THE GREAT PROBLEMS OF SCIENCE

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# The Thermodynamics of the isolated particle

## (or the hidden thermodynamics of particles)

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# PREFACE

*This work is, in its essentials, the editing of the last course that I taught at l'Institut Henri-Poincaré during the winter of 1961-1962 before going on retreat.*

*For about twelve years, I have reprised an attempt to interpret wave mechanics that I proposed without success in 1926-1927 some time after my doctoral thesis under the names of the theory of the pilot wave and then the theory of the double solution. Some prolonged reflections on the subject have now led me to affirm that the current interpretation that is assumed by quantum mechanics does not truly afford a reasonable explanation for certain essential and incontestable experimental facts* ([1])*, and that, as a result, it must be revised by reestablishing the constant localization of the corpuscle in space in the course of time by endowing the wave that accompanies it with the character of physical reality and postulating the existence of an appropriate connection between the wave and the corpuscle.*

*I first reprise my old attempt at reinterpretation in the form that I gave it on another occasion, while meanwhile introducing a certain number of important complements. However, in all of these latter years, I have increasingly been led to think that the hydrodynamical form of that reinterpretation, while being a necessary basis for departure, must be completed by some considerations of a statistical order. Now, in 1946-1948, before having reprised my researches on the reinterpretation of wave mechanics, I studied the old theories of Helmholtz and Boltzmann, which tend to establish a correspondence between mechanical quantities and thermodynamic quantities, and I believe that I see in it the start of the thermodynamics of an isolated particle. Quite recently, following the publication of a paper by Terletsky, I had the idea of trying to utilize the Bohm-Vigier hypothesis of a sub-quantum medium, by conceiving of it as a sort of hidden thermostat, in order to construct the thermodynamics of an isolated particle. The object of the present book is to present that attempt.*

*The first five chapters of the work recall some results that are well-known, but I have insisted on certain points, either because they have been occasionally misinterpreted or because they are very important in what follows. The essential chapters are the last four (chap. VI, VII, VIII, and IX), in which the concepts of random perturbations and statistical thermodynamics are progressively introduced in the context of the hydrodynamical picture, which presents the theory of the double solution in its original form, and which leads to the thermodynamics of the isolated particle and to a theory of fluctuations of the motion of the particle in its wave.*

*I believe that one thus arrives at a quite remarkable and promising form for the reinterpretation of wave mechanics that I believe is necessary. I then emphasize quite strongly that a larger number of young researchers should indeed take an interest in that attempt, because it seems to me that it is along that path that one will achieve the greatest future progress in quantum physics.*

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([1]) *See*, for example, the bibliography [**2**], [**3**], and [**4**].

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CHAPTER I

# REVIEW OF THE PRINCIPLES OF CLASSICAL MECHANICS

**1. Hamilton's principle of stationary action.** – One knows that all of classical mechanics – at least, when the forces are derived from a potential (we leave aside the case in which there is a vector potential, which we shall return to) – can be explained by a general principle of stationary action. In order to state that principle, one introduces a function of the coordinates of $N$ material points, the components of their velocities, and possibly time, namely, the *Lagrange function* $\mathcal{L}(x_1, \ldots, z_N;\ \dot{x}_1, \ldots, \dot{x}_N, t)$, where the dot indicates a derivative with respect to time. There might or might not be constraints, on the condition that they be holonomic, so one can express the coordinates with the aid of $n$ parameters $q_k$ ; if there is no constraint then $n = 3N$, and if there are constraints then $n < 3N$. However, in any event, the Lagrange function is of the form $\mathcal{L}(q_1, \ldots, q_n;\ \dot{q}_1, \ldots, \dot{q}_n, t)$.

In non-relativistic classical mechanics, one gives a precise form to the Lagrange function by setting:

$$\mathcal{L} = T - U, \tag{1}$$

where $T$ is the global kinetic energy and $U$ is the global potential energy of the system, both of which are expressed with the aid of the variables $q_1, \ldots, q_n;\ \dot{q}_1, \ldots, \dot{q}_n, t$.

One can then reduce all of dynamics to the following principle: *If the system starts with a certain configuration that is defined by the values* $q_1^{(0)}, \ldots, q_n^{(0)}$ *of q at the instant* $t_0$ *in order to arrive at another configuration* $q_1^{(1)}, \ldots, q_n^{(1)}$ *at the instant* $t_1$ *then the equations of motion are such that the integral* $\int_{t_0}^{t_1} \mathcal{L}\, dt$ *is stationary for an infinitely small variation of the motion between the initial and final state.* This is Hamilton's principle of stationary action.

One can make this statement more precise by introducing the notion of configuration space. Each configuration of the system is defined by the set of values of the $n$ coordinates $q_1, \ldots, q_n$, and can be, in turn, represented by a point in an $n$-dimensional space in which each point is framed by $n$ coordinates $q_1, \ldots, q_n$. The instantaneous state of the system is thus found to be represented by a point of the configuration space, so that a figurative point starts at a point $A$ at the instant $t_0$ in order to arrive at a point $B$ at the instant $t_1$, after having described a certain trajectory in configuration space. The trajectory of the figurative point is therefore defined by $n$ functions of time $q_1(t), q_2(t), \ldots, q_n(t)$ that define the motion of the system completely. The function $\mathcal{L}(q_1, \ldots, q_n;\ \dot{q}_1, \ldots, \dot{q}_n, t)$ has a well-defined value at each point on the curve $C$, and the curvilinear integral $A = \int_{t_0}^{t_1} \mathcal{L}\, dt$ has a well-defined sense. That integral, which has the physical dimensions

$ML^2T^{-1}$ of energy multiplied by time (or quantity of motion multiplied by length), is called the *action integral*, or more precisely, *the Hamiltonian action integral*.

If the form of the curve $C$ varies infinitely little while fixing its extremities, as well as the instants $t_0$ and $t_1$ , then one will have:

$$(2) \qquad \delta A = \delta \int_{t_0}^{t_1} \mathcal{L}\, dt = \int_{t_0}^{t_1} \delta\mathcal{L}\, dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} \left( \frac{\partial\mathcal{L}}{\partial q_i} \delta q_i + \frac{\partial\mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt ,$$

and since:

$$\delta\dot{q}_i = \delta \frac{dq_i}{dt} = \frac{d}{dt} \delta q_i ,$$

by an integration by parts, one gets:

$$(3) \qquad \delta \int_{t_0}^{t_1} \mathcal{L}\, dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} \left[ \frac{\partial\mathcal{L}}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial\mathcal{L}}{\partial \dot{q}_i} \right) \right] \delta q_i \, dt ,$$

since the $\delta q_i$ are zero at the two extremities of the curve $C$. If the $\int_{t_0}^{t_1} \mathcal{L}\, dt$ is stationary then the right-hand side of equation (3) must be zero, no matter what the $\delta q_i$ are. One then has:

$$(4) \qquad \frac{d}{dt} \left( \frac{\partial\mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial\mathcal{L}}{\partial q_i} \qquad (i = 1, 2, \ldots, n).$$

These are the celebrated "Lagrange equations," in the form that is valid when the forces are derived from a potential and the constraints are holonomic. One then sees that these equations are consequences of Hamilton's principle of stationary action, and that they thus appear to us as the key to the treasures of classical analytical dynamics.

**2. Lagrange momenta. Conservation theorems.** The configuration variables $q_i$ are often called the Lagrange "coordinates." The $\dot{q}_i$ are the corresponding "generalized velocities" that define the motion of the system. If the material points of this system are not subject to any constraints, and if one utilizes rectangular, Cartesian coordinates then the $q_i$ and the $\dot{q}_i$ are the coordinates and components of velocity in the usual sense of the words.

Instead of employing the $\dot{q}_i$, one can employ some quantities $p_i$ that are called "Lagrange momenta," and which are defined by the relations:

$$(5) \qquad p_i = \frac{\partial\mathcal{L}}{\partial \dot{q}_i} \qquad (i = 1, 2, \ldots, n).$$

Equations (5) permit one to express the $\dot{q}_i$ with the aid of the $p_i$ . The variable $p_i$ is called "canonically conjugate" to the variable $q_i$ . If there are constraints, and if one employs rectangular coordinates then one can set:

$$\mathcal{L} = \tfrac{1}{2}\sum_k m_k(\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2) - U(x_1, \dots, z_N, t) \tag{6}$$

so:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_k} = p_{x_k} = \frac{\partial T}{\partial \dot{x}_k} = m_k \dot{x}_k . \tag{7}$$

The quantity $p_{x_k}$ that is canonically conjugate to $x_k$ is then equal to the $x$-component of the quantity of motion of the $k^{\text{th}}$ material point.

In the general case, the Lagrange equations can be written:

$$\frac{dp_k}{dt} = \frac{\partial \mathcal{L}}{\partial q_k} \qquad (k = 1, 2, \dots, n). \tag{8}$$

Therefore, if $\mathcal{L}$ is independent of $q_k$ then $p_k$ will remain constant in the course of motion. In particular, in the case of the absence of constraints and the use of rectangular coordinates, if $U(x_1, \dots, z_N, t)$ does not depend upon one of the variables – say, $x_k$ – then one has $\partial\mathcal{L} / \partial x_k = 0$, and in turn, $p_{x_k}$ = const. One concludes from this that if the component along one of the rectangular axes of the force is zero then the component of the quantity of motion of the material point along that axis is constant. This is the theorem of the conservation of the quantity of motion.

Now, consider the quantity $E$ that is defined by:

$$E = \sum_k p_k \dot{q}_k - \mathcal{L} \tag{9}$$

in the general case. We call it the *energy* of the system. Since we assume that $U$ does not depend upon velocity, and one easily sees that $T$ is a homogeneous, quadratic function of the velocities $\dot{q}_i$ if the constraints do not depend upon time, Euler's theorem on homogeneous functions permits us to write:

$$2T = \sum_{i=1}^n \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_{i=1}^n \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_{i=1}^n \dot{q}_i p_i , \tag{10}$$

so:

$$E = \sum_{i=1}^n p_i \dot{q}_i - \mathcal{L} = 2T - (T - U) = T + U; \tag{11}$$

$E$ is then, in fact, the total energy, which is the sum of the kinetic and potential energy. One finds, moreover, that:

$$\frac{dE}{dt} = \sum_{i=1}^{n}(\dot{p}_i\dot{q}_i + p_i\ddot{q}_i) - \sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right) - \frac{\partial \mathcal{L}}{\partial t}. \tag{12}$$

From the Lagrange equation, the first term in the right-hand side compensates for the third one, and from the definition of the $p_i$ , the second term compensates for the fourth one. What remains is:

$$\frac{dE}{dt} = -\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial U}{\partial t}. \tag{13}$$

If the external forces are constant or zero (viz., a conservative or isolated system) then $U$ does not depend upon $t$ and $E$ = const. This is the theorem of the conservation of energy.

**3. Maupertuis's principle of least action.** – The definition (9) of $E$ permits us to write:

$$dA = \mathcal{L}\,dt = \sum_{i=1}^{n} p_i dq_i - E\,dt. \tag{14}$$

Now, imagine a configuration space-time by adjoining a time dimension to that configuration space. Let $P$ be the point of that space that represents the initial configuration at the initial instant $t_0$ , and let $Q$ be the point that represents the final configuration at the instant $t_1$ . The Hamiltonian action integral is written:

$$A = \int_{t_0}^{t_1} \mathcal{L}\,dt = \int_P^Q\left(\sum_i p_i dq_i - E\,dt\right). \tag{15}$$

This is a curvilinear integral that is taken in configuration space-time along the line that represents the motion of the system between $t_0$ and $t_1$ . Hamilton's principle is then written:

$$\delta A = \delta\int_{t_0}^{t_1} \mathcal{L}\,dt = \delta\int_P^Q\left(\sum_i p_i dq_i - E\,dt\right) = 0, \tag{16}$$

where the $P$ and $Q$ are now fixed under the variation.

From the preceding statement of the principle of stationary action, one can deduce another analogous principle *in the particular case of a field that is constant in the course of time:* viz., Maupertuis's principle of least action.

In the case of constant fields, the energy of the system is a constant; i.e., a first integral. If $A$ and $B$ are the endpoints of the trajectory in configuration space that correspond to $t_0$ and $t_1$ then it is easy to see that one cannot make that trajectory vary while keeping $A$, $B$, $t_0$ , and $t_1$ fixed if the total energy is to remain constant during the variation. One easily sees this in the simple case of a free material point: The trajectory is then a straight line, and if one varies the form of the trajectory while keeping its extremities $A$ and $B$ fixed then one forcibly lengthens it, from the definition itself of the straight line and velocity, and in turn, the energy cannot remain constant if $t_0$ and $t_1$

remain fixed. This is the reason why one cannot deduce Maupertuis's principle directly, where one performs a variation with constant energy, from Hamilton's principle, where one performs a variation with constant $t_0$ and $t_1$ . In order to make that deduction, one must pass through the intermediary of a formula that is often called "the principle of varied action."

In order to find the formula in question, start with the expression (15) for the Hamiltonian action, but suppose that one varies the endpoints $P$ and $Q$, which amounts to varying not only the endpoints $A$ and $B$ in configuration space, but also the end times $t_0$ and $t_1$ . One then obtains the desired formula:

$$\delta A = \delta \int_{t_0}^{t_1} \mathcal{L}\,dt = \int_{t_0}^{t_1} \delta\mathcal{L}\,dt + \left[\sum_{k=1}^{n} p_k \delta q_k - E\,\delta t\right]_0^1 . \tag{17}$$

The integral in the last expression represents the variation of the Hamiltonian action integral that is due to the variation of motion when $A$, $B$, $t_0$ and $t_1$ remain fixed; from Hamilton's principle, it is zero. The bracket represents the variation of action that corresponds to the variation of the points $P$ and $Q$ in configuration space-time, and one has:

$$\delta A = \left[\sum_{k=1}^{n} p_k \delta q_k - E\,\delta t\right]_0^1 . \tag{18}$$

Now, return to configuration space, properly speaking. One can define the integral

$$\mathcal{A} = \int_A^B \sum_{k=1}^{n} p_k\,dq_k \tag{19}$$

in it. This is Maupertuis's action integral. It is taken in configuration space from the point $A$, which represents the initial configuration, up to the point $B$, which represents the final configuration.

In the case of conservative or isolated systems (i.e., constant or zero external actions, resp.), the total energy $E$ of the system is constant, and the integral (19) is independent of time. We have:

$$\int_{t_0}^{t_1} \mathcal{L}\,dt = \int_P^Q \left(\sum_{k=1}^{n} p_k \delta q_k - E\,\delta t\right) = \mathcal{A} - \int_{t_0}^{t_1} E\,dt , \tag{20}$$

so:

$$\delta \int_{t_0}^{t_1} \mathcal{L}\,dt = \delta\mathcal{A} - \int_{t_0}^{t_1} \delta E\,dt - \left|E\,\delta t\right|_{t_0}^{t_1} . \tag{21}$$

Hence, upon replacing the left-hand side with the value that is given by (18), one will get:

$$\delta\mathcal{A} = \left|\sum_{k=1}^{n} p_k \delta q_k\right|_0^1 + \int_{t_0}^{t_1} \delta E\,dt . \tag{22}$$

Now, suppose that one keeps the points *A* and *B* in configuration space fixed during the variation, as well as the value *E* of energy. One then gets:

$$\delta\mathcal{A} = 0. \tag{23}$$

This is Maupertuis's least-action principle, where the variation must be performed while keeping the extreme configurations fixed, along with the value of energy, but not the extreme time points $t_0$ and $t_1$ .

In the particular case where the $q_i$ are the 3*N* Cartesian coordinates of *N* material points of a system that is not subject to any constraints, one has:

$$\mathcal{A} = \int_A^B \sum_{k=1}^{3N} p_k\,dq_k = \int_A^B \sum_{k=1}^{3N} m_k\,(v_{x_k}\,dx_k + v_{y_k}\,dy_k + v_{z_k}\,dz_k)\,, \tag{24}$$

and for just one material point:

$$\mathcal{A} = \int_A^B m(v_x dx + v_y dy + v_z dz) = \int_A^B m\mathbf{v}\cdot d\mathbf{s}\,, \tag{25}$$

in which the integral is then taken from *A* to *B* along the trajectory in three-dimensional physical space.

**4. Hamilton's equations.** – We can take the variables that define the motion of a system of *n* Lagrange variables $q_i$ and corresponding momenta $p_i = \partial\mathcal{L}/\partial\dot{q}_i$ , which form a system of "canonical" variables. We can then express the generalized velocities $\dot{q}_i$ as functions of the $q_i$ , the $p_i$ , and possibly time, by relations of the form:

$$\dot{q}_i = f_i(q, p, t) \qquad (i = 1, 2, \ldots, n). \tag{26}$$

The energy *E* will be expressed as a function of the same variables by means of the "Hamiltonian function" $H(q, p, t)$, in such a way that:

$$E = \sum_{i=1}^{n} p_i\dot{q}_i - \mathcal{L}(q, \dot{q}, t) = H(q, p, t), \tag{27}$$

in which the $\dot{q}_i$ are expressed in the right-hand side as functions of the *q*, the *p*, and *t*. One will then have:

$$\left\{\begin{aligned} \frac{\partial H}{\partial p_k} &= \dot{q}_k + \sum_{i=1}^{n} p_i\frac{\partial\dot{q}_i}{\partial p_k} - \sum_{i=1}^{n}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{\partial\dot{q}_i}{\partial p_k} = \dot{q}_k\,, \\ \frac{\partial H}{\partial q_k} &= \sum_{i=1}^{n} p_i\frac{\partial\dot{q}_i}{\partial p_k} - \frac{\partial\mathcal{L}}{\partial q_k} - \sum_{i=1}^{n}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{\partial\dot{q}_i}{\partial q_k} = -\frac{\partial\mathcal{L}}{\partial q_k} = -\dot{p}_k\,, \end{aligned}\right. \tag{28}$$

from the definition of $p_i$ and the Lagrange equations. One has thus obtained the celebrated system of Hamilton's equations:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad (k = 1, 2, \ldots, n), \tag{29}$$

and one easily infers that:

$$\frac{dH}{dt} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_k}\dot{q}_i + \frac{\partial H}{\partial p_k}\dot{p}_i\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}, \tag{30}$$

so if $U$, and in turn, $H$, do not depend upon time explicitly then $H$ = const, which is the theorem of the conservation of energy.

**5. Classical mechanics and relativistic mechanics.** – We just recalled several points of classical analytical mechanics. The introduction of the principle of relativity by Einstein in 1905 led to a modification of the formulas of classical mechanics. We shall not recall the well-known principles of the special theory of relativity here. In the following chapter, we shall restrict ourselves to summarizing the principles of the relativistic dynamics of a material point, while especially insisting upon the principle of the energy of inertia, which will play a very important role in all of what follows.

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CHAPTER II

# RELATIVISTIC DYNAMICS

**1. General formulas.** – Newtonian mechanics admits the group of Galilean transformations with absolute time – i.e., its equations keep their form when one makes a Galilean transformation – but it does not admit the group of Lorentz transformations, which preserve the equations of electromagnetism, and in particular, those of the propagation of light in vacuo. However, since the transformation formulas of the two groups differ only by terms of order $\beta^2 = v^2 / c^2$, the divergence is very weak for the motions considered in classical rational mechanics, which consists of stellar motions. The Lorentz transformation is found to be confirmed by the identity of interference phenomena in all Galilean reference systems, since the interference phenomena can be observed with infinitely more precision than mechanical phenomena, so it is natural to suppose that the principle of relativity applies to all natural phenomena and that the equations of classical mechanics are not rigorously exact, so they must be modified in such a fashion as to become invariant under the Lorentz transformation.

The essential condition that the relativistic dynamics of a material point must then satisfy, *a priori*, is obviously that of agreeing with classical dynamics whenever $\beta^2$ is negligible compared to unity, because one must necessarily recover the old dynamics as a first approximation in the case of weak velocities with respect to the velocity $c$ of light in vacuo (an approximation that is called "Newtonian"). One is then led to define the basis for relativistic dynamics as a principle of stationary action that reduces to the usual Hamilton principle when one can neglect $\beta^2$ compared to unity. In order to do this, one must adopt a function $\mathcal{L}$ of the coordinates and velocities of the material point such that the equations of relativistic dynamics are all derived from the equation:

$$\delta A = \delta \int_{t_0}^{t_1} \mathcal{L} \, dt = 0, \tag{1}$$

in which the variation is performed while keeping the initial and final positions of the material point and its values at the instants $t_0$ and $t_1$ fixed, and in the classical Hamilton principle. The usual calculation that permits one to pass from Hamilton's principle to the Lagrange equations is applied here, and gives:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}, \ldots, \tag{2}$$

or, upon setting:

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}}, \ldots, \tag{3}$$

$$\frac{dp_x}{dt} = \frac{\partial \mathcal{L}}{\partial x}, \ldots \tag{4}$$

Now, in classical mechanics, one sets:

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z, t), \tag{5}$$

however, here we must choose the function $\mathcal{L}$ differently.

Ordinarily, in the theory of relativity, one represents each "event," which is defined by the set of its four coordinates $x, y, z, t$, by a point in four-dimensional space-time. The motion of a material point is then represented by a continuous sequence of event-points that form what one calls the "world line" of the material point in space-time. When one passes from one Galilean reference system to another, the coordinates of each point of that world line vary, since one must perform a Lorentz transformation on these coordinates. Meanwhile, there exists an invariant quantity that is attached to each element of the world-line. Indeed, let $dx, dy, dz, dt$ be variations of the coordinates in a Galilean system that correspond to a small element of the world-line. Consider the quantity:

$$ds = \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2} = \sqrt{c^2dt^2 - dl^2}\ , \tag{6}$$

$dl$ being the element of the trajectory that is described by the material point in the time $dt$. One of the essential properties of the Lorentz transformation is that the quantity (6) is an invariant of that transformation.

Since $v = \beta c = dl / dt$ is the velocity of the material point, one can write:

$$ds = c\ dt\ \sqrt{1-\beta^2} = c\ d\tau, \tag{7}$$

where $d\tau = dt\ \sqrt{1-\beta^2}$ is the element of proper time of the material point that corresponds to the element $ds$ of its world-line, so $d\tau$ is the time interval that is indicated by a clock that is carried along by the motion of the material point while it displaces by $dl$ during the time $dt$. Formula (7) expresses the "slowing-down of clocks."

In relativistic dynamics, one considers the presence of a material point at the instants $t_0$ and $t_1$ and at the points $x_0$, $y_0$, $z_0$ and $x_1$, $y_1$, $z_1$ in space as defining two event-points in space-time $P(x_0, y_0, z_0, t_0)$ and $Q(x_1, y_1, z_1, t_1)$, and one seeks a principle of stationary action of the form (1), where the integral is taken along the world-line from $P$ to $Q$, and the variation does not affect the points $P$ and $Q$.

One obtains a satisfactory form of the Lagrange function $\mathcal{L}$ by setting:

$$\mathcal{L} = - m_0\ c^2\ \sqrt{1-\beta^2} \tag{8}$$

for a free material point, where $m_0$ is a constant that is called the "proper mass," and which characterizes the material point envisioned. That formula permits one to write:

$$A = \int_{t_0}^{t_1} \mathcal{L}\, dt = - m_0\ c^2 \int_0^1 \sqrt{1-\beta^2}\ dt = - \int_P^Q m_0 c\, ds\ , \tag{9}$$

and the invariance of the action integral $A$ becomes obvious, which is satisfactory. Moreover, if $\beta^2$ is small compared to unity then one will have:

$$\mathcal{L} = -\, m_0\, c^2 + \tfrac{1}{2} m_0\, v^2, \tag{10}$$

and since the constant term has no importance, since it contributes nothing to the variation, we come down to the classical expression $\mathcal{L} = T = \frac{1}{2} m v^2$ for the Lagrange function for a free material point, as we must.

If a material point is subject to a field that is derivable from a potential $U$ then, for the moment, we content ourselves with adding the term $-\,U$ in $\mathcal{L}$ to the "kinetic" term $-\, m_0\, c^2 \sqrt{1-\beta^2}$ , by analogy with classical mechanics, and we write the principle of stationary action in the form:

$$\delta A = \delta \int_P^Q \left(-\, m_0\, c^2 \sqrt{1-\beta^2} - U\right) dt = 0, \tag{11}$$

which gives the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} \tag{12}$$

by the classical argument.

It is easy to calculate the Lagrange momenta $p_x$ , $p_y$ , $p_z$ ; one finds:

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{m_0 \dot{x}}{\sqrt{1-\beta^2}}, \ \ldots \tag{13}$$

If one then defines the "impulse" – or quantity of motion – vector by:

$$\mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1-\beta^2}} \tag{14}$$

then one will have:

$$\frac{d\mathbf{p}}{dt} = -\ \mathrm{grad}\ U. \tag{15}$$

One can, moreover, write $\mathbf{p} = m\mathbf{v}$ by setting $m = \dfrac{m_0}{\sqrt{1-\beta^2}}$; $m$ is called the "mass of motion" of the material point; it increases with the velocity of the point. For an observer that is coupled to the material point, $\beta = 0$ and $m = m_0$ ; the mass $m$ reduces to the proper mass $m_0$ – or "rest mass." When $v$ tends to $c$, $m$ tends to infinity; i.e., the mass of motion increases indefinitely when the velocity approaches $c$. The velocity $c$ is the limiting velocity of all corpuscular motion.

**2. World energy and impulse.** – The general equations that were obtained above permit us to prove, as in classical mechanics, that the quantity:

$$W = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - \mathcal{L} \tag{16}$$

has the total derivative with respect to time:

$$\frac{dW}{dt} = -\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial U}{\partial t}, \tag{17}$$

and that it therefore remains constant if the external field is constant in the course of the motion, and this further leads us to consider $W$ to be the energy of the material point.

In classical mechanics, where $\mathcal{L} = T - U$, we have found that $E = T + U$. Here, since we have set $\mathcal{L} = -m_0 c^2 \sqrt{1-\beta^2} - U$, and we have $\mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1-\beta^2}}$, we find that:

$$W = \frac{m_0c^2}{\sqrt{1-\beta^2}} + U. \tag{18}$$

The total energy of the material point is then the sum of the potential energy $U$ and the term $\frac{m_0c^2}{\sqrt{1-\beta^2}}$ that we must now interpret. For $\beta = 0$ – i.e., for an observer that is coupled to the motion – this term reduces to $m_0c^2$, and represents the proper internal energy of the moving point. For an observer that passes the moving point with the velocity $\beta c$, the kinetic part of the energy $W$ will be $\frac{m_0c^2}{\sqrt{1-\beta^2}} - m_0c^2$, which represents the internal proper energy of the moving point.

Generalizing that result, Einstein arrived at the following statement: *Any mass m is always associated with an amount of energy that equals the product of that mass by the square* $c^2$ *of the velocity of light in vacuo.* Later on, we shall study this principle of the energy of inertia more deeply.

One can call the increase in energy that is due to the motion when the moving point passes from a state of rest to a velocity $\beta c$ its "kinetic energy." One then sets:

$$T = \frac{m_0c^2}{\sqrt{1-\beta^2}} - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right). \tag{19}$$

If $\beta \ll c$ then one sees that $T$ reduces to $\frac{1}{2}m_0v^2$, as it must. Finally, one must set:

$$W = m_0c^2 + T + U = m_0c^2 + E, \tag{20}$$

upon letting $E$ denote the energy $T + U$ that is the sum of the kinetic energy and the potential energy. Therefore, the total energy $W$ of relativistic mechanics is obtained by adding the internal energy $m_0c^2$, which is characteristic of the theory of relativity, to $E$. While $E$ can be positive or negative, $W$ is always positive.

