of the characteristic system. If one makes a change of variables in such a fashion that the integrals are p of the new variables $y_1, y_2, \ldots, y_p$, for example, then the new Pfaff system in which one makes:

$$y_i = c_i, \qquad dy_i = 0 \qquad (i = 1, 2, \ldots, p)$$

has class at most $\gamma - p$; however, it can have a lower class. For example, if the new system has class r then it will be completely integrable.

If S contains only $r + 1$ variables then it will be completely integrable, and in turn, of class r. Hence, S cannot have class $r + 1$. As for one Pfaff equation, its class is always an *odd* number.

16. Canonical forms. Derived systems. Special systems. – One knows the results of Pfaff, Darboux, Frobenius, and Weber that relate to the reduction of a Pfaff form to a canonical form. One even knows that such a form ω of class $2p$ can be reduced to the canonical form:

$$\sum z_i \, dy_i \qquad (i = 1, 2, \ldots, p),$$

where z_i, y_i form a system of $2p$ distinct variables, and a Pfaff form ω of class $2p + 1$ will reduce to the canonical form:

$$dy_{p+1} + \sum z_i\, dy_i \,.$$

The only invariant of a Pfaff form under the most general group of point-like transformations is the class of that form.

An equation $\omega = 0$ will have been reduced to canonical form when one has put into the form:

$$\sum Y_i\, dy_i = 0 \qquad (i = 1, 2, \ldots, p)$$

if ω has class $2p$, and to the form:

$$dy_{p+1} + \sum z_i\, dy_i = 0$$

if ω has class $2p + 1$.

A Pfaff equation of class three can always be reduced to the canonical form:

$$dy_2 - y_2\, dy_1 = 0.$$

For the Pfaff systems that we shall appeal to in what follows, we shall use the following canonical form:

$$\text{(63)} \qquad \begin{cases} dy_1 = 0, & dy_2 = 0, & \cdots, \\ dy_{\rho-1} = 0, & dy_\rho = 0, & dy_{\rho+2} - y_{\rho+2}\, dy_{\rho+1} = 0, \\ dy_{\rho+3} - y_{\rho+1}\, dy_{\rho+1} = 0, & \cdots, & dy_{r+1} - y_{r+2}\, dy_{\rho+1} = 0. \end{cases}$$

For example, we then appeal to the form:

$$\text{(64)} \qquad \begin{cases} dy_2 - y_2\, dy_1 = 0, \\ dy_3 - y_4\, dy_1 = 0. \end{cases}$$

Suppose that the system S has been reduced to the canonical form (63). One then takes the general integral that is represented by the formulas:

$$\text{(65)} \qquad \begin{cases} y_1 = c_1, & y_2 = c_2, & y_\rho = c_\rho, & y_{\rho+1} = a, \\ y_{\rho+2} = \varphi(a), & y_{\rho+3} = \varphi'(a), & \cdots, & y_{r+2} = \varphi^{(r-\rho)}(a), \end{cases}$$

in which $\varphi(a)$ is an arbitrary function.

Let S be a system of r equations in $r + 2$ variables, and then adjoin the equations:

$$\text{(66)} \qquad dx_{r+1} - x_{r+3}\, dx_{r+2} = 0, \quad dx_{r+2} - x_{r+4}\, dx_{r+3} = 0, \quad \ldots, \quad dx_{r+l-1} - x_{r+l+1}\, dx_{r+l} = 0$$

to the equations of the system S, in which $x_{r+3}, \ldots, x_{r+l+1}$ are new variables. Equations (66), along with those of (S), form a system Σ that obviously has the following property: Any solution:

$$x_i = \varphi_i(a) \qquad (i = 1, 2, \ldots, r + 2)$$

to the system S will provide the solution:

$$x_i = \varphi_i(a), \qquad x_{i+3} = \frac{\varphi'_{r+1}(a)}{\varphi_{r+2}(a)} = f_{i+3}(a), \qquad x_i = \frac{\varphi'_{r+2}(a)}{f_{l+1}(a)} = f_{i+1}(a), \qquad \dots$$

of the system Σ, and conversely, one deduces a solution of S from every solution:

$$x_\rho = f_\rho(a) \qquad (\rho = 1, 2, \dots, r + l + 1)$$

of the system Σ; with Cartan, we say that Σ is a *prolongation* of S.

Now consider the system S:

$$(67) \qquad \begin{cases} \omega = \sum a_i \, dx_i, \\ \varpi = \sum b_i \, dx_i, \end{cases} \qquad (i = 1, 2, 3, 4).$$

One can, in general reduce it to the canonical form (64) by a change of variables, as was proved for the first time by Engel [**17**]. S. Lie [**33**] proved the same thing by geometric considerations. Weber [**43**] appealed to the results of Engel and found some more general results that he deduced from the preceding ones. Finally, Cartan give a more direct method for the aforementioned reduction [**7**].

Consider the bilinear covariants:

$$\begin{aligned} \omega' &= \sum a_{ik}(dx_i\, \delta x_k - dx_k\, \delta x_i), \\ \varpi' &= \sum b_{ik}(dx_i\, \delta x_k - dx_k\, \delta x_i), \end{aligned} \qquad (i, k = 1, 2, 3, 4).$$

α'. First suppose that ω' becomes identically zero if one takes into account the equations:

$$(68) \qquad \omega(d) = 0, \qquad \omega(\delta) = 0, \qquad \varpi(d) = 0, \qquad \varpi(\delta) = 0,$$

which we denote by:

$$(69) \qquad \omega' = 0 \qquad (\text{mod } \omega, \varpi)$$

to abbreviate. The equation $\omega = 0$ will have class 3 (general case) or class 1.

1. Class 3: One converts to the canonical form:

$$(70) \qquad \Omega = dy_2 - y_2\, dy_1 = 0$$

by a change of variables, and the system (67) can be replaced by a system of the form:

$$\Omega = dy_2 - y_2\, dy_1 = 0, \quad \Pi = H_1\, dy_1 + H_2\, dy_2 + H_3\, dy_3 = 0,$$

and one will have, by virtue of (70):

$$\Omega' = dy_3\,\delta y_1 - dy_1\,\delta y_3 = 0 \qquad\qquad (\text{mod } \Omega, \Pi),$$

which demands that:

$$H_4 \equiv 0, \text{i.e.,} \qquad \Pi = H_1\,dy_1 - H_3\,dy_3 = 0.$$

It follows from the hypothesis that S is not completely integrable that:

$$\Omega' \neq 0 \qquad\qquad (\text{mod } \Omega, \Pi),$$

and that H_1 and H_2 cannot be zero, and the equation $\Pi = 0$ can be written:

$$dy_3 + \frac{H_1}{H_3}\,dy_1 = 0.$$

The ratio $H_1 : H_3$ necessarily depends upon y_4 , and upon taking y_4 to be the coefficient, the system S will take the form (64).

