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# Mechanics, according to the principles of the theory of extensions

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There is scarcely a realm in which the indispensability of the calculus that was presented in my *Ausdehnungslehre* (of 1844 and 1862) proves to be as persuasive as it does in mechanics. One can say that any simple mechanical concept has a likewise simple associated concept in the calculus. In fact, once I had recognized the first principles themselves, I developed the entire method of calculation further most quickly and fruitfully by resorting to mechanics. Without exception (except for changes of notation here and there), I have already published the methods that I will turn to in this article and the equations that I will arrive at by using them in a paper on the theory of ebb and flow that I submitted at Pfingsten 1840 as a test paper for the scientific board of examiners in Berlin. Very little of it is overlooked in my *Ausdehnungslehre* of 1844. The recent textbooks and articles in mechanics, namely, G. Kirchhoff's *Vorlesungen* (1875, 1876) show me that the presentation of these methods is still just as requisite today as it was thirty-seven years ago when I found the time and opportunity to publish them. In a later article, I think that I will then solve the most important of the problems of mechanics that have not be touched upon by new methods that likewise arise from the theory of extensions.

## § 1. Concepts and laws of the theory of extensions that shall be employed here.

For the sake of clarity, I will give an overview of the theory, to the extent that it shall be applied in this article, but refer to my *Ausdehnungslehren* of 1844 and 1862 for the detailed treatment, which I will denote by $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in the sequel. I start with the notion of *line segment.* I understand this to mean a bounded straight line of definite length and direction; i.e., I regard two line segments to be equal if and only if they have equal lengths and directions. Line segments will be *added* when one continuously lays them one after the other, so the line segment from the initial point of the first one to the end point of the last one is their *sum* ($\mathfrak{A}_1$: § 15-18, $\mathfrak{A}_2$ : 220). *Subtraction* reverts to addition, since one can add the line segment that goes from $B$ to $A$, instead of the one from $A$ to $B$. The concept of *multiplication* or *division* by a number emerges from the general concept of these processes immediately. The fact that the usual rules of calculation are completely valid for all of these processes is proved in the theory of extensions.

The *exterior product* of the line segments $a$ and $b$, which is written $[ab]$, will be defined formally by first requiring that for any product the following relationship to addition is true: i.e., one has:

$$[a\,(b + c)] = [ab] + [ac], \qquad [(a + b)\,c] = [ac] + [bc],$$

and second, that the exterior product of equal line segments is zero:

$$[aa] = 0,$$

but conceptually by requiring that when $a$ is the line segment from the point $A$ to the point $B$ and $b$ is the line segment from $B$ to $C$ or from $A$ to $D$, then $[a\ b]$ will be the *surface space* of the parallelogram $ABCD$, and indeed in the sense that two such surface spaces are equal to each other if and only if they lie in parallel planes, have equal areas, and the perimeters of both of them go through the same sides (right or left) ($\mathfrak{A}_1$: § 28-30, 37, $\mathfrak{A}_2$: 239, *et seq*.) The addition of surface spaces, when they do not lie in parallel planes, is determined completely by the formula:

$$[ab] + [ac] = [a\,(b + c)].$$

The formula:

$$[(a + b)(a + b)] = 0$$

immediately yields the second important law of exterior multiplication, namely:

$$[ab] = -\,[ba].$$

The exterior product of three line segments $a$, $b$, $c$, or of a surface space $[ab]$ and a line segment $c$ is defined formally by requiring that:

$$[abb] = 0$$

and therefore also:

$$[abc] = -\,[acb],$$

and conceptually by associating that symbol to the volume of a parallelepiped that has $a$, $b$, $c$ as the sides that are connected to each other. It will be zero when the three line segments lie in a plane. Furthermore, one has:

$$[abc] = [bca] = [cab] = -\,[acb] = -\,[cba] = -\,[bac].$$

($\mathfrak{A}_1$: § 37, $\mathfrak{A}_2$: 240, *et seq*.)

I understand the term *inner product* $[a \mid b]$ of two line segments $a$ and $b$, whose lengths are $\mathfrak{a}$ and $\mathfrak{b}$, and subtend the angle $\angle\, ab$ to mean the product:

$$[a \mid b] = \mathfrak{a}\mathfrak{b} \cos \angle ab,$$

and for the sake of brevity, write $a^{\underline{2}}$ for $[a \mid a]$, and call this the *inner square* of the line segment $a$ ($\mathfrak{A}_1$: XI, $\mathfrak{A}_2$: 179). I would like to call the of three line segments $e_1$, $e_2$, $e_3$ that are perpendicular to each other and whose lengths and exterior product $[e_1e_2e_3]$ equal one a *normal set*. The rules for inner multiplication then emerge from the concept immediately, namely:

$$[a \mid b] = [b \mid a],$$
$$[a \mid b] = 0 \qquad \text{when } a \text{ and } b \text{ are perpendicular,}$$
$$a^{\underline{2}} = \mathfrak{a}^2, \qquad \text{when } \mathfrak{a} \text{ is the length of } a,$$
$$[a \mid b] = \mathfrak{a}_1\mathfrak{b}_1 + \mathfrak{a}_2\mathfrak{b}_2 + \mathfrak{a}_3\mathfrak{b}_3,$$

when

$$a = \mathfrak{a}_1 e_1 + \mathfrak{a}_2 e_2 + \mathfrak{a}_3 e_3\,, \qquad b = \mathfrak{b}_1 e_1 + \mathfrak{b}_2 e_2 + \mathfrak{b}_3 e_3\,,$$

and $e_1$, $e_2$, $e_3$ is a normal set. Should one wish to express a unit of one normal set as the multiple sum of the units of another one, then the coefficients would emerge immediately as the inner products of the latter three units with the former one; e.g.:

$$e_1 = [e_1 \mid \varepsilon_1]\, \varepsilon_1 + [e_1 \mid \varepsilon_2]\, \varepsilon_2 + [e_1 \mid \varepsilon_3]\, \varepsilon_3\,.$$

In fact, if one sets:

$$e_1 = x\, \varepsilon_1 + y\, \varepsilon_2 + z\, \varepsilon_3$$

then one gets that $[e_1 \mid \varepsilon_1] = x$ immediately through inner multiplication by $\varepsilon_1$, since:

$$[\varepsilon_2 \mid \varepsilon_1] = [\varepsilon_3 \mid \varepsilon_1] = 0, \text{ and } \varepsilon_1^{\underline{2}} = 1,$$

and likewise, $[e_1 \mid \varepsilon_2] = y$, $[e_1 \mid \varepsilon_3] = z$, so $e_1 = [e_1 \mid \varepsilon_1]\, \varepsilon_1 + [e_1 \mid \varepsilon_2]\, \varepsilon_2 + [e_1 \mid \varepsilon_3]\, \varepsilon_3$.

