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# TRANSFORMANTS

## A NEW MATHEMATICAL VEHICLE

## A SYNTHESIS

## OF COMBEBIAC'S TRI-QUATERNIONS AND GRASSMANN'S GEOMETRIC SYSTEM

## THE CALCULUS OF QUADRI-QUATERNIONS

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### PREFACE

The work of Grassmann (*Ausdehnungslehre*) and Combebiac's tri-quaternions, which are based upon Hamilton's quaternions, present themselves as remarkable geometric systems. In his introduction, Combebiac said: "The procedures that were presented in Grassmann's *Ausdehnungslehre* will realize this objective (a geometric analysis that is true for any reference system) if they are subjected to an absolutely systematic set of rules, which is a condition that has not been satisfied, in our view," and since, on the other hand, it is difficult to choose between one or the other system, each of which is brilliant – one cannot have two conflicting truths regarding the same subject – we have been led to establish a synthesis of these two mathematical thoughts.

We would like to believe that we have succeeded in that task by imagining a synthesis that simultaneously encompasses the two preceding ones.

Our system is based upon a group of 32 elements, to which, we have attributed the name of "mu-lambda-group," according to the letters of the alphabet that are adopted there, or the name of the "group of quadri-quaternions."

During our study, we have encountered the new vehicle of "transformant" which is essentially the solution to several mathematical problems, and which, at the same time, one can have recourse to, with great advantage, in other fields of applications.

# CHAPTER I

## Binary transformants.

### § 1. – Origin of transformants.

One knows ([**4**], pp. 68) that when two substitutions:

$$S\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \qquad \text{and} \qquad S\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix}$$

$$\begin{cases} x = \alpha\xi_1 + \beta\eta_1, \\ \xi_1 = \alpha_1\xi + \beta_1\eta, \end{cases} \qquad \begin{cases} y = \gamma\xi_1 + \delta\eta_1, \\ \eta_1 = \gamma_1\xi + \delta_1\eta \end{cases}$$

are effected one after the other that will give the following composed substitution:

$$S\begin{pmatrix}A & B\\ C & D\end{pmatrix} = S\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} S\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix} = S\begin{pmatrix}\alpha\alpha_1 + \beta\gamma_1 & \alpha\beta_1 + \beta\delta_1\\ \gamma\alpha_1 + \delta\gamma_1 & \gamma\beta_1 + \delta\delta_1\end{pmatrix}.$$

These are then represented by complex units $e_n$ ($n$ = 1, 2, 3, 4), and by a composition (i.e., multiplication):

$$S\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} S\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix} \equiv (\alpha\, e_1 + \beta\, e_2 + \gamma e_3 + \delta\, e_4) \cdot (\alpha_1\, e_1 + \beta_1\, e_2 + \gamma_1\, e_3 + \delta_1\, e_4).$$

This gives the following matrix:

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $e_1$ | $e_1$ | $e_2$ | 0 | 0 |
| $e_2$ | 0 | 0 | $e_1$ | $e_2$ |
| $e_3$ | $e_3$ | $e_4$ | 0 | 0 |
| $e_4$ | 0 | 0 | $e_3$ | $e_4$ |

The multiplication becomes clearer with the adoption of complex units that have two indices. The first index indicates the row in $S\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and the second one indicates the column.

The Cayley square is then:

| | $e_{11}$ | $e_{12}$ | $e_{21}$ | $e_{22}$ |
|---|---|---|---|---|
| $e_{11}$ | $e_{11}$ | $e_{12}$ | 0 | 0 |
| $e_{12}$ | 0 | 0 | $e_{11}$ | $e_{12}$ |
| $e_{21}$ | $e_{21}$ | $e_{22}$ | 0 | 0 |
| $e_{22}$ | 0 | 0 | $e_{21}$ | $e_{22}$ |

The simple composition then follows from this that:

$$\boxed{\begin{aligned} &e_{mn} \cdot e_{rs} = e_{ms} \quad \text{if} \quad n = r; \\ &e_{mn} \cdot e_{rs} = 0 \quad\ \ \text{if} \quad n \neq r. \end{aligned}}$$

The enlargement to more units would present no difficulty, because the law of composition would remain the same.

It is to these quantities $e_{mn}$ that we will attribute the name of "transformants."

## § 2. – Relationship between the units $e_{mn}$ and the quaternion units, as well as the imaginary unit.

*Note.* – We denote the roots of unity by the symbol $i_n = \sqrt[n]{-1}$.