2. Class 1: It is completely integrable then. One then writes it in the form:

$$dy_1 = 0 \qquad\qquad \text{or} \qquad\qquad y_1 = \varphi_1\,(x_1, x_2, x_3, x_4).$$

One infers x_1 as a function of y_1, x_2, x_3, x_4, and one converts the system S into the form:

$$dy_1 = 0\,, \qquad\qquad H_2\,dy_2 + H_3\,dy_3 + H_4\,dy_4 = 0,$$

in which the H_i contain y_1 . Hence, the second equation can be regarded as a Pfaff equation in three variables that is not completely integrable and has y_1 as a parameter. One can then put it into the form:

$$dy_3 - y_4\,dy_2 - K\,dy_1 = 0,$$

and the system S, into the canonical form:

$$dy_1 = 0\,, \qquad\qquad dy_3 - y_4\,dy_2 = 0.$$

Finally, if $\omega' = 0$, $\varpi' = 0$ then the system S will be reducible to the form:

$$dy_1 = 0, \qquad\qquad dy_3 = 0.$$

β'. Now suppose that ω' is non-zero by taking equations (68) into account. We shall replace the given system with another equivalent one of the same form $\Omega = 0$, $\Pi = 0$, but in which:

$$\Omega' = 0 \qquad\qquad (\text{mod } \Omega, \Pi).$$

We first remark that one can replace an equation of S with another one of the form:

$$\lambda\,\omega + \mu\,\varpi = 0,$$

in which λ, μ denote arbitrary functions of the x. One knows that:

$$(\lambda\,\omega + \mu\,\varpi)' = \lambda\,\omega' + \mu\,\varpi' \qquad (\text{mod } \omega, \varpi).$$

Hence, in order to have:

$$\Omega' = (\lambda\,\omega + \mu\,\varpi)' = 0 \ (\text{mod } \omega, \varpi),$$

it will suffice that one should have:

$$\lambda\,\omega' + \mu\,\varpi' \qquad (\text{mod } \omega, \varpi). \tag{71}$$

Now, it is easy to determine the ratio λ / μ in such a way that one will have the identity (71), and indeed, for example, suppose that one has introduced the values of dx_3, dx_4, δx_3, δx_3 as the functions of dx_1, dx_2, δx_1, δx_2 that one infers from equations (68) into ω'. One will then have expressions of the form:

$$\omega' = A\,(dx_1\,\delta x_2 - dx_2\,\delta x_1),$$
$$\varpi' = B\,(dx_1\,\delta x_2 - dx_2\,\delta x_1)$$

for ω' and ϖ' when one sees that it will suffice to choose λ, μ in such a fashion that $\lambda A + \mu B = 0$ for one to have the identity (71). One then replaces the system (67) with:

$$\Omega \equiv \sum A_i\,dx_i = 0,$$
$$\Pi \equiv \sum B_i\,dx_i = 0 \qquad \text{or} \qquad \Omega' = 0 \ (\text{mod } \Omega, \Pi).$$

Cartan called the equation $\Omega = 0$, which enjoys an invariant property, the *derived equation* of the given system.

One then reverts to the first case, and one has then reduced that equation to a canonical form.

Therefore: *The reduction of the given system to its canonical form depends uniquely upon the reduction of the derived equation to its canonical form.*

Cartan applied his method to the search for the derived equation to equation (1). Suppose that one has solved it for dz / dx:

$$\frac{dz}{dx} = F\left(x, y, z, \frac{dy}{dx}\right),$$

which is equivalent to the system:

$$\omega = dy - u\,dx = 0, \qquad \varpi = dz - F\,(x, y, z, u) = 0,$$

in which one considers u to be a new variable. One has:

$$\omega' = du\,\delta x - dx\,\delta u, \qquad \varpi' = \frac{\partial F}{\partial u}(du\,\delta x - dx\,\delta u),$$

so

$$\left(\varpi - \frac{\partial F}{\partial u}\omega\right)' = 0,$$

and the derived equation is:

$$\Omega = \varpi - \frac{\partial F}{\partial u}\omega = 0,$$

or even:

(72) $$\Omega = dz - p\,dx - q\,dy = 0,$$

with

(73) $$p = F - u\frac{\partial F}{\partial u}, \qquad q = \frac{\partial F}{\partial u},$$

and one has reduced it to its canonical form – viz., equation (72) – in which p, q are coupled by the relation that results from eliminating u from (73); one thus comes back to the classical method.

The same method was employed by Cartan with the equation:

$$\frac{dz}{dx} = A\left(x, y, z, \frac{dy}{dx}\right)\frac{d^2y}{dx^2} + B\left(x, y, z, \frac{dy}{dx}\right),$$

which reduces to the system:

$$\omega = dy - u\,dx = 0,$$
$$\varpi = dz - A(x, y, z, u)\,du - B(x, y, z, u)\,dx = 0.$$

He then applied it to the calculation of $z = \int y^m \frac{d^2y}{dx^2}$, in which y is an arbitrary function of x and also to the calculation of the quadratures:

$$u = \int \frac{dx}{1+xy}, \quad v = \int \frac{dy}{1+xy},$$

in which x, y are coupled by an arbitrary function (cf., Beudon [**2**]).

Now consider a system S of r equations (62), $\omega_i = 0$, and suppose that one has:

(74) $$\sum l_i\,\omega_i' \equiv 0 \qquad (\text{mod } \omega_1, \omega_2, \ldots, \omega_r),$$

in which l_i denote functions of the variables. One will also have:

(74′) $$\left(\sum l_i\,\omega_i\right)' \equiv 0 \quad (\text{mod } \omega_1, \omega_2, \ldots, \omega_r),$$

and we then say that the equation:

(75) $$\sum l_i \, \omega_i = 0$$

belongs to *the derived system* of (S), which we, following Cartan, define in the following manner:

The derived system of S is composed of all distinct equations of the form (75), *in which* $l_1, l_2, \ldots, l_i$ *are arbitrary functions of the variables such that one has the identity* (74); we denote it by S'. As one sees, it is composed of the set of equations in S such that two arbitrary integral linear elements of S are in involution with each other. Let r' be the number of equations in S'. One can obviously write the equations of S in such a fashion that the r' equations of S' are:

$$\omega_1 = 0, \qquad \omega_2 = 0, \qquad \ldots, \qquad \omega_{r'} = 0.$$

One will have:

$$\omega_1' \equiv \omega_2' \equiv \ldots \equiv \omega_i' \equiv 0 \qquad (\text{mod } \omega_1, \omega_2, \ldots, \omega_r).$$

In order for the system S to be completely integrable, it is necessary and sufficient that the system S' must coincide with S.

Consider a system S of r equations in $r + 2$ variables that is not completely integrable.