Not just minor difficulties are associated with converting the equations that are obtained from this calculus into algebraic equations. One then must only choose an arbitrary coordinate system, assume that three line segments $e_1$, $e_2$, $e_3$ lie on the three coordinate axes, and represent every line segment that enters into an equation as a multiple sum of $e_1$, $e_2$, $e_3$, and thus in the form $\mathfrak{a}_1\, e_1 + \mathfrak{a}_2\, e_2 + \mathfrak{a}_3\, e_3$, while every surface space that occurs in the form $\mathfrak{a}_1\, [e_2\, e_3] + \mathfrak{a}_2\, [e_3\, e_1] + \mathfrak{a}_3\, [e_1\, e_2]$, in which the numbers $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ might be the coordinates of the line segment or surface space, respectively, and one ultimately obtains equations in which either no geometric quantities at all occur any more or they take on the forms:

$$\mathfrak{B}_1\, e_1 + \mathfrak{B}_2\, e_2 + \mathfrak{B}_3\, e_3 = 0 \qquad \text{or} \qquad \mathfrak{B}_1\, [e_2\, e_3] + \mathfrak{B}_2\, [e_3\, e_1] + \mathfrak{B}_3\, [e_1\, e_2] = 0,$$

resp., where the $\mathfrak{B}$ are functions of just the coordinates. Three equations:

$$\mathfrak{B}_1 = 0, \qquad \mathfrak{B}_2 = 0, \qquad \mathfrak{B}_3 = 0,$$

then arise from any such equation.

For differentiation and integration, the ordinary definitions suffice. In mechanics, only spatial quantities enter in addition to time $t$ as independent variables. Hereinafter, it will be especially convenient to denote the differentials differently. I let $\delta$ denote the differential quotient with respect to time, in which only those quantities are regarded as constant that are expressly defined to not change in time, such that when, e.g., the line segment $x$ is represented in a proper series ($\mathfrak{A}_2$: 454):

$$x = a_0 + a_1 t + a_2 t^2 + \ldots$$

in which, $a_0$ , $a_1$, $a_2$, … are line segments that do not change in time, so:

$$\delta x = a_1 + 2a_2 t + \ldots$$

By contrast, I will generally let $d$ denote the differentials of functions of spatial quantities in which the time is held constant. The concept of partial differential quotients of the functions of spatial quantities can be (as was done in $\mathfrak{A}_2$: 436, *et seq.*) established precisely as it is for functions of algebraic quantities. Therefore, I shall choose the indeed somewhat circuitous – but, I believe, easier for the reader – path of reducing to partial differential quotients of functions of algebraic quantities. I shall start with a normal set $e_1$, $e_2$, $e_3$, and express the line segments $x$, $y$, … upon which an algebraic function $f$ shall depend in coordinates with respect to the line segments of each set, namely:

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 , \qquad x = x_1 e_1 + x_2 e_2 + x_3 e_3 , \qquad \text{etc.},$$

so $f$ becomes a function of these coordinates $x_1$ , $x_2$, etc. Now, if $\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}$, etc., are the partial differential quotients with respect to all of the coordinates of the set ($\mathfrak{A}_2$: 436) then I shall understand the partial differential quotients of $f$ with respect to the line segment $x$ – which is written $\dfrac{\partial}{\partial x} f$ – to mean the line segment:

$$\frac{\partial}{\partial x} f = \frac{\partial f}{\partial x} e_1 + \frac{\partial f}{\partial y} e_2 + \frac{\partial f}{\partial z} e_3 .$$

It immediately follows from this that:

$$\left[\frac{\partial}{\partial x} f \mid dx\right] = \frac{\partial f}{\partial x} dx_1 + \frac{\partial f}{\partial y} dx_2 + \frac{\partial f}{\partial z} dz_3 .$$

However, it remains to be shown that $\frac{\partial}{\partial x} f$, whose definition was closely linked to the normal set $e_1$, $e_2$, $e_3$ here, remains completely unchanged when one chooses any another normal set $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$. Let $x = \xi_1\,\varepsilon_1 + \xi_2\,\varepsilon_2 + \xi_3\,\varepsilon_3$, so:

$$x_1\,e_1 + x_2\,e_2 + x_3\,e_3 = \xi_1\,\varepsilon_1 + \xi_2\,\varepsilon_2 + \xi_3\,\varepsilon_3\,.$$

If one inner-multiplies this equation by $e_1$ then one will get:

$$x_1 = \xi_1\,[\varepsilon_1 \mid e_1] + \xi_2\,[\varepsilon_2 \mid e_1] + \xi_3\,[\varepsilon_3 \mid e_1],$$

since one has $e_1^2 = 0$, $[e_1 \mid e_2] = [e_1 \mid e_3] = 0$, and from this, one will get the values of $x_2$ and $x_3$ when one replaces $e_1$ with $e_2$ and $e_3$, resp. Therefore, one will get $\frac{\partial x_1}{\partial \xi_1} = [\varepsilon_1 \mid e_1]$, and in general:

$$\frac{\partial x_r}{\partial \xi_s} = [\varepsilon_r \mid e_s].$$

Now, one has:

$$\frac{\partial f}{\partial \xi_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial \xi_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial \xi_1} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial \xi_1}$$

$$= \frac{\partial f}{\partial x_1}[\varepsilon_1 \mid e_1] + \frac{\partial f}{\partial x_2}[\varepsilon_1 \mid e_2] + \frac{\partial f}{\partial x_3}[\varepsilon_1 \mid e_3],$$

and from this, one obtains $\frac{\partial f}{\partial \xi_2}$, $\frac{\partial f}{\partial \xi_3}$ when one replaces $\varepsilon_1$ with $\varepsilon_2$ and $\varepsilon_3$, resp.; one then gets:

$$\begin{aligned}\frac{\partial f}{\partial \xi_1}\varepsilon_1 + \frac{\partial f}{\partial \xi_2}\varepsilon_2 + \frac{\partial f}{\partial \xi_3}\varepsilon_3 &= \frac{\partial f}{\partial x_1}([\varepsilon_1 \mid e_1]\,\varepsilon_1 + [\varepsilon_2 \mid e_1]\,\varepsilon_2 + [\varepsilon_3 \mid e_1]\,\varepsilon_3) \\ &\quad + \frac{\partial f}{\partial x_2}([\varepsilon_1 \mid e_2]\,\varepsilon_1 + [\varepsilon_2 \mid e_2]\,\varepsilon_2 + [\varepsilon_3 \mid e_2]\,\varepsilon_3) \\ &\quad + \frac{\partial f}{\partial x_3}([\varepsilon_1 \mid e_3]\,\varepsilon_1 + [\varepsilon_2 \mid e_3]\,\varepsilon_2 + [\varepsilon_3 \mid e_3]\,\varepsilon_3) \\ &= \frac{\partial f}{\partial x_1}e_1 + \frac{\partial f}{\partial x_2}e_2 + \frac{\partial f}{\partial x_3}e_3,\end{aligned}$$

and from that, the proof of the theorem on the change of normal set; i.e., the value of the partial differential quotient $\frac{\partial}{\partial x} f$ is independent of the choice of normal set.