One can define a "world-velocity" quadri-vector at each point of the world-line of a material point, which has the components:

$$(21) \qquad \begin{cases} u_1 = \dfrac{dx}{ds} = \dfrac{dx}{dt}\dfrac{dt}{ds} = \dfrac{v_x}{c\sqrt{1-\beta^2}}, & u_2 = \dfrac{dy}{ds} = \dfrac{v_y}{c\sqrt{1-\beta^2}}, \\ u_3 = \dfrac{dz}{ds} = \dfrac{v_z}{c\sqrt{1-\beta^2}}, & u_4 = \dfrac{d(ct)}{ds} = \dfrac{1}{\sqrt{1-\beta^2}}. \end{cases}$$

Upon multiplying this quadri-vector by the invariant $m_0c^2$, one deduces the "world-impulse" quadri-vector $\mathbf{I} = m_0c^2\,\mathbf{u}$ with the components:

$$(22) \qquad \begin{cases} I_1 = \dfrac{m_0 v_x}{\sqrt{1-\beta^2}}, & I_2 = \dfrac{m_0 v_y}{\sqrt{1-\beta^2}}, \\ I_3 = \dfrac{m_0 v_z}{\sqrt{1-\beta^2}}, & I_4 = \dfrac{m_0 c}{\sqrt{1-\beta^2}} = \dfrac{W}{c}. \end{cases}$$

One sees that the three spatial components of the world-impulse are the components of the quantity of motion, while the temporal component is equal to the energy, divided by $c$ (i.e., an abstraction from the potential energy). The quadri-vector $\mathbf{I}$ thus combines the quantity of motion and energy into a single geometric entity.

The kinetic part of the Hamiltonian action:

$$\int -m_0c^2\sqrt{1-\beta^2}\,dt = -\int m_0 c\, ds,$$

which is an invariant, is therefore the circulation of the quadri-vector $\mathbf{I}$ along the world-line. It is easy to verify that by virtue of equation (16), one can write it in the form:

$$-\int (W\,dt - p_x\,dx - p_y\,dy - p_z\,dz).$$

This permits one to consider the integral as being (up to sign) the scalar product in space-time of the world-impulse quadri-vector with the quadri-vector $d\mathbf{s}$ whose components are $dx$, $dy$, $dz$, $dt$, and therefore to write:

$$(23) \qquad A = \int_P^Q \mathcal{L}\,dt = -\int_P^Q (\mathbf{I}\cdot d\mathbf{s})$$

for the free particle.

**3. Relativistic dynamics of the electron in an electromagnetic field.** – We have taken the relativistic Lagrange function to be:

$$\mathcal{L} = -\, m_0\, c^2 \sqrt{1-\beta^2} - U,$$

but the term $U$ is not satisfactory, since the action must be an invariant, and $U$ is not invariant. We shall examine the question more closely by putting ourselves into the case of a point-like charge that displaces in an electromagnetic field (relativistic dynamics of the electron).

We obtain a satisfactory relativistic form for $\mathcal{L}$ by starting with the following remark: The relativistic study of electromagnetic quantities shows that the scalar potential $V$ and the vector potential $\mathbf{A}$ transform like the variables $x$, $y$, $z$, $t$ under a Lorentz transformation; i.e., they form the components of a space-time quadri-vector whose spatial components are $A_x$, $A_y$, $A_z$, and whose temporal component is $V$. It then results that one obtains an action integral $A = \int \mathcal{L}\, dt$ that is invariant if one sets:

$$\mathcal{L} = -\, m_0\, c^2 \sqrt{1-\beta^2} - \varepsilon V + \frac{\varepsilon}{c}(\mathbf{A} \cdot \mathbf{v}), \tag{24}$$

where $\varepsilon$ is the electric charge of the particle. Indeed, if one lets $\mathcal{P}$ denote the space-time quadrivector of “world-potential,” whose components are $A_x$, $A_y$, $A_z$, $V$ then one easily verifies that the expression for the action is written in the obviously invariant way as:

$$A = -\int m_0 c\, ds - \frac{\varepsilon}{c}\int (\mathcal{P} \cdot d\mathbf{s}), \tag{25}$$

where $\mathcal{P} \cdot d\mathbf{s}$ is the space-time scalar product of the two quadri-vectors $\mathcal{P}$ and $d\mathbf{s}$, which is formed according to the rule:

$$(\mathbf{A} \cdot \mathbf{B}) = A_4 B_4 - A_1 B_1 - A_2 B_2 - A_3 B_3 \,.$$

As always, the Lagrange equations are written:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}, \ \ldots \tag{26}$$

and upon setting $p_x = \partial \mathcal{L} / \partial \dot{x}$, …, they further take the form:

$$\frac{dp_x}{dt} = \frac{\partial \mathcal{L}}{\partial x}, \ \ldots \tag{27}$$

One easily finds:

(28) $$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{m_0 \dot{x}}{\sqrt{1-\beta^2}} + \frac{\varepsilon}{c} A_x, \ldots$$

or vectorially:

(29) $$\mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1-\beta^2}} + \frac{\varepsilon}{c} \mathbf{A}.$$

The vector **p**, whose components are the Lagrange momenta, is therefore the sum of the quantity of motion $\frac{m_0 \mathbf{v}}{\sqrt{1-\beta^2}}$ and a sort of "potential quantity of motion" $\varepsilon / c\ \mathbf{A}$.

If one writes the Lagrange equations (27) explicitly, while taking into account the expressions for the electric field **h** and the magnetic field **H** as functions of the potential $V$ and **A**, then one obtains three vectorial equations of the form:

(30) $$\frac{d}{dt}\left(\frac{m_0 \mathbf{v}}{\sqrt{1-\beta^2}}\right) = \mathbf{f},$$

with:

(31) $$\mathbf{f} = \varepsilon\left[\mathbf{h} + \frac{1}{c}\mathbf{v} \wedge \mathbf{H}\right],$$

and one recognizes that the vector **f** is the Lorentz force that acts upon the charge $\varepsilon$ when it is animated with the velocity **v**. Therefore, the derivative with respect to time of the quantity of motion is equal to the Lorentz force, which permits one to recover the well-known dynamics of the electron. One remarks, moreover, that equation (30) is not equivalent to the Newton relation $m\boldsymbol{\gamma} = \mathbf{f}$, which is due to the variation of the mass $m = \frac{m_0}{\sqrt{1-\beta^2}}$ with velocity.

If one calculates the energy $W$ by the formula (16) then one finds:

(32) $$W = \frac{m_0 c^2}{\sqrt{1-\beta^2}} + \varepsilon V,$$

which is satisfactory.

We finally remark that since the quantities $p_x$, $p_y$, $p_z$, and $W / c$ form the four components of a space-time quadri-vector of world-impulse **I**, which is defined by:

(33) $$\mathbf{I} = m_0 c\ \mathbf{u} + \frac{\varepsilon}{c}\mathcal{P},$$

one will always have:

$$\int_P^Q \mathcal{L}\, dt = -\int_P^Q (\mathbf{I} \cdot d\mathbf{s}).$$

**4. Hamilton's equations.** – Since $V$ depend upon $x, y, z, t$, in general, the energy $W$ is a function of $x, y, z, \dot{x}, \dot{y}, \dot{z}$, and $t$. However, since $\dot{x}, \dot{y}, \dot{z}$ can be expressed as functions of $x, y, z, p_x, p_y, p_z, t$, one can write:

$$W = H(x, y, z, p_x, p_y, p_z, t). \tag{34}$$

Upon eliminating $v_x, v_y, v_z$ between the equations (29) and (32), one finds, after some calculations:

$$H(x, y, z, p_x, p_y, p_z, t) = c\sqrt{m_0^2c^2 + \sum_{xyz}\left(p_x - \frac{\varepsilon}{c}A_x\right)^2} + \varepsilon V. \tag{35}$$

The same reasoning that led us from the Lagrange equations to Hamilton's equations in classical mechanics again gives us:

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x}, \ldots, \qquad \frac{dp_x}{dt} = -\frac{\partial H}{\partial x}, \ldots \tag{36}$$

here. The last three equations are the equations of motion; the first three are easy to verify.

**5. Summary of the principle of the energy of inertia.** – While studying the relativistic dynamics of the material point, we encountered the principle of the energy of inertia, according to which there exists the general relation $W = mc^2$ between energy and mass. However, that relation was proved only for a material point. In his first papers on relativity, Einstein was led to generalize that statement to a set of material points, and then to an arbitrary body.

First, consider a set of material points with no interactions, and refer that set to a Galilean reference system (which we denote by 0), such that:

$$\sum \frac{m_0\mathbf{v}_0}{\sqrt{1-\frac{v_0^2}{c^2}}} = 0, \tag{37}$$

where the sum is taken over all the material points. We say that in the system where the total impulse is zero, the set of material points is globally at rest. This system of reference is analogous to the one that linked to the center of gravity of the particles in classical mechanics; we shall call it the "proper" reference system of the set. In this proper system, we have the expression:

$$W_0 = \sum \frac{m_0c^2}{\sqrt{1-\frac{v_0^2}{c^2}}} \tag{38}$$

for the energy, since the particles are non-interacting.

Now, pass to another reference system (denoted by 1) that is animated with respect to the system 0 with the velocity $v = \beta c$. The global energy of the particles in this system will be:

$$W_1 = \sum \frac{m_0 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}}. \tag{39}$$

Now, upon taking the *z*-axis to be the direction of relative motion of the two reference systems 0 and 1, one will have the usual formulas for the addition of velocities:

$$v_{1x} = \frac{v_{0x}\sqrt{1-\beta^2}}{1+\frac{v v_{0x}}{c^2}}, \qquad v_{1y} = \frac{v_{0y}\sqrt{1-\beta^2}}{1+\frac{v v_{0y}}{c^2}}, \qquad v_{1z} = \frac{v_{0z}\sqrt{1-\beta^2}}{1+\frac{v v_{0z}}{c^2}}, \tag{40}$$

from which, one infers that:

$$\frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}}\frac{1+\frac{v v_{0x}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}}\frac{1+\frac{\beta}{c^2}v_{0x}}{\sqrt{1-\beta^2}}. \tag{41}$$

From this, one deduces:

$$W_1 = \sum \frac{m_0 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}} = \frac{W_0}{\sqrt{1-\beta^2}} + \frac{\beta c}{\sqrt{1-\beta^2}} \sum \frac{m_0 v_{0x}}{\sqrt{1-\frac{v_0^2}{c^2}}}. \tag{42}$$

The last term is zero, by virtue of the definition (37) of the proper system, and what remains is:

$$W_1 = \frac{W_0}{\sqrt{1-\beta^2}}. \tag{43}$$

If the set of particles reduces to just one particle of proper mass $M_0$ then one will have:

$$W_1 = \frac{M_0 c^2}{\sqrt{1-\beta^2}}. \tag{44}$$

One can thus say that the system in collective motion with the velocity $\beta c$ behaves like a unit with a proper mass:

(45) $$M_0 = \frac{W_0}{c^2},$$

a formula that expresses the energy of inertia, here.

Moreover, in the reference system 1, the global quantity of motion **g** of the particles has the *x*-component:

(46) $$g_{1x} = \sum \frac{m_0 v_{1x}}{\sqrt{1-\frac{v_1^2}{c^2}}} = \sum \frac{m_0 v_{0x}}{\sqrt{1-\frac{v_0^2}{c^2}}} = 0,$$

and similarly, $g_{1y} = 0$. For $g_{1z}$, one will have:

(47) $$g_{1z} = \sum \frac{m_0 v_{1x}}{\sqrt{1-\frac{v_1^2}{c^2}}} = \left( \sum \frac{m_0 v_{0x}}{\sqrt{1-\frac{v_0^2}{c^2}}} + \sum \frac{m_0 v}{\sqrt{1-\frac{v_0^2}{c^2}}} \right) \frac{1}{\sqrt{1-\beta^2}}.$$

Since the first term is zero, from the definition of the proper system, one has:

(48) $$g_{1z} = \frac{v}{\sqrt{1-\beta^2}} \sum \frac{m_0}{\sqrt{1-\frac{v_0^2}{c^2}}} = \frac{v}{\sqrt{1-\beta^2}} \frac{W}{c^2} = \frac{M_0 v}{\sqrt{1-\beta^2}},$$

and that formula again shows the energy of inertia.

One can remark that the principle of the energy of inertia determines the value of the constant of energy completely, which is left arbitrary in classical mechanics; indeed, one cannot introduce an additive constant into the expression for energy without disrupting its variance completely.

**6. Various extensions of the principle of the energy of inertia.** – The argument that we just developed shows us that the principle of the energy of inertia is valid for a set of non-interacting particles (i.e., the absence of potential energy). In particular, they prove that if one contributes to the heat of a gas that is assumed to be perfect then its mass must increase. We shall now show that the radiation of an energy *W* must also possess a mass that is equal to $W / c^2$, and that, in turn, a body that radiates loses mass, while a body that absorbs radiation acquires a supplementary mass.

We give a proof that is due to Einstein himself. One knows that in electromagnetic theory, one proves that radiation of energy *W* possesses a quantity of motion that equals $W / c$. Therefore, when a body emits radiation of global energy *W*, it takes on a recoil motion with the quantity of motion $W / c$. With Einstein, consider a hollow cylinder, such as the one in Figure 1.

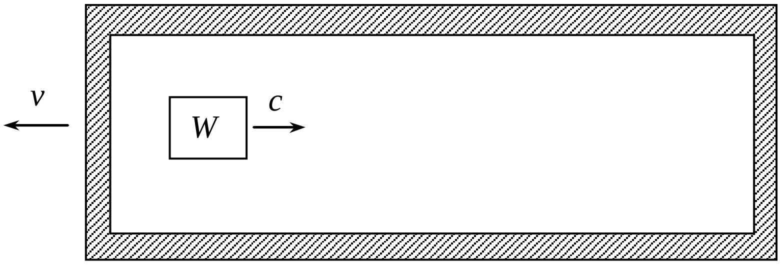

Figure 1.

As a result of that emission, the cylinder must recoil towards the left with a quantity of motion $W / c$. If $M$ is the mass of the cylinder then since its recoil velocity $v$ is small compared to $c$ , its quantity of motion will be $Mv$, and one will have $Mv = W / c$. When the wave train has arrived at the right-hand wall of the cylinder, which we assume to be absorbent, it will be absorbed, and the cylinder will then acquire the quantity of motion $W / c$ to the right, so its motion will stop. However, between the emission and absorption of the wave train, the center of gravity theorem would not be verified if the radiation had no mass. Let $x$ be the global displacement of the cylinder towards the left during the time $t = x/ v = Mcx / W$. During this time, the small wave train will be displaced towards the right by $X = ct = Mc^2x / W$. In order for the center of gravity theorem to be satisfied, one must have that the small wave train has a mass $\mu$ such that:

$$Mx - \mu X = Mx - \mu \frac{Mc^2 x}{W} = 0, \tag{49}$$

so

$$m = \frac{W}{c^2}. \tag{50}$$

This is indeed the expression for the principle of the energy of inertia, and one can, moreover, easily repeat its proof by appealing to the notion of photon.

Another generalization of the principle studied consists of extending to a body that has both kinetic energy and potential energy, and showing that one further has $W = M_0 c^2$. We shall examine how things happen by studying two particularly simple cases.

First, consider a macroscopic body of mass $M_1$ that collides with another body of mass $M_2$ that is originally at rest with a velocity $\beta_1 c$. Suppose that as a result of the collision the two bodies remain coupled and are finally animated with the common velocity $\beta_2 c$ in the same direction as $\beta_1 c$. If we now write the conservation of the quantity of motion and energy in the form:

$$\frac{M_1 c^2}{\sqrt{1-\beta_1^2}} + M_2 c^2 = \frac{(M_1 + M_2)c^2}{\sqrt{1-\beta_2^2}}, \qquad \frac{M_1 \beta_1 c}{\sqrt{1-\beta_1^2}} = \frac{(M_1 + M_2)\beta_2 c}{\sqrt{1-\beta_2^2}}, \tag{51}$$

which appears natural, then we encounter an impossibility, since the two equations in just one unknown $\beta_2$ are incompatible.

However, since the collision is inelastic, it gives off heat, since the single body that is formed from the juxtaposition of the original two bodies is the site of heating, its mass $M'$ is greater than $M_1 + M_2$, in such a way that one must write in place of equations (51):

$$\frac{M_1 c^2}{\sqrt{1-\beta_1^2}} + M_2 c^2 = \frac{M' c^2}{\sqrt{1-\beta_2^2}}, \qquad \frac{M_1 \beta_1 c}{\sqrt{1-\beta_1^2}} = \frac{M' \beta_2 c}{\sqrt{1-\beta_2^2}}, \tag{52}$$

and these two equations in two unknowns $\beta_2$ and $M'$ are soluble. The first one can be written:

$$\frac{M_1 c^2}{\sqrt{1-\beta_1^2}} + M_2 c^2 = M' c^2 + M' c^2 \left[\frac{1}{\sqrt{1-\beta^2}} - 1\right], \tag{53}$$

and this show us that the initial total energy of the two bodies is finally recovered in the form of the energy $M'c^2$ of the two bodies once they collide and their collective kinetic energy. One can further write:

$$M_1 c^2 \left(\frac{1}{\sqrt{1-\beta_1^2}} - 1\right) = [M' - (M_1 + M_2)]\, c^2 + M' c^2 \left(\frac{1}{\sqrt{1-\beta_1^2}} - 1\right), \tag{54}$$

which shows that the initial kinetic energy of the body 1 has served, on the one hand, to communicate to the ensemble the quantity of heat:

$$Q = [M' - (M_1 + M_2)]\, c^2, \tag{55}$$

which has taken the internal energy of the ensemble from the value $(M_1 + M_2)c^2$ to the value $M'c^2$, and on the other hand, to provide the kinetic energy of the final composite body. All of this is very clear.

As another simple example, consider a body that absorbs radiation. This can be a macroscopic body that absorbs a train of waves or a system of atomic levels that absorbs a photon. Let $M_0$ be the initial proper mass of the body, which is assumed to be at rest initially, let $W$ be the energy of the absorbed radiation (or photon), and let $\beta c$ be the recoil velocity that the body has after absorption.

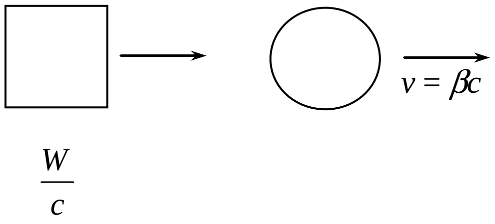

Figure 2.

Once more, if one writes:

(56) $$M_0c^2 + W = \frac{M_0c^2}{\sqrt{1-\beta^2}}, \qquad \frac{M_0\beta c}{\sqrt{1-\beta^2}} = \frac{W}{c}$$

here then one will encounter an impossibility. One must then write:

(57) $$M_0c^2 + W = \frac{M_0'c^2}{\sqrt{1-\beta^2}}, \qquad \frac{M_0'\beta c}{\sqrt{1-\beta^2}} = \frac{W}{c},$$

while assuming that the absorption of radiation increases the proper mass of the body.

The first equation (57) can be written:

(58) $$W = (M_0' - M_0)c^2 + M_0'c^2\left[\frac{1}{\sqrt{1-\beta^2}} - 1\right].$$

This then shows that the energy contributed to the body by the radiation serves:

1. To augment the internal energy of the body by the quantity $(M_0' - M_0)c^2$ (which is heat, in the case of a macroscopic body and internal energy in the case of microphysical entity).
2. To give the body its final energy, so its mass has therefore increased.

The inverse problem of the emission of radiation (or a photon) by a macroscopic or microscopic body is treated similarly, and one arrives at analogous conclusions.

Therefore, the principle of the energy of inertia seems to indeed have a completely general significance. One knows that the exactitude of that principle is confirmed completely by the essential role that it plays in nuclear physics in order to establish the balance of energy in nuclear reactions.

**7. Important remark.** – One of the fundamental ideas that results from the considerations that we just presented is the following one: *If a body receives energy that does not transform into kinetic energy then its mass will increase, and if it loses energy that does not come from its kinetic energy then its mass will diminish.* In other words – and this is the essential point – a variation of the proper mass of a body corresponds to the energy that is received or lost inside of the body in the form of hidden energy, and which, because it is not externally manifested, can be considered to be internal heat. We shall recover this idea, but developed in a more precise fashion, when we study relativistic thermodynamics, and it is upon applying it to particles that we will then arrive at an outline for the thermodynamics of an isolated particle.

CHAPTER III

# NOTIONS FROM STATISTICAL THERMODYNAMICS

**1. Introduction.** – We have seen the importance of the quantity of action in classical and relativistic mechanics. That importance has been underscored by the development of the theory of quanta, which, since the beginning, was led to write that the Maupertuisian action integral $\int \mathbf{p} \cdot d\mathbf{l}$ over an entire period of motion must be equal to an integer multiple of Planck's constant in order to quantize the periodic motions of a corpuscle at the atomic level.

The theory of relativity attaches importance to the Hamiltonian action due to the fact that that quantity is invariant. It is the fundamental invariant of mechanics, just as entropy is, as we know, the fundamental invariant of thermodynamics. That remark prepares us to discover some curious analogies between action and entropy.

However, before making that analysis precise, we must now, after having recalled the principles of mechanics, also study certain aspects of thermodynamics. In what follows, I will suppose that classical thermodynamics is known "in principle," and I will attempt to underscore only the broad ideas of the statistical interpretation of thermodynamics.

**2. Basis for the statistical interpretation of thermodynamics.** – Statistical mechanics, which was first developed by Clausius and Maxwell, and then more completely by Boltzmann and Gibbs, permits one to study the statistical means of very complex systems that are defined by an extremely large number of parameters. Its great success has been to arrive at an interpretation for the laws of thermodynamics in such a fashion that these laws seem to be derived from the fact that thermodynamics always envisions global mean properties of very complex systems whose detailed description can be effected only with the aid of an enormous number of parameters. For example, thermodynamics treats global properties of gases, and in the eyes of atomic physics, a gas is composed of an immense number of molecules or atoms, where the state of each of these elements would be described with the aid of several parameters. The thermodynamic laws of gases are then considered by statistical mechanics to be the global observable result of uncoordinated motions of molecules.

We shall make some of the concepts that are at the basis for the theory of Boltzmann and Gibbs more precise. In classical statistical mechanics, one assumes that the immense number of elements is composed of material bodies that obey the laws of classical mechanics, in such a way that if one knows the positions and velocities of all of these elements precisely at a given moment then one can, in principle, calculate all of their ultimate history rigorously. However, in practice, one cannot observe the evolution of all of the molecules, and one observes only mean statistical effects, so statistical mechanics proposes to determine the laws that govern these effects. Therefore, in that classical theory, one assumes – at least, in principle – the existence of a subordinate determinism, so the "probabilistic" character of the effects thus obtained proves uniquely the impossibility of observing anything but those global effects. The introduction of relativistic dynamics in place of that of Newton changes nothing essential in the foregoing, since that dynamics preserves the fundamental concepts of classical dynamics.

**3. Extension-in-phase and Liouville's theorem.** – In order to develop classical statistical mechanics, we envision a very complex system whose configuration is defined by coordinates $q_1, q_2, \ldots, q_N$ , where the number of them will generally be regarded as very large.

We suppose that our system obeys the laws of classical mechanics, when expressed in the form of Hamilton's equations. If the energy of the system is given by the function $H(q_1, \ldots, q_N ; p_1, \ldots, p_N , t)$ then one can write the canonical equations as:

$$(1) \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2, \ldots, N),$$

where $N$ is the number of coordinates $q_i$ (i.e., the number of degrees of freedom). In the case where the system is isolated or subject to external actions that are independent of time, $H$ is constant.

Since the state of the system is defined entirely by the knowledge of the $q_i$ and $p_i$, one can represent that state by a point in a $2N$-dimensional space that is formed with the aid of the $q_i$ and $p_i$ ; Gibbs gave that space the name of "extension-in-phase." In the course of time, the figurative point of the system describes a trajectory in the extension-in-phase.

Now, imagine very small variations $dq_1, \ldots, dq_N ; dp_1, \ldots, dp_N$ of the coordinates and momenta that start with given values. A small volume element $d\tau$ corresponds to these variations in the extension-in-phase. That volume element possesses two properties that confer great importance upon it. The first of these properties is the following one: *If one performs a change of canonical variables that makes the variables* $q_1, \ldots, q_N ; p_1, \ldots, p_N$ *for which Hamilton's equations* (1) *are verified pass over to the new variables* $Q_1, \ldots, Q_N; P_1, \ldots, P_N$ *for which Hamilton's equations again verified (i.e., a change of canonical variables) then the value of the element* $d\tau$ *remains the same.* I will not give the proof of that theorem here, which results from the manner by which one defines the conjugate variables $p_i$ and $q_i$ . It shows that the volume element $d\tau$ possesses an intrinsic significance that is independent of the choice of canonical variables that serve to define the system.

The second property of the element $d\tau$ is more important for what follows, and it is expressed by "Liouville's theorem." In order to state that theorem, we shall no longer consider just one exemplar of our system, but a great number of different exemplars of that same system. At the instant $t$, each of these exemplars will be represented by a certain point in the extension-in-phase, and there will be a certain number of representative points inside of the element $d\tau$ that is of interest to us. We now fix our attention on the representative points that are thus found inside the element $d\tau$ at the instant $t$. In the course of time, these points will displace, and at the final instant $t'$ one will recover these same representative points by continuity inside of another element $d\tau'$ in the extension-in-phase, an element that will contain all of the representative points, and only them. Liouville's theorem then tells us that $d\tau'$ is equal to $d\tau$. Naturally, $d\tau'$ can have a very different form from $d\tau$ – for example, it can be a long strip that is folded upon itself, while $d\tau$ is a small cube – but the *volumes* of the two elements are the same.

One can prove Liouville's theorem by comparing the motion of representative points in the extension-in-phase to the motion of the molecules of a fluid in a 2*N*-dimensional space. In effect, from that viewpoint, the theorem expresses the idea that a given number of fluid molecules always occupy the same volume in the 2*N*-dimensional space; i.e., the fluid behaves like an incompressible fluid. Now, the incompressibility condition for a fluid is that the divergence of its velocity must be zero at every point. Here, the velocity of the fluid in the 2*N*-dimensional space has 2*N* components that are $\dot{q}_1$, …, $\dot{q}_N$ ; $\dot{p}_1$, …, $\dot{p}_N$ , and the incompressibility condition is written:

$$\sum_{i=1}^{N}\left(\frac{\partial}{\partial q_i}\dot{q}_i + \frac{\partial}{\partial p_i}\dot{p}_i\right) = 0. \tag{2}$$

Now, this equation is obviously satisfied by virtue of Hamilton's equations (1). Liouville's theorem then results.

Liouville's theorem shows us that a uniform distribution of representative points of exemplars of the system considered in the extension-in-phase is maintained indefinitely. It is easy to comprehend that this will inspire us to take the element $d\tau$ of the extension-in-phase as a measure of the probability that the system is found to be represented by a representative point that is situated in $d\tau$ at the instant *t*. Nevertheless, that hypothesis provokes several remarks.

A first remark is that it often happens that the evolution of a mechanical system admits some first integrals; i.e., that certain functions of the *q* and *p* remain constant in the course of that evolution. Therefore, in the usual case of an isolated system, the energy $H(q, p)$ remains constant. When there are first integrals, the representative point is required to move on certain multiplicities in the extension-in-phase that are less than 2*N*-dimensional (for example, on a $2N - 1$-dimensional multiplicity, if there is just one first integral). Therefore, for an isolated system, the representative point must displace in the hypersurface $H = E =$ const., or more precisely, since the energy is always known only with an uncertainty *dE*, it is always contained in a very thin layer that lies between the surfaces $H = E$ and $H = E + dE$. It is only the volume elements of that layer that one could naturally consider to be probability measures.

Another essential remark is that Liouville's theorem does not permit one to only prove rigorously that one can take the element $d\tau$ of the extension-in-phase to be a measure for the probability of the presence of the representative point in that element, although certainly that hypothesis is suggested by the theorem. In order to obtain a satisfactory justification, one must add a postulate that is known by the name of the "ergodic hypothesis." We state it here as: *Let a system that admits energy as its only first integral be uniform and have an initial energy state that is found between E and E + dE. The representative point of the system displaces in the extension-in-phase while remaining in the layer between the hypersurfaces H = E and H = E + dE. We then "assume" that at the end of a sufficiently long time the representative point has "uniformly swept out" all of the layer in question.* One can state this postulate in a form that is somewhat less strict that is called the "quasi-ergodic hypothesis," by which one is content to affirm that in the course of its displacement the representative point passes as close as one desires to any point of the layer in question. If one or the other of these two hypotheses is exact then the probability for the presence for the representative point in an element of the extension-in-phase is proportional to the fraction of the very long time $T$ during which the system moves in that element. One can thus equate the means that are taken over the extension-phase with the means that are taken in time.