If one solves those equations for $dx_1, \ldots, dx_i$ then one will get:

(76) $$\omega_\rho(d) = dx_\rho - (a_\rho \, dx_{i+1} + b_\rho \, dx_{i+2}) = 0 \qquad (\rho = 1, 2, \ldots, r),$$

and the ω_ρ' will be expressed uniquely by means of the binomial ([1]):

$$[dx_{r+1}, dx_{r+2}];$$

i.e.:

(77) $$(\rho = 1, 2, \ldots, r) \; \omega_\rho' = K_\rho \, [dx_{r+1}, dx_{r+2}] \qquad (\text{mod } \omega_1, \omega_2, \ldots, \omega_r).$$

Since S is not completely integrable, none of the K_ρ are zero. Let $K_r \neq 0$. One then infers from the relations (77) that:

$$\left(\omega_i - \frac{K_i}{K_r}\omega_r\right)' \equiv 0 \qquad (\text{mod } \omega_1, \omega_2, \ldots, \omega_r),$$

and the derived system S' will be:

$$\omega_i - \frac{K_i}{K_r}\omega_r = 0 \qquad (i = 1, 2, \ldots, r - 1).$$

([1]) We let $[\omega_1, \omega_2]$ denote the bilinear form $\omega_1(d)\,\omega_2(\delta) - \omega_2(d)\,\omega_1(\delta)$. In that way, we will write $[dx_{r+1}, dx_{r+2}]$ in place of $dx_{r+1}\,\delta x_{r+2} - dx_{r+2}\,\delta x_{r+1}$.

If one takes the ω_i in S to be the combinations:

$$\omega_i - \frac{K_i}{K_r}\,\omega_r = 0 \qquad (i = 1, 2, \ldots, r - 1)$$

then one will have the system S in the form:

$$\omega_i = 0,\ \omega_r = 0, \qquad \text{in which} \qquad \omega_i' = 0 \qquad (\text{mod } \omega_1, \omega_2, \ldots, \omega_r).$$

Look for S''; i.e., the derivative of the system:

$$(78) \qquad \omega_i = 0.$$

If one replaces $dx_1, dx_2, \ldots, dx_{r-1}$; $\delta x_1, \delta x_2, \ldots, \delta x_{r-1}$ with their expressions that one infers from $\omega_i(d) = 0$, $\omega_i(\delta) = 0$ then the ω_i' will become linear combinations of:

$$[dx_r, dx_{r+1}], \quad [dx_r, dx_{r+2}], \quad [dx_{r+1}, dx_{r+2}],$$

or even:

$$[\omega_r, dx_{r+1}], \quad [\omega_r, dx_{r+2}], \quad [\omega_{r+1}, dx_{r+2}].$$

Now, since these ω_i' must be zero when one takes the equation $\omega_r = 0$ into account, what will remain are identities of the form:

$$(79) \qquad \omega_i = L_i\,[\omega_r, dx_{r+1}] + M_i\,[\omega_r, dx_{r+2}] \qquad (\text{mod } \omega_1, \omega_2, \ldots, \omega_{r-1}),$$

or furthermore:

$$(79') \qquad \omega_i' = [\omega_r, L_i\,dx_{r+1} + M_i\,dx_{r+2}] \qquad (\text{mod } \omega_1, \omega_2, \ldots, \omega_{r-1}).$$

Having said that, we distinguish:

1. The general case in which the ratios $L_i : M_i$ are not the same, no matter what the i. The formula (79′) will then show that the $\omega_i' = 0$ reduce to two distinct equations. There are then $i - 3$ distinct relations of the form:

$$\sum l_i\,\omega_i \equiv 0,$$

and consequently, the system S'' is composed of $r - 3$ equations. In that case, we (with Cartan) say that the system S is a *normal* system.

2. The case in which the ratio $L_i : M_i$ is independent of i, so the system S'' is composed of $r - 2$ distinct equations. For our purposes, the essential result is that *the system S can be put into the form:*

$$\text{(I)} \qquad \begin{cases} \Omega_i = dy_i - (A_i\, dy_i + B_i\, dy_{i+1}) = 0, \\ \Omega_i = dy_i - y_{i+2}\, dy_{i+1} = 0, \end{cases}$$

in which the A_i, B_i are independent of y_{r+2} .

In order to prove that, first determine the class of S', or – what amounts to the same thing – the order of the characteristic system. In order to do that, we remark in this case, we can write:

$$L_i\, dx_{i+1} + M_i\, dx_{i+2} = \mu_i\, (a\, dx_{r+1} + b\, dx_{r+2}),$$

and we see that it suffices that an integral linear element of the system S' satisfies the relations:

$$\omega_i = 0, \qquad a\, dx_{r+1} + b\, dx_{r+2} = 0$$

for it to satisfy the equations $\omega_i' = 0$. Hence, the number of equations that define characteristic elements of S' is $r + 1$, and *the class of* S' *is* $r + 1$.

Now let φ_h $(x_1, x_2, \ldots, x_{r+2})$ $(h = 1, \ldots, r + 1)$ be first integrals of the system:

$$\text{(80)} \qquad \omega_i = 0, \qquad a\, dx_{r+1} + b\, dx_{r+2} = 0 \qquad (\lambda = 1, 2, \ldots, r).$$

Set φ_h $(x_1, x_2, \ldots, x_{r+2}) = y_h$, while taking the new variables to be y_h and one of the old variables – x_{r+2}, for example. One can write the equations $\omega_i = 0$ in such a manner that they will contain only y_1 , y_2 , …, y_{r+1} , and one can then give the equations $\omega_i = 0$ of the system the form:

$$dy_i - (A_i\, dy_r + B_i\, dy_{r+1}) = 0,$$

in which the A_i , B_i contain only y_1 , y_2 , …, y_{r+1} .

On the other hand, the equation $\omega_r = 0$ belongs to the system (80), and since the y_h are first integrals of that system, it establishes a linear relation between the dy_h, and consequently, if one takes the equations for S' into account then one can write the equation $\omega_r = 0$ in the form of $dy_r - H\, dy_{r+1} = 0$, in which H cannot contain only y_1, y_2, …, y_{r+1}, since otherwise, the system S would, in fact, be completely integrable. Hence, one can consider H to be a new variable y_{r+2} ; i.e., one can write:

$$\Omega_r = dy_r - y_{r+2}\, dy_{r+1} = 0,$$

and S will be a prolongation of S'.