## § 2. Basic laws of mechanics.

If $x$ means the line segment that points from a fixed point to the moving point then it is immediately clear that $\delta x$ represents the magnitude and direction of the velocity of that point and, in the same way, $\delta^2 x$ represents its acceleration or change in motion (*). From the law of persistence, any change in the motion of a material point must be ascribed to a *cause* that acts upon it. If we let the effect of that cause be equal to the line segment $p$ then we will have the equation:

(1) $$\delta^2 x = p.$$

If this effect is, e.g., constantly equal to the line segment $g$ then we will obtain immediately from integrating the equation $\delta^2 x = g$ that $\delta x = c + gt$, where $c$ is an arbitrary constant line segment (viz., the initial velocity), and from another integration we get $x = b + ct + \frac{1}{2}gt^2$, where $b$ is, once more, a constant line segment (viz., the initial value of $x$). These equations include the usual law of ballistics in its most general form.

The so-called parallelogram law for force can be expressed thus: If the effects of several causes on the point $x$ equal the line segments $p_1$, $p_2$, …, when taken individually, then the simultaneous effect $p$ of all of the causes is equal to the sum of the line segments $p = p_1 + p_2 + \ldots$ Equation (1) is then also true when $p$ is the sum of all effects that the various causes simultaneously exert on the moving point.

I use the term *simple force* to refer to a cause that is seated at a material point, in some way, such that the effect of that cause on another material point depends upon only the mutual position of the points, but is completely independent of the surrounding space. Therefore, if a material point $A$ exerts an effect $BC$ on another point $B$, and one arbitrarily relocates the figure $ABC$ to $A_1B_1C_1$, such that $ABC$ is remains congruent with $A_1B_1C_1$, however, then $B_1C_1$ must be the effect of $A_1$ on $B_1$ . It follows from this that $BC$ must lie in the infinite straight line $AB$. If this were not the case, but $A$, $B$, $C$ were to define a triangle, and one rotated it around the $AB$ axis through an arbitrary angle into the position $ABC_1$ then $A$ would have to exert the effect $BC_1$ on $B$, which contradicts the effect $BC$, so the force can act only to draw them together or push them apart. However, if the points $A$ and $B$ have precisely the same character then one must also be able to switch $A$ with $B$ with changing the effect. Now, if $A$ acts upon $B$ with the effect $BC$ (viz., pushing or pulling), and one rotates $ABC$ around the midpoint of $AB$ to the position $A_1B_1$ such that $A_1$ coincides with $B$ and $B_1$ coincides with $A$, and if we let $C_1$ be the point upon which $C$ falls, then $B_1C$ – i.e., $AC_1$ – must not be regarded as merely the effect of the point $A_1$ on $B_1$, but also the that of the point $B$ on $A$; i.e., the effect must be reciprocal, and the two effects must be equal and opposite to each other. Moreover, if the material points keep the same character then the magnitudes of effects must be a function of only the mutual separation (**). Now, if $A$ and $B$ do not, in fact, have completely the same character, but $A$ exerts the equal and opposite effect on $B$ that $B$ does on $A$, then we will say that $A$ and $B$ are equal *in mass.* Which mass we use as the unit of mass is, in itself, irrelevant.

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(*) It would be simpler to immediately set $x$ equal to the moving point. I will save this for a later article in which the calculations will be presented using points.

(**) It is already implicit in this that I cannot ascribe the forces that the electricity that moving electric currents exert to a simple force.

However, once that unit is established, we can set the force equal to the acceleration that is associated with a mass of 1. Therefore, we can make the endpoint of $x$ in equation (1) a point of mass 1 and set $p$ equal to the force – or sum of the forces – that act upon it. In this sense, we can make equation (1) the fundamental equation of mechanics.

### § 3. Motion of a freely-moving set of material points.

I shall distinguish between *internal* and *external* forces relative to the set. Internal forces are ones by which a point of the union acts upon the other points of the same set, while external forces are the remaining ones. I shall first assume that all points of the set are of equal mass and, in fact, of mass 1. Now, let $x_1, \ldots, x_m$ be the line segments that point from a fixed point to the moving points of the set, let $p_1$ be the sum of all the forces that act upon the first point, etc., so, from (1), one has the $m$ equations $\delta^2 x_1 = p_1$ , …, $\delta^2 x_m = p_m$ , and adding these gives:

$$\delta^2 x_1 + \ldots + \delta^2 x_m = p_1 + \ldots + p_m .$$

Since the internal forces are pair-wise equal and opposite, they drop out under addition, so it follows that we can consider $p_1, \ldots, p_m$ to be external forces here. Now, let $s$ be the line segment that points from the fixed point to the center of mass of the set, and let $y_1, \ldots, y_m$ be the line segments that point from the center of mass to the points of the set, so, by definition, the center of mass will satisfy $y_1 + \ldots + y_m = 0$. However, $x_1 = s + y_1, \ldots, x_m = s + y_m$, so $x_1 + \ldots + x_m = ms$, in which the construction of the center of mass also lies, so:

$$\delta^2 x_1 + \ldots + \delta^2 x_m = \delta^2 (x_1 + \ldots + x_m) = m\, \delta^2 s,$$

and one thus obtains the above equation in the form:

$$\delta^2 s = \frac{1}{m} p , \tag{2}$$

where $p$ is the sum of all external forces and $m$ is the mass of the set. This is the equation of motion for the center of mass. It can likewise be true as the equation for the motion of a point of mass $m$, and from now on, we can also assume points of unequal mass, so, for the sake of simplicity, without sacrificing any generality, we can now continue to use points of mass 1. If we introduce the value $s + y_1$, in place of $x_1$, $\delta^2 s + \delta^2 y_1$, in place of $\delta^2 x_1$, and replace $\delta^2 s$ with the value that was found from (2) in the equation of motion then we will get:

$$\delta^2 y_1 = p_1 - \frac{1}{m} p , \qquad \text{etc.,} \qquad \delta^2 y_m = p_m - \frac{1}{m} p \tag{3}$$

for the equations of the relative motion of an arbitrary set relative to the center of mass of the point.

If one exterior multiplies the equation $\delta^2 x_1 = p_1$ by $x_1$ then one will get $[x_1\ \delta^2 x_1]$ on the left-hand side, but this is the time differential of $[x_1\ \delta x_1]$, since under differentiation the other term $[\delta x_1\ \delta x_1]$ will be zero, from the rules of exterior multiplication. One thus obtains $\delta[x_1\ \delta x_1] = [x_1\ p_1]$. If one constructs the same equations for the other points and adds them then one will obtain, upon applying the summation notation:

(4) $$\delta \sum [x\ dx] = \sum [x\ p].$$

The internal forces also drop out here. If we then let, e.g., $\lambda(x_2 - x_1)$ be the force that the first point exerts upon the second one then the effect that the latter exerts upon the former one will be the opposite one, namely, $\lambda(x_1 - x_2)$, so upon summation, one will have:

$$[x_1\ \lambda(x_2 - x_1)] + [x_2\ \lambda(x_1 - x_2)] = \lambda\,[x_1\ x_2] + \lambda\,[x_2\ x_1] = 0,$$

since $[x_1\ x_2] = -[x_2\ x_1]$. Therefore, all internal forces drop out, and likewise for the forces that are directed from the starting point of the $x$. If $\lambda x_1$ is such a force that acts upon the first point then one will have $[x_1\ \lambda x_1] = 0$. Therefore, if no other external forces that are directed from the starting point of the $x$ act, as such, then equation (4) will express the invariability of the total surface motion $\sum x\ dx$.

