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The development and present state of differential line geometry

(Report to the German Society of Mathematicians that was presented at the annual meeting in Merano on 27 September 1905) (*)

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Textbooks and monographs

Plücker, *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement*, Part I, 1868. Part II, 1869.
Koenigs, "Sur les propriétés infinitésimales de l'espace réglé," Thèse, 1882.
Sturm, *Liniengeometrie in synth. Behandlung*, 3 volumes, 1892, 1893, 1896.
Klein, *Einleitung in die höhere Geom.* (Göttingen lectures), 1893.
Koenigs, *La Géométrie réglée*, Paris, 1895.
Fano, *Lezioni di Geometria della retta*, Roma 1896.
Zindler, *Liniengeometrie mit Anwendungen*, v. I 1902, v. II 1906.
Jessop, *A Treatise on the Line Complex*, 1903.

(*) The oral report that was presented was restricted to the abundance of material that is concerned with the differential geometry of ray congruences.

1. Older investigations. Certain sections of **Malus**'s optics ([1]) can be regarded as the oldest investigation of differential line geometry and line geometry at all ([2]). He expressed the equations of a line as:

$$m\,(z - z') = o\,(x - x'), \qquad n\,(z - z') = o\,(y - y'),$$

in which m, n, o are arbitrary functions of x', y', z', and thus associated every point $P \equiv (x', y', z')$ of space with a line g that went through it, and thus defined a line complex. He then sought the neighboring point of P whose line intersected g and found a second-order cone of directions. He then restricted himself to lines of the complex that emanated from the points of a surface:

$$F(x', y', z') = 0,$$

and thus, to a general line congruence, and found that every such congruence can be regarded as the locus of the intersections of two systems of developable surfaces. He also found the focal surfaces, determined the two focal planes and the angle between them, presented the condition for the two families of developable surfaces to intersect perpendicularly everywhere (or as we now say, for the ray system to be a normal system) and asked which surfaces $F = 0$ one must associate with a given complex in order for the system to be a normal system. He obtained a nonlinear first-order partial differential equation.

One refers to the theorem that a normal system will always go to another such system after arbitrarily many reflections and refractions at separation surfaces between homogeneous media as the *theorem of Malus and Dupin*. **Malus** himself proved that theorem only for a single refraction (or reflection) of rays that started from a point, and even expressed the opinion (*loc. cit.*, pp. 103) that the theorem would no longer be true for several refractions. The error was corrected by **Dupin** and **Cauchy** ([3]).

At the same, from his interest in optics, **Hamilton** ([4]) was led to investigate ray systems. He represented them analytically by giving the direction cosines of a ray as functions of the position in space. However, the functions α, β, γ of x, y, z must be arranged such that they will not change when one replaces x with $x + \rho\,\alpha$. They must then fulfill the differential equations that one obtains ([5]) when one sets the φ in:

$$\alpha\,\varphi_x + \beta\,\varphi_y + \gamma\,\varphi_z = 0$$

([1]) "Optique," Journ. de l'Éc. polyt., t. VII (cah. 14) 1808; pp. 1-44 and 84-129.

([2]) Generally, **Monge** had previously (1796) investigated the normal system of a surface in connection with the theory of surfaces, and in fact remarked that it could be decomposed into ∞^1 developable surfaces in two different ways (Journ. de l'Éc. polyt., cah. 2). However, it was in **Malus** that line manifolds appeared for the first time independently. For the older investigations of line geometry, cf., also **Lie** and **Scheffers**, *Geom. d. Berührungstransf.* pp. 268, *et seq.*

([3]) For the discovery of the Malus-Dupin theorem, cf., **Darboux**, *Théorie des surf. II*, pp. 280, note.

([4]) "Theory of systems of rays," Transact. of the R. Irish Ac. **15** (1828); "First supplement to an essay on the theory of systems of rays," *ibid.* **16** (1830).

([5]) Cf., also **Cauchy**, Proc. of the Lond. Math. Soc. **8** (1877) or Coll. Papers IX, no. 625 and Mess. of Math. (2) **17** (1887) or Coll. Papers XII, no. 876, and furthermore, **Bertrand**, "Mém. sur la Théorie des surf.," J. de Math. **9** (1844).

equal to α, β, γ, in turn. Moreover, one obtains equations from differentiating $\sum \alpha^2 = 1$ that yield:

$$\beta_z - \gamma_y = k\alpha, \quad \gamma_x - \alpha_z = k\beta, \quad \gamma_y - \beta_x = k\gamma,$$

in which:

$$k = \sum \alpha (\beta_z - \gamma_y).$$

For $k = 0$, $\sum \alpha \cdot dx$ will be the complete differential of a function V. That is the case for normal systems, and V = const. will then be the equation for the family of orthogonal surfaces. **Hamilton** found the most important results of **Malus** once more and, in fact, added the equation that is named for him, which we would like to discuss in more detail:

If one considers a fixed ray s of a congruence and a variable neighboring ray s', along with the shortest distance a between them and the foot N of the latter on s, and one lets s' go to s then the limiting position of the plane (s, a) will be determined by the angle α between it and a fixed starting plane through s, and the limiting position of N will be determined by the distance z from a fixed point to s. Now, for a suitable choice of starting plane, z and α will be related by:

$$z = z_1 \cos^2 \alpha + z_2 \sin^2 \alpha \qquad (z_1, z_2 \text{ const.}) \tag{1}$$

for all ruled surfaces of the congruence, and that is "Hamilton's equation." **Hamilton** ([6]) has also already compared the various cross sections of one and the same bundle that was defined by the neighborhood of a ray, and thus came very close to **Kummer**'s measure of density. He already found the *cylindroid* in the first treatise as the locus of the shortest distances from a ray to its neighboring rays.

The discovery of the null system and the ray thread by **Giorgini** (1827) and **Möbius** (1833), as important as it also is, was not concerned with *differential* line geometry, so we shall not discuss it here. By contrast, the concept of the distribution parameter P for a ray of a ruled surface is important for differential geometry: If s' is a neighboring ray, a is the shortest distance between the two, and ω is the angle between them then one will have:

$$P = \lim \frac{a}{\omega}. \tag{2}$$

One can thank **Chasles** for that concept, as well as the discovery of the correlation between the points of a ray of a ruled surface and their associated contact planes ([7]).

Sturm, in his treatise on the theory of vision ([8]) considered an "infinitely-thin ray bundle"; i.e., the neighborhood of a ray s of a normal congruence. Such a neighborhood will be determined completely when one knows the two principal curvatures of a normal surface of the bundle at its point of intersection with s and the orientation of the planes of the associated principal section through s (viz., its azimuth). Now, if a refracting surface

([6]) *Loc. cit.*, (rem. 4), vol. XVI, pp. 59.

([7]) "Sur les surf. engendrées par une ligne droite, etc." Corresp. math. et phys., ed. Quetelet (3) III (Tome XI), 1839.

([8]) Comptes R. XX, pp. 554, 761, and 1238 (1845), transl. in Pogg. Ann. d. Phys. u. Chemie, Bd. 65 (III, Reihe 5), pp. 116-134 and 374-395.

is present then the ray s will be refracted into another s' whose neighborhood can be defined by the three analogous elements, and **Sturm** solved the problem of determining those elements. In that way, he proved (as **Monge** knew already) that the normals to a surface in the neighborhood of a point P could be regarded approximately as the intersections of two lines – viz., the *focal lines* – and said (*loc. cit.*, pp. 376) that the focal lines cut the normal at the point P *perpendicularly*. However, that is not the only possible way of looking at things, and that is why several misunderstandings will arise later in regard to this subject (no. **7**).

2. Kummer's theory of ray systems. The papers of **Malus** and **Hamilton** seem to have been almost forgotten, since **Kummer**, in his treatise "Allgemeine Theorie der geradlinigen Strahlensysteme" ([9]) proved most of the theorems on ray systems that were known at the time in a newer and simpler way and extended them at some points ([10]). Its starting point was the following: One intersects the ray congruence with an arbitrary surface F:

$$x\,(u, v), \qquad y\,(u, v), \qquad z\,(u, v).$$

A ray s of the congruence is then determined when its point of intersection P_0 with F and its direction cosines X, Y, Z are known as functions of u, v. If one denotes:

$$\text{(3)}\qquad \begin{aligned} &\sum X_u^2 = E, \quad \sum X_u X_v = F, \quad \sum X_v^2 = G, \\ &\sum X_u x_u = e, \quad \sum X_u x_v = f, \quad \sum X_v x_u = f', \quad \sum X_v x_v = g \end{aligned}$$

(these quantities might be called Kummer's "fundamental quantities") then the two formulas:

$$\text{(4)}\qquad \begin{aligned} &\sum dX^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ &\sum dx\,dX = e\,du^2 + (f + f')\,du\,dv + g\,dv^2 \end{aligned}$$

will play a role here that is analogous to the two quadratic differential forms of the theory of surfaces: The first one represents the line element as the spherical image of the congruence. The topics that **Kummer** then treated were: The shortest distance to the neighboring rays, limit point ([11]), principal planes, Hamilton's equation, focal points, focal planes, midpoints, focal surfaces, normal systems. The *density* of the ray system at a point P of a ray is defined thus: One lays a plane through P that is perpendicular to s and draws a small closed curve around P in it whose area is f. By means of the spherical map of the system, it will then correspond to a curve on the unit sphere with an area of φ. The density is then:

([9]) J. f. Math, **57** (1860).