Unfortunately, the ergodic or quasi-ergodic hypotheses are certainly not exact. There exist simple cases, such as that of periodic motion, in which they break down. Meanwhile, one can assume that these exceptional cases have a vanishing probability. Nevertheless, the ergodic or quasi-ergodic hypotheses are very difficult to justify rigorously in the classical theory, and it indeed seems that analogous difficulties persist in quantum theories. Boltzmann introduced a hypothesis of "molecular chaos," which invokes the random character of perturbations that the motion of the molecules is subject to as a result of their continual collisions. That hypothesis, to which we shall return, plays a role that is analogous to that of the ergodic hypothesis. We shall have to return to that question.

Without stopping at the difficulties that we just pointed out, we assume that one can adopt a measure of the relative probability of the states of a system that is by represented by the element $d\tau$ of its extension-in-phase to be the magnitude of that element itself.

**4. Entropy and probability. Boltzmann's relation.** – The quantity that was introduced by the development of classical thermodynamics that is the most characteristic and the most mysterious in regard to its physical significance is certainly that of entropy, which is a quantity that always tends to increase under any spontaneous physical transformation. The great success of statistical thermodynamics has been to arrive at the interpretation of entropy as a quantity that measures the degree of probability of the state of the body considered. It is, moreover, easy to determine the nature of that functional relationship between the entropy of a body and the probability of its state. Indeed, if one considers two systems with no mutual interactions whose entropies are $S_1$ and $S_2$ then thermodynamics tells us that the entropy of the global system that is composed of both the systems is $S_1 + S_1$ . On the other hand, if $P_1$ is the probability of a state of the first system and $P_2$ is that of the state of the second system then the probability of the global state of the system 1 + 2 is equal to $P_1P_2$ , from the theorem of composite probabilities. Therefore, if the relation between entropy and probability is of the form $S = f(P)$ then one must have:

$$f(P_1) + f(P_2) = f(P_1P_2). \tag{3}$$

Upon differentiating (3) with respect to $P_1$ , one will have:

$$f'(P_1) = P_2\, f'(P_1P_2),$$

and then upon differentiating with respect to $P_2$ , one will have:

$$f'(P_1P_2) + P_1P_2\, f''(P_1P_2) = 0,$$

a relation that takes the form:

$$f'(x) + x\,f''(x) = 0,$$

from which one infers, by a double integration, that $f(x) = C \log x + D$. The functional relationship between entropy and probability is therefore:

$$S = k \log P + \text{const.}, \tag{4}$$

which one can write, by conveniently normalizing the probability:

$$\boxed{S = k \log P.} \tag{5}$$

This is the famous Boltzmann formula, and as we verify, in order to establish the agreement with classical thermodynamics, one must adopt the numerical value:

$$k = 1.37 \times 10^{-16} \text{ erg} / {}^{\text{o}}\text{K} = 1.37 \times 10^{-23} \text{ J} / {}^{\text{o}}\text{K}$$

for the constant $k$, which is called "Boltzmann's constant."

How must one evaluate the probability $P$ in Boltzmann's formula? The most natural definition consists of saying that $P$ is equal to the number of elementary complexions that realize the state of the body that is being considered, divided by the total number of all possible complexions. However, the latter number is difficult to evaluate, and its introduction will add only a constant in the expression for entropy. One thus agrees to take $P$ to be the number of complexions that realize the state considered without dividing by the total number of possible complexions, which amounts to fixing the arbitrary constant in the entropy in a certain manner, which is shown to be adequate.

In accord with Liouville's theorem, one thus takes the value of $P$ to be a value that is proportional to the volume in the extension-in-phase that corresponds to the state of the system. Here again, one can recognize various possibilities.

Consider a system in an energy state $E$. The hypersurface $E$ = const. is closed and limited to a certain volume $\Phi(E)$ in the extension-in-phase. One can take $\Phi(E)$ to be the probability of the state $E$. A second definition that seems more natural consists of regarding the energy as being defined only up to $dE$, and remarking that the representative point of the system then displaces in a layer that lies between the hypersurfaces $E$ and $E + dE$, whose volume is obviously $\partial\Phi / \partial E\ dE$, which leads one to set $P = \partial\Phi / \partial E$. Finally, a third possible definition starts with the fact that for a given total energy the distribution of the individual energies over the various constituents of the system can vary, and that one of these distributions is the most probable one, which corresponds to a larger domain in the extension-in-phase that the other one does. One can then define $P$ by taking into account the complexions of the total energy $E$ that correspond to that most probable distribution.

The three definitions of $P$ (and, in turn, of $S$) that we just recalled are not equivalent, and in the case of systems with a small number of degrees of freedom ([2]) they can give very different results. Now, one finds – and this is a remarkable circumstance – that for

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([2]) However, in that case, it is doubtful that one can truly speak of the entropy of the body.

the systems with a very large number of degrees of freedom that are usually envisioned by thermodynamics the three definitions are practically equivalent for the application of Boltzmann's formula, which obviates the need to justify one choice or the other in that case. We shall not insist upon the proof of that "insensitivity of Boltzmann's formula," which one can find in a good number of classical works.

**5. Temperature and thermal equilibrium.** – In classical thermodynamics, one defines the variation of entropy by the formula:

$$dS = \frac{dQ}{T} = \frac{dE + d\mathcal{T}}{T}, \tag{6}$$

where $T$ is the absolute temperature, $dQ$ is the quantity of heat received by the body, $dE$ is the variation of its internal energy, and $d\mathcal{T}$ is the work that it does on the environment. If the body does no work then one has $dS = dE / T$. The preceding formulas are valid only if the transformation is "reversible."

Since the entropy generally depends upon not just the energy $E$, but also some other parameters (such as the volume $V$ that is occupied by the body), one concludes from the preceding relation that one can define the absolute temperature of a body by the formula:

$$\frac{1}{T} = \frac{\partial S}{\partial E}. \tag{7}$$

If two bodies 1 and 2 are in contact and can exchange heat (but not macroscopic mechanical work) then the temperatures of the two bodies tend to equalize, and when thermal equilibrium is attained, one has:

$$\left(\frac{\partial S}{\partial E}\right)_1 = \left(\frac{\partial S}{\partial E}\right)_2 . \tag{8}$$

Consider this question of thermal equilibrium from the viewpoint of statistical mechanics. Since the global system 1 + 2 is assumed to be isolated, its total energy is constant, but it can be distributed in various ways between the two systems 1 and 2. The probability of the global state 1 + 2, where the body 1 has energy $E_1$ and the body 2 has the energy $E_2$, is:

$$P = P_1(E_1) \cdot P_2(E_2) = P_1(E_1) \cdot P_2(E - E_1) . \tag{9}$$

The most probable state of the global system 1 + 2 corresponds to the maximum of $P$, which is defined by the relation:

$$\frac{\partial \log P}{\partial E_1} = 0 \quad \text{or} \quad \frac{\partial \log P_1}{\partial E_1} + \frac{\partial \log P_2}{\partial E_1} = 0.$$

Since $dE_2 = - dE_1$ , one thus has:

$$\frac{\partial \log P_1}{\partial E_1} = \frac{\partial \log P_2}{\partial E_2}. \tag{10}$$

Upon multiplying this by $k$ and introducing Boltzmann's relation, this becomes:

$$\frac{\partial S_1}{\partial E_1} = \frac{\partial S_2}{\partial E_2}, \tag{11}$$

and this equation translates into the equality of the temperatures of the two bodies at the moment of equilibrium. One can thus say that the temperature of body is related to the probability $P_m$ of its most probable state by the relation:

$$\frac{1}{kT} = \frac{\partial \log P_m}{\partial E}, \tag{12}$$

which corresponds to the third definition of entropy by the Boltzmann formula that was discussed above.

One can remark that statistical mechanics, which goes further than classical thermodynamics, can define the entropy of an arbitrary state by the relation $S = k \log P$, *even if that state is not an equilibrium state of maximal probability.* This general definition of entropy coincides with that of thermodynamic entropy for states of maximal probability. It then provides the value of entropy in classical thermodynamics, which can be expressed with the aid of any one of the three definitions of $P$ that were previously pointed out for the systems with an enormous number of parameters that are envisioned by classical thermodynamics.

However, Boltzmann's formula also permits one to study the fluctuations of the state of a body around its most probable state. We shall ultimately have to return to the theory of fluctuations thus obtained.

**6. The Boltzmann-Gibbs canonical distribution.** – Once more, consider a system that is composed of the union of two bodies 1 and 2. We suppose that they are very weakly coupled and that they can exchange heat between them. The body 1 can have an arbitrary (i.e., small or large) number of degrees of freedom, but we essentially suppose that the body 2 is a "thermostat" – i.e., an enormous heat reservoir (i.e., uncoordinated molecular energy) – in such a fashion that the body 1 has only a completely negligible chance of taking an appreciable fraction of its energy. In other words, if $E_2^{(m)}$ is the energy of the thermostat when it is in the equilibrium state of maximal probability with the body 1 then one can assume that for all practically realizable states the difference $E_2 - E_2^{(m)}$ is always extremely small when compared to $E_2^{(m)}$. If the probability of the state of the body 2 is $P_2$ then one can always write:

(13) $$\log P_2 = \log P_2^{(m)} + \frac{\partial \log P_2^{(m)}}{\partial E}(E_2 - E_2^{(m)}) + \dots$$

The hypotheses that were made on the nature of the thermostat then permit us to neglect the unwritten terms, which are of higher order than $E_2 - E_2^{(m)}$, and upon calling the absolute temperature of the thermostat $T$, one will have:

(14) $$\log P_2 = \log P_2^{(m)} + \frac{1}{kT}(E_2 - E_2^{(m)}),$$

so

(15) $$P_2 = P_2^{(m)} e^{(E_2 - E_2^{(m)})/kT}.$$

However, if $E_1^{(m)}$ denotes the energy of the body 1 when it is in its most probable equilibrium state with the thermostat then one will have:

$$E_2 - E_2^{(m)} = E_1^{(m)} - E_1 ,$$

by the conservation of energy. Since we have assumed that the interaction of the body 1 with the thermostat is weak, the probability of the state in which the thermostat has the energy $E_2$ and the body 1 has the energy $E_1$ is:

(16) $$P = P_1(E_1) \cdot P_2(E_2) = P_1(E_1) \cdot P_2^{(m)} e^{(E_1^{(m)} - E_1)/kT}.$$

The preceding formula can be further written in the form:

(17) $$P = P(E_1)\ e^{(F - E_1)/kT},$$

where $P(E_1)$ is the total number of configurations of the body 1 that have energy $E_1$ when one imposes no constraint on it regarding the thermostat (e.g., an *a priori* probability).

We have thus obtained the Gibbs "canonical distribution law," which seems to be valid for a system in thermal contact with a thermostat that fixes the temperature.

The constant $F$ that appears in this canonical distribution law is calculated by writing that:

(18) $$\sum_i P(E_i) e^{(F - E_i)/kT} = 1,$$

in which the sum is taken over all possible states of the body, which forms a discontinuous sequence, by hypothesis (if the sequence is continuous then one replaces the summation $\sum_i$ with an integral over $E$). From the preceding equation, one infers that:

(19) $$e^{-F/kT} = \sum_i P(E_i) e^{-E_i/kT} = Z,$$

so

$$F = -\,kT \log Z. \tag{20}$$

The sum (or integral) $Z$ that is defined by the right-hand side of (19) was called the "state sum" by Planck. It plays an essential role in the calculations of statistical thermodynamics.

We remark that the body 1 can be composed of just one molecule. It results from this that the notion of temperature is meaningful for just one molecule when that molecule is found to be in energetic contact with a thermostat of temperature $T$ that *imposes* its temperature upon the molecule.

**7. Important remarks on the subject of the canonical distribution law.** – In chapter IV of his famous work *Elementary Principles of statistical Mechanics,* Willard Gibbs wrote the canonical distribution law in the form:

$$P = e^{(\psi - E)/\theta}, \tag{21}$$

which amounts to setting $kT = \theta$ and $P(E_1)\ e^{F\,/\,\theta} = e^{\psi\,/\,\theta}$ in formula (17). He then introduced what he called "the probability exponent" $\eta$ by setting:

$$\eta = \log P, \qquad P = e^{\eta}, \tag{22}$$

and then he showed that $\eta$ has – up to an additive constant – the property of thermodynamic entropy for the body 1 *with the sign changed* ($^3$). One can then write:

$$\eta = -\,\frac{S_1}{k} + \text{const.}, \qquad P = e^{\eta} = \text{const. } e^{-S_1/k}. \tag{23}$$

Now, the – sign in the exponent of the last formula can be surprising because it seems to be in opposition with the Boltzmann formula (because one will have a tendency to write $P$ = const. $e^{+S_1/k}$). This can give rise to some confusion that has occasionally appeared in very serious works. It is this sign change that explains some apparent anomalies that I will discuss later on.

Meanwhile, the sign change that we just pointed out is easy to explain, because the formula $P(E) = P(E_1) \cdot P(E_2)$ that we started with gives $P$ = const. $e^{S_2/k}$, from the Boltzmann formula when it is applied to the thermostat. However, if the body 1 and the thermostat form, by hypothesis, a system that is isolated from all external action where the exchanges of energy between the two constituents are processes that are assumed to be reversible then there must be conservation of total entropy, which gives:

$$S_1 + S_2 = S_1^{(m)} + S_2^{(m)} = \text{const.},$$

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($^3$) One must suppose that the body 1 has a number of degrees of freedom that is very large in order for one to be able to attribute entropy to it.

and, in turn, $P$ = const. $e^{-S_1/k}$, which is indeed in accord with the Gibbs formula (23).

The preceding remark will play an important role in the considerations that we will ultimately develop on the analogy between action and entropy. We shall add some other remarks there.

For the system that is composed of the body 1 and the thermostat in weak energetic contact, we have found that $P = P_1 \times P_2$ , with:

(24) $$P_2 \approx P_2^{(m)} e^{(E_1^{(m)} - E_1)/kT} ,$$

in which $P_2$ is equal to $e^{S_2/k}$, from Boltzmann's formula. Formula (24) gives us only an approximate value, because in order to obtain it we have neglected the terms of higher order in $E_2 - E_2^{(m)}$. We can set:

(25) $$P_2 \approx e^{(F-E_1)/kT} , \qquad P \approx P_1 \; e^{(F-E_1)/kT} ,$$

and we recover the canonical distribution law.

If the body 1 is complex and possesses a large number of degrees of freedom then one has, by the definition of the state sum $Z$:

(26) $$Z = \sum_i P_i(E_i) e^{-E_i/kT} = e^{-F/kT} ,$$

so, since the probability of the most probable state of 1 is *in this case* infinitely larger than that of all the other possible states:

(27) $$P_1^{(m)} e^{E_1^{(m)}/kT} \approx e^{-F/kT} .$$

Since one can then introduce the notion of entropy for the body 1 with no difficulty, one infers the relation $P_1^{(m)} = e^{S_1^{(m)}/kT}$ from this and Boltzmann's formula, in such a way that formula (25) gives us:

(28) $$F \approx E_1^{(m)} - TS_1^{(m)} .$$

Therefore, for a body that has a very large number of degrees of freedom and is kept at the temperature $T$, $F$ is the free energy of classical thermodynamics. However, if one substitutes the value $F = E_1 - TS_1$ into the second formula (25) then one finds that $P \sim e^{-S_1/kT}$, which seems to be in contradiction with Boltzmann's formula. This is the difficulty that we have already encountered above, and which we resolved by remarking that $S_1 + S_2$ = const., in such a way that one recovers Boltzmann's formula $P \sim e^{S_2/kT}$ for $P_2$, which is satisfactory.

The proportionality of the exponent in the exponential in the canonical law for $S_2$ and $-S_1$ is valid for any sort of body 1. However, in the case where the body 1 possesses a large number of degrees of freedom, we have:

(29) $$P \approx P_1\, e^{S_2/k} = e^{(S_1+S_2)/k} = e^{(S_1^{(m)}+S_2^{(m)})/k} = e^{S_m/k},$$

since $P_1$ is then equal to $e^{S_2/k}$, from Boltzmann's formula, in which $S_m$ is the entropy of the state of maximal probability for the system 1 + 2, here, so we have:

(30) $$P \approx P^{(m)}.$$

This can say that for the system 1 + 2 the state of maximal probability is the only one that is realized in practice, which is satisfactory, when one is given the hypothesis that the body 1 is very complex.

**8. Applications of the canonical distribution law.** – First, consider the case in which the body 1 is a very complex body that is defined by a very large number of parameters (for example, the set of molecules in a gas). We have shown above that in this case $F$ is equal to $E - TS$, and coincides with the thermodynamic potential or free energy of the body. We then envision the opposite case where the body 1 is defined by a small number of parameters. The canonical distribution law is always valid, but $F$ no longer represents free energy.

For example, consider a gas molecule. It is defined by a small number of parameters, but, since one can consider it as being in energetic contact with a thermostat that composed of the set of all other molecules of the gas, which is assumed to be in thermal equilibrium, one can apply the canonical distribution law to it. Now, the *a priori* probability for the coordinates and momenta of that molecule to have values that are found in the interval $x \to x + dx, \ldots, p_x \to p_x + dp_x$ is, from Liouville's theorem, equal to the element:

$$d\tau = dx\, dy\, dz\, dp_x\, dp_y\, dp_z$$

of the extension-in-phase for the molecule.

From the canonical distribution law, the probability for the molecule, which is considered to be in contact with the rest of the gas that forms a thermostat at the temperature $T$, to have its representative point in the element $d\tau$ of its extension-in-phase is $C\ e^{-E/kT}\, d\tau$. It then results that the number of gas molecules whose coordinates lie between $x$ and $x + dx$, …, and whose Lagrange momenta lie between $p_x \to p_x + dp_x$ , … is:

(31) $$dn = C e^{-E/kT}\ dx\, dy\, dz\, dp_x\, dp_y\, dp_z = C\, m^2\ e^{-E/kT}\ dx\, dy\, dz\, dv_x\, dv_y\, dv_z,$$

$$E = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)\,.$$

The constant $C$ is determined by writing that $\int dn = N$, where $N$ is the total number of gas molecules. Formula (31) constitutes the celebrated law for the velocity distribution between the molecules of a gas, and is due to Maxwell.

Since one can unite the elements $dp_x\, dp_y\, dp_z$ in the extension-in-momenta that form a spherical shell between the spheres:

$$\frac{p^2}{2m} = E \qquad \text{and} \qquad \frac{(p+dp)^2}{2m} = E + dE,$$

a shell whose volume is equal to:

$$4\pi p^2\, dp = 2\pi (2m)^{3/2} \sqrt{E}\; dE,$$

one finds that the number of gas molecules per unit volume whose energy lies between $E$ and $E + dE$ is:

(32) $$dn_E = C\, e^{-E/kT} \sqrt{E}\; dE.$$

If the gas is found to be placed in a force field that acts on the molecules then one must take the potential energy into account in the expression for $E$. For example, if the gas is placed in a gravitational field then one will have:

$$E = \frac{p^2}{2m} + mgz$$

($z$ being the altitude of the molecule), and if one integrates the expression for $dn$ over $dp_x\, dp_y\, dp_z$ then one will find that the gas density varies with altitude like $e^{-mgz/kT}$. This is Laplace's famous "barometric law," which is applicable to not only the molecules of gas, but also to the grains of an emulsion in suspension in a liquid. It was by applying that law in the latter case that Jean Perrin determined the value of Avogadro's number ($\mathcal{N} = 6.06 \times 10^{23}$) in his celebrated experiments a half-century ago.

To once more show the importance of a remark that was made in the preceding paragraph, consider the $i^{\text{th}}$ molecule of a gas. If it is in contact with the ensemble of a gas that forms a thermostat at the temperature $T$ then it will have a probability $P = C\, e^{-E/kT}\, d\tau$ of having its representative point in the element $d\tau$ of the extension-in-phase. If it is permissible to attribute an entropy $S_i$ to it then that will be given by formula (23) as $S_i = -k \log P$. Even if the introduction of the entropy $S_i$ for a molecule seems debatable, one will undoubtedly admit more easily that the entropy of the ensemble of gas molecules is given from thermodynamics by the mean value of $S_i$ ; i.e., that:

(33) $$S = \overline{S_i} = -k \sum_i P \log P.$$

Since $P$ is a continuous function of the canonical variables $x, \ldots, p_z$ here, one can write:

(34) $$S = -k \int f \log f\, d\tau.$$

This is a classical formula of Boltzmann in the kinetic theory of gases. However, if we set $S_i = k \log P$, in line with Boltzmann's relation, then we would find that $S = k \int f \log f\, d\tau$, with a difference of sign.

With the aid of the preceding formulas, we can easily find expressions for all of the quantities that characterize perfect gases. We content ourselves by noting that one can thus recover the law of Mariotte-Gay-Lussac in the form:

$$pV = NkT, \tag{35}$$

where $N$ is the number of gas molecules that occupy the volume $V$ at the temperature $T$ at the pressure $p$. Upon applying that formulas to a gram-molecule of a perfect gas for which the number $N$ of molecules is equal to Avogadro's number $\mathcal{N}$, one can write:

$$pV = RT, \tag{36}$$

upon setting $R = k\mathcal{N}$, where $R$ is the "gas constant" that relates to a gram-molecule, and whose well-known experimental value is $R = 8.3 \times 10^7$ ergs / °K. One infers from this that:

$$k = \frac{R}{\mathcal{N}} = \frac{8.3\times10^7}{6.06\times10^{23}} = 1.37 \times 10^{-16}, \tag{37}$$

and this is indeed the value of Boltzmann's constant that we have previously stated.

**9. The equipartition of energy theorem.** – In classical mechanics, the Lagrange momenta appear in the expression for the energy of a point-like molecule by way of their squares. If a material point is referred to a position of equilibrium that is taken to be the origin of the coordinates for a force that is proportional to the elongation then the coordinates and momenta both appear by way of their squares in the expression for energy:

$$E = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{K}{2}(x^2 + y^2 + z^2).$$

In a general fashion, one says that a canonical variable is a "momentoid" if it enters into the expression for energy by way of its square. The simplest example of a momentoid is a Lagrange momentum for a free particle, and the name "momentoid" comes from that fact.

The equipartition of energy theorem can then be stated by saying: *If one of the canonical variables of a system is a momentoid when the system is in thermal equilibrium at the temperature T then the corresponding term in its energy expression has the mean value* $\frac{1}{2}kT$. *Thus, the energy divides into means that are the same for all the momentoids, and if all of the variables are momentoids then it divides equally into means over all the degrees of freedom.* Hence, one has the name of the theorem.

We suppose, for example, that the variable $q_k$ is a momentoid, and prove the theorem for that variable. The energy of the system is of the form:

$$E = \alpha q_k^2 + f(q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, p_1, \ldots, p_N).$$

The mean value of the term $\alpha q_k^2$ in thermal equilibrium at the temperature $T$ will be:

$$\overline{\alpha q_k^2} = \frac{C\int_{2N}\cdots\int \alpha q_k^2\, e^{-\alpha q_k^2/kT} e^{-f/kT} dq_1 \dots dp_N}{C\int_{2N}\cdots\int e^{-\alpha q_k^2/kT} e^{-f/kT} dq_1 \dots dp_N}. \tag{38}$$

Upon dividing the top and bottom by the factor:

$$C\int_{2N-1}\cdots\int e^{-f/kT} dq_1 \dots dq_{k-1}, dq_k \dots dp_N\ ,$$

and upon setting $u = \sqrt{\frac{\alpha}{kT}}q_k$, one finds that:

$$\overline{\alpha q_k^2} = kT\,\frac{\int_0^\infty u^2 e^{-u^2} du}{\int_0^\infty e^{-u^2} du} = \tfrac{1}{2}kT. \tag{39}$$

Q. E. D.

Naturally, the same proof applies to a momentoid of type $p_k$ .

In classical statistical mechanics, it often happens that all of the canonical variables are momentoids, and there is then equipartition of energy between all of the degrees of freedom. The equipartition of energy theorem has given statistical mechanics a large number of exact results, but it has also led to very significant failures that made the introduction of quanta into physics necessary. We shall not elaborate upon these well-known points here.

**10. Relativistic statistical mechanics.** – As we have seen, relativistic mechanics can be developed by starting with a stationary action principle, and one concludes with canonical Hamilton equations of the usual type:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}. \tag{40}$$

This then permits the introduction of the extension-in-phase and the proof of Liouville's theorem, which is a consequence of Hamilton's equations. One can also introduce Boltzmann's relation between entropy and probability, and choose one or the other of the three definitions that were envisioned previously for the number $P$ of complexions that correspond to a given energy state.

Nothing will change in the definition of temperature, or in the canonical distribution that gives the probability of the states of a system in contact with a thermostat, or in the identification of the quantity $F$ with free energy when that system has a large number of degrees of freedom.

All of the preceding conclusions that were obtained thus remain valid as long as one does not introduce an explicit expression for the Lagrange function $\mathcal{L}$, the momenta $p_k$ that are deduced from it, and the Hamiltonian function that gives the energy as a function of the $q_k$ and $p_k$ . However, things are different in the applications where one does introduce expressions for the momenta $p_k$ or the Hamiltonian function. We shall give an example by recalling the case of Maxwell's law.

Upon considering a gas molecule to be in contact with a thermostat that is composed of the rest of the gas, the canonical distribution law leads us to following expression for the number of molecules in the element $d\tau$ of the extension-in-phase where the figurative point is:

$$dn = C\, e^{-E/kT}\, d\tau.$$

That formula remains valid, but since we no longer have:

$$E = \tfrac{1}{2} m v^2 = \frac{p^2}{2m}$$

here, we can no longer infer formula (32) for the energy distribution.

Indeed, here we have the following relations for the energy and quantity of motion:

$$W = \frac{m_0 c^2}{\sqrt{1-\beta^2}}, \qquad \mathbf{p} = \frac{m_0 c \boldsymbol{\beta}}{\sqrt{1-\beta^2}}, \qquad W^2 = p^2 c^2 + m_0^2 c^4. \tag{41}$$

We can then replace $E$ with $W$ in the expression for $dn$, since $W = E + m_0 c^2$, and that replacement of $E$ with $W$ only amounts to modifying the constant $C$. Moreover, a variation $dW$ of $W$ corresponds to a variation $dp$ of $p$ such that:

$$W\, dW = pc^2\, dp.$$

Therefore, a spherical shell in the extension-in-phase has the volume:

$$4\pi \frac{W}{c^2} \sqrt{\frac{W^2}{c^2} - m_0^2 c^2}\, dW .$$

As a result, the number of molecules per unit volume whose total energy $W$ lies between $W$ and $W + dW$ will be:

$$dn_W = C e^{-W/kT}\, W \sqrt{\frac{W^2}{c^2} - m_0^2 c^2}\, dW . \tag{42}$$

If all of the molecules (except for a very small number) have a velocity that is much less than $c$ (which is usually the case for all gaseous materials) then one can recover formula (32) in the following fashion: If we set:

$$W = E + m_0\, c^2 \qquad \text{and} \qquad \frac{E}{m_0 c^2} = \eta$$

then we will have $W = m_0\, c^2\, (1 + \eta)$, and we can write:

$$dn_W = \text{const.}\; e^{-E/kT}\,(1 + \eta)\sqrt{(2+\eta)\eta}\; d\eta, \tag{43}$$

a form for (42) that shows that one indeed recovers formula (32) in the Newtonian approximation, where one has $\eta \ll 1$.