If one exterior multiplies equations (3) by $y_1$, etc., in the same way and adds them then one will obtain, since $\sum\left[y\dfrac{p}{m}\right] = \dfrac{1}{m}[\sum y \cdot p]$ is zero, due to the property of the center of mass, that:

(5) $$\delta \sum [y\ dy] = \sum [y\ p];$$

i.e., the surface equation (4) will also be true when one replaces the fixed starting point of the $x$ with the moving center of mass.

For the further development of equation (6), it is very essential to regard all of the forces $p$ that several points exert upon the point $x_1$ as partial differential quotients with respect to $x_1$ of an algebraic function of all of these points, such that if $U$ is that function then one would let $p = \dfrac{\partial}{\partial x_1}U$. One can then say that the force $p_1$ originates in the tendency (*) of the function $U$ to increase. Namely, the increase that $U$ experiences during an infinitely small displacement $dx_1$ is $\left[\dfrac{\partial}{\partial x_1}U \mid dx_1\right]$; if $dx_1$ maintains the same length then this increase is largest when $dx_1$ has the direction of the first factor, which follows immediately from the formula $[a \mid b] = \mathfrak{ab} \cos \angle\, ab$; i.e., the motion that results from the force $p_1$ assumes the direction in which the function $U$ increases most rapidly; i.e., the tendency will be fulfilled most completely. Likewise, when the point $x_1$ changes its position, but $dx_1$ continually keeps the same length and the direction of the first factor

(*) I gave this idea of a tendency its foundation in the aforementioned paper in the year 1840.

remains $\frac{\partial}{\partial x_1}U$ then the force behaves like the increase of $U$; i.e., like the achievement of the goal that one tends towards. In fact, one can therefore regard the force $p$ as the expression of the tendency of $U$ to increase. It is well-known that $U$ will be called the potential. Finding $U$ poses no difficulty. We first consider the force $p_{21}$ that a point $x_2$ exerts upon another one $x_1$ . From § 2, this force can be regarded as a function of their separation, and it thus behaves like $f(r)$. However, in order to represent the direction, we write it as:

$$p_{21} = \frac{1}{r}\, f(r)\,(x_1 - x_2).$$

Here, $r$ is the length of $x_1 - x_2$ , i.e., $r^2 = (x_1 - x_2)^2$, and when this is differentiated, one will get:

$$r\,dr = [(x_1 - x_2) \mid (dx_1 - dx_2)] \quad \text{or} \qquad dr = \frac{1}{r}(x_1 - x_2) \mid (dx_1 - dx_2),$$

so

$$\begin{aligned} f(r)\,dr &= \frac{1}{r} f(r)\,[(x_1 - x_2) \mid (dx_1 - dx_2)] \\ &= [p_{21} \mid (dx_1 - dx_2)]. \end{aligned}$$

Now, let $\int fr \cdot dr = U_{12}$ so one has $dU_{12} = [p_{21} \mid (dx_1 - dx_2)]$, so $\frac{\partial}{\partial x_1}U_{12} = p_{21}$ , and so one also has $\frac{\partial}{\partial x_2}U_{12} = p_{12}$ . If one has defined a family of quantities $U_{r,\,s}$ between any two points in the same way then their sum $U$ will become a function of these points, and the force that the remaining points of the family exert upon a point $x_1$ will then be equal to $\frac{\partial}{\partial x_1}U$ .

The distinction between external and internal force is likewise important for the introduction of this potential into equation (6). If one lets $V$ be the complete internal potential – i.e., the sum of the potentials between any two points of the set – and let $U$ the total external potential – i.e., the sum of the potentials between any internal point and an external one – then the first of equations (6) will admit a complete integration, while the last one, will admit one only insofar as the external points are unchanging in time. In fact, if one considers the forces $p_{12}$ and $p_{21}$ that the first two points exert upon each other in the sum $\sum [p \mid \delta x]$, with the associated potential $U_{12}$ , so $[p_{12} \mid \delta x_2] + [p_{21} \mid \delta x_1]$ is the associated part of that sum, hence, it is equal to $\left[\frac{\partial}{\partial x_1}U_{12} \mid \delta x_2\right] + \left[\frac{\partial}{\partial x_1}U_{12} \mid \delta x_1\right] = \delta U_{12}$ , and one extends this to all internal forces, then the part of that sum that will arise from this will equal $\delta V$, and equation (6) will assume the form:

$$(7) \qquad \tfrac{1}{2} \sum (\delta x)^2 = V + \int \sum \left(\frac{\partial}{\partial x}U \mid \delta x\right).$$

### § 4. Motion of a constrained moving set.

The constrained motion of a set will be most simply represented by condition equations that the moving points are subjected to. The motion is still not determined by these equations alone. Moreover, one must assume that there are forces that act upon the points of the set, as long as they also move the position only infinitely little away from one that satisfies the equations, and irresistibly move it back to a position that does satisfy these equations. This brings us to the more precise determination of these forces.

Let $L = 0$ be such an equation of condition, so we would like to ascribe the force of tendency that arises from it that would conserve the equation $L = 0$; i.e., a potential that is equal to $L$ or an arbitrary function of $L$, such as $f(L)$. Having assumed this, the force that conserves the tendency $L = 0$ that is produced at $P \cdot x_1$ equals $\frac{\partial}{\partial x_1} f(L) = f'L \frac{\partial}{\partial x_1} f(L)$, if $f'L$ is the derivative of $fL$, or if we denote this derivative by $\lambda$ then the force that the point $x_1$ feels due to that tendency will be $\lambda \frac{\partial}{\partial x_1} L$, the force that $P \cdot x_1$ feels will be $\lambda \frac{\partial}{\partial x_2} L$, etc. Now, if that condition equation $L = 0$ is associated with other ones $M = 0$, etc., then forces $\mu \frac{\partial}{\partial x_1} M$, $\mu \frac{\partial}{\partial x_2} M$, etc., will arise from that, and, from the fundamental law (1), the equation of motion will become:

$$
(8) \qquad \left\{ \begin{aligned} \delta^2 x_1 &= p_1 + \lambda \frac{\partial}{\partial x_1} L + \mu \frac{\partial}{\partial x_1} M + \cdots \\ \delta^2 x_2 &= p_2 + \lambda \frac{\partial}{\partial x_2} L + \mu \frac{\partial}{\partial x_2} M + \cdots \end{aligned} \right.
$$

where

$$
L = 0, \quad M = 0, \ldots
$$

suffice to determine the unknowns $\lambda$, $\mu$.