([10]) One finds a concise presentation of Kummer's theory in **Bianchi**, *Vorl. über Differentialgeom.* (transl. Lukat), chap. X.

([11]) The definition of this and a concept that follows from it will be given in no. **5.**

$$\lim \frac{\varphi}{f}.$$

The concept of the angle of rotation between neighboring rays s and s_1 is peculiar to **Kummer**: If one drops two perpendiculars from two points A, B of s_1 onto s then the angle between them is called the *angle of rotation of the segment AB relative to s.* The examination of the angle of rotation culminates in the following elegant theorem: If one draws the normals to neighboring points of a surface and gives then the lengths at which their angles of rotation will be a right angle then they will be equal to the radii of curvature of the normal sections of the surface that is determined by the two neighboring points.

Kummer's theory was represented by **Gorton** ([12]) with the use of the theory of quaternions and completed by **Hensel** ([13]), who freed the two differential forms of mixed terms simultaneously. An arbitrary surface F whose properties do not depend upon the congruence at all is introduced in that theory. One can avoid that when one gives the rectangular, homogeneous line coordinates of the rays of the system as functions of two independent parameters:

$$q_i = q_i(u, v) \qquad (i = 1, \ldots, 6). \tag{5}$$

Zindler has pursued that path ([14]).

Ever since **Plücker**'s *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement* (1868-69), the work that was done on line geometry piled up so much that the details of its chronology would be inconvenient for the further discussion.

3. Directions in line space. If one determines a line by any four mutually-independent coordinates $u_1, \ldots, u_4$ then the increments $du_1, \ldots, du_4$ (whose ratios alone are relevant) will determine a direction that starts from one of those lines. Naturally, the concept of direction carries over to the case of the superfluous coordinates that are ordinarily employed in line geometry, as **Klein** ([15]) did, who also defined the angle between two direction by analogy to the cosine formula in point space ([16]). A ray, together with a neighboring ray, determines a **Chasles** correlation (no. **1**). The theory of directions is then closely related to the theory of correlations of a line and was treated by **Koenigs** ([17]) in that way. If two correlations are involutory then one says that the corresponding directions in line space are *perpendicular* to each other. In fact, e.g., the following analogy exists: All directions that start from a line of the complex and are contained in the complex are in involution with a direction that is not contained in the complex itself ([18]) (analogue of the surface normal in point space). The analogue of the triply-orthogonal system of surfaces in point space also exists in line space and is called a

([12]) Am. J. of Math. X.
([13]) J. f. Math. **102** (1888).
([14]) *Liniengeom.*, v. II, sect. II (1906).
([15]) "Über Liniengeometrie u. metr. Geom.," Math. Ann. V, § 3.
([16]) *Loc. cit.*, pp. 271, note.
([17]) Thèse, Part I (1882).
([18]) **Klein**, *loc. cit.*, pp. 272.

system of complexes in involution ([19]). **Koenigs** ([20]) has also given a "multiply-orthogonal decomposition," but not generally in a real manner.

Two ruled surfaces determine the same correlation on a common ray when they have the same distribution parameter, central point, and central plane there. They then also contact along an entire generator, and that is the geometric content of the concept of a direction. For that reason, the quantities:

$$z, \alpha, P$$

that were introduced in no. **1** can prove to be the most geometrically-intuitive *direction coordinates* in the neighborhood of a well-defined ray, and as such, will be useful for the examination of the restrictions that the directions that are contained in a complex or a congruence are subjected to. Namely, analogously to the way that ∞^2 directions emanate from a point in space, but only ∞^1 from a point of a surface, ∞^3 directions will emanate from a point of unrestricted line space, but only ∞^2 from a point of a complex, and ∞^1 from a point of a congruence. Therefore, two relations must exist between the three direction coordinates z, α, P for the directions that are contained in a congruence (no. **5**), but only one for the directions of a complex (no. **14**).

Among the directions, one can distinguish the *intersecting* ones, whose representative neighboring rays cut the starting ray s, or – speaking more precisely – whose ruled surfaces have a cuspidal generator at s or whose correlations are singular or whose distribution parameter P vanishes. The analytical way of characterizing this is the vanishing of a quadratic differential form in the du_ν ([21]). If one then interprets the du_ν as homogeneous coordinates of a three-dimensional point space then any direction in the neighborhood of a certain ray will be mapped to a point and intersecting directions will get mapped to a second-order surface F. In that way, the "linear pencils of directions," which get mapped to sequences of points on a line, will play a special role ([22]), and two orthogonal directions will be mapped to points that are conjugate relative to F ([23]).

4. Ruled surfaces. The differential geometry of ruled surfaces has mostly been developed using the methods of the theory of surfaces. That does not belong to the scope of this report, and we shall remark only that **Antomari** ([24]) wrote a comprehensive monograph on ruled surfaces.

We have only to enumerate the results that were obtained by line-geometric methods: A $5-n$-dimensional linear region in a linear complex that contains a ruled surface is determined by n neighboring generators ($n = 2, \ldots, 5$) of that ruled surface. If one passes to the limit by letting the generators converge to one e then one will obtain all linear complexes that contact the ruled surface to order $n - 1$ at e. One finds its analytical

([19]) *Ibidem.*
([20]) Thèse, pp. 74.
([21]) **Koenigs**, Thèse, pp. 18.
([22]) *Ibid.*, pp. 23, *et seq.* It makes no essential difference that **Koenigs** dually mapped a direction to a plane.
([23]) *Ibidem.*
([24]) Thèse, Paris, 1894, at Nony.

representation in **Koeinigs** ([25]). The common rays of all these complexes define the *osculating ruled family* for $n = 3$. The limiting positions of the transversals of four neighboring generators are called *osculating rays* ([26]).

One finds the condition for the *developability* of a ruled surface, which is given by three equations in line coordinates, in **Koenigs** ([27]), in which he also examined the order to which the moment of two lines would vanish when one of them g belongs to a ruled surface, while the other one is conjugate to g in a linear complex that contacts the ruled surface at a neighboring generator to g to some well-defined order. **Koenigs** developed the moment of two neighboring generators of a ruled surface as a power series in the parameter ([28]). From a more detailed discussion of the construction of this formula, one can infer ([29]) that the distance between two neighboring generators will always vanish to odd order when one takes the element of arc length of the spherical image as an independent variable. **Klein** ([30]) published the condition of developability for a ruled surface whose line coordinates are given as functions of a parameter.

Voss ([31]) has investigated skew surface of orders two to four by line-geometric methods, as well as in regard to their differential geometry, and likewise the principal tangents to ruled surfaces ([32]).

5. The neighborhood of a ray in a congruence. Analogous to the way that all curves that start from a point on a surface in point space have only ∞^1 directions there, all ruled surfaces that start from a ray in a congruence have only ∞^1 directions, as well. As was mentioned already in no. **3**, two relations must then exist between the three direction coordinates z, α, P. One of them is Hamilton's equation (1), which gives z as a function of α. The other one, which gives P as a function of α, first appeared in **Mannheim** ([33]), although **Cesàro** ([34]) referred to both equations (6) as Hamilton's formulas ([35]). For a suitable choice of starting element of the direction coordinates, the aforementioned relations can be written thus ([36]):

$$z = c \sin 2\alpha, \quad P = P_0 + c \cos 2\alpha,$$

([25]) *Géom. reglées*, chap. IV.

([26]) **Voss**, Math. Ann. VIII and **Koenigs**, Thèse, pp. 97, *et seq.*

([27]) Thèse, pp. 30,

([28]) *Géom. reglées*, pp. 63 (1895). This book is a reprinting of treatises that were published in the Ann. de la Fac. Toulouse **3**, **6**, **7** (1889-93).

([29]) **Zindler**, *Liniengeom.*, Bd. II, § 3, 4. The fact that the distance will vanish to at least third order when it vanishes to higher than first order is a theorem that has been known for some time, and which **Koenigs** (*Géom. reglées*, pp. 63) attributed to **Bouquet**.