3. The case in which all of the L_i , M_i are zero.

The system S' is completely integrable, and one can replace it with:

$$dy_i = 0 \qquad (i = 1, 2, \ldots, r - 1),$$

in which the y_i are functions of $x_1, x_2, \ldots, x_{r+2}$ give the integrals of the equations of S'. Substitute the variables $y_i, x_r, x_{r+1}, x_{r+2}$ for the $x_1, x_2, \ldots, x_{r+2}$, and take into account that $dy_i = 0$. $\omega_r = 0$ will then become a Pfaff equation in three variables in which the y_i are considered to be parameters; i.e., one will have a form:

$$A\,dx_r + B\,dx_{r+1} + \Gamma\,dx_{r+2}$$

for ω_r, in which A, B, Γ contain the y_i like the parameters, and the equation $\omega_r = 0$ can be reduced to the canonical form:

$$dy_r - y_{r+2}\,dy_{r+1} = 0.$$

One finally has the canonical form:

$$\text{(II)} \qquad dy_i = 0, \qquad dy_r - y_{r+2}\,dy_{r+1} = 0$$

for the system S.

Now suppose that we find ourselves in the second case. Since S' will have class $r + 1$, and it will be composed of $r - 1$ equations, we can start with S and proceed as before when we started with S.

One then confirms that S'' decomposes into $r - 2$ equations, and that three cases are possible:

α'. $S^{(3)}$ is composed of $r - 4$ equations, and S' is then a normal system. Since S is the prolongation of S', one sees that S will be the prolongation of a normal system.

β'. $S^{(3)}$ is composed of $r - 3$ equations, and S'' has class r.

γ'. $S^{(3)}$ is composed of $r - 2$ equations; the system S'' is then its proper derivative. Hence, the system S'' will be completely integrable.

In a general fashion, let α_i denote the number of linearly-independent equations that the system $S^{(i)}$ is composed of, let γ_i be its class, and set $\alpha_i - \alpha_{i+1} = \mu_i$. Suppose that $\gamma_\rho - \alpha_\rho = 2$. From the preceding, one will then have $\mu_\rho = 1$, and $\mu_{\rho+1}$ will be either 2, 1, or 0. If $\mu_{\rho+1} = 2$ then the system $S^{(\rho)}$ will be, by definition, a normal system. If $\mu_{\rho+1} = 0$ then one will also have $\mu_{\rho+2} = \mu_{\rho+3} = \ldots = 0$, and the system $S^{(\rho+1)}$ will be its proper derivative. The same thing will be true for $S^{(\rho+2)}$, etc. Hence, the system $S^{(\rho+1)}$ is completely integrable.

If:

$$\mu_1 = \mu_2 = \ldots = \mu_{i-k-1} = 1, \qquad \text{and} \qquad \mu_{i-h} = 2$$

then one will have:

$$\alpha_1 = r - 1, \qquad \alpha_2 = r - 2, \qquad \ldots, \qquad \alpha_{i-h} = h$$

and

$$\alpha_{i-h+1} = h - 2,$$

$$\gamma_1 = r + 1, \qquad \gamma_2 = r, \ \ldots, \qquad \gamma_{i-h-1} = h + 3,$$

and then $S^{(r-h-1)}$ will be a normal system of $h + 1$ equations and class $h + 3$. As one saw above, $S^{(r-h-2)}$ will be a prolongation of $S^{(r-h-1)}$, and one can take a system of variables such that the equations of $S^{(r-h-1)}$ contain only the variables $y_1, y_2, \ldots, y_{h+3}$ and their differentials, and the last equation of the system $S^{(r-h-2)}$ will be $dy_{h+2} = y_{h+4}\, dy_{h+2}$. The system $S^{(r-h-3)}$ is obtained by combining the equations of the system with one more equation that one can write in the form:

$$dy_{h+1} - y_{h+5}\, dy_{h+1} = 0.$$

Upon continuing in the same manner, one will easily see that one can choose the variables $y_1, y_2, \ldots, y_{r+1}, y_{r+2}$ in such a way that the equations of S take the form:

$$\text{(81)} \qquad \begin{cases} \omega_1 = 0, \qquad \omega_2 = 0, \quad \cdots, \qquad \omega_{h+1} = 0, \\ dy_\rho - y_{\rho+1}\, dy_{\rho+1} = 0, \qquad (\rho = h+2, \ldots, r). \end{cases}$$

Hence, S is a prolongation of the normal system $S^{(r-h-1)}$. We say that *S is a special system if there are no values of i ($i = 1, 2, \ldots, r - 2$) for which μ_i are not equal to zero or one.* Therefore, *if a system S of r equations of class $r + 2$ is not normal then a prolongation of a normal system will be a special system* [**10**, **20**].

17. Explicitly integrable systems. Theorem of E. Cartan. – The question of the existence of an explicit general integral of a system S is linked with that of the reduction of a system S to a canonical form.

Let S be a special system, and let:

$$\mu_1 = \mu_2 = \ldots = \mu_{i-\rho-1} = 1, \qquad \mu_{i-\rho} = \mu_{i-\rho+1} = \ldots = \mu_{i-1} = 0.$$

From the preceding, the system $S^{(r-\rho)}$ is completely integrable, and the system $S^{(r-\rho-1)}$ is composed of $\rho + 1$ equations, and it has class $\rho + 3$. We can then argue as in the third case. From that argument, one can put the system into the form:

$$\text{(82)} \qquad dy_1 = 0, \qquad \ldots, \qquad dy_\rho = 0, \qquad dy_{\rho+2} - y_{\rho+3}\, dy_{\rho+1} = 0.$$

Take the $y_1, y_2, \ldots, y_{\rho+3}$ to be the new variables, along with $r - \rho - 1$ of the old variables $x_{\rho+4}, \ldots, x_{r+2}$, for example. $S^{(r-\rho-2)}$ is then composed of $\rho + 2$ equations, $\rho + 1$ of which are the (82), and the other one can be put into the form:

$$\omega_{\rho+2} = H\, dy_{\rho+1} + K\, dy_{\rho+2} + \sum T_\nu\, dx_\nu, \qquad (\nu = \rho + 4, \rho + 5, \ldots, r + 2).$$

By expressing the idea that the equation $[dy_{\rho+1}, dy_{\rho+2}] = 0$ is a consequence of equations $\omega_{\rho+2}(d) = 0$, $\omega_{\rho+2}(\delta) = 0$, one will now see that $\omega_{\rho+2} = 0$, which is written:

$$\text{(83)} \qquad dy_{\rho+1} - y_{\rho+4}\, dy_{\rho+1} = 0,$$

in which $y_{\rho+4}$ is a new variable.