Suppose that we have three substitutions $S$, $S_1$, $S_2$ ([**4**], pp. 79), with the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$, $\alpha_1$, $\beta_1$, $\gamma_1$, $\delta_1$, and $\alpha_2$, $\beta_2$, $\gamma_2$, $\delta_2$, resp., and the relation:

$$SS_1 = S_2$$

$$\begin{cases} \alpha = D + i_2 C, \\ \delta = D - i_2 C, \end{cases} \qquad \begin{cases} \beta = -B + i_2 A, \\ \gamma = \ \ B + i_2 A. \end{cases}$$

$\alpha_1 = D_1 + i_2\, C_1$ , etc., $A$, $B$, $C$, $D$ ($A_1$, …, $A_2$, …, resp.) are real expressions.

Then:

$$\begin{aligned} A_2 &= (AD_1 + DA_1) + (BC_1 - CB_1), \\ B_2 &= (BD_1 + DB_1) + (CA_1 - AC_1), \\ C_2 &= (CD_1 + DC_1) + (AB_1 - BA_1), \\ D_2 &= -AA_1 - BB_1 - CC_1 + DD_1, \end{aligned}$$

and

$$A_2^2 + B_2^2 + C_2^2 + D_2^2 = (A^2 + B^2 + C^2 + D^2) \cdot (A_1^2 + B_1^2 + C_1^2 + D_1^2).$$

For the quaternions ([**1**], pp. 56):

$$q = w + \rho \quad = w + x\, i + y\, j + z\, k,$$
$$r = w_1 + \rho_1 \ = w_1 + x_1\, i + y_1\, j + z_1\, k\,.$$

Upon changing the letters, in order to avoid collisions, one finds that:

$$\begin{aligned}(w^2 + x^2 + y^2 + z^2) \cdot (w_1^2 + x_1^2 + y_1^2 + z_1^2)\\ &= (ww_1 - xx_1 - yy_1 - zz_1)^2\\ &+ (wx_1 + w_1x + yz_1 - zy_1)^2\\ &+ (wy_1 + w_1y + zx_1 - xz_1)^2\\ &+ (wz_1 + w_1x + xy_1 - yx_1)^2,\end{aligned}$$

which is an algebraic formula that was derived by Euler.

A comparison with the preceding one gives:

$$D_2 \equiv ww_1 - xx_1 - yy_1 - zz_1,$$

and consequently:

$$\begin{array}{llll} D \equiv w, & A \equiv x, & B \equiv y, & C \equiv z, \\ D_1 \equiv w_1, & A_1 \equiv x_1, & B_1 \equiv y_1, & C_1 \equiv z_1\,, \end{array}$$

and similarly:

$$\begin{cases} \alpha = w + i_2 z, \\ \delta = w - i_2 z, \end{cases} \qquad \begin{cases} \beta = -y + i_2 x, \\ \gamma = \phantom{-}y - i_2 x, \end{cases}$$

$$\begin{cases} \alpha_1 = w_1 + i_2 z_1, \\ \delta_1 = w_1 - i_2 z_1, \end{cases} \qquad \begin{cases} \beta_1 = -y_1 + i_2 x_1, \\ \gamma_1 = \phantom{-}y_1 - i_2 x_1. \end{cases}$$

Since:

$$(D_2 + A_2\, i + B_2\, j + C_2\, k) = (D + A\, i + B\, j + C\, k) \cdot (D_1 + A_1\, i + B_1\, j + C_1\, k),$$

and, on the other hand:

$$\begin{aligned} & [(D + i_2\, C)\ e_{11} + (-B + i_2\, A)\ e_{12} + (B + i_2\, A)\ e_{21} + (D - i_2\, C)\ e_{22}] \\ \times\ & [(D_1 + i_2\, C_1)\, e_{11} + (-B_1 + i_2\, A_1)\, e_{12} + (B_1 + i_2\, A_1)\, e_{21} + (D_1 - i_2\, C_1)\, e_{22}] \\ =\ & [(D_2 + i_2\, C_2)\, e_{11} + (-B_2 + i_2\, A_2)\, e_{12} + (B_2 + i_2\, A_2)\, e_{21} + (D_2 - i_2\, C_2)\, e_{22}]\,, \end{aligned}$$

one can identify:

$$D\,(e_{11} + e_{22}) + C\, i_2\,(e_{11} - e_{22}) + B\,(e_{21} - e_{12}) + A i_2\,(e_{12} + e_{21}) = D + C\, k + B\, j + A\, i\,;$$

thus:

$$\boxed{\begin{aligned} e_{11}+e_{22} &= 1, \\ i_2(e_{11}-e_{22}) &= k, \\ e_{21}-e_{12} &= j, \\ i_2(e_{12}+e_{21}) &= i. \end{aligned}}$$

It is necessary that the left-hand sides of the four equations must obey the same laws as the right-hand sides; i.e.:

| | | |
|---|---|---|
| 1) $ij = \;\; k$, | 2) $jk = \;\; i$, | 3) $ki = \;\; j$, |
| 4) $ji = -k$, | 5) $kj = -i$, | 6) $ik = -j$, |
| 7) $i^2 = -1$, | 8) $j^2 = -1$, | 9) $k^2 = -1$. |