([30]) Math. Ann. V, pp. 293.

([31]) *Loc. cit.*, pp. 26.

([32]) Math. Ann., Bd. 12.

([33]) Liouv. J. (2) **17** (1872), pp. 126 or *Géom. Ciném.*, pp. 284 (1894).

([34]) Natürl. Geom. (transl. Kowalewski), § 211.

([35]) Admittedly, one finds a formula for P (which is dp / ds, in the reference) as a function of an independent parameter t in **Kummer**, J. f. Math., Bd. 57, pp. 200. However, t no longer has the direct meaning of an azimuth for the ray in question, and for that reason, the formula admits no simple geometric interpretation, so it will not be discussed any further, either.

([36]) **Cesàro**, *loc. cit.*, § 212.

in which P_0 and c are constants, or also (after rotating the starting plane through 45°):

$$z = \tfrac{1}{2}(A - B)\sin 2\alpha, \qquad (6)$$

$$P = A\sin^2\alpha + B\cos^2\alpha,$$

in which A and B are constant. If one eliminates α then one will obtain ([37]):

$$(P - A)(P - B) + z^2 = 0. \qquad (7)$$

It follows from equations (6) that: The central points of all ruled surfaces that start from a regular ray s of a congruence fill up a finite segment of length $A - B$ on s, in which A and B are the extreme values that the distribution parameter can assume. The endpoints of that *limiting segment* are called *limit points.* Two values of α and two values of z belong to $P = 0$. They determine the cuspidal planes and the cuspidal points of the ruled surfaces of the congruence that have a cuspidal generator at s. The cuspidal points are called *focal points* of the ray s and the cuspidal planes are called *focal planes.* The midpoint of the segment between the focal points, which is called the *focal segment*, also bisects the limiting segment and is called the *midpoint* of the ray. The extreme positions of the central points that belong to two mutually-perpendicular planes as central planes ($\alpha = \pm \pi / 4$) are called the *principal planes.* All of these concepts were presented by the authors that were named in nos. **1** and **2**. The planes $\alpha = 0, \pi / 2$ might be called *curvature planes*. The central planes, in fact, attain P for them, so the curvature at the central point of the associated ruled surface also attains an extreme ([38]). In fact, the determination of the invariants A and B from the representation (5) and the general position of the ray relative to the coordinate system is completely analogous to the calculation of the principal radius of curvature at a point of a surface ([39]). The curvature planes are the bisecting planes, just as the wedge (*Keil*) of the focal planes is also the wedge of the principal planes. The plane that is perpendicular to the midpoint on s is called the *middle plane*, the planes that are perpendicular to s at the limit points are called *limit planes.* If φ is the angle between the focal planes then the focal segment will have the length $(A - B)\sin\varphi$.

If one seeks all rays in the neighborhood of a ray that define the same angle with s then one will obtain a "ruled surface *of constant inclination*," and if one seeks all of them that have the same distance from s then one will obtain a "ruled surface *of constant distance*." The former has order four ([40]), while the latter has order eight ([41]).

For $A = B$, P will be constant, and the limiting points will coincide; such a ray is called *isotropic*. For $A = - B$, the limit points will coincide with the focal points, and the focal plane with coincide with the principal planes, as well as being perpendicular to each

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([37]) **Cesàro**, Natürl. Geom. § 211. One will also find this equation in geometric form in **Mannheim**, *Géom. cinema.*, pp. 281, Theorem **8′**.

([38]) The value of this curvature is, in fact, $-1 / P^2$; cf., e.g., **Zindler**, *Liniengeom.*, Bd. II, § 3.

([39]) *Ibidem*, § 22.

([40]) **Hensel**, J. f. Math., Bd. 102.

([41]) **Zindler**, *Liniengeom.*, Bd. II, § 23.

other; such a ray is called a *normal ray*. The midpoints, the limit points, the principal planes, and the curvature planes are always real. One calls the ray *hyperbolic*, *elliptic*, or *parabolic* according to whether the focal points and focal planes are real, not real, or coincide, respectively. The first case will occur when A and B have unequal signs, the second, when they have equal signs, and the third, when A or B is zero. The elliptic rays split into two groups depending upon the signs of A and B, namely, the "right-wound" and "left-wound" neighborhoods.

There are *singular* rays, for which the foregoing theorems are not true. They have still been studied only a little, but one can find a discussion of "p-fold" rays, from which $2p$ intersecting directions will emanate, in **Koenigs** ([42]) and **Weiler** ([43]). A simple case of special rays are the *cylindrical* ones ([44]), for which the limit points are at infinity.

6. The surfaces that are linked with a congruence. The geometric locus of the midpoints of all rays of a congruence is called the *middle surface* of the congruence, the locus of focal points is called the *focal surface,* and the locus of limit points is the *limit surface;* the envelope of the middle planes is called the *middle envelope*, and that of the limit planes is called the *limit envelope.* The focal surface consists of two sheets, which can define the same surface analytically, and similarly for the limit surface and the limit envelop. The middle surface, limit surface, and focal surface were defined by **Kummer** ([45]), and the middle envelope, by **Ribacour** ([46]).

a) *The limit surfaces.* The first limit surface of a ray system that was actually found and investigated is that of the axis congruence of order three and class two of a linear two-dimensional manifold of linear complexes. It is, at the same time, the congruence of shortest distances between any two rays of a second-order ruled family. **Waelsch** ([47]) has found that the focal surface of that congruence is identical with the limit surface, although it is not a normal congruence, since a limit point of any ray will, at the same time, be a focal point of another ray. The surface is of order six and class four, and was also investigated further by **Joly** ([48]) and especially by **Study** ([49]) (the latter appealed to the use of elliptic functions).

The limit surfaces of a ray net (the definition of that word is in no. **7**) are of order ten ([50]) and reduce to order six in special cases, namely, in the case of nets of revolution, rectangular nets, and parabolic ones. A model of the last case appears in the publication of **Schilling**.

([42]) Thèse, pp. 50, *et seq.*

([43]) Zeit. f. Math. u. Phys., Bd. 31.

([44]) **Zindler**, *Liniengeom.*, Bd. II, § 34.

([45]) J. f. Math., Bd. 57, pp. 203.

([46]) "Étude des Élassoïdes ou surf. à courbure moyenne null," Mém. cour. et des sav. étr. Acad. Belg. **44** (1882), § 2. The author employed the method of "perimorphism" that is employed in **Darboux**'s method of moving trihedra, as well as the methods of "natural geometry" that were employed by later authors, namely, **Cesàro**.

([47]) Wiener. Sitz. **95** (1887) or Nova acta der Leop. Acad. Halle **52** (1888).

([48]) Trans. of the Irish Acad. **30** (1895).

([49]) *Geom. d. Dynamen*, pp. 479, *et seq.*

([50]) **Zindler**, *Liniengeom.*, Bd. II, § 26.

b) *The focal surfaces.* One or both sheets of the focal surface can reduce to curves. That case was investigated by **Sturm** and classified in terms of the general theory ([51]). If one overlooks that case then one will have the following: One can arrange their rays in the neighborhood of a congruence in which the focal points are real into a family of ∞^1 developable surfaces in two ways. The ridge lines (*Gratlinien*) of each family fill up each sheet of the focal surface, and both sheets will contact each ray of the congruence at the two focal points. Each focal plane contacts the one sheet at a focal point and is the osculating plane to the ridge line that lies in the other sheet at the other focal point. The focal surface is then also the envelope of the focal planes. On each sheet, the ridge lines of the one family of developable surfaces will be conjugate to the curves at which the developables of the other family contact that sheet ([52]). **Voss** ([53]) represented the focal surface in symbolic form for a congruence that is given as the intersection of two complexes. Far-reaching investigations have been made of the singularities in the focal surfaces, although more along an enumerative-algebraic direction ([54]).

Waelsch has determined the inflection tangents of the focal surface ([55]). If Q_1, Q_2 are the focal points of a ray s, E_1, E_2, the associated focal planes, etc., such that E_1 is the osculating plane of the ridge line that contacts s at Q_1 then E_1 will contact the focal surface at Q_2, and E_2 will contact it at Q_1. Now, **Waelsch** (*loc. cit.*) referred to the lines of the pencils (Q_1, E_1) and (Q_2, E_2) as *central rays* of s and showed that when one moves Q_1 on the focal surface, the central rays of all neighboring points will lie in a linear complex, namely, the *auxiliary complex* of the point Q_1 . The other focal point Q_2 also has an auxiliary complex. These two complexes (like any two linear complexes with the same two singular rays of the pencil) determine a double ratio δ that is also called the "double ratio of the ray s." For the normal congruences of Weingarten surfaces, one has

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([51]) Mitteil. der Hamburger math. Ges. **2** (1890) or *Liniengeom.*, Bd. II, pp. 12, *et seq.*. Cf., also **Weiler** in ([43]).