One likewise confirms that $S^{(r-\rho-3)}$ is composed of equations (82), (83), and the equation $dy_{\rho+1} - y_{\rho+5}\, dy_{\rho+1} = 0$, and so on. One finally arrives at the canonical form:

(84)
$$\begin{cases} dy_1 = 0, \quad dy_2 = 0, \quad \ldots, \quad dy_\rho = 0, \\ dy_{\rho+2} - y_{\rho+2}\, dy_{\rho+1} = 0, \quad \ldots, \quad dy_{i+1} - y_{i+2}\, dy_{\rho+1} = 0. \end{cases}$$

Therefore:

a′. Any special system of r equations in $r + 2$ variables reduces to the canonical form (84).

b′. The number of equations of the form $dy_i = 0$ in the canonical form (84) coincides with the number r, where $r - \rho$ is the smallest of the indices for which μ_i is equal to 0. In other words: All of the systems of r equations in $r + 2$ variables for which the number ρ has the same value can be reduced to the same canonical form. The canonical form (84) to which the special system S is reduced indicates that the system S is explicitly integrable. Indeed, in order to get the explicit general integral of the system S that gives the integral multiplicities M_1 :

(85)
$$\begin{cases} y_1 = c_1, \quad \ldots, \quad y_\rho = c_\rho, \\ y_{\rho+1} = a, \quad y_{\rho+2} = \varphi(a), \quad y_{\rho+3} = \varphi'(a), \quad \ldots, \quad y_{i+2} = \varphi^{(r-\rho)}(a), \end{cases}$$

it will suffice to set $y_{\rho+1} = a$, $y_{\rho+2} = \varphi(a)$, in which φ denotes an arbitrary function of a. Consequently, if the change of variables that converts the system S to canonical form (84) is defined by the formulas:

$$x_i = f_i\,(y_1, y_2, \ldots, y_{i+2}) \qquad (i = 1, 2, \ldots, r + 2)$$

then one will get the general integral of S from the formulas:

(86)
$$x_i = f_i\,[c_1, c_2, \ldots, c_\rho, a, , \varphi'(a), \ldots, \varphi^{(r-\rho)}(a)]\,.$$

Therefore: *Any special system has an explicit general integral. Conversely: If a system of r equations in r* + 2 *variables is explicitly integrable then it will be a special system.* In order to prove that, one remarks that if a system S that is the prolongation of a normal system Σ admits an explicit integral then (Σ) will likewise admit an explicit general integral, since that would result from the form (81) to which a system that is the prolongation of a normal system would reduce. One then proves that it is impossible for an explicitly integrable system Σ to be a normal system, which then implies the beautiful theorem of Cartan: *The necessary and sufficient condition for a system S of r equations in r* + 2 *variables to have an explicit general solution is that is should be a special system.*

After the special systems, the simplest systems are the normal systems of r equations in $i + 2$ variables, whose second derivative is a special system of $r - 3$ equations. One easily sees that such a normal system can be converted into the form:

$$dy_2 - y_3\, dy_1 = 0, \qquad \dots, \qquad dy_r - y_{i+1}\, dy_1 = 0, \qquad dy_{i+2} - F\, dy_1 = 0,$$

in which F is an arbitrary function of $r + 2$ variables y_i ; hence, the general solution is:

$$y_1 = a, \qquad y_2 = \varphi(a), \qquad y_3 = \varphi'(a), \dots, \qquad y_{r+1} = \varphi^{(i-1)}(a),$$

and y_{i+2} is given by integrating the differential equation:

$$dy_{r+2} = F\,[a, \varphi(a), \varphi'(a), \dots, \varphi^{(i-1)}(a), y_{r+2}]\; da.$$

18. Consequences of the theorem of E. Cartan. – If a Monge system reduces to a special Pfaff system then by virtue of Cartan's theorem, it will have an explicit general solution. Hence, it would be interesting to *look for the Monge systems that reduce to special systems.*

From what we saw above, any Pfaff system of two equations in four variables is a special system.

Suppose one has the system:

$$(87) \qquad f_i\,(dx_1, dx_2, \dots, dx_{n+1}) = 0 \qquad (i = 1, 2, \dots, n - 1),$$

in which the f_i do not refer to the x. One replaces it with an equivalent system of $n - 1$ equations in $n + 1$ variables of the form:

$$(88) \qquad \frac{dx_k}{dx_{n+1}} = \varphi_k\left(\frac{dx_1}{dx_{n+1}}\right) \qquad (k = 2, 3, \dots, n).$$

Upon introducing a new variable:

$$x_{n+2} = \frac{dx_1}{dx_{n+1}},$$

one will arrive at a system S of n Pfaff equations in $n + 2$ variables:

$$\omega_1 = dx_1 - x_{n+2}\, dx_{n+1} = 0, \qquad \omega_k = dx_k - \varphi_k\,(x_{n+2})\, dx_{n+1} = 0.$$

One easily sees that the system S', which is composed of $n - 1$ equations, has the same form as S, which has $r + 1$ variables, and if S' is not completely integrable then one will see moreover that S'' has the same form and n variables, and so on. Hence, S will be a special system. Consequently, it will admit an explicit general solution. One can generalize that result. Consider a system of q equations ($q < n - 1$):

$$F_i\,(dx_1, dx_2, \dots, dx_{n+1}) = 0 \qquad (i = 1, 2, \dots, q).$$

If one adjoins $n - q - 1$ equations of the same form:

$$F_{q+1}(dx_1, \ldots, dx_{n+1}) = 0, \quad F_{q+2}(dx_1, \ldots, dx_{n+1}) = 0, \quad \ldots, \quad F_{n-1}(dx_1, \ldots, dx_{n+1}) = 0$$

to it then the system, thus-completed, will have the preceding form, and the general integral of S will have an explicit form with several arbitrary functions.

One can look for the cases in which a system of three equations in five variables is a special system. In particular, suppose that one has a system of the form:

(89) $$\omega_1 = dx_2 - x_3\, dx_1 = 0, \quad \omega_2 = dx_3 - x_4\, dx_1 = 0, \quad \omega_3 = dx_4 - f(x_1, x_2, x_3, x_4)dx_1 = 0.$$

One seeks to determine the function f in such a manner that S is a special system. If one forms S' then one will have:

$$\omega_1' = 0, \qquad \omega_2' = [dx_1, dx_2], \qquad \omega_3' = f_{x_4}' [dx_1, dx_4] \qquad (\text{mod } \omega_1, \omega_2, \omega_3).$$

One will then have:

$$\omega_3' - f_{x_4}' \omega_2' = 0,$$

and the equations of S' will be:

$$\omega_1 = 0, \qquad \omega_3 - f_{x_4}' \omega_2 = 0,$$

or rather:

$$\varpi_1 = \omega_1 = dx_2 - x_3\, dx_1 = 0, \qquad \varpi_2 = dx_3 - f_{x_4}'\, dx_4 - (f - x_4\, f_{x_4}')\, dx_1 = 0,$$

so

$$\varpi_1' = [dx_1, dx_3],$$
$$\varpi_2' = f_{x_4^2}' [dx_3, dx_4] + x_4\, f_{x_4^2}' [dx_1, dx_4] + \lambda\, [dx_1, dx_3] \qquad (\text{mod } \varpi_1, \varpi_2).$$

In order for the system S'' to have the form of one equation, it is necessary and sufficient that one should have:

$$f_{x_4^2}' = 0;$$

i.e., that f must be linear with respect to x_4. If one sets:

$$x_1 = x, \qquad x_2 = y, \qquad x_5 = z$$

then one will have:

$$x_3 = y', \qquad x_4 = y'', \qquad \frac{dx_5}{dx_1} = z',$$

and the system S will reduce to the second-order Monge equation:

(90) $$z' = f(x, y, y', y'', z).$$

Conversely, any equation (90) will reduce to a system of the form (89). Hence, in order for (90) to be explicitly integrable, it is necessary that f should be linear in y''' that condition is sufficient, moreover [**12**].