Now, let $dx_1$, $dx_2$, … be arbitrary displacements of the points $x_1$, $x_2$, …, which, however, satisfy the equations $dL = 0$, $dM = 0$, etc. If one inner multiplies the equations above by $dx_1$, $dx_2$, etc., and adds them then, since one has:

$$
\left[ \frac{\partial}{\partial x_1} L \mid dx_1 \right] + \left[ \frac{\partial}{\partial x_2} L \mid dx_2 \right] + \ldots = dL = 0,
$$

the terms in $\lambda$, $\mu$, … will drop out, and one will obtain:

$$
(9) \qquad \sum [(\delta^2 x - p) \mid dx] = 0,
$$

which will be true for all displacements that satisfy the equations $dL = 0$, $dM = 0$, …

## § 5. Equilibrium and mean motion.

If the forces that act upon a set depend upon only the positions of the points of the set and not otherwise upon time then equilibrium is possible, and the formulas for equilibrium are then included in the equations above, when one only sets the accelerations and velocities of all points of the set equal to zero, by which, equilibrium between the forces is then demanded. However, if these equations are fulfilled then the initial position of the points of the set or their initial velocities can be such that no equilibrium arises, and, in particular, that for very minor deviations of the initial state from a state of stable equilibrium, oscillations about that state will occur. These properties of equilibrium and its perturbation by infinitely small oscillations arise only for the case in which the forces depend upon time in such a way that a state of *mean motion* can exist, in which, if this state of mean motion is a stable one then small oscillations can once more occur that do not exceed a certain maximum in the course of time.

I will use the term "mean motion of a set" that is impelled by given (time-varying) forces to refer to that motion for which, under all of the motions that depend upon the various initial states, the smallest motion comes about – or stated more precisely – for which the sum of the animating forces that are active during a sufficiently long time is a minimum. If $T = \frac{1}{2} \sum (\delta x)^2$ (the points of the set are always thought of as having equal masses) is the animating force, so $T\,\partial t$ is the animating force that is active during the time element $\partial t$, then $\int T\,dt$, when taken between the limits $t = 0$ and $t = t$, will be the total animating force that is active during this time. Therefore, for the mean motion, that integral shall, for a sufficiently large $t$, be less than it is for any other motion of the system, and also remain smaller that that when $t$ increases from there on arbitrarily. For linear equations of motion, I shall couple the concept of mean motion with that of the *mean integration* of linear differential equations of arbitrary order, so I shall choose second-order differential equations as an example. Let $n$ numerical quantities $u_1, \ldots, u_n$ be thought of as dependent upon an independent numerical quantity $t$, and let the differential of those quantities with respect to $t$ be denoted by $\delta$, and let that dependency be partially determined by the $n$ equations:

$$(10)^* \qquad \begin{cases} \delta^2 u_1 + a_{1,1}\delta u_1 + \cdots + a_{1,n}\delta u_n + b_{1,1}\delta u_1 + \cdots + b_{1,n}\delta u_n = f_1 t, \\ \qquad\qquad\qquad \vdots \\ \delta^2 u_n + a_{n,1}\delta u_1 + \cdots + a_{n,n}\delta u_n + b_{n,1}\delta u_1 + \cdots + b_{n,n}\delta u_n = f_n t, \end{cases}$$

where the $a$ and $b$ are constant numerical quantities.

The general integration of these equations is well-known. Therefore, in order to clearly single out the mean integration, it will be necessary to present the general integration lucidly. It is first clear that one can decompose the $ft$ into arbitrary terms, take the general integrals that relates to these terms individually, and then add the integrals so obtained. I decompose the $f\,t$ into exponential terms whose exponents are proportional to time, and then present the equations to be integrated in the form:

$$(10)\qquad \begin{cases} \delta^2 u_1 + a_{1,1}\delta u_1 + \cdots + a_{1,n}\delta u_n + b_{1,1}\delta u_1 + \cdots + b_{1,n}\delta u_n = g_1 e^{\kappa t}, \\ \qquad\vdots \\ \delta^2 u_n + a_{n,1}\delta u_1 + \cdots + a_{n,n}\delta u_n + b_{n,1}\delta u_1 + \cdots + b_{n,n}\delta u_n = g_n e^{\kappa t}, \end{cases}$$

where $g_1, \ldots, g_n$ . One also thinks of the $u$ as represented by such terms. Two types of these terms then emerge, namely, ones with the exponential quantity $e^{\kappa t}$ and ones with $e^{\lambda t}$, where $\lambda$ is different from $\kappa$, but still to be found. The former terms define the mean integral and can be found immediately, while the latter ones depend upon the solution to an equation of degree $2n$. The mean integration gives $u_1 = y_1\, e^{\kappa t}, \ldots, u_n = y_n\, e^{\kappa t}$, where the $y_1, \ldots, y_n$ are determined precisely by the $n$ equations:

$$(11)\qquad \begin{cases} \kappa^2 y_1 + \kappa a_{1,1} y_1 + \cdots + \kappa a_{1,n} y_n + b_{1,1} y_1 + \cdots + b_{1,n} y_n = g_1, \\ \qquad\vdots \\ \kappa^2 y_n + \kappa a_{n,1} y_1 + \cdots + \kappa a_{n,n} y_n + b_{n,1} y_1 + \cdots + b_{n,n} y_n = g_n, \end{cases}$$

unless the determinant of the coefficients of the $y_1, \ldots, y_n$ should be zero, which we will discuss below. By contrast, if one were to set $u_1 = y_1\, e^{\lambda t}, \ldots, u_n = y_n\, e^{\lambda t}$, where $\lambda$ is not equal to $\kappa$, then this would yield a system of equations that would correspond to (11) above, with the difference that $\lambda$, $z$ would enter in place of $\kappa$, $y$, and the right-hand sides would be zero. It follows from this that the determinant of the coefficients of $z_1, \ldots, z_n$ should be zero. That gives the $2n^{\text{th}}$-degree equation for $\lambda$ that was mentioned above. For each of the $2n$ values $\lambda_1, \ldots, \lambda_{2n}$ that satisfies this $2n^{\text{th}}$-degree equation, the associated ratios of the $z$ are determined, and with that the general integration is complete. It is only when $\kappa$ is equal to one of the $2n$ values $\lambda_1, \ldots, \lambda_{2n}$ that the aforementioned case will occur in which the $y_1, \ldots, y_n$ of the mean integration become infinite or undetermined; in this case, one can first make $\kappa$ differ from the value of $\lambda$ by infinitely little, and then determine the mean integration that relates to this $\kappa$. The mean integration always remains independent of the solution of the equation of degree $2n$. However, in order to be able to go over to the equations of motion, we must give equations (10) still another form. Then, since the terms $g\, e^{\kappa t}$, which should represent the forces, become infinite with $t$ for real $\kappa$, they do not correspond to the case in nature under this assumption. One therefore replaces $g\, e^{\kappa t}$ with the two terms $c \cos \kappa t + c' \sin \kappa t$; i.e., $\frac{c - c'i}{2} e^{i\kappa t} + \frac{c + c'i}{2} e^{-i\kappa t}$. These two terms differ only by the sign of $i = \sqrt{-1}$. If one now replaces $g\, e^{\kappa t}$ in (10) with one of them, $\frac{c - c_1' i}{2} e^{i\kappa t}$, etc., then one must replace $g_1$ in (11) with $\frac{c - c_1' i}{2}$, etc., and furthermore, replace $i\kappa$ with $-\kappa^2$ and $\kappa^2$, and then the $y$ that are determined from (11) will become imaginary – say, $v + wi$ – so one will get $u_1 = (v_1 + w_1$