([52]) Most of this is in **Kummer**, J. f. Math., Bd. 57, pp. 205, *et seq.*

([53]) Gött. Nachricht, 1873.

([54]) Math. Ann., Bd. 9.

([55]) "Zur Infinitesimalgeom. der Strahlenkongruenzen," Wiener Sitz. **109**, II (1891). This treatise was based upon the following representation of congruence: Let:

$$\sum a_k\, x_k = (ax) = 0$$

be the equation of a line, where x, as the Kleinian coordinates, will satisfy the relation $(xx) = 0$, and analogously for the a. If the a depend upon two parameters u, v then a ray congruence will be defined, and if one denotes:

$$\sum a_k\, x_k = a\,(u, v, x)$$

then one will have the equation:

$$0 = a + b\,du + c\,dv + \tfrac{1}{2}(e\,du^2 + 2f\,du\,dv + g\,dv^2) + \ldots$$

in the neighborhood of a ray a. In this, $b, \ldots, g$ are derivatives of a with respect to u, v, and thus linear in the x. Therefore, $b = 0, \ldots, g = 0, \ldots$ will be the equations of linear complexes. The differential-geometric properties in the neighborhood of a, and especially the projectively-invariant properties, will be expressed systematically by them.

$\delta = 1$, and for second-order surfaces, one has $\delta = 9$. If d is the length of the limit segment and r_1, r_2, r_1', r_2' are the radii of curvature of the two sheets of the focal surface then [56]:

$$\delta = \frac{r_1\, r_2\, r_1' r_2'}{d^4}.$$

Demoulin has found a relation between δ, the angle between the focal planes, and the torsions of the two ridge lines [57].

c) *The remaining surfaces* have not been investigated very much for general congruences (cf., also no. **10**, *a*). According to **Waelsch** [58] and **Study** [59], the middle surface of the axis congruence that was mentioned under *a*) is a (fourth-order) Steiner surface, and at the same time, the basepoint surface relative to the midpoints of the entire figure. The middle envelopes of the hyperbolic and elliptic ray nets are equilateral, hyperbolic paraboloids [60].

7. The contacting ray net. The intersection of two linear complexes is a ray system of order and class one, and (according to **Sturm**) is briefly called a *ray net.* The regular ray nets also consist of the totality of lines of intersection of two rays – namely, the *focal lines* – and are called hyperbolic, elliptic, or parabolic according to whether the focal lines are real and distinct (i.e., skew), complex-conjugate (of the second kind), or coincident, respectively. Among the hyperbolic nets, the "rectangular" ones, whose focal lines cross at right angles, are distinguished, while among the elliptic ones, one has the *nets of revolution*, whose focal lines meet the infinitely-distant spherical circle. The elliptic nets have two real, isotropic rays (in analogy with the focal points of an ellipse) [61]. The singular ray nets are the rays of a bundle or a field.

The ray nets (or "nets) are the simplest ray congruences, and one can aspire to represent the neighborhood of an arbitrary congruence approximately in such a way that one replaces it with a suitable-chosen ray net [62] (which is the analog of the contact plane in surface theory). To that end, we define: Two ray congruences *contact* each other at a common ray *s* when they have in common all directions in line space that start at *s* and are contained in them. One then addresses the problem of exhibiting the contacting net for a ray *s* of a congruence. Ordinarily, one formulates that problem less precisely as: Suppose that one is given two lines that are intersected by all rays of the congruence in a neighborhood of *s*. Such lines are called *focal lines* [63] of the neighborhood of *s*, or, more briefly, of the ray *s* itself. **Monge** and **Sturm** already found (no. **1**) a pair of focal

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[56] *Loc. cit.*, pp. 203.
[57] Comptes R. **130** (1900), 1701.
[58] Wiener Sitz. **95**, II (1887).
[59] *Geom. d. Dynamen*, pp. 475.
[60] **Sturm**, *Liniengeom. I*, pp. 167.
[61] **Zindler**, *Liniengeom., Bd. II*, § 24.
[62] **Klein**, Math. Ann. V, pp. 289.
[63] The term "focal lines" was also employed for the ridge lines of developables [**Bianchi**, Ann. di. Mat. (2) XV].

lines for normal systems, which they assumed to be perpendicular to s. **Kummer** ([64]) investigated cross sections of the surrounding infinitely-thin bundle of rays for general rays systems and found that they reduced to straight lines at the focal points. Since he considered only cross sections of the bundle, he also obtained only focal lines b_1, b_2 that were perpendicular to s, and some of the authors that followed him were of the opinion that this would be an essential property of any useful focal lines. However, one can, in fact, consider any line of the pencils (s, b_1) and (s, b_1) to be as good as a focal line, as **Klein** ([65]), **Weingarten** ([66]), and **Matthiessen** ([67]) made clear. One can express this precisely as: One can choose any ray in the pencils (Q_1, E_2) and (Q_2, E_1) arbitrarily, except for s. Both of those rays, as focal lines, determine a ray net that contacts the congruence, and one will get all contacting nets in that way, so there will be ∞^2 of them then ([68]). Those two pencils are then called *focal pencils* ([69]); their rays are nothing but focal lines. The contacting net whose focal lines are perpendicular to s might be called the *principal net* of the ray s. For the investigation of the neighborhood of s, it is most convenient if it is not distinguished among the contacting nets, as long as one is dealing with "first-order properties" ([70]). The principal net of an isotropic ray s is a net of revolution. Thus, the rays in the neighborhood of s can be arranged approximately into coaxial hyperboloids of rotation.

For the principal net, the ray s considered is, at the same time, the principal ray (which cuts the focal lines perpendicularly). Since the line at infinity belongs to the location in the principal net that is perpendicular to s, any two planes of that location will cut the net into affine fields. **Möbius** ([71]) derived the most essential results of **Kummer** by starting with that. He was followed by **Frischauf** ([72]) and **Zech** ([73]), who pursued the constructive aspect of the theory, and later, by **Bobek** ([74]). Since two affine fields are determined by three pairs of corresponding points, in order to determine an "infinitely-thin ray bundle" (**Zech**, *loc. cit.*), it suffices to give two neighboring rays, in addition to the starting ray; i.e., the entire manifold of directions of a congruence is determined by two directions, which should be self-explanatory from the analytical conception of a direction. **Mannheim** has also given a constructive theory of infinitely-thin ray bundles ([75]).

If one goes on to the higher-order properties then one can find distinguished contacting nets, or, what amounts to the same thing, distinguished focal lines. The tangents to the focal surfaces that are conjugate to s are such lines. If one lays intersecting planes through them then the cross section of the bundle will vanish to order

([64]) *Loc. cit.*, pp. 222.
([65]) Math. Ann. V, pp. 289.
([66]) J. f. Math., Bd. 98.
([67]) Schlömilch's Zeit. f. Math., Bd. 29, Suppl. and Acta math. IV.
([68]) **Klein**, *loc. cit.*
([69]) **Waelsch**, Wiener Sitz., Bd. 100, II
([70]) With **Koenigs**, we might say "n^{th}-order properties" to mean ones that depend upon the derivatives up to order n, inclusive, of the coordinates with respect to the independent variables.
([71]) "Geom. Entwicklung der Eigenschaften unendlich dünner Strahlenbündel," Ber. d. sächs Ges. **14** (1862), or Ges. Werke, IV.
([72]) Schlömilch's Zeit. f. Math., Bd. 16.
([73]) *Ibidem*, Bd. 17.
([74]) Wiener Sitz. **83**, II (1881).
([75]) *Géom. ciném.*, pp. 279, *et seq.*

three ([76]). The two auxiliary complexes (no. **6**, *b*) have a ray net in common whose focal lines are distinguished "principal lines" ([77]). We mention some results of **Koenigs** ([78]): If one considers all ruled surfaces in a congruence that have a well-defined direction and go through a ray *s* then their osculating ruled families for *s* will fill up a ray net – viz., the *osculating net* of the direction in question. The osculating nets of all directions of *s* define a quadratic complex. Any linear complex that contains a contacting ray net of a congruence is called a *contacting linear complex.* If one chooses a line *g* of the congruence, a neighboring line *s*, and a linear complex *C* that contacts *s* then the moment of *g* relative to its conjugates in *C* will vanish to order four, in general.