CHAPTER V

GENERAL THEOREMS ON THE CORRESPONDENCE BETWEEN PARTIAL DIFFERENTIAL EQUATIONS AND MONGE EQUATIONS.

19. Monge equations and systems in involution of various types. –

α'. We remark that in some particular cases the theories that relate to systems in involution of first-order partial differential equations can correspond to theories that relate to the integration of Monge systems. One sees such a correspondence in Goursat's method, in which Monge systems correspond to systems that are called *associated*, and since those associated systems are in involution, one has explicit general solutions to the corresponding Monge systems [**47**].

β'. We have already encountered (Chap. II) a correspondence between systems in involution of linear second-order partial differential equations and a second-order Monge equation, and we saw that in the case that was studied, the integration of the second-order Monge equation reduced to the integration of a first-order Monge equation [**28**].

γ'. Suppose one has a nonlinear system in involution. By means of a complete integral, one can make it correspond to two Monge equations of the form:

(91) $$\Phi_i\left(a_1, a_2, a_3, a_4; \frac{da_2}{da_1}, \frac{da_3}{da_1}, \frac{da_4}{da_1}\right) = 0 \qquad (i = 1, 2),$$

such that in order to obtain the general integral of the system in involution, one must obtain the most general expressions for the four functions a_1, a_2, a_3, a_4 of one variables that verify the two relations (91).

γ'. Cartan has shown how one can link the theory of Monge equations in five-dimensional space to the study of certain systems in involution of three second-order partial differential equations in one unknown function of three independent variables when the three families of two-dimensional characteristics coincide and that single family depends upon eight parameters. Cartan has shown that the integration of such a system in involution reduces to the integration of a system of five completely-integrable Pfaff equations and one Monge equation:

(92) $$F\left(X_1, X_2, X_3, X_4, X_5; \frac{dX_2}{dX_1}, \frac{dX_3}{dX_1}, \frac{dX_4}{dX_1}, \frac{dX_5}{dX_1}\right) = 0.$$

He also showed that conversely, under certain conditions, any Monge equation of the form (92) will give rise to a system in involution of the stated type. If two such systems in involution that lead to the same nonlinear Monge equation then they can be converted into each other by a contact transformation [**9**].

20. Sheaves of infinitesimal transformations. Derived sheaves. Vessiot's theory. – One has Vessiot [**39**, **41**] to thank for a new theory of general problems in integration. His theory opened up a new path to the study of indeterminate differential systems, and it correlates with Cartan's theory of the Pfaff problem. Vessiot's theory is found to be based upon the notion of the correspondence between a system of differential equations and a system of linear partial differential equations.

First, let S be a system of ordinary differential equations. As one knows, one can make it correspond to a linear partial differential equation E in such a way that the solutions to E are first integrals of S, and conversely. One then has a sort of *duality* between S and E. One can say that S, E are correlative. Then consider, with Cartan, a completely-integrable system S of s Pfaff equations in n variables:

(93) $$\omega_i = \sum a_{k,l}(x_1, x_2, \ldots, x_n)\, dx_k = 0 \qquad (i = 1, 2, \ldots, s;\ k = 1, 2, \ldots, n).$$

Choose $n - s$ linear differential forms $\varpi_1, \varpi_2, \ldots \varpi_{n-s}$ arbitrarily that are mutually-independent and independent of the forms $\omega_1, \omega_2, \ldots \omega_s$. One can obviously express dx_1, $dx_2, \ldots, dx_n$ as functions of $\omega_1, \omega_2, \ldots \omega_s$; $\varpi_1, \varpi_2, \ldots \varpi_{n-s}$, in one and only one manner. One can then express any total differential:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

linearly in terms of $\omega_1, \omega_2, \ldots \omega_s$; $\varpi_1, \varpi_2, \ldots \varpi_{n-s}$, where the coefficients are linear and homogeneous in $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$ and distinct; i.e., they are linearly-independent forms in the $\frac{\partial f}{\partial x_i}$. One then has an identity of the form:

(94) $$df = Z_1\, \omega_1 + Z_2\, \omega_2 + \ldots + Z_s\, \omega_s + X_1\, \varpi_1 + X_2\, \varpi_2 + \ldots + X_{n-s}\, \varpi_{n-s}\,.$$

The system of $n - s$ partial differential equations:

$$X_\rho = 0 \qquad (\rho = 1, 2, \ldots, n - 1)$$

admits s independent first integrals of the completely-integrable system S as solutions, and one knows that the system S *corresponds to a complete system*

(95) $$X_1 = 0, \qquad X_2 = 0, \qquad \ldots, \qquad X_{n-s} = 0,$$

and conversely. Hence, the two systems S, E correlate, and the integration of one of them will imply the integration of the other one. One can further say that if one is given a system S of s Pfaff equations in n completely-integrable variables and a complete system E of $n - s$ linear partial differential equations then there will exist a duality between S and E such that the integration of each of them will imply the integration of the other one *if one has an identity of the form* (94), where $\varpi_1, \varpi_2, \ldots, \varpi_{n-s}$ are new linear functions in the $dx_1, dx_2, \ldots, dx_s$ and $Z_1, Z_2, \ldots, Z_{n-s}$ are functions of the new forms in $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$.

Vessiot considered an *arbitrary* Pfaff system and extended the notion of *duality* to that notion. Let $\omega_1, \omega_2, \ldots, \omega_s, \varpi_1, \varpi_2, \ldots, \varpi_{n-s}$ be n independent Pfaff expressions in the dx, and let f be an indeterminate function. One can write an identity of the form (94). The linear operations $X_1, \ldots, X_{n-s}, Z_1, \ldots, Z_s$ are all well-defined then. However, if one is given only the Pfaff system $\omega_1 = 0, \omega_2 = 0, \ldots, \omega_s = 0$ then one can replace the ω_i in the identity (94) with other linear expressions in ω_i; i.e., the ω_i are defined only up to a linear substitution, and one can choose $\varpi_1, \varpi_2, \ldots, \varpi_{n-s}$, arbitrarily, in such a way that if one makes a first choice then one can, in turn, replace them with other expression that are linear in ω_i and ϖ_j. Such replacements of the ω_i and ϖ_j have the effect of replacing $X_1, \ldots, X_{n-s}$ with homogeneous linear combinations of the form:

$$(96) \qquad X = \lambda_1 (x_1, \ldots, x_n) X_1 + \ldots + \lambda_m (x_1, \ldots, x_n) X_m \qquad (m = n - s),$$

and can give forms for $Z_1, \ldots, Z_s$ that are entirely arbitrary in $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$.