**11. Application to a photon gas.** – Consider the opposite case to the Newtonian approximation, in which almost all of the gas molecules have a velocity that is very close to $c$. That is what happens at any temperature when the proper mass of the molecules tends to zero. This case can be compared to that of black-body radiation, because black-body radiation can be considered to be a photon gas, where photons have a proper mass of zero or possibly just extraordinarily small. Since one will then have $W \gg m_0\, c^2$ for almost all of the molecules, one will find, from (42), that:

$$dn_W = C\, e^{-W/kT}\, W^2\, dW. \tag{44}$$

For photons, one sets $W = h\nu$, and one will find:

$$\rho_\nu\, d\nu = h\nu\, dn_\nu = \text{const.}\; e^{-h\nu/kT}\, \nu^2\, d\nu \tag{45}$$

for the energy density of a photon gas that corresponds to the frequency interval $\nu \to \nu + d\nu$. Now, this form for the spectral density is the one that Wien had proposed long ago, and which is, in fact, valid only for large values of the quotient $\nu / T$. The Wien spectral law is therefore the form that Maxwell's law takes for a gas whose molecules have vanishing proper mass, as I have pointed out in an article in *Journal de Physique* in 1922.

However, in fact, the true spectral density of black-body radiation is given by Planck's law:

$$\rho(\nu)\, d\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{k\nu/kT} - 1}\, d\nu\,. \tag{45, bis}$$

It was the introduction of quanta and the transformations that they are subject to into statistical mechanics that explained the difference between the real Planck law and Wien's law. We shall not dwell upon this well-known question.

It is curious to note what happens for the distribution of *velocities* in the case of molecules of vanishing proper mass. Since $W = m_0 c^2 / \sqrt{1-\beta^2}$, in order for such a molecule to have an appreciable energy, it is necessary that its velocity be extremely close to $c$. Therefore, $W$ remains essentially zero when the velocity increases from 0 to $c - \varepsilon$, $W$ increases from an extremely small value to infinity when $v$ increases from $c - \varepsilon$ to $c$. Therefore, the law of distribution of energies (44) preserves a bell shape that is analogous to the classical Maxwell law, while the law of distribution of velocities is represented by a curve with a spike in the immediate neighborhood of $v = c$. That is what is illustrated in Figure 3.

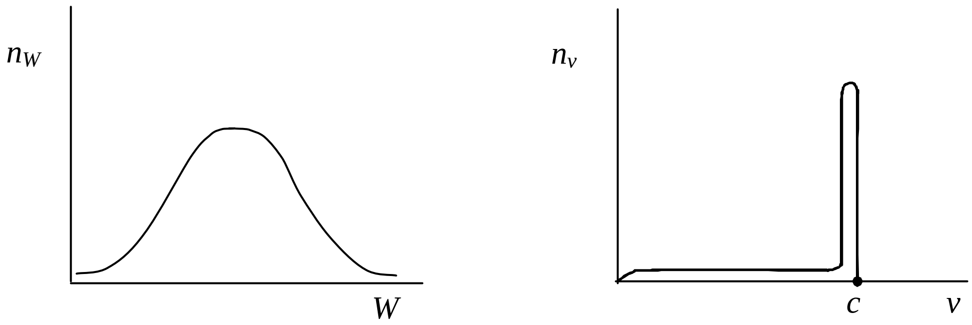

Figure 3.

When $m_0$ tends to zero, the entire curve of the velocity distribution tends to squeeze up against the vertical line $v = c$, in a sense.

**12. Mean value of $p_k\ \dot{q}_k$ in relativistic statistical mechanics.** – In relativistic statistical mechanics, one can no longer prove the equipartition of kinetic energy over the degrees of freedom, because then the kinetic energy of the material point:

$$E_0 = m_0 c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) = c\sqrt{p^2 + m_0^2 c^2} - m_0\, c^2$$

is no longer a quadratic form in the $p_k$ . However, we shall show that in relativistic statistical mechanics, one has:

$$\overline{p_k \dot{q}_k} = kT. \tag{46}$$

Indeed, one can write:

$$\overline{p_k \dot{q}_k} = \frac{\int_{2N}\cdots\int p_k \dot{q}_k e^{-W/kT} dq_1 \ldots dp_N}{\int_{2N}\cdots\int e^{-W/kT} dq_1 \ldots dp_N} = \frac{\int_{2N}\cdots\int p_k \frac{\partial W}{\partial p_k} e^{-W/kT} dq_1 \ldots dp_N}{\int_{2N}\cdots\int e^{-W/kT} dq_1 \ldots dp_N}, \tag{47}$$

from Hamilton's equations, and since $W$ always tends to infinity at the same time as $p_k$ , an integration by parts easily provides formula (46).

That formula, which is valid in relativistic statistical mechanics in a general fashion, gives the formula (4):

(48) $$\bar{E}_k = \tfrac{1}{2} N\, kT,$$

in the Newtonian approximation, where $2E_k = \sum_k p_k \dot{q}_k$, and this formula reduces to the classical expression for the equipartition of energy, but in relativistic dynamics:

$$2E_k \neq \sum_k p_k \dot{q}_k \,.$$

One must remark that formula (48) is valid only in the proper system of the body considered, in such a way that it is preferable to write it in the form:

(49) $$\overline{p_k^0 \dot{q}_k^0} = kT_0 \,,$$

in which the index 0 says that the quantities are evaluated in the proper system.

For a molecule, one can always (even in relativistic theory) write:

(50) $$\overline{\frac{1}{2}\sum_{k=1}^{3} p_{0k}\dot{q}_{ok}} = \tfrac{3}{2} kT_0 \,,$$

and since:

$$p_{0k} = \frac{m_0 \dot{q}_{0k}}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad \text{with} \qquad v^2 = \sum_{k=1}^{3} \dot{q}_{0k}^2 \,,$$

one gets:

(51) $$\overline{\frac{1}{2}\frac{m_0 v^2}{\sqrt{1-\dfrac{v^2}{c^2}}}} = \tfrac{3}{2} kT_0 \,.$$

In relativistic dynamics, the quantity under the line of the mean in (51), which we shall call the "pseudo-kinetic energy," is not equal to the kinetic energy. It only agrees with it in the Newtonian approximation, and formula (51) gives back the classical expression for the equipartition of energy. We shall confirm that the pseudo-kinetic energy plays an important role in relativistic thermodynamics.

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(4) In order to avoid any confusion with temperature, here we shall denote the kinetic energy by $E_k$.

CHAPTER IV

# NOTIONS FROM RELATIVISTIC THERMODYNAMICS

**1. Relativistic invariance of entropy.** – The Hamiltonian action is the fundamental invariant of mechanics. We shall now see that that entropy is the fundamental invariant of thermodynamics. In order to understand the invariant character of entropy, it suffices to recall that, according to Boltzmann, the entropy of a macroscopic state is proportional to the logarithm of the number of complexions that realize that state. Entropy is therefore expressed by a *number* whose invariance seems obvious. In order to confirm that intuition, we remark that, on the one hand, the definition of entropy by Boltzmann's formula involves an *integer* number of complexions and that, on the other hand, the transformation of entropy under a change of Galilean reference system must be expressed by a *continuous* function of the relative velocity of the reference systems. It then necessarily results that this continuous function is constant and equal to unity (since it is equal to 1 when the reference systems coincide), and it then follows that entropy is an invariant.

One can also reason in a different way: Consider a body that passes from a state of rest in a reference system 1 to a state of motion with a velocity $v$ by being accelerated adiabatically and with constant pressure by a body that is immobile in the system considered. The set of two bodies evolves adiabatically, its entropy is constant, and since the body that produces the acceleration keeps constant entropy, the same is true for the accelerated body. Therefore, $S_1 = S_2$ , where the indices 1 and 2 refer to the initial and final state of the accelerated body, respectively. Now, let a system that is originally coupled to the accelerated body be put into a state of uniform motion. The state 2 that refers to that system is identical with the state 1 that refers to the original system. One thus has $S_2' = S_1$ , and as a result $S_2' = S_2$ , a formula that expresses the invariance of entropy.

**2. Relativistic variance of temperature.** – Deducing the relativistic variance of temperature demands some very delicate reasoning. We shall give the argument that seems the most instructive.

Consider a body $C$ that is found in a state of thermodynamic equilibrium with an absolute temperature $T_0$ , and which possesses an invariant volume $V_0$ when envisioned in a Galilean reference system $R_0$ that is linked to it. This can be, for example, a gas that is enclosed in a rigid container of volume $V_0$ at the temperature $T_0$ . Let $M_0$ be the total proper mass of the body $C$.

We now place ourselves in a Galilean reference system where the body $C$ possesses a velocity $v = \beta c$ of uniform translation, and suppose that in this reference system a heat source provides the quantity of heat $Q$ to $C$. We shall show the essential point that in order for the body $C$ to conserve its velocity $\beta c$, a certain amount of work $A$ must be done on it at the same time as the quantity of heat $Q$.

Indeed, since the body keeps the velocity $\beta c$, by hypothesis, its energy, which is $\frac{M_0 c^2}{\sqrt{1-\beta^2}}$, can increase as a result of receiving a quantity of heat $Q$ and a quantity of work $A$ only if its proper mass varies and passes from its initial value $M_0$ to a final value $M_0 + \Delta M_0$ . In other words, the heat and work that is absorbed by the body $C$ in motion will have increased its internal energy, which must make its proper mass increase, from the principle of inertia for energy.

The principle of conservation of energy permits us to write:

$$\frac{\Delta M_0 c^2}{\sqrt{1-\beta^2}} = Q + A\,. \tag{1}$$

If $F$ denotes the force that has been exerted on the body $C$ in order to communicate the work $A$ in the reference system $R$ then the derivative of the quantity of motion with respect time must be equal to $F$ at each instant, which gives:

$$\frac{(M_0 + \Delta M_0)v}{\sqrt{1-\beta^2}} - \frac{M_0 v}{\sqrt{1-\beta^2}} = \int F\,dt = \frac{1}{v}\int F\,v\,dt = \frac{A}{v}\,, \tag{2}$$

since $v$ is constant, by hypothesis, and $A = \int F\,v\,dt$. One thus has:

$$\frac{\Delta M_0}{\sqrt{1-\beta^2}} v^2 = A, \tag{3}$$

and one sees that the work done on the body in the process envisioned is equal to the increase in the pseudo-vis viva $\frac{\Delta M_0 v^2}{\sqrt{1-\beta^2}}$ (viz., twice the pseudo-kinetic energy). Finally, upon comparing (3) and (1), we obtain:

$$A = \frac{\beta^2}{1-\beta^2} Q\,, \tag{4}$$

so

$$Q = \Delta M_0\, c^2 \sqrt{1-\beta^2} = -\,\Delta\mathcal{L}, \tag{5}$$

$\Delta\mathcal{L}$ being the variation of the Lagrange function of the body in the process envisioned that is due to the variation of the proper mass. Therefore, as we have stated, in order for the body to preserve the constant velocity $v = \beta c$ in the reference system $R$ when it receives the quantity of heat $Q$, it is necessary that this input of heat be completed by an input of work $A$ that is given by (4). In that relation, we easily infer, moreover, that:

(6) $$\Delta M_0 = \frac{\sqrt{1-\beta^2}}{c^2}\,(A + Q) = \frac{Q}{c^2\sqrt{1-\beta^2}}\,.$$

One sees that all of these considerations finally describe the principle of the energy of inertia, which permits one to envision variations of the proper mass of a body that result from the variation of its internal energy.

Now, return to the reference system $R_0$ . Since the body keeps an invariant form there, no work is done in this system during the process envisioned. Seen in that system, the operation that is performed must therefore consist uniquely in the input of a quantity of heat $Q_0$ to the body $C$, such that:

(7) $$\frac{Q_0}{c^2} = \Delta M_0 = \frac{1}{\sqrt{1-\beta^2}}\frac{Q}{c^2}\,.$$

During the passage from $R_0$ to $R$, the quantity of heat transforms according to the law:

(8) $$Q = Q_0\sqrt{1-\beta^2}\;,$$

and since the entropy $S = \int \frac{dQ}{T}$ is invariant, it then results that the absolute temperature of the body must transform according to the law:

(9) $$T = T_0\sqrt{1-\beta^2}\;.$$

This is the fundamental formula that gives the transformation of temperature when one passes from $R_0$ to $R$.

We further remark that the important formula (5) can be obtained in the following fashion: We start with the definition of energy $W = \sum_{i=1}^{3} p_i\dot{q}_i - \mathcal{L}$, when applied to the body $C$ in its translational motion. Since the $\dot{q}_i$ are constant, we have:

(10) $$dW = \sum_{i=1}^{3} \dot{q}_i\,dp_i - d\mathcal{L}.$$

Now:

(11) $$\sum_{i=1}^{3} \dot{q}_i\,dp_i = \sum_{i=1}^{3} \dot{p}_i\,dq_i = dA,$$

because $\dot{p}_i = f_i$ , so $dW = dA - d\mathcal{L}$, and upon integrating this over the entire process imagined, one gets:

(12) $$\Delta W = A - \Delta\mathcal{L}.$$

Since $\Delta W = A + Q$, from the conservation of energy, one has:

$$Q = -\Delta\mathcal{L}. \tag{13}$$

One then infers that:

$$\Delta S = \frac{Q}{T} = -\frac{\Delta\mathcal{L}}{T} \qquad \text{or} \qquad \frac{1}{T} = -\frac{\partial S}{\partial\mathcal{L}}. \tag{14}$$

**3. Extension to the case where the volume of the body *C* varies.** – We shall repeat the preceding reasoning while supposing that the volume of the body *C* varies. We appeal to the fact that in the relativistic theory of elasticity the pressure in an isotropic body is an invariant, in such a way that we can set $p = p_0$.

We place ourselves in the reference system *R*. While the velocity of the body *C* remains constant, one provides it with a quantity of heat *Q* and work *A* while its volume varies by $\Delta V$. The body *C* is assumed to be isotropic, and its internal pressure is equal to *p*. The conservation of energy permits us to write:

$$\Delta W = \frac{\Delta M_0 c^2}{\sqrt{1-\beta^2}} = Q + A - p\,\Delta V, \tag{15}$$

and since one always has the relation (3), which is proved as before, one easily finds that:

$$A = \frac{Q - p\Delta V}{1-\beta^2}\,\beta^2. \tag{16}$$

One deduces from (15) and (16) that:

$$\Delta M_0 = \frac{Q - p\Delta V}{c^2}\,\frac{1}{\sqrt{1-\beta^2}}. \tag{17}$$

On the other hand, in the reference system $R_0$, where $A = 0$, one has:

$$\frac{\Delta W}{c^2} = \Delta M_0 = \frac{Q_0 - p_0 \Delta V_0}{c^2}, \tag{18}$$

and comparing this with (17) gives:

$$Q - p\,\Delta V = (Q_0 - p_0\,\Delta V)\sqrt{1-\beta^2}. \tag{19}$$

Moreover, upon substituting (16) into (15), one gets:

(20) $$\Delta W = \frac{Q - p\Delta V}{1-\beta^2} = \frac{T\Delta S - p\Delta V}{1-\beta^2},$$

whereas, in the system $R_0$, one gets:

(21) $$\Delta W_0 = T_0\,\Delta S_0 - p_0\,\Delta V_0\,.$$

We then find that:

(22) $$\left(\frac{\partial W}{\partial S}\right)_V = \frac{T}{1-\beta^2}, \qquad \left(\frac{\partial W_0}{\partial S}\right)_{V_0} = T_0\,,$$

and since $dS = dS_0$ and $dW = \frac{dW_0}{\sqrt{1-\beta^2}}$, one comes back to formula (9) for the transformation of temperature.

**4. The "inverse of temperature" quadri-vector.** – Various authors – notably, Tolman and Eckart, van Dantzig and Bergmann – have proposed giving a tensorial variance to temperature by considering the inverse of the temperature to be the temporal component of a quadri-vector whose spatial components will be zero in the proper system of the body. One will indeed then have:

(23) $$\frac{1}{T} = \frac{1}{T_0\sqrt{1-\beta^2}},$$

in accord with formula (9).

For example, one can define a quadri-vector $\boldsymbol{\theta}$ on spacetime by:

(24) $$\theta^i = \frac{U^i}{T_0},$$

where $U^i$ is the "world velocity" of the proper system of the body *C*, because then one will indeed have $\theta^4 = 1 / T$.

In my opinion, despite the elegance of that representation, its exactitude remains doubtful, since one can hardly see what the physical sense would be of the spatial components of the quadri-vector $\boldsymbol{\theta}$. One can make the same representation for the volume of a body that transforms according to:

$$V = V_0\sqrt{1-\beta^2}$$

from the Lorentz contraction, like temperature. One sets:

$$\xi^i = \frac{U^i}{V_0}, \qquad \text{so} \qquad \xi^4 = \frac{1}{V}.$$

However, the physical sense of the components $\xi^1$, $\xi^2$, and $\xi^3$ does not appear to be obvious, and it is doubtful whether such a representation of the Lorentz contraction means anything. This can lead to distrust of the representation (24), and one contents oneself with the transformation law (9).

**5. Refinement and extension of the formula** $Q = -\Delta\mathcal{L}$. – We proved formula (13) by supposing that the velocity of the body $C$ remains constant. We shall free ourselves of that hypothesis and show that one can always write the expression for the quantity of heat that is provided to a body whose proper mass varies as:

$$\delta Q = -\delta_{M_0}\mathcal{L}, \tag{25}$$

where $\delta_{M_0}\mathcal{L}$ represents the variation that the Lagrange function of the body is subject to when its proper mass varies, *while all of the other variables that $\mathcal{L}$ depends upon remain constant.* That extension of formula (13) will play an important role in the rest of our presentation.

First, take the case of the motion of a body in the absence of an external field, for which we can set $\mathcal{L} = -M_0c^2\sqrt{1-\beta^2}$, and recall the proof in paragraph 2, but without assuming that the velocity is constant. One has:

$$\delta W = \delta \frac{M_0c^2}{\sqrt{1-\beta^2}} = \frac{\delta M_0c^2}{\sqrt{1-\beta^2}} + \frac{M_0c^2\beta\,\delta\beta}{(1-\beta^2)^{3/2}} \tag{26}$$

and

$$\begin{aligned} Fv\,\delta t &= \delta A = \delta \frac{M_0v}{\sqrt{1-\beta^2}}\,v \\ &= \frac{\delta M_0v^2}{\sqrt{1-\beta^2}} + \frac{M_0v\,\delta v}{\sqrt{1-\beta^2}} + \frac{M_0v^2\beta\,\delta\beta}{(1-\beta^2)^{3/2}}. \end{aligned} \tag{27}$$

One infers from this that:

$$\begin{aligned} \delta W - \delta A &= \frac{\delta M_0(c^2-v^2)}{\sqrt{1-\beta^2}} - \frac{M_0v\,\delta v}{\sqrt{1-\beta^2}} + \frac{M_0\beta\,\delta\beta(c^2-v^2)}{(1-\beta^2)^{3/2}} \\ &= \delta M_0c^2\sqrt{1-\beta^2}. \end{aligned} \tag{28}$$

One thus has:

$$\delta Q = \delta W - \delta A = \delta M_0c^2\sqrt{1-\beta^2} = -\delta_{M_0}\mathcal{L}, \tag{29}$$

even when the velocity varies.

We now pass on to the more general case of a body (a particle, for example) that possesses an electric charge $\varepsilon$ and displaces in an electromagnetic field that is defined by the potentials $V$ and **A**. We then have:

$$W = \frac{M_0 c^2}{\sqrt{1-\beta^2}} + \varepsilon V. \tag{30}$$

Introduce the quantity:

$$\mathcal{F} = \frac{M_0 v^2}{\sqrt{1-\beta^2}} + \frac{\varepsilon}{c}\mathbf{A}\cdot\mathbf{v} = \left(\frac{M_0 \mathbf{v}}{\sqrt{1-\beta^2}} + \frac{\varepsilon}{c}\mathbf{A}\right)\cdot\mathbf{v} = \mathbf{p}\cdot\mathbf{v}. \tag{31}$$

This quantity, which reduces to the pseudo-vis viva for **A** = 0, is the natural generalization for $\mathbf{A} \neq 0$. Finally, we have:

$$\mathcal{L} = -M_0 c^2 \sqrt{1-\beta^2} - \varepsilon V + \frac{\varepsilon}{c}\mathbf{A}\cdot\mathbf{v} \tag{32}$$

here.

We then see that:

$$W = \mathcal{F} - \mathcal{L}, \tag{33}$$

and we deduce from this, while compensating for the terms in **A**, that:

$$\delta W = \delta\mathcal{F} - \delta\mathcal{L} = \delta\frac{M_0 c^2}{\sqrt{1-\beta^2}} + M_0 c^2\,\delta\sqrt{1-\beta^2} + \varepsilon\,\delta V + \delta M_0 c^2\sqrt{1-\beta^2}\ . \tag{34}$$

One easily verifies that it is equivalent to write:

$$\delta W = M_0 c^2\,\delta\frac{1}{\sqrt{1-\beta^2}} + \varepsilon\,\delta V + \frac{\delta M_0 v^2}{\sqrt{1-\beta^2}} + \delta M_0 c^2\sqrt{1-\beta^2}\ . \tag{35}$$

The first two terms in the right-hand side of (35) represent the work that is done on the body whose proper mass remains constant, while the third term represents the work done on it that corresponds to the increase in its proper mass. In summation, the first three terms in question thus represent the total work that is done on the body during a time interval $\delta t$. Since we must always have:

$$\delta W = \delta A + \delta Q,$$

we must have that the last term in the right-hand side of (35) is equal to $\delta Q$, which indeed gives us formula (25) again.

## CHAPTER V

# ANALOGIES BETWEEN MECHANICAL QUANTITIES AND THERMODYNAMIC QUANTITIES

### (Helmholtz's theory and Boltzmann's formula for periodic systems)

**1. Generalities.** – In the foregoing, we have summarized the well-known interpretation of thermodynamic quantities with the aid of statistical mechanics, an interpretation in which both the laws of mechanics and the concepts of the calculus of probabilities intervene simultaneously. The success of that interpretation, which is, above all, due to the magnificent work of Boltzmann and Gibbs, has made some forget some other attempts that were made in the same era by Helmholtz and Boltzmann himself, attempts in which one tries to recover certain concepts and laws of thermodynamics with the aid of only mechanical considerations without introducing any probabilistic ideas. Since the first law of thermodynamics, in which one assumes that heat is an energy of molecular agitation, immediately comes down to the mechanical theorem of the conservation of energy, it is essentially the interpretation of the second law of thermodynamics and the notion of entropy that it is related to it that forms the object of the theories of Helmholtz and Boltzmann on the subject.

These attempts at mechanical, but not statistical, explanations of the second law of thermodynamics remain incomplete, and lead one to only some very fragmentary results that apply to only some specialized models. The success of the statistical interpretation of thermodynamics has had the result of making people abandon them, and, due in part to the work of Ehrenfest on adiabatic invariance, they do not seem to have been the object of any new research for some sixty years. They are nonetheless interesting, and it might be that there is something very profound that is hidden behind the analogies that they reveal.

**2. Helmholtz's theory.** Helmholtz started with some very general considerations regarding a mechanical system that is defined by Lagrange variables $q_i$. He supposed that this system is subject to internal forces that are derived from a potential $U$ and external forces whose work done on the coordinate $q_i$ is denoted by $\mathcal{A}_i\, dq_i$. One will always denote the absolute temperature by $T$ here and the kinetic energy by $E_{\text{kin}}$.

The Lagrange equations for the system are written:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \mathcal{A}_i, \tag{1}$$

and Helmholtz, who wrote long before the birth of the theory of relativity, utilized the classical definition of the Lagrange function:

$$\mathcal{L} = E_{\text{kin}} - U, \tag{2}$$

$E_{\text{kin}}$ being a function of the $q_i$ and $\dot{q}_i$ that is homogeneously quadratic in the $\dot{q}_i$.

Helmholtz then introduced the fundamental hypothesis that the parameters $q_i$ divide into two categories: The one consists of $q_a$ that vary quite slowly, and the other consists of $q_b$ that vary quite rapidly. This hypothesis is certainly suggested by the molecular conception of matter, in which the $q_b$ would be, for example, the coordinates of the gas molecules and the $q_a$ would be the infinitely slower variables that determine the external configuration of the system.

Moreover, Helmholtz further assumed that the potential energy $U$ depends only upon the $q_b$ and that the coordinates $q_a$ enter in only by way of their derivatives $\dot{q}_b$ in the expression for $E_{\text{kin}}$ and therefore $\mathcal{L}$. This permits one to write:

$$\frac{\partial \mathcal{L}}{\partial q_b} = 0, \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_b}\right) = \dot{p}_b = \mathcal{A}_b . \tag{3}$$

for any $q_b$ .

By definition, the elementary work done on the coordinate $q_b$ will be:

$$dQ_b = \mathcal{A}_b \, dq_b = \dot{p}_b \dot{q}_b dt = \dot{q}_b dp_b . \tag{4}$$

As for the parameters $q_a$ , since they are, by hypothesis, slowly-varying, their contributions to the terms in $d / dt$ can be neglected, and one will have:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_a}\right) = 0, \qquad -\frac{\partial \mathcal{L}}{\partial q_a} = \mathcal{A}_a . \tag{5}$$

To commence, Helmholtz then considered systems that he called “monocyclic,” for which there is only one parameter $q_b$ that varies rapidly, and he proved the following curious result: *For a monocyclic system, for which one can set* $Q_b = Q$*, the quotient* $dQ / E_{\text{kin}}$ *is an exact differential.*

Indeed, let $q$ be the unique parameter of the system that varies rapidly. We have $dQ = \dot{q}\, dp$, and as a result:

$$\frac{dQ}{\frac{1}{2} p\dot{q}} = \frac{2dp}{p} = 2d\,(\log p). \tag{6}$$

Now, in classical mechanics one has:

$$2\, E_{\text{kin}} = \sum_i \frac{\partial E_{\text{kin}}}{\partial \dot{q}_i} \dot{q}_i , \tag{7}$$

because $E_{\text{kin}}$ is a homogeneously-quadratic function of $\dot{q}_i$. Now, only the coordinate $q$ is rapidly-varying, in such a way that:

(8) $$2\,E_{\text{kin}} \approx \frac{\partial E_{\text{kin}}}{\partial \dot{q}}\dot{q} = p\dot{q}\,,$$

and as a result:

(9) $$\frac{dQ}{E_{\text{kin}}} = 2d\,(\log p) = \text{exact diff.}$$

This is Helmholtz's theorem.

However, $dQ$, which is the work done by the rapidly-varying parameter, is analogous to the energy that is provided to a gas molecule and can thus be assimilated into a quantity of elementary heat. Upon defining the absolute temperature $T$ as being proportional to the kinetic energy, one can set:

(10) $$\frac{dQ}{T} = dS,$$

and that relation reverts to the definition of entropy in thermodynamics. Moreover, Helmholtz's theory remains valid in relativistic theory, since then it is $\frac{1}{2}p\dot{q}$ that one must consider to be proportional to the temperature, in such a way that the relation (6) again leads to the formula (10).

However, the case of monocyclic systems is too specialized, and there is good reason to consider the case of polycyclic systems that involve several rapidly-varying variables $q_b$ . Whether the system is monocyclic or polycyclic, one must, moreover, with Helmholtz, distinguish the "complete systems" from the "incomplete systems," the latter being the ones for which the work $\mathcal{A}_a\,dq_a$ that corresponds to the variation of at least one of the slowly-varying parameters $q_a$ is zero. Let $q_c$ be the slowly-varying parameters that enjoy that property. One has:

(11) $$\frac{\partial \mathcal{L}}{\partial q_c} = 0$$

for $q_c$ .

Since, by hypothesis, $\mathcal{L}$ does not depend upon the $q_b$ and the $\dot{q}_a$ are negligible, the relations (11) couple the $q_a$ , the $\dot{q}_a$ , and the $q_c$ . Since there are just as many relations as there are $q_c$ , they provide an expression for the latter as functions of the $q_a$ and $\dot{q}_b$ . One can thus eliminate the $q_c$ and define the state of the system as a function of the $q_a$ and $\dot{q}_b$ .