$i)\, e^{i\kappa t}$, …, $u_n = (v_n + w_n\, i)\, e^{i\kappa t}$. If one then replaces $g$ in (10) with $\frac{c+c'i}{2} e^{-i\kappa t}$ then this will give values for $u_1$, …, $u_n$ that differ from the ones above only by the signs of $i$. Let them be denoted by $u_1'$, …, $u_n'$, so $u_1' = (v_1 - w_1\, i)\, e^{-i\kappa t}$, …, $u_n' = (v_n - w_n\, i)\, e^{-i\kappa t}$; one then gets $u_1 + u_1' = 2v_1 \cos \kappa t - 2w_1 \sin \kappa t = a_1 \cos \kappa t + b_1 \sin \kappa t$, when one sets $2v_1 = a_1$ and $-2w_1 = b_1$.

It must now be proved that the motion that is determined by linear equations of the form (10) provides the mean integration, as well as the mean motion as it was defined above. For the motion of a set of $m$ equally-massive points in space the $n$ in equations (10) and (11) will equal $3m$, so the equation in $\lambda$ will then be of degree $6m$. We assume perpendicular coordinate axes. The total animating force then becomes $T = \frac{1}{2} \sum \delta u^2$, so $\int T\, dt = \frac{1}{2} \sum \int \delta u^2\, dt$, in which the sum extends over $u_1$, …, $u_{3m}$. Now, for the general integration, $u_1$ consists, in part, of terms from the mean integration, which are of the form $a_1 \cos \kappa t + b_1 \sin \kappa t$, and, in part, of $6m$ terms of the form $ze^{\lambda_1 t}$; therefore, $\delta u_1$ includes terms of the form $\kappa b_1 \cos \kappa t - \kappa a_1 \sin \kappa t$ and ones of the form $\lambda_1 z e^{\lambda_1 t}$, and $\delta u^2$ then contains the squares of these terms and twice the products of each pair of them. One sees immediately that the terms of the form $\lambda_1 z e^{\lambda_1 t}$ become infinite for infinite $t$ when $\lambda_1$ is real, so $\sqrt{T dt}$ is certainly smaller when these terms are absent than when they are present. We can then omit these terms for the proof of the mean motion, and the same thing is true when $\lambda_1 = \alpha + \beta\, i$, and $\alpha$ is non-zero. Only terms for which $\lambda_1 = \beta\, i$ are then to be considered. Another value of $\lambda$ might then be $\lambda_2 = -\beta\, i$, and the real terms in $u_1$ that would arise from this would be of the form $p \cos \beta t + q \sin \beta t$, so the ones in $\delta u_1$ would be of the form $\beta (q \cos \beta t - p \sin \beta t)$, and thus of the form that corresponds to the terms of the mean integration. If we first consider the squares – e.g., $(\kappa b_1 \cos \kappa t - \kappa a_1 \sin \kappa t)^2$, so in $T\, dt$ one will have the terms $\frac{1}{2} \kappa^2 (\kappa b_1 \cos \kappa t - \kappa a_1 \sin \kappa t)^2\, dt$ – then this would give:

$$\tfrac{1}{2} \kappa^2\, [b_1^2 (1 + \cos 2\kappa t)\, dt + a_1^2 (1 - \cos 2\kappa t)\, dt - 2a_1\, b_1 \sin 2\kappa t\, dt].$$

When integrated, will this give $\frac{1}{2} \kappa^2 (a_1^2 + b_1^2)\, t + P$, where $P$ provides nothing but finite periodic terms. If we further consider the doubled product of two such terms – e.g., $\kappa(b_1 \cos \kappa t - a_1 \sin \kappa t)$ and $\beta(p \cos \beta t - p \sin \beta t)$ then that would give the term:

$$\kappa\beta\, dt\, (b_1 q \cos \kappa t \cos \beta t + a_1 p \sin \kappa t \sin \beta t - b_1 p \cos \kappa t \sin \beta t - a_1 q \sin \kappa t \cos \beta t)$$

$$= \kappa\beta\, dt \times \left[ \frac{b_1 q + a_1 p}{2} \cos(\kappa + \beta) t + \frac{b_1 q - a_1 p}{2} \cos(\kappa - \beta) t - \frac{b_1 q + a_1 p}{2} \sin(\kappa + \beta) t - \frac{b_1 q - a_1 p}{2} \sin(\kappa - \beta) t \right],$$

so when $\kappa$ is not equal to $\beta$, this will produce only finite periodic terms. Now, we can assume that $t$ is large enough that the periodic terms vanish when compared to the terms of the form $\frac{1}{2}\kappa^2(a_1^2+b_1^2)\,t$, etc. One then gets $\int T\,dt = \frac{1}{4}\sum \kappa^2\,(a^2+b^2)\,t + \frac{1}{4}\sum \beta^2\,(p^2+q^2)t$, where the first sum relates to all terms of the mean integration, while the second sum relates to the remaining ones. Here, $a$ and $b$ are unchanging values, while $p$ and $q$ can be zero, so for a sufficiently long $t$ the integral $\int T\,dt$ is smallest when the $p$, $q$ are all zero; i.e., the integral is the mean one. It is thus proved that for linear differential equations the motion of the mean integration likewise yields the mean motion.

I shall call a term of the form $a\cos\kappa t + b\sin\kappa t$, where $\kappa$ is positive, but $a$ and $b$ might be numbers or line segments, an *elliptic term* and $\kappa$, its *indicator*. In fact, if $a$ and $b$ were line segments of unequal direction here, and $a\cos\kappa t + b\sin\kappa t$ were represented as the line segment $r$ that starts from a fixed point then in time $2\pi/\kappa$ its endpoint would describe an ellipse, and in fact, in such a way that the line segment $r$ itself would describe equal spaces in equal times, namely, in the time $dt$, it would describe the space $\frac{1}{2}[a\ b]\ \kappa\,dt$; the line segments $a$ and $b$ are conjugate semi-axes of the ellipse. In fact, if one sets $\cos kt = u$, $\sin kt = v$ then that radius will become $r = ua + vb$ and $u^2 + v^2 = 1$, which is the equation for the ellipse with the conjugate semi-axes $a$ and $b$. Furthermore, in the time element $dt$, the endpoint of $r$ describes the line segment $\delta r \cdot dt$ – i.e., $(b\cos\kappa t - a\sin\kappa t)\,\kappa\,dt$ – and $r$ itself describes the surface space $\frac{1}{2}[r\ \delta r]\,dt$ – i.e., $\frac{1}{2}[(a\cos\kappa t + b\sin\kappa t)\,(b\cos\kappa t - a\sin\kappa t)]\,\kappa\,dt$. Taking the exterior product gives the value $[a\ b]$, since $[a\ a] = [b\ b] = 0$, $[a\ b] = -[b\ a]$ and $\cos^2\kappa t + \sin^2\kappa t$ is equal to 1, so the surface space that is described in the time element $dt$ equal to $\frac{1}{2}[a\ b]\ \kappa\,dt$.