8. Applications to geometrical optics. Some of the studies that were mentioned already in no. **7** (viz., **Mathiessen**, **Ahrendt**) were of interest to optics. If a refracting surface and an infinitely-thin ray bundle are given then the important problem for optics is to find the refracted bundle. One can call this problem "the Sturm problem," for the sake of brevity, although **Sturm** (no. **1**) considered only the normal bundle. It was solved by **C. Neumann** ([79]) in such a form that the incident bundle is given by the principal ray and the two focal lines that are normal to it, and one seeks the analogous elements in the refracted bundle. A geometric solution goes back to **Mannheim** ([80]). The case of the sphere as the refracting surface was examined thoroughly by **Reusch** ([81]), **Lippich** ([82]), **Neumann** ([83]).

We have already mentioned the **Malus-Dupin** theorem in no. **1**. In more recent times, **Ribacour** ([84]), **Jamet** ([85]), **Gorton** ([86]), and **Bianchi** ([87]) have given proofs of it. **Demoulin** has extended it to the case of infinitely-many refracting surfaces ([88]); i.e., to continuously-curved light paths. **Levi-Civita** has shown ([89]): The property of being a normal system is the only property that is invariant under refraction. Two congruences that are either both normal or both not normal are always derivable from each other by a finite number of refractions. One refraction will suffice for normal congruences, while two will suffice for the other ones, in general; in the latter case, two ruled surfaces of congruences can be associated with each other arbitrarily, ray-wise.

([76]) **Ahrendt**, Schlömilch's Zeit., Bd. 36.
([77]) **Waelsch**, Wiener Sitz., Bd. 100, II, pps. 164 and 169.
([78]) Thèse, pp. 100, *et seq.*
([79]) Ber. d. sächs. Ges., Bd. 32, 1880.
([80]) Atti della Acc. dei Lincei, 1885-86 or *Géom. ciném.*, pp. 550, *et seq.*
([81]) Pogg. Ann. d. Phys. u. Chemie **130** (1867).
([82]) Wiener Denkschr. **38** (1877).
([83]) Cf., footnote 79.
([84]) J. de Math. (4) **7**, pp. 103.
([85]) Ann. d. la fac. de sc. Marseille, 1900.
([86]) Am. J. of Math., XIII.
([87]) *Vorl. üb. Differentialgeom.*, § 143. Cf., also **Bruns**, "Das Eikonal," Abh. d. sächs. Ges. XXI (1895).
([88]) Comptes R., t. 129.
([89]) Atti della Acc. dei Lincei (5) **9** (1900).

9. The ruled surfaces in a congruence. The contact planes of four ruled surfaces of a congruence through the same ray have the same double ratio for all points of the ray ([90]).

a) If one always seeks the directions in a congruence for which $P = 0$ then one will obtain the *developable* of the congruence (cf., no. **6**, *b*). **Klein** ([91]) has exhibited its differential equation for the case in which the congruence is given as the intersection of two complexes.

b) If one always seeks the directions in a congruence for which z is an extreme then one will get the *principal surfaces* ([92]). They are then the surfaces whose central point is always a limit point; i.e., whose line of striction lies in the limit surface ([93]).

c) If one always seeks the directions in which P is an extreme then one will obtain ruled surfaces that might be called *curvature surfaces*, since they are analogous to the lines of curvature in the theory of surfaces. They are, at the same time, the ones whose lines of striction lie in the middle surface, and were first introduced, as such ("rigate medie"), and almost simultaneously, by **Burgatti** ([94]) and **Cifarelli** ([95]).

One finds the differential equation of all three kinds of surfaces for the parametric representation (5) of congruence in **Zindler** ([96]).

d) The contacting principal nets depend upon two constants (apart from their positions in space), one of which determines merely the form of the net (for hyperbolic nets, it is the angle between the focal lines), while the other one determines the magnitude (for hyperbolic nets, it is the distance between the focal lines). One can then distinguish between ruled surfaces of *constant neighboring magnitude* and *constant neighboring form* in a congruence ([97]). The former are, at the same time, the ones for which the congruence has the same density at all of its central points ([98]).

e) One understands the *moment of two lines* to mean the product of their shortest distance and the sine of their angle. If the line coordinates of a line are given as functions of a parameter t – say, the arc length of the spherical image – then one can express the moment M of two neighboring lines of the ruled surface as functions of t and Δt. It vanishes to order two. If one then defines:

([90]) The analogous theorem for general congruences of curves is in **Darboux**, *Th. des surf. II*, pp. 3.

([91]) Math. Ann. V, pp. 292.

([92]) Deviating from customary usage, **Ribacour** understood "surfaces principales" to mean developables [in the paper that was cited in ([46])].

([93]) For example, **Bianchi**, *Vorl. über Differentialgeoem.*, § 139.

([94]) Atti della Acc. dei Lincei (5) **8** (1899).

([95]) Ann. di. Mat. (3) **2** (1899).

([96]) *Liniengeom.*, Bd. II, §§ 27, 28.

([97]) **Zindler**, *loc. cit.*, § 31.

([98]) The surfaces of equal density that **Kummer** introduced (*loc. cit.*, pp. 214) consist of all points in space at which the ray system has equal density. Thus, the line of striction of any ruled surface of constant neighboring magnitude will lie on a surface of equal density.

$$J = \int \sqrt{M} \cdot dt$$

for two rays of the ruled surface then one will obtain a finite value, in general. If two rays of a congruence are given then one can pose the problem of linking them with the ruled surface of the congruence for which J has a minimum. Following **Koenigs** ([99]), in analogy to what one does in the theory of surfaces, one calls any ruled surface in the congruence for which the first variation of J vanishes a *geodetic ruled surface*.

That concept can be extended analogously to complexes and all of line space; e.g., the usual screw surfaces are geodetic ruled surfaces in line space ([100]).

f) The ruled surfaces of a linear congruence and their principal tangent curves were examined by **Pittarelli** ([101]).

10. Special congruences.

a) *Isotropic congruences.* A congruence that consists of nothing but isotropic rays (no. **5**) is called *isotropic*. For such a congruence, both sheets of the limit surface coincide with the middle surface, in which the lines of striction of all it its ruled surfaces also lie ([102]). The principal surfaces and the curvature surfaces will be undetermined. The focal surfaces go through the spherical circle at infinity. The middle envelope of an isotropic congruence is a minimal surface ([103]). Conversely, ∞^3 isotropic congruences belong to a minimal surface ([104]). If one knows an orthogonal, isometric curve net on a sphere then one derive an isotropic congruence from it by a simple construction ([105]). Such a thing is determined already by one of its ruled surfaces ([106]). There are also isotropic congruences of revolution and screw congruences ([107]).

b) *Normal congruences***.** A congruence that consists of nothing but normal rays (no. **5**) is called a *normal congruence*, since it is the normal system of ∞^1 (parallel) surfaces F. It focal surface (which coincides with the limit surface) is the central surface or evolute surface (viz., the locus of both principal curvature midpoints) of each surface F ([108]). The ridge lines of the developables are geodetic lines on the focal surface, since it follows immediately from no. **6** that their osculating planes are perpendicular to the focal surface ([109]). **Monge** already treated ([110]) the problem of finding a normal congruence on one

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([99]) Thèse, pp. 86, *et seq.*
([100]) *Ibidem*, pp. 89.
([101]) Rend. dell' Acc. dei Lincei (5) **3** (1894).
([102]) **Ribacour** in the paper that was cited in ([46]).
([103]) *Loc. cit.*, § 26 [also **Cosserat**, Mém. de l'Ac. Toulouse (9) **4** (1892)].
([104]) **Ribacour**, *loc. cit.*, § 34.
([105]) *Loc. cit.*, § 27.
([106]) *Loc. cit.*, § 35. In this paper, one finds numerous theorems about the connection between isotropic congruences and minimal surfaces; cf., also his treatise in J. de Math. (4) **7** (1891).
([107]) **Zindler**, *Liniengeom.*, Bd. II, § 29.
([108]) **Kummer**, *loc. cit.*, § 9.
([109]) **Monge** already knew this theorem from **Kummer** (*loc. cit.*, pp. 229). The converse goes back to **Beltrami** ("Ricerche di Anal. applic.," Giorn. di Mat., v. 2 and 3).

sheet of the focal surface when one is given the other sheet, and considered the special case in which the given sheet is a developable surface. **Saussure** ([111]) and **Study** ([112]) investigated the normal congruences of developable surfaces. The middle envelope of a normal congruence is also called the *evolute middle surface* of any orthogonal surfaces F that belongs to the congruence.