Now, let $\mathcal{L}'$ be the expression for $\mathcal{L}$ when one expresses it with the aid of only the $q_a$ and $\dot{q}_b$ . From (11), one has:

(12) $$\left\{ \begin{aligned} \frac{\partial \mathcal{L}'}{\partial q_a} &= \frac{\partial \mathcal{L}}{\partial q_a} + \sum_c \frac{\partial \mathcal{L}}{\partial q_c}\frac{\partial q_c}{\partial q_a} = \frac{\partial \mathcal{L}}{\partial q_a}, \\ \frac{\partial \mathcal{L}'}{\partial \dot{q}_b} &= \frac{\partial \mathcal{L}}{\partial \dot{q}_b} + \sum_c \frac{\partial \mathcal{L}}{\partial q_c}\frac{\partial q_c}{\partial \dot{q}_b} = \frac{\partial \mathcal{L}}{\partial \dot{q}_b}. \end{aligned} \right.$$

One must then have

$$(13) \qquad -\frac{\partial \mathcal{L}'}{\partial q_a} = \mathcal{A}_a$$

for the slowly-varying parameters $q_a$ and:

$$(14) \qquad \mathcal{A}_b = \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{q}_b}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_b}\right) = \dot{p}_b$$

for the $q_b$ , and one recovers, after eliminating the $q_c$ :

$$(15) \qquad dQ_b = \dot{q}_b dp_b \ .$$

All of the equations keep the same form for the incomplete systems as they have for complete systems. In particular, one always has that $dQ / E_{\text{kin}}$ is an exact differential for monocyclic systems, even incomplete ones.

Nevertheless, Helmholtz insisted upon the fact that the incomplete systems differ from the complete systems on one important point: The kinetic energy is, in the Newtonian approximation, a homogeneous, quadratic function of the $\dot{q}_i$ whose coefficients can depend upon the $q_i$ , but when one replaces the $q_c$ as functions of the $q_a$ and the $\dot{q}_b$ , the kinetic energy can cease to be quadratic in the $\dot{q}_b$ , and can even be of odd degree in the $\dot{q}_b$ , and therefore of odd degree with respect time, a circumstance that is important because the reversibility with respect to time then disappears. This case presents itself, for example, for a rotator that is endowed with a governor: The energy of the rotator is not proportional to the square of the angular velocity, because its moment of inertia varies with the velocity. Helmholtz sought to infer an interpretation of irreversible thermodynamics from this fact, but Henri Poincaré, who had presented Helmholtz's theory on another occasion in the last chapter of his treatise on thermodynamics, gave an argument to prove that one does not really arrive at an explanation for the existence of irreversibility.

For Helmholtz, the $q_b$ correspond to visible molecular motions. As the $q$ vary, the energy of the system varies by:

$$dE = \sum_a \mathcal{A}_a dq_a + \sum_b \mathcal{A}_b dq_b \ .$$

Now, the first term is the work that is done on the system that corresponds to the visible motions, while:

$$\sum_b \mathcal{A}_b dq_b = \sum dQ_b = dQ$$

is the total energy that is provided by the motion of the molecules, and thus, the heat that is received by the system. One thus has:

$$dQ = dE + d\mathcal{T}, \tag{16}$$

where $d\mathcal{T}$ is the work *done by* the system *on* the environment, and this is an expression of the first law of thermodynamics, translated into the equivalence of heat and work. Moreover, since we have found that $dQ / E_{\text{kin}}$ is an exact differential for monocyclic systems, we have – at least, in this case – recovered the two fundamental principles of thermodynamics that are valid for reversible transformations.

It is obvious that the results of Helmholtz, thought interesting, are quite restrictive, since they apply only to monocyclic systems. Moreover, with Helmholtz, we have assumed that the potential energy does not depend upon the rapidly-varying parameters $q_b$ : This hypothesis is exact for perfect gases, whose molecules all have no mutual interactions, but it is no longer true for real gases and, *a fortiori*, for liquids and solids. We shall verify later on that Boltzmann, inspired by Helmholtz's theory, generalized it to the polycyclic systems whose potential energy depends upon some rapidly-varying parameters $q_b$ , but he was obliged to assume that that these systems are periodic and to introduce means taken with respect to time over a period of motion. In conclusion, we note that the notion of probability is not introduced into the Helmholtz theory anywhere.

**3. The canonical schema for thermodynamics, after Helmholtz.** – Independently of the interesting, but insufficient, considerations that we just recalled, Helmholtz proposed a curious "canonical schema" for thermodynamics.

Start with the classical relation:

$$dE = dQ - p\,dV = T\,dS - p\,dV = T\,dS + \sum_i \mathcal{A}_i\,dq_i\,, \tag{17}$$

and, with Helmholtz, introduce a variable $\varepsilon$ such that, by definition, the temperature is the "velocity" $\dot{\varepsilon}$ that corresponds to that variable. We then set:

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = T. \tag{18}$$

If $\mathcal{E}$ denotes the generalized force that corresponds to $\varepsilon$ then one will have:

$$dE = E\,d\varepsilon - p\,dV = \mathcal{E}\,\dot{\varepsilon}\,dt - p\,dV, \tag{19}$$

so, by indentifying this with $dS = \dfrac{dE + p\,dV}{T}$:

$$\mathcal{E}\,\dot{\varepsilon}\,dt = T\,dS \tag{20}$$

and taking (18) into account, we get:

(21) $$\mathcal{E} = \dot{S}\,.$$

We suppose that $\varepsilon$ is a variable of the type called “cyclic;” i.e., such that $\mathcal{L}$ does not depend upon $\varepsilon$ (so $\partial\mathcal{L} / \partial\varepsilon = 0$).

Now, envision an extremely slow reversible process for which $\dot{V} \approx 0$; we have $\partial\mathcal{L} / \partial\dot{V} = 0$, and the Lagrange equations for the variables $\varepsilon$ and $V$ are:

(22) $$-\frac{\partial\mathcal{L}}{\partial V} = -p, \qquad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varepsilon}}\right) = \mathcal{E},$$

which gives us:

(23) $$p = \frac{\partial\mathcal{L}}{\partial V}, \qquad \frac{\partial\mathcal{L}}{\partial\dot{\varepsilon}} = p_\varepsilon = \int \mathcal{E}\,dt = \int \dot{S}\,dt = S,$$

hence:

(24) $$p_\varepsilon = S.$$

From its general definition, the energy will then be given by:

(25) $$E = \dot{\varepsilon}\frac{\partial\mathcal{L}}{\partial\dot{\varepsilon}} + \dot{V}\frac{\partial\mathcal{L}}{\partial\dot{V}} - \mathcal{L} = TS - \mathcal{L},$$

and one infers from this:

(26) $$\mathcal{L} = TS - E = -F,$$

in which $F = E - TS$ is the free energy.

The correspondence (26) between the Lagrange function and the free energy (with the sign changed) is very interesting: It plays a significant role in Planck’s old work on black-body radiation and in various works on electrostatics and electromagnetism.

For $T$ = const., one will have:

(27) $$d\mathcal{T} = p\,dV = -d(E - TS) = -dF.$$

Helmholtz’s canonical schema for thermodynamics essentially starts with the introduction of a variable $\varepsilon$ such that the temperature is its derivative with respect to time, but the significance of the variable $\varepsilon$ remains mysterious. In the thermodynamics of the isolated particle, we will recover the relation (18), while giving a precise sense to the variable $\varepsilon$.

**4. Boltzmann’s theory for periodic systems. Preliminary formula.** – More precise than the preceding theory is Boltzmann’s theory, which establishes an analogy between mechanical quantities and thermodynamic quantities in the case of periodic systems. In order to develop that theory, it is necessary to first examine what the principle of stationary action becomes when one applies it to certain periodic systems.

We first suppose that the system envisioned is defined by $N$ Lagrange coordinates $q_i$ . Like Helmholtz did in the theory that we previously analyzed, we assume that the $q_i$ are divided into two categories: The one consists of Helmholtz's $q_b$ , which are rapidly-varying and correspond to molecular motions; let $q_1, \ldots, q_r$ be these coordinates. The other ones $q_i$ are of Helmholtz's type $q_a$ and vary slowly; they correspond to the constraints to which the system is subject; we denote them by $q_{r+1}, \ldots, q_N$ .

We let $U^*$ denote the potential energy that corresponds to the constraints. The Lagrange function will be:

$$\mathcal{L} = \mathcal{L}_1 - U - U^*, \tag{28}$$

where $U$ is the potential energy that corresponds to the molecular motions, and $\mathcal{L}_1$ is the kinetic term, which is equal to the kinetic energy of the molecules in the Newtonian approximation, since the kinetic energy that corresponds to coordinates of the type $q_a$ is negligible, at least in very slow processes.

Let $A$ be Maupertuisian action integral that corresponds to the motion of the molecules; i.e., to the parameters of type $q_b$ . One has:

$$A = \int \sum_{k=1}^{r} p_k dq_k \ . \tag{29}$$

The total Maupertuisian action, taking into account the variations of the constraints – i.e., the variation of the $q_a$ – is:

$$\mathcal{A}^* = \mathcal{A} + \int \sum_{k=r+1}^{N} p_k dq_k \ . \tag{30}$$

The theory of Maupertuisian action that was presented in the first chapter then gives the formula:

$$\delta\mathcal{A}^* = \int \delta E^* \, dt + \left| \sum_{k=1}^{N} p_k \delta q_k \right|_0^1 . \tag{31}$$

where $E^*$ is the total energy of the system, which is the sum of the energy $E$ of the molecular motions and the potential energy $U^*$. (One thus has $E^*$ = kinetic energy of the molecules + $U + U^* = E + U^*$).

Now, the formula for varied action, when one varies the limits of $q$, gives us:

$$\delta\mathcal{A}^* = \delta\mathcal{A} + \delta \int \sum_{k=r+1}^{N} p_k dq_k = \delta\mathcal{A} + \int \delta \sum_{k=r+1}^{N} p_k dq_k + \left| \sum_{k=r+1}^{N} p_k \delta q_k \right|_0^1 . \tag{32}$$

Substitute this value for $\delta\mathcal{A}^*$ into the preceding equation, while taking into account the relation:

$$\delta E^* = \delta E + \delta U^*,$$

and one gets:

(33) $$\delta\mathcal{A} = \int\left[\delta E + \delta U^* - \delta\sum_{k=r+1}^{N} p_k\dot{q}_k\right]dt + \left|\sum_{k=1}^{r} p_k\delta q_k\right|_0^1 .$$

Now, there exist some important cases in which the terms in $p_k\ \delta q_k$ disappear. That is what happens, for example, when the values of the coordinates $q_{r+1}$ , …, $q_N$ of the type $q_a$ remain constant in the varied motion and when, moreover, the current motion and the varied motion are both periodic. The first hypothesis entails that the $\dot{q}_a$ are zero in both the natural motion and the varied motion, in such a way that:

$$\delta\sum_{k=r+1}^{N} p_k\dot{q}_k = 0.$$

The second hypothesis entails that $\sum_{k=1}^{r} p_k dq_k$ has the same value at the two extremities of the unvaried trajectory, since these two extremities coincide, as is illustrated in Figure 4.

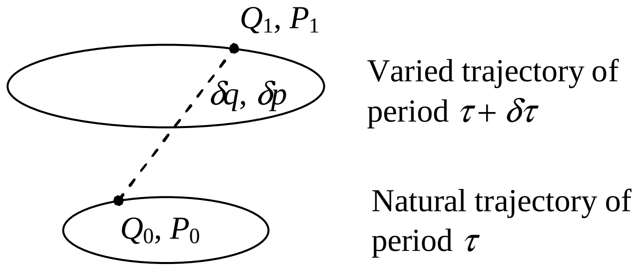

Figure 4.

What then remains is:

(34) $$\delta\mathcal{A} = \int_0^{\tau}(\delta E + \delta U^*)\,dt ,$$

with:

(35) $$\mathcal{A} = \oint\sum_{k=1}^{r} p_k dp_k ,$$

$\tau$ being the period of the natural motion. This is the formula that we need.

**5. Boltzmann's formula for periodic systems.** – In 1897, Boltzmann recalled the earlier work of Clausius and Szily (1872) by using formula (34) in order to obtain a very interesting formula that was employed later on by Ehrenfest in his theory of adiabatic invariants.

In order to prove Boltzmann's formula (which should not be confused with the relation $S = k \log P$), we shall recall some hypotheses that were made above: If a system that is defined by $r$ rapidly-varying "molecular" coordinates of Helmholtz type $q_b$ and $N$ –

$r$ slowly-varying constraint coordinates of Helmholtz type $q_a$ then the variation is performed with constant values of the $q_a$ , and periodicity of the natural motion, as well as the varied motion.

In a general manner, consider a trajectory $AB$ that corresponds to a state of the system and a trajectory $CD$ that corresponds to a neighboring state (Fig. 5).

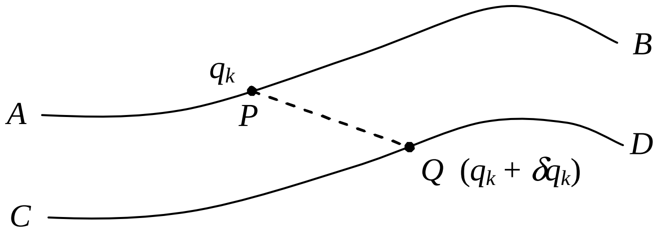

Figure 5.

In order to pass from $P$ to $Q$, one must act upon all of the molecules by giving them energy; i.e., by providing heat. That heat serves to augment the internal energy and to accomplish an external work $\delta\mathcal{T} = \delta U^*$. Therefore:

$$\delta Q = \delta E + \delta U^* . \tag{36}$$

Now, suppose that one passes very slowly from $P$ to $Q$ by means of a reversible transformation, and remark that the coordinates $q_r$ , …, $q_N$ have values on the trajectories $AB$ and $CD$ that are different, but constant and close to each other, in such a way that the first hypothesis that was assumed in the preceding paragraph is found to be verified. Let $AD$ be the curve that represents the slow passage from one trajectory to the other in a time $t_1 - t_0$ .

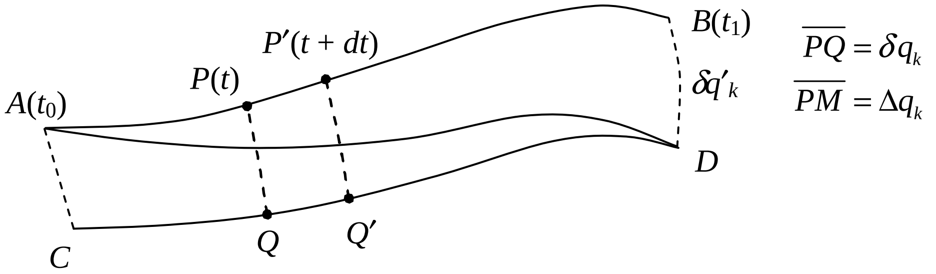

Figure 6.

One has approximately:

$$\Delta q_k = \frac{t - t_0}{t_1 - t_0}\,\delta q_k ,$$

and in a time $dt$, $\Delta q_k$ varies by:

$$d\,\Delta q_k = \frac{dt}{t_1 - t_0}\,\delta q_k\,.$$

During the same time *dt*, the heat that is provided to the system is:

$$d\,\delta Q = \frac{dt}{t_1 - t_0}\,\delta Q\,,$$

and the work that is done is:

$$d\,\delta U^* = \frac{dt}{t_1 - t_0}\,\delta U^*.$$

Therefore, for the total transformation *AMD*, one has:

$$\text{(37)} \qquad \Delta Q = \int_{t_0}^{t_1} d\delta Q = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \delta Q\,dt = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} (\delta E + \delta U^*)\,dt\,.$$

Now, introduce the hypothesis that the system is periodic with period $\tau$ and choose the time interval to be equal to $\tau$. Upon conferring formula (34) that was obtained above, one gets:

$$\text{(38)} \qquad \delta Q = \frac{\delta \mathcal{A}}{\tau} = \frac{1}{\tau}\delta\oint\sum_{k=1}^{r} p_k\,dq_k = \nu\,\delta\mathcal{A}, \qquad \text{with} \quad \nu = \frac{1}{\tau}.$$

This formula is general – i.e., it is as valid for relativistic mechanics as it is for classical mechanics. If we limit ourselves to the Newtonian approximation then we have $2E_{\text{kin}} = \sum_{k=1}^{r} p_k \dot{q}_k$ and as a result:

$$\text{(39)} \qquad \delta Q = \frac{1}{\tau}\delta\int_0^{\tau}\sum_{k=1}^{r} p_k\,dq_k = \frac{2}{\tau}\delta(\bar{E}_{\text{kin}}\tau)\,,$$

where $\bar{E}_{\text{kin}}$ is the mean value of the kinetic energy for a cycle of motion ([5]).

We have thus obtained the curious formula of Boltzmann, which has a good number of applications in the context of the old theory of quanta from around 1920-1925. It was upon starting with that formula that Ehrenfest developed his beautiful theory of adiabatic invariants, which has, in turn, been transposed into wave mechanics. Léon Brilluoin has formerly devoted some beautiful presentations of the theory of adiabatic invariants and gave numerous examples of some diverse applications of Boltzmann's formula (39) to phenomena. Notably, he deduced a very simple way of deducing the formula that Wien proved by a thermodynamic argument for the spectral distribution of black-body radiation ([6]).

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([5]) Francis Fer recently communicated a new proof of Boltzmann's formula (39) to me that seems more rigorous than the one that was given above. *See* bibliography [**12**].

([6]) *See*, in particular, bibliography [**1**], chapter VII, and note attachment 2.

CHAPTER VI

# THE ORIGIN OF WAVE MECHANICS AND ITS INTERPRETATION BY THE THEORY OF THE DOUBLE SOLUTION

**1. Cyclic frequency and wave frequency.** – The author formerly arrived at the first principles of wave mechanics by reflecting upon the difference between the relativistic transformations of the frequency of a clock and the frequency of a wave. This question was of great interest to us, and since it was generally silently passed over in the treatises on quantum mechanics, we shall stop to discuss it briefly.

We know that the transformation formulas for energy and temperature when one passes from the proper system of a body to another Galilean system that is animated with the velocity $\beta c$ with the respect to the first one are:

$$W = \frac{W_0}{\sqrt{1-\beta^2}}, \qquad T = T_0 \sqrt{1-\beta^2} \,. \tag{1}$$

We see that, whereas the energy is smaller in the proper system than it is in the other one, the opposite situation is true for temperature. The difference between the two types of transformation is the same as the one that exists for the frequency of a wave and the frequency of a clock (i.e., the wave frequency and cyclic frequency, resp.).

In a reference system $R_0$ , consider a periodic process that varies sinusoidally in the course of time. It will be represented by a tracking variable that has the expression:

$$q_0 = a_0 \sin 2\pi\nu_0 \, t_0 \,, \tag{2}$$

with a convenient choice of time origin $t_0$ . The process in question can be, for example, the motion of a clock fixed in $R_0$ and then the variable $q_0$ can be the projection of the extremity of a hand of the clock onto its face.

The theory of relativity tells us that that for an Galilean observer who sees the clock displace with the velocity $\beta c$ , the cyclic motion of that clock seems slowed down by the ratio $\sqrt{1-\beta^2}$ , in such a way that the tracking variable will have an expression of the form:

$$q = a_0 \sin 2\pi\nu_c \, t, \tag{3}$$

where $\nu_c$ is the "cyclic frequency" of the clock for that observer. From the relativistic formula for the slowing-down of clocks, one has:

$$\nu_c = \nu_0 \sqrt{1-\beta^2} \,. \tag{4}$$

Now imagine that an immobile clock of frequency $\nu_0$ is placed at every point of the reference system $R_0$ . Thus, a periodic phenomenon of frequency $\nu_0$ will be found to be defined at each point of $R_0$ , and the *totality* of corresponding variables $q_0$ , which are assumed to all be in phase, will define a stationary wave in $R_0$ whose expression will be:

$$\Psi_0 = a_0 \sin 2\pi\nu_0\, t_0 \tag{5}$$

at any point of $R_0$ .

We pass to another Galilean reference system $R$ that is animated with respect to $R_0$ with the velocity $\beta c$ and take the $z$-axis in $R$ to be the direction of the relative velocity of $R$ with respect to $R_0$ . The Lorentz transformation shows that in the reference system $R$ the stationary wave that is defined by the totality of all $\Psi_0$ takes the form of a progressive wave with the expression:

$$\Psi = a_0 \sin 2\pi\nu_0\, \frac{t - \frac{\beta}{c} z}{\sqrt{1-\beta^2}} = a_0 \sin 2\pi \left( \nu t - \frac{z}{\lambda} \right), \tag{6}$$

with

$$\nu = \frac{\nu_0}{\sqrt{1-\beta^2}}, \qquad V = \frac{c}{\beta}, \qquad \lambda = \frac{V}{\nu} = \frac{c\sqrt{1-\beta^2}}{\nu_0 \beta}. \tag{7}$$

The expression for $\Psi$ gives the distribution of phases of the clock as it is observed by the observer $R$: That distribution is then a wave that propagates along $Oz$ with the frequency $\nu$ and phase velocity $V$.

The formula:

$$\nu = \frac{\nu_0}{\sqrt{1-\beta^2}} \tag{8}$$

shows how the "wave frequency" transforms when one passes from the Galilean system $R_0$ , where the wave is stationary, to the system $R$, where it is progressive. That fundamental formula is well-known and plays a great role in many important problems of the theory of relativity; for example, the theory of the Doppler effect.

Now, the comparison with formula (1), (4), and (8) makes it clear that the energy and the wave frequency, on the one hand, and the temperature and cyclic frequency, on the other, transform in the same way. The first of these two facts permits us to suppose that the relation $W = h\nu$ between the energy of the corpuscle and the frequency of the wave that wave mechanics associates with it is valid in all of the Galilean systems if it is always realized in the proper system $R_0$ in the form:

$$W_0 = h\nu_0 = m_0\, c^2 \tag{9}$$

that makes the proper mass $m_0$ correspond to the proper frequency $\nu_0$ . This is the point of departure for wave mechanics.

In the first place, one can be surprised to see that the quantity of heat and the temperature, which transform like a cyclic frequency according to the formulas:

$$Q = Q_0\sqrt{1-\beta^2} \qquad \text{and} \qquad T = T_0\sqrt{1-\beta^2}\ ,$$

do not transform like the energy does. Indeed, one knows that the heat is a form of energy (viz., the energy of uncoordinated molecular motions) and that, furthermore, statistical thermodynamics has accustomed us to considering the temperature of a body as being proportional to the mean kinetic energy of its molecules. However, if one refers to the argument in chapter IV that permitted us to establish the relativistic variance of temperature then one sees that the global energy that is provided by a body divides into heat and work, and that this division is imposed by the fact – which was unknown in the theories before relativity – that an input of energy is capable of varying the proper mass of the body. The connection thus established, as a consequence of the principle of inertia and energy, between the internal energy of a body and the quantity of heat that it receives permits us to explain why the formula for the transformation is not the same for the energy and the quantity of heat. Furthermore, as far as the proportionality between the temperature and the mean kinetic energy of the molecules is concerned, one must remark that it is true only in the proper system of the body, and that even in that case it is valid only in the Newtonian approximation; this results from formulas that were given in paragraph 12 of chapter III. Therefore, nothing demands that the temperature should have the same relativistic transformation formula as energy.

**2. Wave mechanics in the geometrical optics approximation.** – The point of departure has been to associate the uniform, rectilinear motion of a free corpuscle with the propagation of a monochromatic plane wave that one can represent by the complex formula:

$$\psi = a\ e^{2\pi i\left(\nu t - \frac{z}{\lambda}\right)}, \tag{10}$$

where the $z$-axis is taken in the direction of propagation, which is the direction of motion of the corpuscle. If one defines the frequency $\nu_0$ of the stationary wave that is associated with the corpuscle in the proper system by formula (9) then the transformation formulas that were given in the preceding paragraph permit one to write:

$$\psi = a\ e^{2\pi i(Wt - pz)/h}, \tag{11}$$

with:

$$W = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \qquad \text{and} \qquad p = \frac{m_0 v}{\sqrt{1-\beta^2}},$$

where $W$ and $p$ are related to $\nu$ and $\lambda$ by the formulas:

$$W = h\nu, \qquad p = \frac{h}{\lambda}. \tag{12}$$

One sees that if one sets:

(13) $$\psi = a\ e^{i\varphi/\hbar}, \quad \text{with} \quad \hbar = \frac{h}{2\pi},$$

where $a$ and $\varphi$ are real, then the phase $\varphi$ coincides with the Hamiltonian action $A$ (with the sign changed), which is equal to $Wt - pz$.

If the corpuscle is subject to a force field then its trajectory is, in general, a curve, but one can always write its associated wave in the form:

(14) $$\psi(x, y, z, t) = a(x, y, z, t)\ e^{i\varphi(x,y,z,t)/\hbar}.$$

In the particular case where the field is permanent and where the propagation of the wave can be described by the geometrical optics approximation, one has:

(15) $$\psi = a(x, y, z)\ e^{-iA(x,y,z,t)/\hbar} = a(x, y, z)\ e^{-i\left(Wt - \int \mathbf{p}\cdot d\mathbf{s}\right)/\hbar},$$

$d\mathbf{s}$ being the trajectory element of the corpuscle, and one further has $\varphi = -A$. One again finds that $p = h / \lambda$, but here $p$ and $\lambda$ are functions of $x, y, z$. One then sees that Maupertuis's principle of least action, which is expressed by:

$$\delta \int_A^B \mathbf{p} \cdot d\mathbf{s} = 0,$$

coincides with Fermat's principle, when it is applied to a ray of a wave, which is expressed by:

$$\delta \int_A^B \frac{ds}{\lambda} = 0.$$

It then results that the rays of the wave then coincide with the possible trajectories of a corpuscle, and that one can preserve the fundamental idea of the localization of the corpuscle in space by assuming that is displaces along *one* ray of the wave. The energy and the quantity of motion of the corpuscle are then defined by the formulas:

(16) $$W = \frac{\partial \varphi}{\partial t}, \qquad \mathbf{p} = -\operatorname{grad} \varphi,$$

which shows the identity of the phase $\varphi$ and the Jacobi function $S$. Moreover, one then easily proves that if one supposes that all of the possible initial positions of the corpuscle before entering into the force field are equally probable then the probability of finding the corpuscle in the element $d\tau = dx\, dy\, dz$ of physical space is equal to:

$$a^2(x, y, z)\, dt = |\ \psi(x, y, z, t)\ |^2\, d\tau.$$

Unfortunately, the clear picture of the association of the wave and corpuscle no longer persists after the geometrical optics approximation is applied. Indeed, one can always find an expression of the form (14) for the wave, but $\psi$ is no longer simply related to the Hamiltonian action that is defined by point mechanics (classical or relativistic). One can no longer define rays in the sense of geometrical optics, and the notion of ray seems to be lost. These circumstances have led most theoreticians to assume that the corpuscle does not have a well-defined position at each instant, but is spread over the "potential state" in all the extent of the wave, while the wave itself is no longer an abstract representation of probabilities. One preserves the picture that is obtained in the geometrical optics approximation only by the following postulate: *The probability for the corpuscle to manifest its presence at the instant t in the volume element $d\tau$ is given by* $|\psi(x, y, z, t)|^2 \, d\tau$, but this postulate becomes completely arbitrary.

**3. The concept of a pilot wave.** – In the era around 1927-1928 in which this very abstract interpretation was developed, I sought to find another one that was more in accord with very concrete concepts that had guided me in my early work.

I started with the three following postulates:

1. The corpuscle must be localized in space at each instant and describe a continuous trajectory in the course of time.

2. The wave of wave mechanics must have physical reality and propagate in space in the course of time.

3. In order to interpret the phenomena in the optics of light and the optics of electrons, it is necessary to suppose that the corpuscle is intimately related to its wave, in such a way that the motion of the corpuscle is, in some way, guided by the propagation of the wave (7).