We can now express the law for the mean motion in our case as follows: If the motion of a set of points were represented by linear differential equations then the elliptic terms that are present in the expression for force would correspond to the elliptic terms of the same indicators in all line segments that point from a fixed point to the moving points, and indeed the coefficients of these terms would be determined completely by the given equations, and outside of these terms no others will emerge from the mean motion.

I now remark that the stability or instability of the mean motion can be most simply derived from the principles that were developed above.

## § 6. Application to the theory of ebb and flow.

Here, we also consider a system that is subject to ebb and flow to be a set of $m$ points whose masses are 1. Equation (3) in § 3 is then valid for the motion relative to the center of mass, namely, $\delta^2 y_1 = p_1 + q_1 - \frac{1}{m}p$, …, $\delta^2 y_m = p_m + q_m - \frac{1}{m}p$, in which I have, in fact, separated the internal forces $q_1$, etc., from the external ones $p_1$, etc, and have set $p_1 + \ldots + p_m = p$. Now, let the system be subjected to a uniform rotation around a fixed axis that goes through the center of mass, and assume, as is permissible in the theory of ebb and flow, which is considered in the first approximation here, that the points are only slightly separated from the position in which they were assumed to have uniform rotation. Furthermore, let $n$ be the angular velocity of that rotation, so $nt$ is the rotation during the time $t$. Let a line segment $a$ be assumed to lie in the rotational plane (and thus perpendicular to the axis), so it moves under the rotation through an angle $nt$ into $a\cos nt$

+ $a'$ sin $nt$, where $a'$ is perpendicular to $a$ in the rotational plane in the positive sense of rotation and has the same length as $a$. From a well-known analogy, we denote this line segment $a'$ by $ai$, where $i$ is the planimetric representation of $\sqrt{-1}$. $a$ then moves to $a$ (cos $nt$ + $i$ sin $nt$) = $a\,e^{int}$, and one will then have $\delta(ae^{int}) = a\,e^{int} \cdot in$. Now, let $in = \alpha$, where $\alpha$ represents the angular velocity, up to its magnitude and direction. $a$ then moves as a result of that rotation into $ae^{\alpha t}$, and $\delta a\,e^{\alpha t}$ becomes $a\,e^{\alpha t}$, $\delta^2 a\,e^{\alpha t} = a\,e^{\alpha t}\,\alpha^2$, where $\alpha^2 = -n^2$, moreover. In this sense, now let $y_1 = (x_1 + u_1)\,e^{\alpha t}$, where $x_1$ is unchanging in time and $u_1$ is infinitely small. One then has:

$$\begin{aligned}\delta y_1 &= \delta u_1\,e^{\alpha t} + (x_1 + u_1)\,e^{\alpha t}\alpha,\\ \delta^2 y_1 &= \delta^2 u_1\,e^{\alpha t} + 2\delta u_1\,e^{\alpha t}\,\alpha + (x_1 + u_1)\,e^{\alpha t}\alpha^2,\\ &= [\delta^2 u_1 + 2\delta u_1\alpha + (x_1 + u_1)\,\alpha^2].\end{aligned}$$

However, when the entire system rotates through $at$, the internal forces will also rotate through the same angle, and we can then write $q_1'e^{\alpha t}$, instead of $q_1$. If we multiply by $e^{-\alpha t}$ then we will obtain the equation:

$$\delta^2 u_1 + 2\delta u_1\alpha + (x_1 + u_1)\,\alpha^2 = q_1' + \left(p_1 - \frac{1}{m}p\right)e^{-\alpha t}.$$

However, $q_1'$ depends upon the mutual separation of the points, so here it will depend upon $x_1 + u_1 - (x_r + u_r)$; i.e., on $x_1 - x_r + (u_1 - x_r)$, where $u_1 - u_r$ is infinitely small compared to $x_1 - x_r$. Thus, $q_1'$ splits into two terms, the first of which does not contain $u$, while the other one is a linear function of $u$. Let the former be denoted by $q_1''$ and the latter, by $\varphi_1$, so we can split the equations above into two sets of equations, namely:

$$x_1\alpha^2 = q_1'', \ldots, x_m\alpha^2 = q_m'', \tag{12}$$

which determine the equilibrium state, and:

$$\left\{\begin{array}{c} \delta^2 u_1 + 2\delta u_1\alpha + u_1\alpha^2 - \varphi_1 = \left(p_1 - \frac{1}{m}p\right)e^{-\alpha t},\\ \vdots \\ \delta^2 u_m + 2\delta u_m\alpha + u_m\alpha^2 - \varphi_m = \left(p_m - \frac{1}{m}p\right)e^{-\alpha t},\end{array}\right. \tag{13}$$

which have precisely the form of the equations that were treated in § 5, and their mean integration then gives the motion of the ebb and flow. All that is left for us to do is to develop the right-hand sides of these equations (13) into elliptic terms. We first assume there is only *one* celestial body, and indeed let it be close to spherical, while the distance from its center to the center of mass of the system is infinitely small when compared to the dimensions of the system. The attraction that a ball exerts upon another point due to its gravitation is the same as it would be if its mass were concentrated at its center. Let $L$ be this attraction at the distance 1, so at a distance $e$, it will be equal to $L / e^2$. Now, let $r$

be the line segment from the center of mass of the system to the center of the ball at time $t = 0$, and let $\rho$ be the length of $r$, so at that point in time $p_1$ is, when ignoring the terms of higher order, due to the smallness of the magnitude and direction, equal to $\frac{L(r-x_1)}{(r-x_1)^3}$ or $\frac{L(r-x_1)}{(r^2-2[r\,|\,x_1])^{3/2}}$. When developed, that gives:

$$p_1 = \frac{L}{\rho^3}\left(r - x_1 + \frac{3[r\,|\,x_1]}{\rho^2}r\right).$$

Since the $x$ are taken from the center of mass, so $\sum x = 0$, one then gets $\frac{1}{m}p = \frac{L}{\rho^3}r$. As a result, at time $t = 0$, the right-hand side of equation (13) will equal $\frac{L}{\rho^3}\left(\frac{3[r\,|\,x_1]}{\rho^2}r - x_1\right)$. Now, let the separation of the celestial body and its declination be assumed to be constant in the course of a day, while its right ascension changes by $\beta t$ in a time $t$, so at time $t$, first, $r$ goes to $r\,e^{\beta t}$, and second $x_1$ goes to $x_1\,e^{\alpha t}$, and the right-hand side of equation (13) becomes $\frac{L}{\rho^3}\left(\frac{3[r\,e^{\beta t}\,|\,x_1 e^{\alpha t}]}{\rho^2}re^{\beta t} - x_1 e^{\alpha t}\right)e^{-\alpha t}$. Now, from the concept of the inner product, the value of this does not change when the two factors rotate around the same axis and through equal angles – e.g., through the angle $-\beta t$ – and when one sets $\alpha - \beta = \gamma$, one gets that the right-hand side of equation (13) is equal to:

$$\frac{L}{\rho^3}\left(\frac{3[r\,|\,x_1 e^{\gamma t}]}{\rho^2}re^{-\gamma t} - x_1\right).$$