If one considers expressions such as $X_j f$ to be symbols of infinitesimal transformations then one can say that X will give an infinitesimal transformation for a determination of the $\lambda_1, \lambda_2, \ldots, \lambda_m$. The preceding remarks led Vessiot to make any system of Pfaff equations correspond, not to a system of transformations $X_1, X_2, \ldots, X_m$, but to the set of infinitesimal transformations that are given by formula (96), in which the λ_j are arbitrary functions of the variables $x_1, x_2, \ldots, x_n$ and the X_j are assumed to be distinct. Vessiot called such a set a *sheaf of infinitesimal transformations.* The X_j, which are assumed to be distinct, constitute a basis for the sheaf. One can obviously take m other distinct, but arbitrary, transformations to be a basis that defines the sheaf. That would amount to performing a homogeneous linear substitution of $X_1, X_2, \ldots, X_m$ whose coefficients are arbitrary functions of the $x_1, x_2, \ldots, x_n$.

Let $\{X_1, X_2, \ldots, X_m\}$ be a sheaf. Associate $n - m$ arbitrary transformations $Z_1, Z_2, \ldots, Z_{n-m}$ with the transformations of the basis $X_1, X_2, \ldots, X_m$, in such a manner that the set $X_1, X_2, \ldots, X_m, Z_1, Z_2, \ldots, Z_{n-m}$ consists of distinct elements. We will get an identity of the form (94), in which $\varpi_1, \varpi_2, \ldots, \varpi_m, \omega_1, \omega_2, \ldots, \omega_{n-m}$ are n independent Pfaff expressions. It will then result that the infinitesimal displacements that the system $\omega_i = 0$ ($i, = 1, 2, \ldots, s$) satisfies are precisely the ones that correspond to the various infinitesimal transformations of the sheaf $\{X_1, X_2, \ldots, X_m\}$, and that any integral multiplicity of the sheaf $\{X_1, X_2, \ldots, X_m\}$ is an integral multiplicity of the Pfaff system $\omega_i = 0$, and conversely.

Conversely, if one is given a system of Pfaff equations then one can make it correspond to an equivalent sheaf of transformations. With Vessiot, we say that a sheaf $\{X_1, X_2, \dots, X_m\}$ and a system $\omega_1 = \omega_2 = \dots = \omega_s = 0$ that correspond to each other are then *correlated with each other* or *dual* to each other. That correspondence makes one see that the theory of systems of Pfaff equations corresponds to a theory of sheaves of infinitesimal transformations by a sort of duality.

A p-dimensional multiplicity is called an *integral* of a sheaf of transformations if it is invariant under p distinct transformations of that sheaf. A family of integral multiplicities such that one and only multiplicity of that family passes through each point of space is called a *complete integral*. Any complete p-dimensional integral is provided by a complete system of p equations $U_1 f = 0$, $U_2 f = 0$, …, $U_p f = 0$ whose left-hand sides are transformations of the sheaf. The transformations $U_1, U_2, \dots, U_p$ define a *complete sub-sheaf* of a certain sheaf F. In Vessiot's theory, one considers complete integrals in place of isolated integral multiplicities.

Cartan used the properties of the bilinear covariants $\omega_i' = \delta\omega_i(d) - d\omega_i(\delta)$ in the problem of integrating a Pfaff system, while Vessiot used the Jacobi brackets:

$$(X_i f, X_h f) = X_i (X_h f) - X_h (X_i f) \qquad (i, h = 1, 2, \dots, m),$$

which are infinitesimal transformations that are covariant to the transformations $\{X_1, \dots, X_m\}$. If they all belong to the sheaf then the sheaf will be called *complete*, and with Vessiot, we will write:

$$(X_i, X_h) = 0 \qquad (\text{mod } X_1, X_2, \dots, X_m)$$

in order to express the idea that the brackets are expressed as homogeneous linear functions of the $X_1, \dots, X_m$. When a sheaf is not complete, the brackets (X_i, X_h) will be expressed as homogeneous linear functions of the $X_1, X_2, \dots, X_m$ and some other infinitesimal transformations $Z_1, Z_2, \dots, Z_{m'}$ that one can choose in such a manner that the $X_1, X_2, \dots, X_m, Z_1, Z_2, \dots, Z_{m'}$ are distinct. Vessiot called that sheaf $\{X_1, X_2, \dots, X_m, Z_1, Z_2, \dots, Z_{m'}\}$ the *derived sheaf* of the sheaf $\{X_1, X_2, \dots, X_m\}$; i.e., the set of brackets $(X_i f, X_h f)$ of the transformations for the sheaf F, taken two at a time, constitutes a sheaf F' that contains F; it is the *derived sheaf* to F.

One has some identities-congruences for the brackets (X_i, X_h) that are called the *structure formulas*, which have the form:

$$(X_i, X_h) = \sum c_{i,h,j} Z_j \quad (\text{mod } F) \qquad (i, h = 1, 2, \dots, m; j = 1, 2, \dots, m').$$

The nature of the sheaf from the standpoint of its integration depends essentially upon its structure. It is by such structural comparisons that one recognizes that one can pass from one sheaf to another by a change of variables. A complete sheaf is a sheaf that is identical to its derived sheaf F'. Similarly, F' has a derived sheaf, and so on. A very interesting property is that the degree of the latter derivative of a sheaf of infinitesimal transformations is equal to the minimum number of effective variables to which one can reduce that sheaf by a change of variables.

The reduction of certain systems of Pfaff equations to canonical forms plays a great role in Cartan's theory. In Vessiot's theory, one also considers canonical forms or types, and one has the problem of the reduction of a sheaf to a canonical or semi-canonical form. Vessiot studied such a problem in the particular case in which the derived sheaf F' has degree $m + 1$, where m denotes the degree of F. One then knows that the structure formulas have the simple form:

$$(X_i , X_h) \equiv c_{i,k}\, Z \qquad (\mathrm{mod}\ F) \qquad (i, k = 1, 2, \ldots, m),$$

in which Z is an arbitrary transformation of the derived sheaf (that does not belong to F).