Guided in part by a hydrodynamical representation of the propagation of a wave $\Psi$ that had just been developed by Madelung, and which I have taken up again, I suppose that the corpuscle, which is always localized in its wave, follows one of the streamlines of the hydrodynamical picture in question. This immediately led me to suppose that if the corpuscle occupies the position $x$, $y$, $z$ at the instant $t$ then it possesses an energy $W$ and a quantity of motion $\mathbf{p}$ there that are given by formulas (16):

$$W = \frac{\partial \varphi}{\partial t}, \qquad \mathbf{p} = -\operatorname{grad} \varphi,$$

*even apart from the geometrical optics approximation.*

Since relativistic dynamics establishes the relation:

$$\mathbf{p} = \frac{W\mathbf{v}}{c^2}, \qquad \text{so} \qquad \mathbf{v} = \frac{\mathbf{p}c^2}{W}$$

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(7) In a remarkable article that appeared in 1953, M. Renninger shows that in the case of photons there exist absolutely convincing experimental reasons to assume these three postulates (bibliography [**2**]). *See* also [**4**], third reference.

for the free corpuscle, one can infer that:

$$\mathbf{v} = -\,c^2\,\frac{\operatorname{grad}\varphi}{\dfrac{\partial\varphi}{\partial t}} \tag{17}$$

in this case.

This fundamental formula, which I have called “the guidance formula,” in some way imposes a well-defined motion of the corpuscle in its wave. In the Newtonian approximation, where $W \approx m_0\,c^2$, it takes the simple form:

$$\mathbf{v} = -\frac{1}{m}\operatorname{grad}\varphi, \tag{18}$$

and it seems to be an extrapolation of the Jacobi formula:

$$\mathbf{v} = -\frac{1}{m}\operatorname{grad} S,$$

which is valid in the geometrical optics approximation that brings one back to classical mechanics.

In these formulas, I have become aware of a quite interesting interpretation. I was previously led to assimilate the totality of local values of a propagating wave to the totality of small clocks that are carried along by the motion of the wave. If the corpuscle is constantly localized inside the wave then that would lead us to represent it as a much larger clock that displaces in the medium of small clocks. The idea then makes us assume that this large clock must displace in such a fashion that the time that it indicates remains constantly equal to that of the small clocks that immediately surround it. In other words, the corpuscle must displace in such a fashion that its internal oscillation remains constantly *in phase* with the progressive wave into which it is incorporated.

It is easy to verify that one thus recovers the guidance formula. Indeed, if the corpuscle displaces by $d\mathbf{s}$ in physical space during the time $dt$ then the persistence of phase agreement between the internal vibration and the surrounding wave obviously demands that:

$$\frac{1}{h}\left(\frac{\partial\varphi}{\partial t} + \operatorname{grad}\varphi\cdot\frac{d\mathbf{s}}{dt}\right)dt = v_c\,dt. \tag{19}$$

Since one has:

$$\frac{1}{h}\frac{\partial\varphi}{\partial t} = v, \qquad v = \frac{v_0}{\sqrt{1-\beta^2}}, \qquad v_c = v_0\,\sqrt{1-\beta^2}\,, \tag{20}$$

one gets:

$$1 + \frac{\operatorname{grad}\varphi\cdot\mathbf{v}}{\dfrac{\partial\varphi}{\partial t}} = \frac{v_c}{v} = 1-\beta^2, \tag{21}$$

and this equation is verified precisely if one attributes the value (17) to **v**; i.e., if one assumes the guidance formula.

The preceding formulas have been written down by supposing that the corpuscle is not subject to any external field. In the more general case of a corpuscle that is subject to a field, one will obtain a relativistic theory by supposing that one is dealing with a corpuscle of electric charge $\varepsilon$ that is subject to an electromagnetic field that is derived from a scalar potential $V$ and a vector potential **A**, and upon adopting the Klein-Gordon wave equation, so one deduces that:

$$(23) \qquad \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t} - \varepsilon V\right)^2 - \left(\operatorname{grad}\varphi + \frac{\varepsilon}{c}\mathbf{A}\right)^2 = m_0^2 c^2 .$$

For the guidance formula, one finds:

$$(23) \qquad \mathbf{v} = -\, c^2 \,\frac{\operatorname{grad}\varphi \cdot \mathbf{v} + \varepsilon V_0}{\dfrac{\partial\varphi}{\partial t} - \varepsilon V},$$

which naturally brings us back to formula (17) for $\mathbf{A} = V = 0$.

It is easy to show that this formula still expresses the idea that the internal vibration of the corpuscle remains constantly in phase with the surrounding wave. It suffices to set:

$$(24) \qquad \left\{ \begin{aligned} & h\nu_0 = \frac{\partial\varphi}{\partial t_0} = m_0 c^2 + \varepsilon V_0 , \\ & h\nu_c = h\nu_0 \sqrt{1-\beta^2} = m_0 c^2 \sqrt{1-\beta^2} + \varepsilon\left(V - \frac{\mathbf{v}\cdot\mathbf{A}}{c}\right). \end{aligned} \right.$$

This phase agreement demands that the relation (19) must always be verified, which gives:

$$(25) \qquad \left(\frac{\partial\varphi}{\partial t} - \varepsilon V\right) + \left(\operatorname{grad}\varphi + \frac{\varepsilon}{c}\mathbf{A}\right)\mathbf{v} = m_0 c^2 \sqrt{1-\beta^2}$$

here. Upon replacing $\sqrt{1-\beta^2}$ in the right-hand side with $\dfrac{m_0 c^2}{\dfrac{\partial\varphi}{\partial t} - \varepsilon V}$, one easily verifies that formula (25) is verified if one adopts the guidance law (23), while taking equation (22) into account.

**4. The quantum potential.** – The interpretation of wave mechanics that we just sketched out was first presented in the form of the theory of the pilot wave. In it, one considers, in short, a homogeneous wave of wave mechanics as having a real physical existence, and one (arbitrarily) imposes that the corpuscle considered, which always has a well-defined position inside the wave, is obliged to follow one of the streamlines of the propagation of the wave, in accord with the guidance formula.

One can then see that this obliges one to assume that the corpuscle is subject to, along with the action of a potential of classical type that translates into the presence of an external field, the action of a new type of potential: "the quantum potential." The "quantum force" that is derived from this potential translates into the existence of an action that the surrounding wave exerts upon the corpuscle, since that would seem necessary for the interpretation of interference and diffraction phenomena in a theory that assumes the localization of a particle in space.

In the case of the non-relativistic Schrödinger equation, the quantum potential has the expression:

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta a}{a}\,. \tag{26}$$

In the case of a corpuscle without spin that obeys the Klein-Gordon equation, one can develop all of the dynamics of the corpuscle in Lagrangian and Hamiltonian form, as I did in some other presentations (*see* [**3**], chap. X). One is then led to attribute a proper mass to the corpuscle that varies with its position in the wave and is given by the formula:

$$M_0 = \sqrt{m_0^2 + \frac{\hbar^2}{c^2}\frac{\Box a}{a}}\,, \tag{27}$$

and *in the proper system* it is the quantity $M_0\,c^2$ whose gradient, with its sign changed, is the quantum force. One can then define the quantum potential by setting:

$$Q_0 = M_0\,c^2 - m_0\,c^2, \qquad Q = Q_0\sqrt{1-\beta^2}\,. \tag{27 bis}$$

In the Newtonian approximation, where $\beta \ll 1$ and $\Box a \approx -\Delta a$, this quantum potential indeed reduces to the expression (26), as one easily verifies, and one can then define the quantum potential by:

$$Q = M_0\,c^2 - m_0\,c^2.$$

I will no longer insist upon this concept of "guidance dynamics" here, even in the more complex form that it takes in the context of the Dirac equations for the electron with spin.

**5. The theory of the double solution.** – When I studied this interpretation of wave mechanics in 1926-1927, it seemed to me that the true theory, which would exceed the provisional viewpoint of the pilot wave, must establish a more intimate link between the corpuscle and the wave. Developing this idea under the name of the "theory of the double solution," I affirmed that the homogeneous wave $\Psi$, which is already the usual one from the wave mechanics of that era, provides only a representation of the probabilities, and that the true physical wave of the corpuscle must be a wave $u$ that involves a very high concentration of amplitude that will be the corpuscle in the strict sense of the word. Outside of that region, the wave *u* will reduce to a homogeneous wave that coincides roughly (up to a normalization constant) with the homogeneous wave $\Psi$ of the usual wave mechanics. The corpuscle will be incorporated into the wave *u*, which constitutes a bunched field of the type that Einstein imagined for the representation of the corpuscle as a local singularity of the field.

After having abandoned this difficult attempt for quite some time, I then resurrected it some dozen years later with the aid of a very small number of collaborators and great progress was made along this path. The usual wave $\Psi$, although a subjective and natural representation of the probabilities, is meanwhile related to the wave *u* in such a fashion that the corpuscle seems to describe one of the lines that is defined by the guidance formula by starting with the propagation of the wave $\Psi$, since this motion can perhaps be interpreted by supposing that the true equation of propagation of the wave *u* is nonlinear, while that nonlinearity usually manifests itself only in the very small singular region of very high field concentration, as I have, moreover, discussed, and which I shall review here ([8]).

The first of these points relates to the proof of the guidance formula. I could give two proofs by starting with the usual linear equations, but while adding a very arbitrary hypothesis of phase matching. One can replace that hypothesis with that of the agreement between the streamlines of the external wave with the streamlines that are internal to the very small region of high field values. They – i.e., the corpuscle – are thus found to be imprisoned in a very slender tube of streamlines of the external field, and the guidance formula results immediately. The proof then takes on a form that is very close to the one that Georges Darmois gave on another occasion in order to show that the motion of a particle in a gravitational field is represented by a geodesic in spacetime. As Einstein showed in the case of general relativity, the success of that proof is certainly linked to the nonlinear character of the field equations (here, the equations of propagation of the wave *u*).

A second point upon which I would like to insist is the following: In the usual quantum theory, there exist phenomena, such as Bohr’s quantum transitions, that one declares must completely elude any description in terms of space and time, and this seems to be true in the context of the linear equations that one utilizes. However, if one assumes that they can be introduced from the nonlinearity in the wave equations then the question changes in character. Indeed, one can demand to know whether these phenomena, which were declared to be impossible to describe, do not, in reality, correspond to some very rapid transitory states of a nonlinear character. Andrade e Silva, Fer, Leruste, and Lochak have carried out some very interesting research in this direction by appealing to some properties of nonlinear equations, and in particular to the theory of limit cycles ([9]).

**6. Remarks on the subject of the hydrodynamical aspect of the preceding concepts.** – We have arrived at a sort of hydrodynamical picture for the propagation of the wave $a\,e^{i\varphi/\hbar}$ in wave mechanics. One obtains it by considering a fluid that has the

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([8]) *See* bibliography [**4**].

([9]) Bibliography [**5**].

given density $\rho$ at each point and each instant that is given in the case of the Schrödinger equation by:

$$\rho = a^2(x, y, z, t) = |\,\Psi(x, y, z, t)\,|^2, \tag{28}$$

and whose local velocity **v** will be defined at each instant by the guidance formula that we studied previously. It then results from the equation of propagation that this fictitious fluid is conserved in such a way that the hydrodynamical equation of continuity:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\mathbf{v} = 0 \tag{29}$$

is constantly satisfied.

Without being able to take such a picture literally, one can nonetheless represent the corpuscle as a sort of granule that is carried along by the fluid flow and describes one of the streamlines, just like the grains of lycopodium that the hydraulic engineers disperse on the surface of a flowing liquid so that the trajectory of each of these grains will materialize the form of a streamline.

It is useful to make a small remark here on the subject of the expression "the corpuscle follows a streamline of the hydrodynamical flow that corresponds to the propagation of its wave." The guidance formula imposes that the corpuscle must have its velocity tangent to the streamline on which it is found at the instant in question. If the flow is permanent – i.e., it does not vary in the course of time – then the streamlines do not deform, and the trajectory coincides with one of the streamlines. If, on the contrary, the flow does not have a permanent character then the streamlines do deform in the course of time, and the trajectory of the corpuscle, although tangent to a streamline at any instant, no longer coincides with a one of the streamlines. This situation is sufficiently well-known in hydrodynamics that it is pointless to elaborate upon it any further.

If one studies the motion of the corpuscle that is defined by the guidance formula then one perceives that as a result of the quantum force, even when the external forces are zero, the energy and quantity of motion of the corpuscle do not remain constant. In the hydrodynamical picture, the quantum force can thus be regarded as a pressure that the fluid exerts on the corpuscle (10).

Meanwhile, one can prove that the energy and quantity of motion are conserved in the mean – i.e., for an infinitude of corpuscles that are distributed in space with the density $\rho$. We limit ourselves to giving a simple proof here that concerns energy in the context of the non-relativistic theory.

One can express the global conservation of energy $W = \int w\,\rho\,d\tau$, either by writing:

$$\frac{\partial}{\partial t}\int \rho w\,d\tau = \int\left(\frac{\partial \rho}{\partial t}w + \rho\frac{\partial w}{\partial t}\right)d\tau = 0, \tag{30}$$

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(10) This analogy was made precise by Jean-Louis Destouches in the work that he has pursued in recent years. *See*, notably, bibliography [**6**].

or by remarking that the mean energy of a particle varies by $\frac{Dw}{Dt}dt$ in the time $dt$, and writing that the mean value of the variations of energy during that time is zero, which leads one to write:

$$\int \rho \frac{Dw}{Dt} d\tau = \int \rho \left( \frac{\partial w}{\partial t} + \mathbf{v} \cdot \operatorname{grad} w \right) d\tau = 0. \tag{31}$$

Upon supposing that $\rho$ is zero at infinity, which is always physically realized because the wave trains are always limited, one can prove that formulas (30) and (31) are equivalent. Indeed, upon taking the continuity equation into account, equation (30) can be written:

$$\int \left( \rho \frac{\partial w}{\partial t} - w \operatorname{div} \rho \mathbf{v} \right) d\tau = 0, \tag{32}$$

and since $\rho$ is zero at infinity, an integration by parts shows that (32) is equivalent to (31).

If one then utilizes formula (31) then we shall apply the non-relativistic form of the theory of guidance by setting:

$$\rho = a^2, \qquad w = \frac{\partial \varphi}{\partial t}, \qquad \mathbf{v} = -\frac{1}{m} \operatorname{grad} \varphi, \tag{33}$$

and we take into account the equation (viz., the generalized Jacobi equation) that is deduced from the Schrödinger equation:

$$\frac{\partial \varphi}{\partial t} = \frac{1}{2m} \operatorname{grad}^2 \varphi + Q, \tag{34}$$

where $Q = -\frac{\hbar^2}{2m}\frac{\Delta a}{a}$. Equation (31) then gives us:

$$\int a^2 \left( \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{m} \operatorname{grad} \varphi \cdot \operatorname{grad} \frac{\partial \varphi}{\partial t} \right) d\tau = 0, \tag{35}$$

but, upon differentiating (34) with respect to time, one sees that the parenthesis in equation (35) is equal to $\partial Q / \partial t$, in such a way that it becomes:

$$\int a^2 \frac{\partial}{\partial t} \left( \frac{\Delta a}{a} \right) d\tau = \int \left( a \Delta \frac{\partial a}{\partial t} - \Delta a \frac{\partial a}{\partial t} \right) d\tau = 0. \tag{36}$$

Since the functions $a$ and $\partial a / \partial t$ are zero at infinity, a double integration by parts shows that equation (36) is indeed verified, and consequently that the mean energy indeed remains constant.

I have, moreover, given a more general proof (11) of the conservation in the mean of energy and the quantity of motion of a corpuscle in the context of the relativistic theory of Klein-Gordon.

We shall return to the significance of the conservation in the mean of the energy and quantity of motion in the following chapter after we have introduced the Bohm-Vigier hypothesis of the sub-quantum medium.

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(11) *See* bibliography [**3**], pp. 270-273.

## CHAPTER VII

# THE INTRODUCTION OF THERMODYNAMICAL CONCEPTS INTO WAVE MECHANICS

**1. The Bohm-Vigier sub-quantum medium.** In the preceding chapter, we obtained a hydrodynamical picture for the interpretation of wave mechanics from the theory of the double solution. That picture corresponded precisely to the ideas that I developed in 1926-1929 and have reprised since 1951. However, in these latter years I have gradually recognized that it constitutes only a first approximation and that it must be completed by the introduction of new hypotheses that involve statistical thermodynamics.

The first step along that path was taken in 1954 by Bohm and Vigier ($^{12}$), when they introduced the hypothesis of the existence of a "sub-quantum medium," which is a hidden medium that is, in some way, deeper than the microscopic level, such that all particles at the microscopic level will be in permanent contact with it and with which they can constantly exchange energy and quantity of motion.

What led Bohm and Vigier to adopt this hypothesis? Outside of some general considerations, it was the desire to give a satisfactory justification to the role of the probability of the presence of a corpuscle at a point at a given instant, which wave mechanics attributes to the quantity $|\Psi|^2$.

We have seen that the guidance formula, by establishing the necessity for the corpuscle to follow one of the streamlines of hydrodynamic convection that correspond to the propagation of the wave, has led us to assume that if all of the probabilities of the initial positions of the corpuscle in the wave are considered to be proportional to $|\Psi(x, y, z, t)|^2$ then the probability for the corpuscle to be found at the instant $t$ in an element $d\tau$ of space is equal to $|\Psi(x, y, z, t)|^2$, at least in the non-relativistic approximation. This conclusion is deduced from the continuity equation, where $\mathbf{v}$ is the velocity that is defined by the guidance formula. However, if one examines this proof more closely then one perceives that it raises a difficulty that is completely analogous to the one that is presented, as we have seen, in statistical mechanics when, after proving Liouville's theorem, one seeks to deduce that the probability of the presence of a representative point of a system in the element $d\tau$ of the extension-in-phase is proportional to that element $d\tau$ ($^{13}$). We have seen that in order to arrive at a justification for this conclusion, one must add either an ergodic hypothesis to Liouville's theorem or a hypothesis of a more physical character in the form of molecular chaos.

We take the latter viewpoint. If the mechanical evolution of a system proceeds regularly with no perturbation then it will generally be justified for us to assume that the same tube of unperturbed trajectories in the extension-in-phase fills up all of that extent. However, one can suppose that the motion of the system is subject to some constant,

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($^{12}$) Bibliography [**7**].

($^{13}$) The difficulty that presents itself here is illustrated very clearly by the fact that from the guidance formula an electron in a hydrogen atom in the state $s$ must remain immobile at a point of the atom in such a way that, without the Bohm-Vigier hypothesis, one would not at all see how one could realize the probability of presence by $|\Psi|^2$.

random perturbations, which one can regard as, for example, due to the continual interaction of the system with an external system. (If the system considered reduces to a molecule in a gas then this will amount to the continual interaction of that molecule with the other molecules of the gas.) The representative point of the system will then pass constantly from one tube of unperturbed trajectories to another one, and after a sufficiently long time has passed (which can be extremely short at our level) one can consider the representative point (and, in turn, the element $d\tau$ that contains it) as having successively traversed all of the segments of the unperturbed trajectory and having thus swept out the entirety of the extension-in-phase, which will then justify the principle that serves as the basis for all of statistical mechanics.

Bohm and Vigier introduced an analogous hypothesis in the new interpretation of wave mechanics in order to justify the role that is played by $|\Psi|^2$ of the probability of the presence. Here, it is the product $\rho\, d\tau$ that is conserved in physical space along a streamline, by virtue of the continuity equation; i.e., if one assumes the guidance formula then it is along a tube of unperturbed trajectories of the particle. In order for one to be able to deduce that $\rho\, d\tau$ gives the probability of presence for the corpuscle in the element $d\tau$, one must have that the same tube of trajectories winds about indefinitely in the portion of physical space that is accessible to the corpuscle in such a manner as to fill it up completely. Now, there is no reason for this to be true, in general ([14]).

In the paper cited above, Bohm and Vigier have presented a theoretical justification for the statistical interpretation of $|\Psi|^2$ by assuming the hypothesis of continual random perturbations to the motion of the corpuscle in a manner that is analogous to the hypothesis that Boltzmann introduced in the context of statistical mechanics. If one assumes that these perturbations are representable as the momentary appearance of small random perturbing potentials into the wave equation then the equation of continuity will remain valid during the periods of perturbation and the quantity $\rho\, d\tau$ will be conserved along a tube of trajectories, even in the perturbed portions of the tube. One can thus consider an element $d\tau$ of the fluid in the hydrodynamical representation of wave mechanics as successively traversing (and in an extremely short time at our level) all of the segments in the unperturbed current tube in such a fashion as to sweep out uniformly, and with conservation of $\rho\, d\tau$, all of the region in physical space that is accessible to the corpuscle, and this will permit one consider the quantity $\rho = |\Psi|^2$ to be measuring the probability of the local presence of the corpuscle.

Such is the general train of reasoning that was developed by Bohm and Vigier in their paper, and this reasoning can be reprised with the aid of the theory of Markov chains. The authors have considered the continual, random perturbations to which the corpuscle is subjected as being the result of random reactions that are exerted upon them by a deep and hidden medium that they have called the "sub-quantum medium."

**2. Comparison with the motion of a granule convected by a hot fluid.** – The concept of a sub-quantum medium, which seems to explain a large number of facts that are asserted in microphysics, and which is introduced in a disguised form in certain results of the quantum theory of fields (for example, when one attributes physical

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([14]) *See* the note on the preceding page.

properties to the vacuum, such as "vacuum polarization"), is of great importance in the problem of establishing a relationship between mechanical quantities and thermodynamic quantities. Indeed, it is inconceivable that one could introduce quantities that have a thermodynamic significance into the dynamics of an isolated particle – i.e., one that is separate from any other microphysical system – since those quantities would seem to be linked essentially to the random variations of a system of great complexity, which cannot be the case for an isolated corpuscle. However, the question arises in a different context if one assumes that a corpuscle at the microphysical level that appears to be isolated is nonetheless always in energetic contact with a deep and hidden medium that possesses a complex and random structure. Now, the introduction of the hypothesis of the existence of a sub-quantum medium leads to the idea that any particle at the microphysical level can be considered as being constantly in contact with a sort of hidden thermostat, and it then becomes possible to attribute a temperature to it, as well as an entropy that is related to its motion. An entirely new horizon then opens up before our eyes.

This leads us to revisit the hydrodynamical picture of the motion of a corpuscle that we previously envisioned, while modifying it in a very profound way. We have compared the motion of a corpuscle that is defined by the guidance formula to the motion of a granule (e.g., a grain of lycopodium) that is carried along a streamline by the convection of a fluid. However, if the fluid is hot – I would like to say that it is not at absolute zero – then the granule will regularly follow the streamline only if it is sufficiently massive as to resist the random collisions of the fluid molecules. If it is very light then it will be as if all the fluid molecules are animated with a Brownian agitation that is superimposed on the regular motion that is imposed by the general convection of the fluid.

Up to a difference in scale, the corpuscle will thus be comparable to a granule in suspension in a hot fluid, which is animated with a Brownian motion that is due to is due to its interactions with the invisible fluid molecules, and to which, for that reason, one can apply the concepts of thermodynamics. If the fluid is collectively immobile then the Brownian motion of the granule will make it jump from here to there without continuous motion. On the contrary, if the fluid is animated with a collective motion then the granule, which would be carried along a streamline regularly without the Brownian motion, will continually pass from one streamline to another as a result of the Brownian motion. This is, moreover, what must happen for the fluid molecules themselves: Indeed, each streamline represents the trajectory of a molecule when abstracted from the Brownian motion, and the set of streamlines gives only a statistical picture of the global motion of the molecules. One now perceives in what manner the introduction of a sub-quantum medium can lead us to modify the hydrodynamical picture that we constructed for the motion of a microphysical particle.

**3. First attempt at establishing a correspondence between entropy and action, and between temperature and frequency.** – The ideas that we just developed lead us naturally to envision the establishment of relations between the dynamical quantities that characterize the motion of a corpuscle (conceived in the manner of the theory of the double solution) and thermodynamic quantities such as entropy and temperature. Several authors have suggested analogous ideas on other occasions. Eddington, in his celebrated book *Space, Time, Gravitation* (pp. 219 of the French translation), sketched out – in very vague terms, moreover – a reconciliation of the two fundamental invariants of relativity, namely, entropy and action.

For fifteen years, the study of the old theories of Helmholtz and Boltzmann has led me to search for a way to establish a correspondence between entropy and action and between cyclic frequency and temperature. I attempted that in a note to the *Comptes rendus de l'Académie des Sciences* (t. 233, 1946, pp. 248) and in my course during the school year 1948-1949. I likewise spoke about it in an article to *Cahiers de Physique* (nos. 31-32, January, 1948, pp. 1). In that era, I imagined two different ways of establishing that correspondence, but here I will recall only the second of them, because it seems to be the most interesting to me, and it is the one that agrees the most with what I will present in Chapter VIII.

The Boltzmann formula for periodic systems that we wrote down in Chapter V in the form:

$$\delta Q = \frac{1}{\tau}\,\delta A = \nu\,\delta A \tag{1}$$

is valid only in the proper system of a body that is the site of a periodic process of frequency $\nu$ and period $\tau$. The quantity $\mathcal{A}$ then represents the cyclic integral of the Maupertuisian action, when taken over an entire period of motion. We must then write, in a more precise fashion:

$$\delta Q_0 = \nu_0\,\delta\mathcal{A}_0\,, \tag{2}$$

with

$$\mathcal{A}_0 = \int_0^\tau \sum_k p_k^0\, d\overset{*}{q}{}_k^0\, dt\,, \tag{3}$$

where the index 0 indicates that the quantities are evaluated in the proper system of the body.

We pass to a Galilean reference system in which the body is animated with the velocity $\beta c$. Upon multiplying the preceding formula by $\sqrt{1-\beta^2}$, we obtain:

$$\delta Q_0 = \nu_c\,\delta\mathcal{A}_0\,. \tag{4}$$

Since $dQ = T\,dS$, one is led to set:

$$\nu_c = CT \qquad \text{and} \qquad \mathcal{A}_0 = \frac{S}{C}\,,$$

where $C$ is a constant that, for obvious reasons, can be naturally set equal to $k / h$. One thus obtains the relations:

$$h\nu_c = kT, \qquad \frac{\mathcal{A}_0}{h} = \frac{S}{k} \tag{5}$$

between the cyclic frequency and the temperature, on the one hand, and between the cyclic integral of the Maupertuisian action and the entropy, on the other. Since $\nu_c$ and $T$ transform in the same way when one passes from the proper system to the other Galilean system, the first relation in (5) is satisfactory from that standpoint. However, the same is not true for the second relation in (5), because $S$ is an invariant, while $\mathcal{A}$ is not.

One can fix the second relation in (5) by considering a corpuscle to be a periodic system that conceived in the manner of wave mechanics as the site of a periodic process of frequency $\nu_0 = W_0 / h$ and proper period $\tau_0 = h / W_0$ , with $W_0 = m_0 c^2$. One will then set, by definition:

$$(6) \qquad \mathcal{A}_0 = \int_0^{\tau_0} \mathcal{L}_0 dt_0 \,,$$

and, by reason of the relativistic invariance of the Hamiltonian action, one will have:

$$(7) \qquad \mathcal{A} = \int_0^{\tau_0} \mathcal{L}\, dt = \mathcal{A}_0$$

in any Galilean system, which naturally leads one to replace the second relation in (5) with:

$$(8) \qquad \frac{\mathcal{A}}{h} = \frac{S}{k},$$

and that is more satisfactory.

I did not take this step fifteen years ago. Nevertheless, I had glimpsed the possibility of the thermodynamics of an isolated particle when I wrote in 1948, in my article in *Cahiers de Physique:* "Here is the beginning of the thermodynamics of a material point that one might seek to develop in the context of wave mechanics. It is very difficult to say where this path will lead, and we must be content to have indicated the point of departure." What stopped me from going further in that era was the fact that I had not yet returned to my research on the theory of the double solution and the fact that I was not thinking about a sub-quantum medium.

It was only in 1961, while reflecting on a recent work of Terletsky, that I first perceived the possibility of introducing relations between frequency and temperature, and entropy and action into the theory of the double solution, whose existence I had hinted at fifteen years before.

**4. On a paper by Terletsky.** – In the course of a visit to Paris at l'Institut Henri-Poincaré, professor Terletsky published two very interesting articles in the *Journal de Physique* in 1961 (15). In the first of the papers and in the second part of the second one, he envisioned the hypothesis that there can exist particles of imaginary mass that are animated with velocities greater than light in vacuo, particles that constitute a sort of hidden thermostat that is analogous to the Bohm-Vigier sub-quantum medium. Despite the interest independent interest in the argument that Terletsky developed on the subject,

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(15) Bibliography, [**8**].