Let the length of $x_1$ equal $\mu\rho$, so one will have $[r\,|\,x_1\,e^{\gamma t}] = \mu\rho^2 \cos\varphi$, where $\varphi$ is the angle between $r$ and $x_1\,e^{\gamma t}$. Let $\eta$ be the angle that the axis $a$ makes with $r$, let $\vartheta$ be the angle that it makes with $x_1$, and let $\omega_1$ be the angle that plane $ar$ makes with the plane $ax_1$, so:

$$\cos\varphi = \cos\eta\cos\vartheta + \sin\eta\sin\vartheta\cos(\omega_1 + \gamma t),$$

and we get that the expression above equals:

$$\frac{L}{\rho^3}\{3m\,[\cos\eta\cos\vartheta + \sin\eta\sin\vartheta\cos(\omega_1 + \gamma t)]\,re^{-\gamma t} - x_1\},$$

where one can replace $r$ with $r_1 + r_2$, where $r_1$ lies on the axis and $r_2$ lies on the equator, so one can replace $re^{-\gamma t}$ with $r_1 + r_2\,e^{-\gamma t}$. If one then replaces $\cos(\omega_1 + \gamma t)$ with its value [*sic*] $\tfrac{1}{2}[e^{i(\omega_1+\gamma t)} - e^{-i(\omega_1+\gamma t)}]$ then one will see at that point that the entire expression will consist of three elliptic terms with indicators 0, $\gamma$, and $2\gamma$, where $\gamma$ is the apparent angular velocity of the celestial body, so $2\pi/\gamma$ is its apparent orbital period. If a second celestial body enters in, as is the case for ebb and flow, which influences the motion, and whose apparent orbital period is $2\pi / \gamma'$, then two elliptic terms with indicators $\gamma$ and $\gamma'$ will enter in. If we denote these five elliptic terms for the first point by $p_{1,0}$, $p_{1,\gamma}$, $p_{1,2\gamma}$, $p_{1,\gamma'}$, $p_{1,2\gamma'}$ then the first of equations (13) will become:

$$\delta^2 u_1 + 2\delta u_1 \alpha + u_1 \alpha_1 - \varphi_1 = p_{1,0} + p_{1,\gamma} + p_{1,2\gamma} + p_{1,\gamma'} + p_{1,2\gamma'} . \tag{14}$$

Since only the mean motion is of interest for ebb and flow, this will imply that one likewise has a sum of five elliptic terms with the same indicators for $u_1$, so:

$$u_1 = u_{1,0} + u_{1,\gamma} + u_{1,2\gamma} + u_{1,\gamma'} + u_{1,2\gamma'} , \tag{15}$$

if $u_{1,0}$ , $u_{1,\gamma}$, etc., represent elliptic terms with the indicators 0, $\gamma$, etc, and one has corresponding equations for every other point. The first term $u_{1,0}$ gives the line segment that says how much the mean position of the first point that is demanded by the celestial body deviates from the equilibrium position. The other four terms give the motion of the points about its mean position. From this, we get the main theorem for the theory of ebb and flow:

*The motion that any point of the ocean executes under ebb and flow is obtained by the interference of four elliptic motions, two of which have the same period, like the apparent orbital period of the sun and the moon, and the other two of which have a period that is half as large.*

Due to its form $a \cos \gamma t + b \sin \gamma t$, where $a$ and $b$ are line segments, any elliptic term will contain six algebraic constants, so the four elliptic terms will collectively contain 24. If these 24 constants are found by observation for a point of the ocean then the motion of the point will be determined precisely. However, should only the height, and therefore only the falling and rising, be determined then one could observe only the projections of any line segment $a$, $b$, etc. onto the vertical line, so one would then obtain eight constants, in agreement with Laplace (*méc. cél.*, IV, 3). The 24 constants are, in principle, determined by the internal forces (e.g., gravitation and electricity), and are thus determined theoretically only when the character of the system is given completely.

If one assumes that matter is distributed continuously in space instead of the $m$ points then one must replace $x_1, \ldots, x_m$ in equations (12) with a variable line segment $x$, and the equation would become:

$$x\alpha^2 = q'', \tag{12*}$$

where $q''$ would be a function of $x$. This equation will determine the equilibrium of the system. The $u_1, \ldots, u_m$ in equations (13) and (14) will then have to be replaced with the quantity $u$ that depends upon $x$, and equation (14) will become:

$$\delta^2 u + 2\delta u \cdot \alpha + u \cdot \alpha^2 - \varphi = p_0 + p_\gamma + p_{2\gamma} + p_{\gamma'} + p_{2\gamma'} , \tag{14*}$$

where $u$, $p_0$ , $p_\gamma$ , … are functions of $x$ and $\varphi$ is a function of $x$ and $u$ that is linear in $u$. Equation (15) then becomes:

$$u = u_0 + u_\gamma + u_{2\gamma} + u_{\gamma'} + u_{2\gamma'} , \tag{15*}$$

where the elliptic terms are likewise functions of $x$; e.g., one might have $u_\gamma = a_x \cos \gamma t + b_x \sin \gamma t$, where $a_x$, $b_x$ are functions of $x$.

If one would like to consider the upper surface of the ocean as it looks at any time during the ebb and flow then one will need only to restrict $x$ to the points of the upper surface. Equation (12*) will then become the equation for the upper surface in equilibrium (viz., when the external forces are set to zero). We can think of this equations as being represented such that $x$ becomes a function of its direction $\xi$; i.e., a function of a line segment $\xi$ that has the same direction as $x$, but whose length is equal to1.

That is the essential idea behind polar coordinates. The equation of the upper surface at time $t$ is then derived easily, since $y = x + u$ and $u$ is known. If one takes the (inner) square of this equations then one will get $y^2 = x^2 + 2[x \mid u]$, since we can omit the last term $u^2$ as being of higher order in smallness. Now, if $z$ is the projection of $u$ onto $x$ (or onto $\xi$) then one will get:

(16) $$y^2 = x^2 + 2xz$$

as the equation of the upper surface at time $t$. Here, $z$ consists of five elliptic terms with the indicators 0, $\gamma$, $2\gamma$, $\gamma'$, $2\gamma'$, but these elliptic terms have a form here such that there coefficients are not line segments, but numerical quantities that depend upon $\xi$.

Ebb and flow was determined only in the first approximation in the above. Should one aspire to a higher approximation, then one would have to take the theory that developed here as the foundation for it, and then treat the second approximation in a corresponding way to what one does in the theory of secular perturbation of the planets.

Stettin, 31 March 1877.