Just as one considered the canonical form (63) for a Pfaff system, here one considers the canonical form:

$$X = \frac{\partial f}{\partial x} + x_1 \frac{\partial f}{\partial x_0} + x_2 \frac{\partial f}{\partial x_1} + \cdots + x_{\rho+1} \frac{\partial f}{\partial x_\rho}, \frac{\partial f}{\partial x_{\rho+1}}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_r}.$$

Knowing a complete integral in that case will permit one to reduce the given sheaf F to a canonical form by a change of variables. One then looks for the other complete integrals on the basis of that canonical form. Vessiot also introduced the notion of the *prolongation of a sheaf*, which he used for the study of the problem of integrating the sheaf. He gave the theorem for the explicit integration of the special systems that corresponded to theorem of Cartan, and constructed a theory that correlated with Cartan's. He showed that in the case considered, if the degrees of the F', F'', …, *increase by one unit* when one passes from each of those derived sheaves to the following one then the general solution of the problem of integrating F (for $s = 1$) will be given by explicit formulas. One will then have the equivalent of Cartan's theorem with somewhat more general hypotheses.

21. Duality between Monge equations and nonlinear partial differential equations. – Let a system of Monge equations be given, and form the equivalent Pfaff system. From Vessiot's theory, one can pass from that system to a sheaf of infinitesimal transformations. One then sees that we can study the Monge systems by means of Vessiot's theory. Moreover, that theory shows the way to the construction of a theory of the Monge problem by appealing to a duality *between Monge equations and nonlinear partial differential equations.* Here, we have indicated far too few of the aspects of Vessiot's very important theory. However, on first glance, one can distinguish that a vast field of research has been opened up by that method.

CHAPTER VI

THE MONGE PROBLEM IN SEVERAL INDEPENDENT VARIABLES.

22. Goursat's theory. – We shall envision some Monge equations in two unknown functions of an arbitrary number of independent variables. Goursat pointed out a class of such equations for which we can explicitly express the two functions by means of independent variables, arbitrary functions of those variables, and their partial derivatives.

He considered the equation:

$$(97) \qquad F(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}; P_1, P_2, \ldots, P_{n+1}) = 0, \quad P_h = \frac{\partial x_{n+1}}{\partial x_h} \qquad (h = 1, 2, \ldots, n + 1),$$

and an integral multiplicity M_{n+1}, for which $x_{n+2} = \varphi(x_1, \ldots, x_{n+2})$, as well as an arbitrary multiplicity M_n :

$$(98) \qquad x_{n+1} = f_1(x_1, \ldots, x_{n+2}), \qquad x_{n+2} = f_2(x_1, \ldots, x_{n+2}).$$

If the multiplicity (98) is contained in an integral M_{n+1} then one will have:

$$f_2(x_1, \ldots, x_{n+2}) = \varphi[x_1, x_2, \ldots, x_n, f_1(x_1, x_2, \ldots, x_n)]$$

and

$$(99) \qquad q_i = P_i + P_{n+1}\, p_i, \qquad \text{with} \quad p_i = \frac{\partial f_1}{\partial x_i}, \qquad q_i = \frac{\partial f_2}{\partial x_i} \qquad (i = 1, 2, \ldots, n).$$

Upon eliminating $P_1, P_2, \ldots, P_n$ between (97), (99), one will deduce that:

$$(100) \qquad F(x_1, x_2, \ldots, x_{n+2};\; q_1 - p_1 P_{n+1}, \ldots, q_n - p_n P_{n+1}, P_{n+1}) = 0,$$

and one defines P_{n+1} at a point x_i of M_n. Any holomorphic root in the domain (D) of that point will give an integral M_{n+1} that is holomorphic in (D). The conclusion breaks down for a root that simultaneously satisfies the condition:

$$(101) \qquad \frac{\partial F}{\partial P_1} p_1 + \ldots + \frac{\partial F}{\partial P_n} p_n - \frac{\partial F}{\partial P_{n+1}} = 0.$$

Finally, upon eliminating P_{n+1} from (100), (101), we will have:

$$(102) \qquad \Phi(x_1, \ldots, x_{n+1}, x_{n+2}; p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n) = 0,$$

which is the differential equation of the singular multiplicities of (97). Those multiplicities are analogous to the integral curves for a Monge equation in three variables. One integrates it explicitly in the following manner: Let $V(x_1, \ldots, x_{n+2}; a_1, \ldots, a_{n+1}) = 0$

be a complete integral of (97). The singular multiplicities M_n are represented by the equations:

$$V = 0,\ \frac{\partial V}{\partial a_1} = 0,\ \ldots,\ \frac{\partial V}{\partial a_n} = 0,$$

$$H \equiv \begin{vmatrix} \dfrac{\partial^2 V}{\partial a_1^2} & \dfrac{\partial^2 V}{\partial a_1\,\partial a_2} & \cdots & \dfrac{\partial^2 V}{\partial a_1\,\partial a_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 V}{\partial a_n\,\partial a_1} & \dfrac{\partial^2 V}{\partial a_n\,\partial a_2} & \cdots & \dfrac{\partial^2 V}{\partial a_n^2} \end{vmatrix} = 0,$$

in which one replaces a_{n+1} with an arbitrary function $f(a_1, a_2, \ldots, a_n)$. Conversely, let one be given an equation of the form (102). Goursat gave the necessary and sufficient conditions for that equation to define the singular multiplicities of a first-order partial differential equation. The ratio $\frac{\partial \Phi}{\partial p_i} : \frac{\partial \Phi}{\partial q_i}$ must be independent of i if one takes into account the equation itself. Those conditions are not sufficient in every case. However, as Goursat proved, if they are satisfied then one can always integrate equation (102) by explicitly expressing the variables and the two unknown functions by means of n auxiliary parameters of an arbitrary function of those parameters and their derivatives up to second order. In his proof, Goursat replaced (102) with a system of two Pfaff equations in $3n + 1$ variables:

$$\omega_1 = dx_{n+1} - \sum p_i\, dx_i = 0 \qquad (i = 1, 2, \ldots, n),$$
$$\omega_2 = dx_{n+2} - f\, dx_n - \sum q_i\, dx_i = 0 \qquad (j = 1, 2, \ldots, n-1),$$

when he supposed that (102) was written in the form:

$$q_n = f(x_1, x_2, \ldots, x_{n+2}\,;\, p_1, p_2, \ldots, p_n\,;\, p_1, p_2, \ldots, p_{n-1}),$$

and he sought the integrals of that system.

Goursat started with equation (97). He remarked that one can start from a system in involution of first-order partial differential equations in one unknown function and generalize the preceding results [**26**].

One finds that generalization in a very important work in which Goursat applied some methods that related to semi-linear systems and gave a certain number of well-defined types of Monge equations of the first class (54).

Goursat also pointed out some particular cases in which a Monge equation in two independent variables of the form:

$$\sum A_{ik}\, dz_i\, dz_k = 0 \qquad (i, k = 1, 2, 3, 4)$$

admit an explicit solution (27).

He further studied the problem of integrating a system that is composed of two equations of the preceding form (55).

One easily distinguishes that Goursat's theory points to an extended field of research.

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