I prefer to not introduce the hypothesis of the existence of particles of imaginary mass here, and I will be content to say some words on the first part of his second paper.

Terletsky envisioned a set of fields $\Psi_1(x, y, z, t)$, …, $\Psi_N(x, y, z, t)$ that were analogous to the wave of wave mechanics, and he defined the total action of that field by the space-time integral of a certain Lagrange function. Moreover, he supposed that this first system of fields is in a state of weak energetic interaction with a second system of fields that he considered as defining a thermostat, and by some calculations into which functionals entered, he deduced a formula that he considered to be the analogue of the Gibbs canonical distribution law, but to me it rather seems as if it must be assimilated into the Boltzmannian definition of entropy $S = k \log P$.

In all of that attempt, Terletsky defined the global action of these $N$ fields by the space-time integral, but then he showed, in a very interesting manner, how the hypothesis that the fields obey some nonlinear equations permits one to reduce that definition to the usual definition of the Hamiltonian action of a corpuscle by an integral that is taken along its world-line. For this, he first recalled that a linear theory of fields always leads to what he called "the ultraviolet catastrophe;" i.e., to the fact that the integral that gives the total energy of the field is divergent in the limit of very high frequencies. One knows that this inadmissible consequence appears in physics in the theory of black-body radiation, and it was in order to avoid it that Planck introduced the notion of the quantum of action for the first time in 1900. Today, one often eliminates that difficulty by introducing an arbitrary cut into the integral at a high frequency (i.e., a cutoff) that eliminates the waves of very high frequency that are responsible for the divergence of the integral: however, this process is not satisfactory, because it is entirely arbitrary.

Now, Terletsky remarks that things are completely different for nonlinear fields because then if there naturally exist solutions in the case of weak amplitudes that have the character of the classical solutions in linear theory very approximately, then, as the work of Terletsky and his students has shown, there can also exist solutions that present very small regions of high concentration for the field that are stable and which have the character of Einstein's "bunched fields." There are then extremely slender world-tubes where the field takes on very high values. In turn, the action integral that was originally considered by Terletsky reduces *very approximately* to a sum of integrals that are taken along the world-tubes in question, and one thus recovers the classical notion of a Hamiltonian action that is linked to the motion of the corpuscles.

It is almost pointless to emphasize here how the ideas of Terletsky are in agreement with the concepts of the theory of the double solution. Moreover, the author adds the following remark, which can be useful in the study of certain difficult problems that present themselves in the new interpretation of wave mechanics: "Outside of solutions with regions of high concentration of the field, the other solutions have the form of quasi-linear wave packets that rapidly spread out into all of space, and whose amplitude consequently tends to zero."

Having thus returned to the usual notion of Hamiltonian action of a particle, Terletsky then sought to deduce the canonical distribution law. However, as he had not introduced the notion of a thermostat that would impose a well-defined temperature on the fields, his conclusions do not seem very clear to me, and I think that they must receive a very different interpretation from the one that the author proposes, an interpretation that we have reduced to the formulas $h\nu_c = kT$ and $A / h = S / k$ that were envisioned above.

Taken as a whole, this very curious paper of Terletsky has sparked my interest, and it has drawn my attention to the possibility of introducing the thermodynamic quantities of temperature and entropy into the reinterpretation of wave mechanics by the double solution, which are linked to the characteristics of the corpuscle that take the form of cyclic frequency and Hamiltonian action. That is what led me to develop the "thermodynamics of the isolated particle" that I will now present.

CHAPTER VIII

# THE THERMODYNAMICS OF THE ISOLATED PARTICLE

### (or the hidden thermodynamics of particles)

**1. Fundamental formulas. –** The reflections that inspired me while reading the work of Terletsky, and which brought me back to old ideas that I had on that subject around 1946-1948 have led me to try to establish a “thermodynamics of the isolated particle” in some recent notes ([16]), which can also be called the “hidden thermodynamics of particles,” since it results from the continual interaction of particles with a hidden thermostat, which can naturally be identified with the sub-quantum medium of Bohm and Vigier. I began by introducing the entropy of an isolated particle, but then I preferred to reason by introducing the entropy of the hidden thermostat. The reason for this is that it is scabrous to define an entropy for the particle, because it is a very simple system that involves only a small number of degrees of freedom: On the contrary, the hidden thermostat is certainly a very complex system, so it is legitimate to speak of its entropy and the use of entropy will permit us to follow a path that was traced out by Einstein on another occasion in his theory of fluctuations.

In order to develop a new thermodynamics, we first assume that we can apply the formula from relativistic thermodynamics:

$$\delta Q = -\delta_{M_0}\mathcal{L} \tag{1}$$

to an isolated particle at the microphysical scale upon supposing that the proper mass $M_0$ of the particle is variable.

Moreover, in accord with my old ideas from 1946-1948, we also assume that the particle that is in permanent energetic contact with the hidden thermostat can be considered as having a temperature $T$ that is defined by the formula:

$$kT = h\nu_c = h\nu_0\sqrt{1-\beta^2} = m_0 c^2, \tag{2}$$

which has the desired relativistic covariance, and where $m_0$ is the usual constant proper mass that is attributed to the particle.

We shall define the entropy $S$ of the hidden thermostat that is in energetic contact with the particle. We take our inspiration from the method that was formerly employed by Einstein in his celebrated work on fluctuations, so we write that entropy in the form:

$$S = S_0 + S(M_0), \tag{3}$$

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([16]) Bibliography [**9**].

where $S_0$ is the part of that entropy that is independent of the fluctuating value of the proper mass $M_0$ of the particle, while $S(M_0)$ is the very small part of that entropy that depends upon the value of $M_0$ . We will then have:

$$\delta_{M_0} S = \delta S(M_0) = -\frac{\delta Q}{T} = \frac{\delta_{M_0}\mathcal{L}}{T}. \tag{4}$$

The – sign that appears before $\delta Q$ is due to the fact that $\delta Q$ is the heat that is *given up* by the hidden thermostat to the particle. Now, we can write the Lagrange function of the particle in the form:

$$\mathcal{L} = -M_0 c^2 \sqrt{1-\beta^2} + \ldots, \tag{5}$$

where the unwritten terms do not depend upon $M_0$ . By using (2) and (5), we thus obtain:

$$\delta S(M_0) = -k\frac{\delta M_0}{m_0}, \tag{6}$$

which finally gives us:

$$S = S_0 - k\frac{M_0}{m_0}, \tag{7}$$

a fundamental formula in which the invariance of the right-hand side is quite evident.

We have thus obtained the two fundamental formulas of the thermodynamics of the isolated particle, which are valid in all Galilean reference systems, namely:

$$\boxed{kT = h\nu_c,} \qquad \boxed{S = S_0 - k\frac{M_0}{m_0}.} \tag{8}$$

We can remark that formula (4) leads us to the relation $\frac{1}{T} = \frac{\delta S}{\delta \mathcal{L}}$, in place of the relation $\frac{1}{T} = -\frac{\delta S}{\delta \mathcal{L}}$ that was found in Chapter IV. However, we should not be surprised at this, since $S$ is referred to the thermostat here and $\mathcal{L}$, to the particle. If it is legitimate to introduce the entropy $S_1$ for the particle, as I did in my note on August 1961, then one will have:

$$\delta S_1 = -\delta S \qquad \text{and} \qquad \frac{1}{T} = -\frac{\delta S_1}{\delta \mathcal{L}}.$$

**2. Analogies between the thermodynamics of an isolated particle and Helmholtz's canonical schema.** – We would like to compare our new thermodynamics with the canonical schema of Helmholtz that we presented in paragraph 3 of Chapter V.

We know that the phase $\varphi$ of the wave that is associated with the particles is equal to $-A$, so one can write:

$$\varphi = -A = h\int_0^t \nu_c\,dt\,, \tag{9}$$

so:

$$\dot{\varphi} = -\dot{A} = h\nu_c\,, \tag{10}$$

in which the dot denotes the derivative with respect to time. Therefore, if we set:

$$\varepsilon = \frac{1}{k}\varphi = -\frac{1}{k}A\,; \tag{11}$$

then the first formula in (8) gives us:

$$T = \frac{h\nu_c}{k} = \frac{\dot{\varphi}}{k} = \dot{\varepsilon}\,. \tag{12}$$

We thus recover the fundamental relation that was assumed by Helmholtz as the basis for his canonical schema, according to which the temperature is the derivative with respect to time of a certain quantity $\varepsilon$.

On the other hand, when the hidden thermostat gives up an energy $dU$ to the particle by communicating a work $dA$ to it (using the notations employed in Chapter IV), we write, with Helmholtz:

$$-\,d_{M_0}U = -\mathcal{E}\,d\varepsilon + dA, \tag{13}$$

and relativistic thermodynamics will then give us:

$$-\,d_{M_0}U = dQ + dA, \tag{14}$$

with:

$$dQ = -T\,d_{M_0}S\,, \qquad dA = d_{M_0}\mathcal{F}\,, \tag{15}$$

where $\mathcal{F} = \dfrac{M_0 v^2}{\sqrt{1-\beta^2}}$ is the pseudo-vis viva of the particle. We specify that in formulas (13) and (14), $-\,d_{M_0}U$ represents the reduction in the internal energy of the hidden thermostat when $M_0$ increases. From (13) and (14), taking (12) into account, one deduces the second formula of Helmholtz's canonical schema:

$$\mathcal{E} = \dot{S}\,. \tag{16}$$

Since $\varepsilon$ in the schema is, by hypothesis, a cyclic variable – so $\partial\mathcal{L} / \partial\varepsilon = 0$ – the Lagrange equation that relates to the variable $\varepsilon$ also gives the third Helmholtz formula:

$$\frac{\partial\mathcal{L}}{\partial\dot{\varepsilon}} = p_\varepsilon = S, \tag{17}$$

which can be verified directly from formula (4), because it gives, up to a constant:

$$\mathcal{L} = TS = \dot{\varepsilon} S\,.$$

On the other hand, since one no longer has the relation:

$$U = \dot{\varepsilon}\frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}} - \mathcal{L} = TS - \mathcal{L}$$

here, whose relativistic variance would not be correct, one can no longer attribute the free energy to the Lagrange function. However, since $T$ is constant, one can write (14) in the form:

$$d_{M_0}(U - TS) = - d_{M_0}\mathcal{F}\,, \tag{18}$$

which leads us to attribute that part of the free energy of the hidden thermostat that depends upon $M_0$ to the pseudo-vis viva of the particle, with the sign changed.

**3. The second law of thermodynamics and Hamilton's principle of least action. –** In the usual applications of Hamilton's least action principle in analytical mechanics, one starts with a "natural" motion – i.e., a motion that conforms to the laws of mechanics. One supposes that in the course of that motion the particle that starts at a point $A$ in space at the time $t_0$ arrives at a point $B$ at the time $t_1$, and then one *imagines* a "varied" motion that is fictitious and infinitely close to the natural motion, while imposing upon that varied motion the constraint that the point $A$ and $B$ and the instants $t_0$ and $t_1$ remain the same as in the natural motion. In other words, one makes the form of the world-line that represents the natural motion in space-time vary slightly, while keeping the extremities of that world-line fixed.

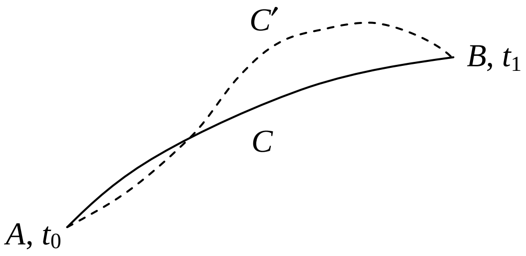

Figure 7.

Hamilton's principle then tells us that the natural motion is characterized by the equation:

$$\int_{t_0}^{t_1} [\delta \mathcal{L}]_{M_0}\, dt = 0, \tag{19}$$

where $[\delta\mathcal{L}]_{M_0}$ is the variation of $\mathcal{L}$ when one keeps the proper mass $M_0$ constant and equal to its normal value $m_0$. Moreover, since Hamilton's principle is a principle of *least* action, we can also write ([17]):

$$\int_{t_0}^{t_1} [\delta\mathcal{L}]_{M_0}\, dt > 0. \tag{20}$$

We shall now introduce a new idea that seems interesting. If one assumes, as we did, that the proper mass of the particle can be subject to fluctuations then it becomes possible to consider varied motions that are no longer like the purely fictitious motions that we imagined, but like motions that can have real meaning under the action of certain instantaneous fluctuations of the proper mass during the interval of time $t_0 \rightarrow t_1$.

Having assumed this hypothesis, we must be able to determine the fluctuated motion *AC′B* by means of the equation:

$$\int_{t_0}^{t_1} \delta(\mathcal{L} + \delta\mathcal{L})\, dt = \int_{t_0}^{t_1} (\delta\mathcal{L} + \delta^2\mathcal{L})\, dt = 0 \tag{21}$$

when one applies Hamilton's principle to it. However, the proper mass is no longer assumed to be constant here, so we must write:

$$\delta\mathcal{L} = [\delta\mathcal{L}]_{M_0} + \delta_{M_0}\mathcal{L}, \qquad \delta^2\mathcal{L} = [\delta^2\mathcal{L}]_{M_0} + \delta^2_{M_0}\mathcal{L}, \tag{22}$$

upon letting $\delta^2_{M_0}\mathcal{L}$ denote the set of terms in $\delta^2\mathcal{L}$ that depend upon the variation of the proper mass. We assume (and we shall justify this later on) that the term in $\delta^2_{M_0}\mathcal{L}$ in (21) is negligible with respect to the other ones, and what remains is:

$$\int_{t_0}^{t_1} \{[\delta\mathcal{L}]_{M_0} + \delta_{M_0}\mathcal{L} + [\delta^2\mathcal{L}]_{M_0}\}\, dt = 0. \tag{23}$$

Since the first integral is zero, by virtue of (19), we obtain:

$$-\int_{t_0}^{t_1} \delta_{M_0}\mathcal{L}\, dt = -(t_1 - t_0)\, \overline{\delta_{M_0}\mathcal{L}} = \int_{t_0}^{t_1} [\delta^2\mathcal{L}]_{M_0}\, dt > 0, \tag{24}$$

in which $\overline{\delta_{M_0}\mathcal{L}}$ is the temporal mean between $t_0$ and $t_1$. Then, since $t_1 - t_0$ is positive and $-\delta_{M_0}\mathcal{L}$ is the quantity of heat that is given to the particle by the hidden thermostat, one sees that in the temporal mean that quantity of heat, which is constantly zero on the natural trajectory, is positive on the fluctuated trajectory. It then results that the entropy $S$ is reduced in mean when one passes from the motion *ACB* to the motion *AC′B*. The entropy is therefore maximal on the natural trajectory with respect to the fluctuations, subject to the conditions of the Hamiltonian variation, and one can say that the natural

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([17]) On the condition that one does not have any kinetic focal point relative to the point *A* between *A* and *B*. See C. R. Acad. Sc., **257** (1963), pp. 1430.

motion is more probable than the varied motion. A very remarkable relation between the principle of least action and the second law of thermodynamics can thus appear ($^{18}$).

It remains for us to show that we have good reason to neglect the term in $\delta^2_{M_0}\mathcal{L}$ in (23). Now, one sees that $\delta_{M_0}\mathcal{L}$ is of the same order as $\delta^2_{M_0}\mathcal{L}$ in equations (24); i.e., it is of second order with respect to the Hamiltonian variations, in such a way that $\delta^2_{M_0}\mathcal{L}$ is of third order, and can be neglected.

In its beginnings, wave mechanics had to establish a relationship between the action of a corpuscle and the phase of its associated wave that would permit one to identify the principle of Maupertuis with Fermat's principle. Pursuing the same type of identification, the preceding theory attaches the principle of least action to the second law of thermodynamics and the increase in entropy.

**4. Remarks on the relation** $h\nu_c = kT$. – We would now like to make some remarks on the first of the fundamental formulas (8) in our new thermodynamics.

First, note that it raises the following difficulty: Since the frequency $\nu_c$ is characteristic of the particle, the temperature $T$ must also depend upon it. Now, it seems natural to attribute a unique temperature $T$ to the sub-quantum medium, independently of the nature of the various kinds of particles that are in energetic contact with it. One can seek to eliminate this difficulty by imagining that the hidden thermostat is formed from sets of particles (i.e., probably the bunched field), where each set contains "hidden" particles of the same nature and has its proper temperature. When the particle is considered at the microphysical scale, it will be – perhaps as a result of a phenomenon of resonance type – in interaction with the hidden particles of the same nature as it, and this is why one can have:

$$T_0 = \frac{h\nu_0}{k} = \frac{m_0c^2}{k}$$

in the proper system of the particle. Admittedly, this hypothesis can seem very artificial, but it is curious to confirm that I was already led to assume it when, in the course of research of a completely different nature, I sought to deduce the wave equation with a mass term from a universal wave equation without a mass term of the type that was envisioned by Heisenberg ($^{19}$).

We must once more emphasize another very delicate point: If one considers the temperatures $T$ and $T_0$ as being referred to the particle itself then the relation:

$$T = T_0\sqrt{1-\beta^2}$$

that results automatically from formula (8) is satisfied because it corresponds to the relativistic transformation of temperature. However, if one would like to refer the temperature to the hidden thermostat then the relation:

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($^{18}$) In figurative terms, one can say that the natural trajectory follows a curve along the bottom of a valley of negentropy.

($^{19}$) *See*, bibliography [**10**], pp. 99-103.

$$m_0\, c^2 = kT_0$$

shows that the hidden thermostat must possess the same temperature $T_0$ in the proper system of every particle of mass $m_0$ . It results from this that one must attribute the same temperature $T_0$ to the hidden thermostat in any Galilean system – at least, as far as particles of proper mass $m_0$ are concerned. This strange property shows that the sub-quantum medium cannot be associated with an ordinary macroscopic thermostat, whose proper system would define a privileged reference system, which would be contrary to the principle of relativity, moreover. Rather, it seems that one can associate it with a "Dirac ether," whose properties appear identical for all Galilean observers (20), or to a "Terletsky ether." [**8**].

The preceding remarks show that one must arrive at a specification of the structure – which is certainly very specialized – of the sub-quantum medium of Bohm and Vigier in such a fashion that the existence of that medium is not in contradiction with the principle of relativity. Terletsky's idea, according to which one can assume that this medium is formed from particles of imaginary mass that displace with a velocity that is greater than velocity of light, can perhaps be useful in solving this problem, but I think that it is premature to restrict the study of that difficult question here (*see* [**18**]).

It seems interesting to us to note that the formula $m_0\, c^2 = kT_0$ leads one to attribute a very high value to the temperature of material particles. Upon employing C.G.S. units and degrees Kelvin, it gives, in effect, $T_0 \sim 10^{37}\, m_0$ . For the electron, one finds that $T_0$ is of the order of ten billion degrees absolute, and for much heavier particles one will find temperatures that are even more elevated. Therefore, every material particle will be found in constant energetic contact with a "hidden heater" that will be a site of extremely elevated temperatures, and which will be present everywhere in what we call the "vacuum." Other considerations have already led certain authors (e.g., Lanczos, Bohm) to an analogous conclusion.

**5. The relations $\bar{M}_0 = m_0$ and $\overline{S(M_0)} = -k$.** – We shall now infer some very interesting conclusions from the second formula in (8), which defines the entropy $S$.

First, consider a particle that is not subject to any external field. From Boltzmann's formula $S = k \log P$, the probability of the state of a particle whose fluctuating proper mass has the value $M_0$ is proportional to $e^{S/h}$, so, from the second formula in (8), it is proportional to $e^{-M_0/m_0}$ . One concludes from this that:

$$\bar{M}_0 = \frac{\int_0^\infty e^{-M_0/m_0} M_0\, dM_0}{\int_0^\infty e^{-M_0/m_0}\, dM_0} = m_0\,. \tag{25}$$

Therefore, the constant proper mass $m_0$ that is usually attributed to the particle appears to us as being the mean value of its true instantaneous proper mass, which fluctuates.

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(20) On the Dirac ether, *see* my article in *Journal de Physique* [**4**], pp. 975.

We can make this idea more precise in the following fashion: When one abstracts from the interactions between the particle and the sub-quantum medium, the theory of guidance leads one to define the variable proper mass of the particle in its proper system by the formula:

$$M_0 c^2 = \sqrt{m_0^2 c^4 + \frac{\hbar^2}{c^2}\frac{\Box a}{a}} = m_0 c^2 + Q_0 ,$$

where $Q_0 = M_0 c^2 - m_0 c^2$ is the previously-defined quantum potential, and one verifies that in the Newtonian approximation it has the known value $-\frac{\hbar^2}{2m_0}\frac{\Delta a}{a}$. The potential $Q$ translates into the interaction between the corpuscle and its wave; it is therefore a quantity at the microphysical level that does not explicitly involve the sub-quantum medium. If one would like to take the interactions of the particle with the sub-quantum medium into account then it would be natural to add a "fluctuating" quantum potential $Q_f$ to the right-hand side of the preceding equation in order to represent that interaction. One would then write:

$$M_0 c^2 = m_0 c^2 + Q_0 + Q_f .$$

As Terletsky has suggested by some interesting arguments in his important papers, there is good reason to think that the energetic exchanges between the sub-quantum medium and the particles must reduce to fluctuations with zero mean, which leads one to set $\bar{Q}_f = 0$. The last equation then gives:

(25 *bis*) $$\overline{M_0}\, c^2 = m_0 c^2 + \bar{Q}_0 ,$$

and if the quantum potential is zero then one recovers the relation $\overline{M_0} = m_0$. The usual proper mass $m_0$ will then result in continual energetic exchanges between the particle and the hidden thermostat.

If we introduce formula (25) into the evaluation of the mean value of entropy which is, from (8), $\bar{S} = S_0 - k\bar{M}_0 / m_0$, then we would find:

(26) $$\bar{S} = S_0 - k \qquad \text{or} \qquad \overline{S(M_0)} = -k.$$

Moreover, formulas (26) can be recovered by the following reasoning: We place ourselves in the proper system of the particle and assume initially that it has a proper mass $M_0$ that is zero, in such a way that $W_0$ and $S(M_0)$ are also zero. If the hidden thermostat provides the quantity of heat $dQ_0$ to the particle then its proper mass will increase by $dM_0 = dQ_0 / c^2$. If the thermostat has furnished the quantity of heat $\delta Q_0$ such that the proper mass $M_0$ has attained its mean value $m_0$ then one will have:

(27) $$\delta Q_0 = m_0 c^2 = kT_0 .$$

The entropy will have diminished by:

$$\delta S = -\frac{\delta Q_0}{c^2} = -k. \tag{28}$$

The term $S(M_0)$ in the expression for $S$ will then pass from the initial value 0 to its value $-k$, and we recover the result (26).

We add that the proper mass $M_0$ can vary from 0 to $+\infty$ during the fluctuations; i.e., $\delta M_0 = M_0 - m_0$ can vary from $-m_0$ to $+\infty$, although very large values for $\delta M_0$ are naturally quite improbable.

**6. Comparison with the Einstein's method for the study of fluctuations.** – Recall the general principle of the method that was employed by Einstein on another occasion for the study of fluctuations.

Consider a complex system whose state depends upon a certain parameter $\varepsilon$, along with a very large number of other parameters. In order to find the probability of a fluctuation in the state of the complex system that is due to a variation of the parameter $\varepsilon$, one can, following Einstein, proceed in the following manner:

Let $S_0$ be the maximal entropy of the system in its most probable state, and let $S(\varepsilon)$ be its entropy for a certain value of the parameter $\varepsilon$. One can write the Boltzmann relation in the form:

$$S(\varepsilon) = k \log \frac{P(\varepsilon)}{P_0} + S_0 , \tag{29}$$

where $P_0$ and $S_0$ are two constants. If we set $S = S_M$ then we must set $P_0 = P_M$ because $S = S_M$ must correspond to the state of maximal probability $P_M$. We will then have:

$$P(\varepsilon) = P_M \, e^{-\frac{S_M - S(\varepsilon)}{h}} . \tag{30}$$

Naturally, the more that $S(\varepsilon)$ is small and distant from $S_M$, the smaller that the probability $P(\varepsilon)$ of the fluctuating state will be. It results from (30) that the mean value of $S_M - S(\varepsilon)$ is:

$$\overline{S_M - S(\varepsilon)} = k. \tag{31}$$

We can apply the preceding formalism to the particle in contact with the hidden thermostat, which is a very complex system, by taking the parameter $\varepsilon$ to be the variable proper mass $M_0$ of the particle, and upon setting:

$$S(\varepsilon) = S_0 = k \frac{M_0}{m_0} , \tag{32}$$

$S$ will be maximal for $M_0 = 0$, so $S_M = S_0$, and we recover the proportionality of $P(M_0)$ with $e^{-M_0/m_0}$ and the formula $\overline{S(M_0)} = -k$.

In view of a comparison that will be made in the following paragraph, we recall that in the era (around 1910) when Jean Perrin carried out his celebrated measurements of Avogadro's constant by studying the distribution in height of granules in suspension in an emulsion under the action of a gravitational field, Smoluchovsky devised his theory of the phenomenon by using Einstein's method. The probability for a granule of mass $m$ to have an altitude $z$, which is measured from the base of the receptacle that contains the emulsion, is given by Laplace's barometric law in the form:

$$P(z) = \text{const. } e^{-\frac{mgz}{kT}} . \tag{33}$$

Upon setting $S(0) = 0$, the corresponding entropy is:

$$S(z) = k \log P(z) = -\frac{mgz}{T} . \tag{34}$$

One easily finds:

$$\overline{z} = \frac{\int_0^\infty z\,P(z)\,dz}{\int_0^\infty P(z)\,dz} = \frac{kT}{mg} , \tag{35}$$

so:

$$\overline{S} = -\frac{mg}{T}\frac{kT}{mg} = -k. \tag{36}$$

The entropy of the granule, when normalized as we did, fluctuates from its maximal value that is equal to 0, when the granule is at the base of the receptacle at the altitude $z = 0$, up to (theoretically) $S = -\infty$ for $z = \infty$, with the mean value $\overline{S} = -k$.

**7. Overview of the results obtained up to now.** – We would now like to summarize in a few words the picture to which the theory of the double solution has finally led us, now that is has been completed with the hypothesis of the existence of a sub-quantum medium that plays the role of the hidden thermostat.

The corpuscle is conceived of as being a very localized inhomogeneity (a small region of very high values of the wave field) inside the wave, whose equation of propagation contains the proper mass of the particle. In the absence of perturbations (if the proper mass has the constant value $m_0$), the corpuscle will regularly describe one of the streamlines of the propagation of the wave, which conforms to the guidance principle. However, the wave and its corpuscle propagate, one might say, "on the surface" of the sub-quantum medium that plays the role of a hidden thermostat, and as a result of the energetic exchanges that take place between the hidden thermostat and the particle its proper mass is subjected to continual fluctuations that make it traverse a whole series of segments of the streamlines in the hydrodynamical picture of the propagating wave. The corpuscle is then animated inside the wave, which constitutes a very small singular region, with a sort of Brownian agitation, and that is what introduces the probability into the predictions of wave mechanics and quantum physics.

The image of a particle that follows, quite wisely, a streamline that conforms to the guidance formula has exactly the same value as the one that is used in hydrodynamics when one considers a fluid molecule as following a trajectory in its motion that is constantly tangent to one of the streamlines of hydrodynamic convection. However, in one case as in another, one must, in reality, superimpose a disordered thermal agitation over this theoretical motion that makes the unit – whether corpuscle or fluid molecule – constantly jump from one streamline to another. Finally, it is only the totality of the streamlines that gives a statistical picture of the motion of an infinitude of exemplars of the unit, but this does not at all imply that each unit does not have well-defined position and motion at each instant.