Verifications:

1) $i_2\,(e_{12}+e_{21})\,(e_{21}-e_{12}) = i_2\,(e_{11}-e_{22})$,
2) $(e_{21}-e_{12})\,i_2\,(e_{11}-e_{22}) = i_2\,(e_{21}+e_{12})$,
3) $i_2\,(e_{11}-e_{22})\,i_2\,(e_{21}-e_{12}) = i_2\,(e_{12}-e_{21}) = e_{21}-e_{12}$,
4) $(e_{21}-e_{12})\,i_2\,(e_{12}+e_{21}) = i_2\,(e_{22}-e_{11}) = -\,i_2\,(e_{11}-e_{22})$,
5) $i_2\,(e_{11}-e_{21})\,(e_{21}-e_{12}) = i_2\,(-\,e_{21}-e_{12}) = -\,i_2\,(e_{12}+e_{21})$,
6) $i_2\,(e_{12}+e_{21})\,i_2\,(e_{11}-e_{22}) = -\,(e_{21}-e_{12})$,
7) $[i_2\,(e_{12}+e_{21})]^2 = (e_{21}-e_{12})\,(e_{21}-e_{12}) = -\,(e_{22}+e_{11})$,
8) $(e_{21}-e_{12})^2 = (e_{21}-e_{12})\,(e_{21}-e_{12}) = -\,e_{11}-e_{22}$ ,
9) $[i_2\,(e_{11}-e_{22})]^2 = -\,(e_{11}-e_{22})\,(e_{11}-e_{22}) = -\,(e_{11}+e_{22})$.

Let $e_{mn}$ be expressed, inversely, in terms of $i$, $j$, $k$. One will have:

$$\boxed{\begin{aligned} &1)\;\; e_{11} = \tfrac{1}{2}(1-i_2k), \\ &2)\;\; e_{22} = \tfrac{1}{2}(1+i_2k), \\ &3)\;\; e_{21} = \tfrac{1}{2}(j-i_2i), \\ &4)\;\; e_{12} = -\tfrac{1}{2}(j+i_2i). \end{aligned}}$$

Here, as well, the right-hand sides must obey the law of $e_{mn}\cdot e_{rs}$ . For example:

1) $e_{11}\cdot e_{22} = 0$, 2) $e_{11}\cdot e_{12} = e_{12}$ .

Indeed:

1) $\tfrac{1}{2}(1-i_2k)\cdot\tfrac{1}{2}(1+i_2k) \;=\; \tfrac{1}{4}(1-i_2k+i_2k+i_2^2k^2) \;=\; 0$,
2 $\tfrac{1}{2}(1-i_2k)\cdot\tfrac{1}{2}(-j-i_2i) \;=\; \tfrac{1}{4}(-j+i_2kj-i_2i+i_2^2ki)$

$$= \tfrac{1}{4}(-j - i_2\, i - i_2\, i - j) = -\tfrac{1}{2}(j + i_2\, i), \text{ etc.}$$

Another example gives the surprising result that:

1) $e_{12} \cdot e_{22} = e_{12}$ ,
2) $e_{22} \cdot e_{12} = 0.$

The product is equal to the first factor, and the inversion of the same factor is equal to zero. Effectively, one has:

$$1)\quad -\tfrac{1}{2}(j + i_2\, i) \cdot \tfrac{1}{2}(1 + i_2\, k) = -\tfrac{1}{4}(j + i_2\, i + i_2\, jk + i_2^2 ik)$$
$$= -\tfrac{1}{4}(j + i_2\, i + i_2\, i + j) = -\tfrac{1}{2}(j + i_2\, i),$$

$$2)\quad -\tfrac{1}{2}(1 + i_2\, k) \cdot \tfrac{1}{2}(-j - i_2\, i) = -\tfrac{1}{4}(j + i_2\, kj + i_2\, i + i_2^2 ki)$$
$$= -\tfrac{1}{4}(j - i_2\, i + i_2\, i - j) = 0.$$

## § 3. – The reciprocal value and the conjugate of a transformant.

Let $t = \alpha\, e_{11} + \beta\, e_{12} + \gamma e_{21} + \delta\, e_{22}$ be a transformant.

The determinant of the transformant = determinant of the substitution $S\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha\delta - \beta\gamma = \Delta\, t$.

$R\, t = t^{-1}$ is to be found.

We have ([**4**], pp. 68, 63):

$$S\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot S\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = S\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore:

1) $\alpha\alpha_1 + \beta\gamma_1 = 1$, so *a*) $\alpha_1 = \dfrac{\delta}{\alpha\delta - \beta\gamma}$,

2) $\alpha\beta_1 + \beta\delta_1 = 1$, so *b*) $\beta_1 = \dfrac{-\beta}{\alpha\delta - \beta\gamma}$,

3) $\gamma\alpha_1 + \delta\gamma_1 = 1$, so *c*) $\gamma_1 = \dfrac{-\gamma}{\alpha\delta - \beta\gamma}$,

4) $\gamma\beta_1 + \delta\delta_1 = 1$, so *d*) $\delta_1 = \dfrac{\alpha}{\alpha\delta - \beta\gamma}$.