Normal congruences, as well as general congruences, were examined many times in connection with and using the methods of the theory of surface, especially in connection with the problem of the bending of surfaces. That exceeds the scope of this report; let it only be remarked in regard to that subject that one finds a dense overview of all the essentials of the aforementioned theory in **Bianchi**'s *Vorlesungen über Differentialgeometrie* ([113]).

c) *Parabolic congruences.* There are congruences whose rays are all parabolic (no. **5**), so both sheets of their focal surfaces will coincide ([114]). They are called "special" or *parabolic* and consist of one system of principal tangents to the focal surface ([115]). The limit segment is $1 / \sqrt{-K}$ if K is the curvature of the focal surface ([116]). **Fano** has investigated the third-order parabolic congruences ([117]).

d) **Bianchi** ([118]) examined the congruences for which the limit segment, as well as the focal segment, are constant (the former = a) and found that the focal surfaces have constant curvature $-1 / a^2$. For that reason, he called them *pseudo-spherical* congruences and determined the ones that are contained in a linear complex.

e) **Ribacour** ([119]) called a triply-orthogonal system for which the trajectories of the one family of surfaces are circles a *cyclic* system. A congruence that is defined by the axes of such circles of called *cyclic*. **Bianchi** ([120]), **Cosserat** ([121]), and **Tzitzeica** ([122]) studied those congruences.

f) **Guichard** concerned himself with congruences whose middle surfaces were planes ([123]) and with ones whose developables cut out conjugate curves from the middle surface, as well as the middle envelopes. If these two surfaces coincide in *one* surface

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([110]) In the version of it for which the one sheet is the central surface of a given surface [Journ. de l'Éc. polyt., **6** (cah. 13) (1806)]; he arrived at a first-order partial differential equation.

([111]) Am. J. of Math., v. 18.

([112]) *Geom. d. Dynamen.*, pp. 306.

([113]) Chap. IX-XIII.

([114]) **Kummer**, *loc. cit.*, § 8, conclusion.

([115]) **Klein**, Math. Ann. V, pp. 290. A degenerate case of these congruences in is **Zindler**, *loc. cit.*, prob. 36.

([116]) **Bianchi**, *Vorl. über Differentialgeom.*, pp. 267.

([117]) Atti della Acc. Torino, 37.

([118]) Ann. di Mat (2) **15** (1887).

([119]) J. de Math. (4) **7**, pp. 229.

([120]) Ann. di. Mat. (2) **18**, **19**, also *Vorl. über Differentialgeom.*, chap. XIII.

([121]) Ann. de la fac. de sc. Toulouse, VII.

([122]) Ann. de l'Éc. norm. (3) **16** (1899).

([123]) Comptes R., **114**, pp. 729.

then it will be a minimal surface, and the congruences will consist of its normals ([124]). He also studied the congruences whose developables determined lines of curvature on the focal surfaces ([125]). **Thybaut** ([126]) and **Bianchi** ([127]) investigated congruences whose focal surfaces were minimal surfaces.

11. Congruences that are linked with one or more surfaces.

a) The axis of a cylinder of rotation that goes through the indicatrix of a point on a surface is called the *optical axis* of that point. The optical axes of all points of a surface define a congruence that was exhibited by **Cosserat** ([128]). If it is a normal congruence then the given surface will have constant curvature.

b) Let the rays of a congruence be associated with the points of a surface in a one-to-one way, and let every ray have a fixed position in regard to the contact plane to the point that is associated with it. A bending of the surface will correspond to such a congruence, and one can ask when a normal congruence goes to another such congruence in that way ([129]). That will happen, above all, when any ray goes through the point that corresponds to it ([130]) (**Beltrami**'s theorem) or lies in the contact plane (**Ribacour**'s theorem). In addition, it is possible only when the surface can be developed into a surface of rotation whose line element can be brought into a certain form. When a fixed line is chosen in the contact plane to any point of the surface, **Bianchi** answered the question ([131]): "When can the congruence remain the normal congruence of a minimal surface or a surface of constant curvature under the bending of the surface?

c) ∞^2 ray congruences are given by any surface in such a way that a ray shall go through every point of the surface that defines the same angles with the axes of the distinguished trihedron everywhere ([132]). If one directs one's attention to a particular point of the surface then any line through it will determined one such congruence of rays. The limit points of these various congruences on the rays of the bundle define a fourth-order surface, while the focal points define one of order three.

d) If one couples each point of a surface with the pole of its contact plane relative to a second-order surface then one will obtain the ray system of *projective normals* ([133]); one calls its focal surface the "projective central surface."

([124]) *Ibidem*, **112** (1891) pp. 1424; also **Petot**, *ibidem*, **113**, pp. 841.
([125]) Ann. de l'Éc. norm. (3), **6**.
([126]) Ann. de l'Éc. norm. (3) **14**, and also Thèse, 1897.
([127]) Ann. di Mat. (3) **10** (1904).
([128]) Ann. de la fac. de sc. Toulouse **8** (1894).
([129]) **Dall'Aqua**, Atti de Ist. Ven. **60** (1901).
([130]) See also **Cifarelli**, Giorn. di Mat. **36**.
([131]) Rend. dell' Acc. dei Lincei (5) **91**; cf., also **Pseborski**, Samml. der Mitt. der math. Ges. in Charkow (2) **7** (1902).
([132]) **Lilienthal**, Math. Ann., Bd. 31, in which the tangents to the congruences that are defined by lines of curvature were examined in particular. **Zeeman** treated the question of when a congruence, thus-defined, is a normal congruence [Nieuw Arch. voor wiskunde (2), **4**]
([133]) **Voss**, Abh. d. Münchner Akad. **16** (1887).

e) A congruence can be given as the tangent congruence to a surface. **Zeeman** ([134]) and **Waelsch** took that starting point, which determines the second focal point of the ray, in particular, and considers the Liouville surfaces to be starting surfaces ([135]).

f) If two surfaces are related to each other (in single-valued or multi-valued way) then a congruence will be defined by the connecting lines of corresponding points, and **Voss** ([136]) investigated its focal surfaces.

g) **Wilczynski** considered the congruence of all osculating ruled families of a ruled surface ([137]).

12. The equations of Cesàro and others. It is known that three relations exist between the six fundamental quantities of order one and two in the theory of surfaces, which are ordinarily referred to as the **Codazzi** equations. Analogously, four relations exist between **Kummer**'s fundamental quantities that appear in the differential forms (4) in no. **2**, or the equivalent constructs in other representations, which were first derived by **Cesàro** ([138]), and then later by **Fibbi** ([139]), **Cifarelli** ([140]), and **Burgatti** ([141]). The last one showed that a single congruence will belong to given fundamental quantities when these relations are fulfilled (up to its position in space); the direction cosines of its rays will be found by Riccati equations.

One can pose the problems of finding a congruence from the given spherical images of:

α) The developable surfaces,
β) The principal surfaces,
γ) The curvature surfaces.

The first problem was treated by **Guichard** ([142]) and **Cosserat** ([143]), the second, by **Bianchi** ([144]), the third by **Burgatti** ([145]) and all three by **Eisenhardt** ([146]). The problem of finding all surfaces on which the developables of a given congruence cut out a

([134]) Nieuw Archief vor wiskunde (2) **4** (1900).
([135]) Wiener Sitz. **102**, II (1893); cf., also **Pell**, Am. J. of Math. **20** (1898).
([136]) Math. Ann., Bd. **30**; cf., also **Panelli**, Mem della Acc. dei Lincei (4) **6** (1890).
([137]) Trans. of the Amer. Math. Soc. **4**.
([138]) Rend. dell' Acc. Napoli (2) **8** (1894) or *Geom. intrinseca* (§ 215 in the German translation); moreover, it is included implicitly in the more general equations that **Lilienthal** found almost simultaneously for the congruences of curves. (*Grundl. einer Krümmungslehre der Kurvenscharen*, Leipzig, 1986).
([139]) And two more general ones for congruences in a space of constant curvature, Ann. della Sc. norm. di Pisa, VII.
([140]) Ann. di Mat. (3) **2.**
([141]) Atti della Acc. die Lincei (5) **8**.
([142]) Ann. de l'Éc. norm. (3) **6**.
([143]) Ann. de la fac. de sc. Toulouse, VII.
([144]) *Vorl. über Differentialgeom.*, § 146.
([145]) Cf. **133**.
([146]) Trans. of the Am. Math. Soc. **3**.

conjugate system was solved by **Darboux** ([147]), along with the inverse problem and some related ones.

The theory of ray congruences is more general than the entire theory of surfaces (since a *special* ray system is coupled with any surface, namely, the normal system), but on the other hand, it is included in the more general theory of congruences of curves that was developed by **Darboux** ([148]), **Lilienthal** ([149]), **Levi-Civita** ([150]), **Dall'Aqua** ([151]), and **Eisenhardt** ([152]) (but admittedly not to the same level of detail as the theory of ray congruences).