If we now return to the experiments of Jean Perrin then we can say that a granule in an emulsion that is subject to the force of gravity has a vertical trajectory that starts on the base of the receptacle and remains immobile for its natural trajectory, but the thermal agitation of the hidden molecules of the fluid in which it is suspended constantly shifts it to the right, to the left, up, and down, in such a way that it always has a non-zero probability of being found at a height $z$ above the base of the receptacle that is given Laplace's law. Likewise, in our present conception of wave mechanics, the particle has a natural trajectory that is the one that is associated with it by the guidance formula, but it is constantly found to be projected here or there by the agitation that comes from its contact with the sub-quantum medium and that is what finally gives a non-zero probability of being found at no particular point of the wave that is equal to $|\psi|^2$.

If these ideas are shown to be indeed correct then a remarkable premonition of Einstein will then have been realized. In 1905, which was the same year in which he laid down the basis for the theory of relativity and discovered the corpuscular aspect to light, he also studied the theory of Brownian motion in great depth, and he seems to have always sensed that the intervention of probabilities into wave mechanics indicated the existence of a sort of continual Brownian motion of microphysical particles. Now, what one calls Brownian motion is what one also calls fluctuations in thermodynamics (21).

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(21) I point out that in my note to the *Comptes rendus de l'Académie des Sciences* on 30 July 1962, I defined the entropy $S$ of the hidden thermostat with the aid of the Hamiltonian integral of the particle, when taken over an internal period $\tau_c = \frac{h}{m_0 c^2 \sqrt{1-\beta^2}}$ of it, by setting:

$$\delta S = \frac{k}{h}\,\delta_{M_0} \int_0^{\tau_c} \mathcal{L}\,dt \,. \tag{a}$$

In the case of material particles (but not in that of photons, even if one considers their proper mass to be slightly non-zero), the proper period $\tau_0 = h/m_0 c^2$ is extremely small, in such a way that it seems legitimate to consider the mass $M_0$ as being constant during that very short duration. Formula ($a$) is then practically equivalent to the definition (8):

$$S = S_0 - k\,\frac{M_0}{m_0} \tag{b}$$

that was adopted above. For material particles, formula ($b$), which is simpler than ($a$), thus seems to be equivalent to ($a$). However, the case of photons, which raises some particular difficulties, must be the object of a special examination.

CHAPTER IX

# STABILITY OF STATES, ENTROPY, AND FREE ENERGY

**1. Quantum transitions and the monochromatic state "prerogative."** – Since the appearance in 1913 of Bohr's theory of the atom, one has attributed a character to quantum transitions, which make a quantum system pass from one stationary state to another, that one might qualify as mystical. Indeed, one renounces the thought of forming any sort of picture of them, and Bohr did not hesitate to assert that any description of them in space and time would be "transcendental." This is what led Schrödinger to say, ironically, that in the present quantum theory one minutely describes the stationary states, in which nothing happens, but refuses to describe the transitions, where something does happen.

The idea that was introduced by the theory of the double solution that wave mechanics must, in the final analysis, rest upon nonlinear equations permits one to think that if the quantum transitions escape any description in the present theory then that is because they constitute essentially nonlinear processes. They will be transitory processes of extremely short duration that are analogous to the ones that one has already encountered in several nonlinear theories in mechanics and physics when there is a brief passage from one limit cycle to another. This very attractive idea was already envisioned some years ago by Čap and Destouches, and has been recently reprised by Fer, Lochak, Andrade e Silva, and Leruste, who have published some papers on that subject that are of great interest (22).

Now, when Lochak and Andrade e Silva were informed of my first note on August 1961 on the thermodynamics of the isolated particle, after my having justifiably remarked that my formulas that are deduced from the relation $dS = dQ / T$ apply only to reversible processes, they suggested to me that the very brief transitory states that they envisioned could have an irreversible character and could be accompanied by a very brief change in entropy (or free energy), and that the passage from one stationary state to another could involve the crossing of a valley of entropy (or mountain of free energy).

I was then led to reflect upon these interesting questions more deeply. In order to comprehend the orientation of my thoughts on the subject, I will start with the remark that in the usual theory one accords a sort of "prerogative" to the states that one can qualify as "monochromatic." I must specify that I intend the phrase "monochromatic states" to mean, on the one hand, the stationary states of the quantum system that are represented by a proper Hamiltonian function and are associated with a stationary wave of well-defined frequency, but also, on the other hand, in the case of particles in progressive motion, to the states that are associated with groups of waves that behave like a monochromatic plane wave in all of their extension. At the beginning of Bohr's theory of the atom, one considered the atom as necessarily being always found in a stationary state, and when, later on, one translated Bohr's theory into the language of wave mechanics, one assumed that the states that were represented by a superposition of proper functions had only a very fleeting existence, and that the atom was always grasped by

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(22) *See* bibliography [**5**] and [**11**].

observations that were made in a stationary state that was represented by one of the proper functions. In the quantum theory of fields, the same preoccupation is manifested by the fact that the "occupation numbers" are, in general, referred to monochromatic plane waves.

In some very penetrating articles that were dedicated to a critique of the present quantum concepts, Schrödinger was, with good reason, surprised at this prerogative that was accorded to monochromatic states. He thought that it was unjustified, because, *a priori*, a superposition state has a more general character than a monochromatic state (viz., the function $\Psi = \sum_i c_i \Psi_i$ is more general than the function $\Psi = \Psi_k$). Meanwhile, the success of the hypothesis that the monochromatic states effectively have a prerogative hardly permits one to doubt, contrary to Schrödinger's opinion, that this prerogative is unjustified. How can we explain that?

The idea that seems to me to be capable of affording the desired explanation is that the superposition states will have a probability that is much weaker than the monochromatic states, which will then be, in some way, unstable, and that the quantum transitions, which are very rapid nonlinear processes, always tend to bring particles or systems back to a more stable, monochromatic, state. It is obvious that from the thermodynamic viewpoint the stability of a state must be attached to an entropy maximum or to a free energy minimum. In order to see this state of affairs more clearly, we shall study a certain number of special cases. To commence, we shall first examine two cases in which one deals with an isolated system that does not exchange energy with the external medium, and we shall verify by these examples that, in such a case, and conforming to the Boltzmann relation $S = k \log P$, it is the maximum entropy that corresponds to the most probable state.

**2. The case of a free particle in the Newtonian approximation.** – I shall first recall the definition of the quantum potential. In general relativistic case, it is the quantity $M_0c^2$ that is given by:

$$M_0c^2 = \sqrt{m_0^2 c^4 + \hbar^2 c^2 \frac{\Box a}{a}}\,, \tag{1}$$

which plays the role in the proper system of the particle of quantum potential whose gradient, with the opposite sign, gives the quantum force. Since the quantum potential $Q$ must transform like a quantity of heat under a change of Galilean reference system, and it is defined only up to a constant, we can set:

$$Q_0 = M_0\,c^2 - m_0\,c^2, \qquad Q = Q_0\,\sqrt{1-\beta^2}\,. \tag{2}$$

In the Newtonian approximation, we easily recover the expression:

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta a}{a}\,. \tag{3}$$

Now, consider a particle that displaces freely without being subject to any force. Its associated wave can be a group of waves that are assimilated to a monochromatic plane wave, and its motion is then uniform and rectilinear, or else formed from a more general superposition of monochromatic plane waves, as in interference or diffraction phenomena, and then the guidance formula would assign a complicated motion to it. I will show that under these conditions a superposition state has a mean entropy that is less than that of a monochromatic state.

Indeed, for a monochromatic state, since the amplitude $a$ of the wave is constant, the potential $Q$ that is given by (3) is zero, while in a superposition state, the mean value of $Q$ over all positions of the particle in the wave will be:

$$(4) \qquad \bar{Q} = -\frac{\hbar^2}{2m}\int a^2 \frac{\Delta a}{a} d\tau = -\frac{\hbar^2}{2m}\int a \Delta a \, d\tau .$$

Since $a$ is always zero at infinity, an integration by parts gives:

$$(5) \qquad \bar{Q} = \frac{\hbar^2}{2m}\int (\text{grad}\, a)^2 \, d\tau > 0;$$

i.e., from (2), $\bar{M}_0 > m_0$ . Therefore, in a superposition state, the mean value of the proper mass $M_0$ is greater than its normal mean value $m_0$ . It then suffices to recall that we have defined the entropy $S$ by the relation:

$$(6) \qquad S = S_0 - k \frac{M_0}{m_0}$$

in order to see that the entropy $S_m$ of a monochromatic state and the mean value $\bar{S}_s$ of a superposition state will have the values:

$$(7) \qquad S_m = S_0 - k, \qquad \bar{S}_s = S_0 - k \frac{\bar{M}_0}{m_0} < S_m ,$$

which indeed shows that the monochromatic state, which has an entropy that is greater than the superposition state, must have a higher probability and greater stability ([23]).

**3. The case of the collision of two particles.** – The case that we just studied was very simple because we considered an isolated particle. We shall now envision the more complicated case of the collision of two particles, in which we always take the Newtonian approximation. The problem was treated using wave mechanics in 1927 by Max Born, and it was at that moment that he introduced the probabilistic interpretation of the new mechanics for the first time.

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([23]) For the extension of this argument to the case of the Dirac electron, see the Appendix.

We assume that at the initial instant the two particles are sufficiently distant from each other for there to be practically no interaction, and that each of them is carried by a train of waves that is assimilated into a monochromatic plane wave. The wave function in configuration space then has a form $\Psi_i$ that is equal to the product of the individual wave functions of the two particles. The interaction begins when the particles approach, and the calculation that one does in the usual theory can, in our view, be interpreted in the following fashion: There is first a linear and causal evolution of the Schrödinger wave $\Psi$ in configuration space. The function $\Psi$ becomes a superposition of the Fourier components of the form $\Psi = \sum_l c_l \Psi_l$ , which corresponds to the totality of the correlated individual wave propagations $v_1$ and $v_2$ in physical space. In the classical calculation that was developed by Born, one then assumes that everything happens as if, at the end of the collision, a process is briefly produced that has destroyed the phase relations between the Fourier components $\Psi_l$ , and which has the ultimate result that each of the particles is, at the end of the interaction, attached to a train of waves that are assimilated into a monochromatic plane wave, while the global energy and quantity of motion of the system are found to be conserved. The final form $\Psi_f$ of the wave function in configuration space therefore becomes equal to one of the functions $\Psi_l$ , which is the product of the final wave functions of the two particles. In summation, there is a passage from the initial state $\Psi_i$ to one of the final states $\Psi_l = \Psi_k$ , a passage that has, *a priori*, a probability of occurrence $| c_k |^2$, from the laws of probabilities in wave mechanics. The passage therefore comes about in two stages: The first one is relatively slow, linear, and causal, and is described quite well by the usual equations of wave mechanics with the aid of the evolution of the function $\Psi$, while the second one is a very brief one that is, according to our conception of things, undoubtedly nonlinear, and involves a brief, but important, exchange of energy and quantity of motion between the two particles. The description of the second stage is completely absent from the usual linear theory, and is simply postulated with no attempt at interpretation.

We shall now attempt to prove that the initial state and the final state, which are both representable by trains of waves that are assimilated to monochromatic plane waves, have an entropy that is greater than that of the intermediate superposition state. Up to the present, the difficulty in proving this is that we have developed the hidden thermodynamics only for a unique, isolated particle, and that we must now generalize the formulas that were obtained to the case of a system of particles. While reserving a deeper study of the question for later, it seems natural to define, in the Newtonian approximation, the temperature $T$ of a system of two particles of masses $m_1$ and $m_2$ and the total proper mass of the system by the formulas:

$$kT = (m_1 + m_2)\, c^2 \tag{8}$$

and

$$M_0\, c^2 = (m_1 + m_2)\, c^2 + Q, \tag{9}$$

where $Q$ is the quantum potential of the system that is defined by starting with the amplitude $a$ of the wave $\Psi$ in configuration space, namely:

$$Q = -\,\hbar^2 \sum_{i=1}^{2} \frac{1}{m_i} \frac{\Delta_i a}{a}. \tag{10}$$

Now, we have the following formulas for the energy $W$ and the Lagrange function $\mathcal{L}$ of the system, while taking into account the mass terms:

$$\begin{cases} W = \quad (m_1 + m_2)c^2 + E_k + V + Q, \\ \mathcal{L} = -(m_1 + m_2)c^2 + E_k - V - Q, \end{cases} \tag{11}$$

where $E_k$ is the total kinetic energy of two particles, and $V$ is the potential energy of interaction. We eliminate $E_k$ , which depends upon the motion of the two particles that is defined by the guidance formula, by subtracting $W$ from $\mathcal{L}$, which gives:

$$\mathcal{L} = -\,2(m_1 + m_2)\,c^2 - 2Q + W - 2V. \tag{12}$$

Since $W$ – viz., the total energy of the system – is a constant and $V$ does not depend upon $M_0$ , we can write:

$$\mathcal{L} = -\,2M_0\,c^2 + \ldots, \tag{13}$$

where the unwritten terms do not depend upon $M_0$ .

Now, we know that $\delta S = \delta_{M_0}\mathcal{L} / T$ , and upon taking formulas (8) and (9) into account, we easily find:

$$S = S_0 - \frac{2k\,M_0}{m_1 + m_2} = S_0 - 2k - \frac{2k}{(m_1 + m_2)c^2}\,Q. \tag{14}$$

For the initial “monochromatic” state and for the final “monochromatic” state, we have $Q = 0$, since $a$ is constant, so:

$$S_m = S_0 - 2k. \tag{15}$$

The term $-\,2k$ is appropriate, since the two particles are then independent and each give a contribution to the total entropy that is equal to $-\,k$. For the superposition state, we find, from (14):

$$\overline{S}_s = S_0 - 2k - \frac{2k}{(m_1 + m_2)c^2}\,\overline{Q}, \tag{16}$$

where $\overline{Q}$ is the mean value of $Q$, calculated in configuration space where the probability of the presence of the representative point of the system at each point is given by $a^2$. Taking into account the expression (10) for $Q$, a calculation that is entirely analogous to the one that we made in physical space in order to obtain formula (5) shows us that $\overline{Q} > 0$. The comparison of formulas (15) and (16) then shows us immediately that:

$$\bar{S}_s < S_m . \tag{17}$$

We thus recover our preceding conclusions on the instability of superposition states, and we see that the brief, transitory process that corresponds to the passage from the state $\Psi = \sum_l c_l \Psi_l$ to the final state $\Psi_k$ is accompanied by a brief augmentation of the entropy, conforming to the ideas of Lochak and Andrade e Silva. The initial and final states are monochromatic ones that correspond to the value $S_0 - 2k$ of the entropy, so we can represent the situation by the diagram in Figure 8.

In this diagram, the solid curve *AB* schematically represents the first stage of the collision, which corresponds to the causal, linear evolution of the function Ψ that is usually calculated, while the dotted line *BC* represents the brief – and undoubtedly nonlinear – transition, which, after crossing a valley of entropy, brings the system from the superposition state *B* to the monochromatic final state *C* with a brief augmentation of entropy.

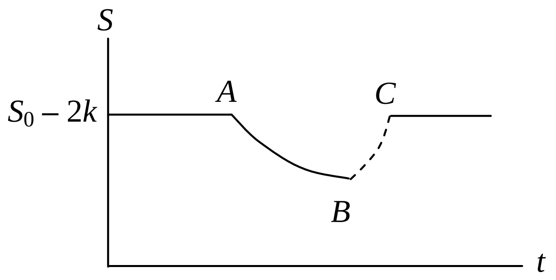

Figure 8.

**4. Introduction of free energy into the hidden thermodynamics of particles. –** We just studied two cases in which a particle or system is isolated and does not exchange any energy with the external medium, while being in contact only with the hidden thermostat. We then found that the stability of the states corresponded to a maximum of entropy, and that, without a doubt, the same thing should be true for all other cases of this type. However, the case of systems that can exchange energy with the external medium must be different. In his very interesting note on April 1963, Lochak insisted upon the fact that in order for a quantum system to be capable of exchanging energy with the external medium the stability of the quantum states must correspond to a minimum of a "free energy" function of the form $F = U - TS$. The quantum states of the system that, except for the state of least energy, are only metastable will correspond to a series of small bowls of free energy on the slope of a mountain of free energy. This conception seems exact to us, but there is good reason to make the definition of that free energy more precise, since it is not precisely the same as it is in the usual thermodynamics.

Let $\mathcal{Q}$ denote the quantity of heat that is given to a particle by the hidden thermostat, and let $Q$ denote the quantum potential of that particle. The formulas for our thermodynamics give us:

(18) $$S = S_0 - k\,\frac{M_0}{m_0} = S_0 - k - \frac{kQ}{m_0 c^2} = S_0 - k - \frac{Q}{T}$$

in the proper system of the particle, so:

(19) $$\delta\mathcal{Q} = -\,T\,dS = \delta Q.$$

It is essential to note that, despite their equality, the quantities $\delta\mathcal{Q}$ and $\delta Q$ have very different physical senses: Indeed, $\delta\mathcal{Q}$ is the quantity of heat that is exchanged between the hidden thermostat and the particle, while $\delta Q$ is the variation in the proper mass of the particle that results from it. The heat $\mathcal{Q}$ can circulate only between the hidden thermostat and the particle, and cannot intervene directly in the dynamics of the particle or its exchanges of energy with the environment, or else the thermostat would not be hidden.

The energy of the particle can be written:

(20) $$W = m_0\,c^2 + E_k + V + Q,$$

where $V$ is the classical potential energy and $E_k$ is the kinetic energy $\frac{M_0 c^2}{\sqrt{1-\beta^2}} - M_0\,c^2$, which reduces to $\frac{1}{2}\,m_0\,v^2$ in the Newtonian approximation, $v$ being the velocity that is defined by the guidance formula. Let $\mathcal{T}$ be the external work done – i.e., the energy that is given to the external medium by the particle. If we set, by definition:

(21) $$U = E_k + V$$

then conservation of energy imposes that we must write:

(22) $$\delta\mathcal{T} = -\,\delta W = -\,\delta(U + Q),$$

and if we adopt:

(23) $$F = U - TS$$

as our definition of free energy then we will have:

(24) $$\delta\mathcal{T} = -\,\delta(U + Q) = -\,\delta(U - TS) = -\,\delta F.$$

We thus recover the classical formula $\delta\mathcal{T} = -\,\delta F$; this entails that the most stable state corresponds to the minimum value of $F$.

However, we must remark that the definition (23) that we adopted for free energy is not identical to the one that is adopted in the usual thermodynamics. Indeed, in that definition, one considers a body that contains a *total* mechanical and thermal energy $U$, and which can exchange heat and work with the environment. One is then led to write:

$$\delta U = \delta \mathcal{Q} - \delta \mathcal{T}, \tag{25}$$

$\delta \mathcal{Q}$ being the heat that is *received* by the body *from* the environment here, and $\delta \mathcal{T}$ being the work that the body *does on* the environment.

Upon setting:

$$F = U - TS, \tag{26}$$

one obtains:

$$\delta \mathcal{T} = -\,\delta(U - TS) = -\,\delta F \tag{27}$$

for a reversible transformation. However, things are different in the problem that we are studying. We have a particle whose energy is given by formula (20). That particle is in contact with the external medium, but it can receive or provide only work with it – i.e., ordinary mechanical energy. It is likewise found to be in contact with the hidden thermostat, with which it can receive or provide only heat, with a corresponding variation in its proper mass. Conservation of energy must be valid for the exchanges of energy between the particle and the external medium, which is an abstraction made from the presence of the hidden thermostat. Its presence will only be manifested in our calculations by the necessity of taking the mean over the position of the particle.

It was these circumstances that led us to take the definition (21) for $U$, which is different from the usual definition of thermodynamics that we just recalled, since our quantity $U$ does not represent the totality of the energy in the system considered. Moreover, when we set $F = U - TS$, we obtained a definition of free energy that differs from the one in classical thermodynamics, first of all, because $U$ is defined the same way in the two theories, and also because $S$ is no longer the entropy of the system considered (the particle does not contain heat in the form of disordered calorific energy), but that of the hidden thermostat, which is a heat reservoir for it.

Briefly, the essential reason for the difference that exists between formulas (24) and (27), which appear to be the same, is that a particle is not a macroscopic body that contains heat in the form of internal molecular agitation.

**5. Examples of applications of the preceding formulas.** – We shall now give some applications of our definition of free energy.

a) *Electron in a hydrogen atom.* – As an example of the application to a quantum system, we consider the simple case of an electron in a hydrogen atom, while limiting ourselves to the completely stationary case, in which the guiding velocity $v$ is zero. One then has:

$$V = -\frac{e^2}{r}.$$

Now, I have proved in my book on the theory of measure ([**4**], pp. 76-77) that in the theory of the double solution the virial theorem takes the form:

$$2\overline{(E_k + Q)} - \overline{(\mathbf{r}\cdot \mathrm{grad}\,V)} = 0, \tag{28}$$

which gives:

$$V + 2\overline{Q} = 0 \tag{29}$$

here, with $E_k = 0$ and $-\overline{(\mathbf{r}\cdot\mathrm{grad}\,V)} = V$. The means are taken with $|\,\Psi\,|^2$; i.e., over the Bohm-Vigier perturbations.

As in the quantum states $V + Q$ = const., no matter what the fluctuating position of the particle is in its wave, we will have:

$$\delta(V + Q) = \delta(\overline{V} + \overline{Q}) = \tfrac{1}{2}\delta\overline{V} = -\delta\overline{Q} = T\ \delta\overline{S}\ . \tag{30}$$

One easily sees that for the transitions with the emission of energy, one has $\delta\overline{V} < 0$ and $\delta\overline{Q} > 0$. Hence, for these transitions:

$$\delta T = -\ \delta(\overline{V} + \overline{Q}) = -\tfrac{1}{2}\delta\overline{V} = \delta\overline{Q}\,. \tag{31}$$

The reduction of the potential energy thus simultaneously compensates for the production of external work and the augmentation $\delta Q$ of the proper mass energy. One sees that here the conservation of energy obligates the entropy $S$ to diminish. The stability of the states will thus be determined by the reduction of the free energy $F$, as we have defined it, and not by the augmentation of the entropy $S$, which is in accord with the concepts of Lochak.

Of course, if there is a supply of external energy ($\delta T < 0$) then $V + Q$ augments, where the augmentation of $V$ simultaneously accounts for the supply of external energy and the reduction of mass energy $Q$ that accompanies an augmentation in entropy $S$.

b) *Case of a linear harmonic oscillator.* – In the case of stationary states of a linear harmonic oscillator, the phase $\varphi$ of the wave can depend upon only time, and the guiding velocity $\mathbf{v} = -\,1/m$ grad $\varphi$ is always zero. As for the potential energy, it has the form $V = \tfrac{1}{2}Kx^2$. One thus has $U = V = \tfrac{1}{2}Kx^2$. Now, the virial theorem (28) gives us:

$$\overline{Q} = \overline{V} \tag{32}$$

here. One can thus write:

$$U + Q = V + Q = \overline{V} + \overline{Q} = 2\overline{V} = 2\,\overline{Q} \tag{33}$$

and

$$\delta T = -\ \delta(V + Q) = -\ \delta(U - TS) = -\ 2\delta\overline{V} = -\,2\,\delta\overline{Q}\,. \tag{34}$$

When the oscillator emits energy into the environment, the two potentials $V$ and $Q$ diminish in mean by the same quantity. Contrary to what happens in the case of the hydrogen atom, the reduction of $V$ is accompanied by an augmentation of $S$, but there is always a reduction of $F = U - TS$.

c) *Case of a particle that does not exchange energy with the environment.* – We shall study the case of a particle that is placed outside of any field ($V = 0$), and which exchanges no energy with the environment. Its variations in energy can then come only from variations in the proper mass that are due to the heat that is absorbed or provided by the hidden thermostat, and here, as we have seen, we must expect to see that stable states correspond to maxima of the entropy $S$.

We shall again compare two states of the particle with the same energy: The one is the state $m$, for which the wave $\Psi$ can be assimilated into a monochromatic plane wave, while the other is the state $s$, for which the wave $\Psi$ is a superposition of monochromatic plane waves of the same frequency. In the state $m$, $Q$ is zero and $F_m$ reduces to the kinetic energy. In the state $s$, $Q$ is non-zero, and one can prove that $\bar{Q} > 0$. Since the energy $E_k + Q$ remains constant, we have:

$$F = E_k + Q = \bar{E}_c + \bar{Q} > 0$$

in both cases. However, since, by hypothesis, external work $\mathcal{T}$ is zero, we have:

(35) $$\delta_{sm}F = \delta_{sm}E_k + \delta_{sm}Q = 0,$$

for the transition $s \rightarrow m$, so:

(36) $$\delta_{sm}\bar{Q} = -\,T\delta_{sm}\bar{S} = -\,\bar{Q}_s > 0,$$

and, in turn:

(37) $$\delta_{sm}\bar{S} > 0, \qquad \delta_{sm}\bar{E}_k > 0.$$

Therefore, during the transition $s \rightarrow m$ there is a simultaneous augmentation of the entropy $S$ and kinetic energy. For a particle that exchanges no energy with the environment, the "monochromatic states" $m$ are more probable than the superposition states and correspond to a maximum of entropy, as we have already seen in paragraph 3.

## 6. Conclusions.

– We can summarize the preceding by saying: "In the case of a particle or system of particles that do not exchange energy with the environment, the stability of a state corresponds to a maximum of entropy. In the case of a system that can give or absorb mechanical energy from the environment (such as a quantum atom, which can emit or absorb a photon, as well as exchange energy with an external particle under a collision), the stability of a state corresponds to a minimum of a suitably-defined *free energy* function."

One can further envision other cases: For example, that of a quantum system in contact with an external macroscopic thermostat with temperature $\theta$. One knows that under these conditions, from the Boltzmann-Gibbs canonical distribution law, the quantum energy state $E_n$ has a probability of being realized that is proportional to $e^{-E_n/k\theta}$. Since, to our way of thinking, any particle is found to be in energetic contact with the hidden thermostat, one is led to conceive of two kinds of thermodynamics that simultaneously come into play: An "external" thermodynamics that is due to exchanges of heat between the system and the macroscopic thermostat and an "internal" thermodynamics that is due to exchanges of heat with the hidden thermostat. In the note that was cited in paragraph 4, Lochak has insisted upon this point by making an interesting remark on the subject, but the problem obviously demands to be examined more closely.

In conclusion, we would like to draw attention to the close link that exists between, on the one hand, our hidden thermodynamics of particles and the conclusions that we have inferred, and on the other hand, the notion of quantum potential and variable proper mass that are characteristic of the dynamics of guidance and the theory of the double solution. It seems probable to us that the theory of the double solution, when completed with the thermodynamics that we sketched out in this volume, will be called upon to play an important role in the future developments of quantum physics.

# APPENDIX

## On the instability of superposition states in the case of the Dirac electron

On page 200 of the work cited in number [**3**] of the bibliography, I gave an expression for the variable proper mass of the Dirac electron, namely:

(1) $$M_0 = \frac{m_0}{\Psi^+\Psi c}\sqrt{-j_\mu j^\mu}\ ,$$

where $j_\mu$ is the "current-density" quadri-vector of the Dirac theory. Now, one has $j_\mu = \rho_0 u_\mu$, with $u_\mu u^\mu = -c^2$, which permits one to write:

(2) $$M_0 = \frac{m_0}{\Psi^+\Psi}\rho_0 .$$

If one defines the two well-known invariants of the Dirac theory by:

(3) $$\Omega_1 = \Psi^+ \Psi = \Psi^+\gamma_4\Psi, \qquad \Omega_2 = \Psi^+\gamma_1\gamma_2\gamma_3\gamma_4\Psi$$

then one of the classical Pauli-Koffink relations gives:

(4) $$\rho_0^2 = -j_\mu j^\mu = \Omega_1^2 + \Omega_2^2 .$$

From (2), (3), (4), one thus has:

(5) $$M_0 = m_0\sqrt{1+\frac{\Omega_2^2}{\Omega_1^2}}\ .$$

However, in Dirac's theory, the invariant $\Omega_2$ is zero for monochromatic plane wave, while it is non-zero for a superposition. Since the entropy of the hidden thermostat was defined in a general fashion by the formula $S = S_0 - M_0 / m_0$, one quickly sees that one has:

(6) $$S_1 < S_m .$$

Here again, one arrives at the result that the superposition state is less stable than the monochromatic state, and one sees the simplicity by which this conclusion was deduced from formula (5).

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