13. Types of complex rays. Singularity surfaces. For the differential-geometric investigation of line complexes, one must employ the following representations: An equation between homogenous (tetrahedral or rectangular) Plücker line coordinates:

$$F(p_1, \ldots, p_6) = 0 \tag{8}$$

with the condition:

$$\sum_{i=1}^{3} p_i\, p_{i+3} = 0, \tag{8a}$$

or between Klein coordinates:

$$F(x_1, \ldots, x_6) = 0 \tag{9}$$

with the condition:

$$\sum_{i=1}^{6} x_i^2 = 0, \tag{9a}$$

or between any four mutually-independent coordinates:

$$f(u_1, \ldots, u_4) = 0: \tag{10}$$

or the "parametric representation":

$$p_\lambda = p_\lambda(u, v, w) \qquad (\lambda = 1, \ldots, 6), \tag{11}$$

in which u, v, w are independent variables, and the functions p_λ must satisfy the relation (8a) identically. In (8) and (9), the functions F are assumed to be homogeneous (and irreducible; let the degree be n), in (11), the p can be assumed to be tetrahedral coordinates for projective properties, while for metric properties, it is preferable to assume that they are rectangular, homogeneous, Plücker coordinates. The representations will be more general, as long as one regards them directly as also being transcendental line complexes.

If a ray p of a complex, with coordinates p_i, fulfills the relation:

([147]) *Théorie des surf. II*, chap. I; cf., also **Cosserat**, *loc. cit.*
([148]) *Théorie des surf. II*.
([149]) *Grundlagen einer Krümmungenslehre der Kurvenscharen*, Leipzig, 1896.
([150]) Atti della Ac. dei Lincei (5) VIII.
([151]) Ann. di Mat. (3) **6.**
([152]) Trans. of the Amer. Math. Soc. **4**.

$$\sum_{\nu=1}^{3} F_\nu F_{\nu+3} = 0, \tag{12}$$

as with (8), or the relation:

$$\sum_{\nu=1}^{6} F_\nu^2 = 0, \tag{13}$$

as with (9), then the ray p will be called *singular*, and otherwise *regular*. The complex cone of a point P on a complex ray p has a contact plane β along p. If one moves P to p then that plane will rotate around p through a regular ray β, and the pencil of planes β will be projective for such a point sequence, so it will define a correlation of p with it ([153]).

If $s \equiv (s_i)$ is a singular ray then one sets ([154]):

$$F = q_{i+3}$$

then the q_ν can also be regarded as coordinates of a ray q by means of (12). Depending upon whether q is or is not identical with s, s will be called a *higher* or *ordinary* singular ray, resp. Among the former, the *double rays* are the simplest, while among the latter case, s and q will intersect at a point S, so they will also determine a plane σ – viz., the *singular plane* that belongs to s – while S is called the *singular point* that belongs to s. A higher singular ray is a double ray (ray of regression, multiple ray) of all points of the complex cone, while an ordinary singular ray is one for only the complex cone of S; dually, an ordinary singular ray is a double tangent only for the complex curve of σ. For all planes of the pencil β (with the exception of σ), S will be associated with s as the contact point of its complex curves (and an analogous dual statement).

The singular rays of a complex define a complex [which is the intersection of (8) and (12)] when not all rays are singular ([155]), namely, the *singularity congruence*. The surface that is enveloped by singular planes is identical with the locus of singular points ([156]) and is called the *singularity surface*. It was represented by **Clebsch** ([157]) using symbolic methods, and is one sheet of the focal surface of the singular congruence; the other sheet is called the *accessory* surface. The two points at which an ordinary singular ray s contacts the focal surface are separated harmonically from the two points at which the complex curve is contacted by the associated plane σ of s as its double tangent ([158]). The singularity surface of a quadratic complex is, in general, a **Kummer** surface (of

([153]) For quadratic complexes, by **Plücker**, *Neue Geom.*, art. 228 (1868), and more generally, by **Pasch** (Habilitationsschrift, Giessen, 1870), and for the representation (10), by **Koenigs**, Thése, art. 29.

([154]) **Pasch**, J. f. Math., Bd. 76.

([155]) This case occurs only when the complex consists of the tangents to a surface or the secants to a curve; **Cayley** (Coll. Papers, vol. IV, nos. 284 and 294 or Quart. J., t. III) and **Klein** (Math. Ann. V) have given the analytical way of characterizing them. **Koenigs** address the fundamental form (expression for the moment of two neighboring rays) of singular complexes, Comptes R., t. 100, pp. 847.

([156]) For quadratic complexes, by **Plücker**, *Neue Geom.*, art. 320, and more generally, by **Pasch**, Habil-schrift, and J. f. Math., Bd. 76.

([157]) Gött. Nachr. 1872 and Math. Ann. V.

([158]) **Pasch**, J. f. Math., Bd. 76, pp. 164.

degree four) with 16 nodes, and it was studied by **Plücker** ([159]), **Klein** ([160]), **Rohn** ([161]), **Reye** ([162]). **Weiler** ([163]) gave its many degenerate cases for the 48 types of quadratic complexes.

14. The neighborhood of a ray in a complex. Contacting linear complexes. Since only ∞^2 directions emanate from a ray in a complex (no. **3**), one relation must exist between the three direction coordinates z, α, P. It was first found by **Koenigs** ([164]) and can be written:

$$P = z \tan \alpha - m \qquad (m = \text{const.}) \tag{14}$$

for a suitable choice of coordinate system. One can obtain an intuitive picture of the distribution of complex rays in the neighborhood of a regular or ordinary singular ($m = 0$) ray from this equation ([165]). The neighborhoods of all complex rays for which m has the same sign are similar to each other in the same sense that one speaks of for the conformal map of similarity into the smallest components. One finds the calculation of the "neighboring magnitude" m in **Zindler** ([166]).

Any complex that includes the directions of advance that start from p, as well as the given one, is called *contacting*. A ray has ∞^1 contacting linear complexes that define a pencil of complexes ([167]). The associated ray net will be parabolic for a regular ray (i.e., it has coincident focal lines), while for an ordinary singular ray, all of the contacting linear complexes will be singular, and their axes define a pencil of rays (S, σ) ([168]).

Zindler ([169]) examined the neighborhood of a regular ray for the representative form (11) with the help of an illustrative pencil of conic sections that gives the distribution parameter P as the quotient of two ternary, quadratic differential forms in the representation. Any complex of the pencil $\mathfrak{B}$ is osculating for ∞^1 directions of the complex; i.e., it contains the osculating hyperboloids of all ruled surfaces of the complex that start from such a direction of p ([170]).

15. Distinguished directions in a complex. Among the directions that start from a ray in a complex, there are distinguished ones, such as the three mutually-perpendicular *principal directions,* for which the pencil $\mathfrak{B}'$ of contacting complexes that correspond to

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([159]) *Neue. Geom.*, pp. 307, *et seq.*
([160]) Math. Ann. II, pp. 213, *et seq.*; V, pp. 293, *et seq.*; Gött. Nachr. 1871.
([161]) Math. Ann., Bd. 15 and 18.
([162]) J. f. Math., Bd. 97.
([163]) Math. Ann., Bd. 7; cf., also **Segre**, Math., Bd. 23.
([164]) Thése, art. 47.
([165]) **Zindler**, Verh. des III intern. Mathematiker-Kongr., 1904 or *Liniengeom.*, Bd. II, § 41.
([166]) *Liniengeom.*, Bd. II, § 42.
([167]) **Plücker**, *Neue Geom.*, art. 300.
([168]) **Plücker**, *loc. cit.*; **Klein**, Math. Ann. V, pp. 285, *et seq.*
([169]) *Liniengeom.*, Bd. II, § 48; there is also a method for examining the neighborhood of a double ray there in § 57.
([170]) **Koenigs**, Thése, art. 91.

the neighboring rays of such a direction, have a complex in common with $\mathfrak{B}$, namely, the *principal complex* ([171]). If one always follows a principal direction then one will arrive at a *principal surface* of the complex. There are four points on a complex ray p for whose complex cone p will be a ray of inflection ([172]). By following the corresponding *inflection directions*, one will arrive at some distinguished developable surfaces of the complex, namely, the *inflection surfaces* ([173]). **Zindler** ([174]) gave some other distinguished directions and linear pencils of directions (no. **3**), and among them, one finds the pencil of isotropic directions, whose directions lie in the same way that they do in the neighborhood of an isotropic ray of the congruence.

16. Differential equations of a complex. Distinguished decompositions. Lie ([175]) called a homogeneous differential equation in x', y', z':

$$\Omega\,(x, y, z; x', y', z') = 0, \tag{15}$$

in which x, y, z are thought of as functions of an independent variable t, and the prime means the derivative with respect to t, a **Monge** equation. In particular, if Ω is linear in x', y', z' then (after multiplying by dt) we will get a total differential equation that is also called a **Pfaff** equation. If a triple of functions x, y, z satisfies equation (15) then the corresponding curve will be called an *integral curve* of the Monge equation. It will determine an elementary cone for any point of space – i.e., cone of directions of advance that are defined by the integral curves that go through that point itself.

A Monge equation belongs to any line complex. Let:

$$F\,(p_1, \ldots, p_6) = 0$$

be the equation of an algebraic complex in rectangular, homogeneous, Plücker coordinates, namely:

$$p_1 = x_2 - x_1, \qquad p_2 = y_2 - y_1, \qquad p_3 = z_2 - z_1,$$
$$p_4 = y_1\,x_2 - y_2\,x_1, \qquad \ldots$$

The associated Monge equation will then read:

$$F\,(x', y', z', yz' - zy', \ldots) = 0. \tag{16}$$

However, conversely, a complex belongs to a Monge equation only when ∞^3 lines occur amongst its integral curves. The condition for that is just the one that the Monge equation can be put into the form (16), in which F must be a homogeneous function ([176]).

([171]) **Klein**, Math. Ann. V, pp. 271; **Koenigs**, Thèse, art. 92, *et seq.*
([172]) **Voss**, Math. Ann. IX.
([173]) **Koenigs**, Thèse, art. 92.
([174]) *Liniengeom.*, Bd. II, § 48.
([175]) *Geom. d. Berührungstransf.*, pp. 178.
([176]) **Lie**, *loc. cit.*, pp. 252, *et seq.*

The theory of line complexes also has a close connection with the theory of first-order partial differential equations; confer **Lie** ([177]) or **Jessop** ([178]) on this.

Klein ([179]) addressed the search for the parabolic congruences of a complex. The question of the normal congruences of a complex was already raised by **Malus** (in a somewhat different form) ([180]) and once more by **Transon** ([181]). The question of isotropic congruences of a complex was treated by **Cosserat** ([182]), and led back to the question of whether two certain partial differential equations have a common solution. For the representation (11), the same problem leads to a total differential equation ([183]) whose integrability likewise resolves the possibility of decomposing the complex into ∞^1 isotropic congruences. The ray thread contains no isotropic congruence; **Picard** ([184]) found its normal congruences.

17. Ruled surfaces and curves in a complex. A curve whose tangents all belong to a complex is called a *curve* of the complex. The tangents themselves define a developable surface in the complex. All complex curves with a common line element have the contact plane of the complex cone of that element as their common osculating plane ([185]), while all curves of a null system (i.e., ray thread) that go through the same point have the same torsion ([186]). For complexes of higher degree, **Demoulin** considered complex curves with a common line element ([187]) and found that a linear relation exists between their curvature and torsion at the points of that element.

One can require that an integral curve of a Monge equation (16) should have a smaller curvature at each of its points then all other integral curves that contain it. One calls such a curve a *straightest* line of the Monge equation ([188]). The problem of finding the shortest integral curve between two points for a given Monge equation does not always lead to a shortest line; there will then be ∞^4 shortest lines, but only ∞^3 straightest lines. The difference between them will become clearest in the case of line complexes, where the lines are, at the same time, the straightest ones. **Liebmann** ([189]) examined the shortest lines of the ray thread. This example is interesting in the context of the calculus

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([177]) *Loc. cit.*, sect. II, chap. 7 and sect. III.

([178]) *A Treatise on the Line Complex*, Cambridge, 1903, chap. 18.

([179]) Math. Ann., Bd. V, pp. 290; one will also find **Lie**’s contribution to that problem in the remarks there. For the parabolic congruences of the ray thread, cf., **Lie**, Christ. Vidensk. Forh., 1882; **Peter**, diss. Leipzig 1895) (or Archiv for Math. og Naturv. 17) and **Lagally**, diss. München 1903.

([180]) Cf., no. **1**.

([181]) J. de l’Éc. polyt., **22** (1861), cah. 38.

([182]) Toulouse, Mém. (9) **4** (1892).

([183]) **Zindler**, *Liniengeom.*, Bd. II, pp. 303.

([184]) Thèse, art. 21 (1877).

([185]) **Lie**, Christ. Vidensk. Vorh. 1883 or *Geom. d. Berührungstransf.*, pp. 303.

([186]) **Lie**, Christ. Vidensk. Vorh. 1883 or *Geom. d. Berührungstransf.*, pp. 231. **Mehmke** employed this theorem for the investigation of the torsion of third-order space curves (Mitt. d. math.-naturw. Vereins in Württemberg, IV, 1891).

([187]) Comptes R. **124** (May 1897), pp. 1077; the corresponding theorem on pp. 308 of *Geom. d. Berührungstrans.* is incorrect.

([188]) **Voss** (Math. Ann., Bd. 23) has developed the differential geometry of the Pfaff equations (viz., the “point-plane system”).

([189]) Math. Ann., **52** (1899).

of variations, where it represents the simplest intuitive case in which an auxiliary condition appears in the form of a non-integrable total differential equation.

Picard ([190]) investigated the ruled surfaces (together with their principal lines) and curves in a ray thread: He found, among other things, that the oscillation points of such a curve are at the same time inflection points ([191]) and determined all of the ruled surfaces with algebraic principal tangent points ([192]) that are contained in a thread. **Steinmetz** ([193]) also treated the algebraic curves in a null system. **Lie** ([194]) determined the curves in a tetrahedral complex.

18. Lie's transformation ([195]). If one lets the equation of a line be:

$$r\,z = x - \rho, \qquad s\,z = y - \sigma,$$

and that of a sphere be:

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = R^2$$

then one can consider r, s, ρ, σ to be the coordinates of a line, and x', y', z', R to be the coordinates of the sphere. When one sets the quantities in the one quadruple equal to any functions of the other ones, one will obtain a map of the sphere manifold to the line space. In order to obtain a distinguished map, **Lie** ([196]) set:

$$x' = \rho + s, \qquad i\,y' = \rho - s, \qquad z' = \sigma - r, \qquad \pm R = \sigma + r;$$

thus:

$$\rho = \tfrac{1}{2}(x' + i\,y'), \qquad s = \tfrac{1}{2}(x' - i\,y'), \qquad \sigma = \tfrac{1}{2}(z' \pm R), \qquad r = -\tfrac{1}{2}(z' \mp R).$$

Any line corresponds to a sphere, while any sphere corresponds to a line. The lines of a pencil of rays correspond to spheres that contact at a point ([197]), so a surface element will again correspond to another one. A surface f corresponds to another one F, and the principal tangents to f will correspond to those of the corresponding surface element that contacts the spheres that have the principal radii of curvature as their radii (viz., the "principal spheres") ([198]). A ruled surface corresponds to a spherical envelope (viz., a *tubular surface*), and in particular, a principal tangent curve of f and the associated developable surface of such a tubular surface that contacts F along a line of curvature. One can then say that the principal tangent curves get mapped to the lines of curvature.

([190]) Thèse, 1877; cf., also **Voss**, Math. Ann., Bd. 12 and Lie, *ibid.*, Bd. 5, pp. 179.
([191]) *Loc. cit.*, pp. 7.
([192]) *Loc. cit.*, pp. 30.
([193]) Am. J. of Math. **14** (1892).
([194]) *Geom. d. Berührungstrans.*, pp. 326.
([195]) Cf., also **Klein**, *Vorl. über höhere Geom.*; **Lie** and **Scheffers**, *Geom. d. Berührungstrans.*, chap. 10; **Jessop**, *Line Complex*, chap. XIII.
([196]) Math. Ann. V, pp. 171.
([197]) *Loc. cit.*, pp. 172.
([198]) *Loc. cit.*, pp. 177.

This transformation was applied to the investigation of principal tangent curves of the Kummer surface ([199]), and for the investigation of surfaces with spherical lines of curvature ([200]). It was generalized by **Duporcq** ([201]) and **Bricard** ([202]). One can, in fact, associate the lines with second-order surfaces that circumscribe such a fixed surface (instead of the spherical circle at infinity). **Lie**'s transformation is a contact transformation; **Goursat** ([203]) found another one for which normal congruences go to other ones.

([199]) **Lie**, *loc. cit.*, pp. 178; **Klein**, Gött. Nachr., 1871.
([200]) **Lagally**, Diss. München, 1903.
([201]) Bull. de la Soc. math. de France, t. 27.
([202]) Nuov. Ann. (4) **5** (1905).
([203]) Comptes R., t. 129.