One then finds that:

$$\boxed{t^{-1} = \alpha_1 e_{11} + \beta e_{12} + \gamma_1 e_{21} + \delta_1 e_{22} = \frac{\delta e_{11} - \beta e_{12} - \gamma e_{21} + \alpha e_{22}}{\alpha\delta - \alpha\gamma}.}$$

One can refer to the numerator of this fraction as "conjugate" to $t$: "$t$" = $Kt$.

This gives us the right to make the following reflection:

If one sets $\delta = \alpha$, $\gamma = -\beta$ then $t$ will become:

$$t' = \alpha(e_{11} + e_{22}) + \beta(e_{11} - e_{21}) = \alpha - \beta j = q,$$
$$Kt' = \alpha(e_{11} + e_{22}) - \beta(e_{11} - e_{21}) = \alpha + \beta j = Kq,$$

according to Hamilton. In addition:

$$\Delta t' = \alpha^2 + \beta^2 = qKq = Kq \cdot q = (Tq)^2. \qquad \text{([1], pp. 30)}$$

One thus has:

$$t^{-1} = \frac{Kt}{\Delta t}.$$

Upon multiplying the left and right sides by $t$, one will find that:

$$1 = \frac{tKt}{\Delta t} = \frac{Kt \cdot t}{\Delta t}, \qquad \Delta t = tKt = Kt \cdot t.$$

One can easily verify:

$$K \cdot Kt = K^2\, t = t,$$

and

$$K(t\, t_1) = Kt_1 \cdot Kt \qquad \text{([1], pp. 32).}$$

Thus:

$$K(tKt) = K^2\, t \cdot Kt = tKt.$$

From this, $tKt$ can only be an ordinary quantity. It is equal to simply $\Delta t$.

## § 4. – Operator and operand (multiplier and multiplicand).

It is not necessary to identify the operand with the operator, as Hamilton did ([**1**], pp. 40).

In what follows, we will employ the transformants $e_{mn}$ as operators and the $e_m$ (with just one index, which are Grassmann's symbols) as operands.

The $e_m$ represent pure geometric forms (points, lines, planes, etc.), and the transformants become dynames (reflections, rotations, and other motions).

We then stipulate the rule:

$$\boxed{\begin{array}{l} e_{mn} \cdot e_r = e_m \;\text{ if }\; r = n, \\ e_{mn} \cdot e_r = 0 \;\;\text{ if }\; r \neq n, \end{array}}$$

and conversely:

$$\boxed{\begin{array}{l} e_r \cdot e_{mn} = e_n \;\text{ if }\; r = m, \\ e_r \cdot e_{mn} = 0 \;\;\text{ if }\; r \neq m. \end{array}}$$

When $\rho = x\, e_1 + y\, e_2$ changes into $\rho' = x'\, e_1 + y'\, e_2$, there will exist a difference only in the case when the operator is found on the right or left.

*N. B.* – In the sequel, we will always multiply on the left, by analogy with $y = \varphi(x)$.

In the case where the transformants are found on the right of $\sum e_m$, one can put them on the left side without changing the result if one transposes the two indices of each term. In symbols:

$$t = \alpha\, e_{11} + \beta\, e_{12} + \gamma e_{21} + \delta\, e_{22}\,,$$
$$Wt = \alpha\, e_{11} + \beta\, e_{21} + \gamma e_{12} + \delta\, e_{22}\,, \qquad \rho = x\, e_1 + y\, e_2.$$

One then has:

$$t_1(\rho\, t) = (t_1 Wt)\, \rho\,.$$

*N. B.* – Generally, $t_1(\rho\, t) \neq (t_1\rho)\, t$. Therefore, the operation is not associative. By contrast, $(t_1\, t)\, \rho = t_1\, (t\, \rho)$; likewise, $(t_2\, t_1)\, t = t_2\, (t_1\, t)$.

Thus:

$$(t_1\, t)\, \rho = \rho\, W(t_1\, t) = t_1\, (t\, \rho) = (t\rho)\, Wt_1 = (\rho\, Wt)\, Wt_1 = \rho\, (Wt\; Wt_1).$$

Consequently:

$$W(t_1\, t) = Wt \cdot Wt_1\,.$$

When the operation $W$ is performed on a product, it will be distributive and transposing.

The transformations that change the $\sum e_{mn}$ are transformants of degree two; in symbols, $\sum e_{mn,\, rs}$. They have the same relationship with the transformants of degree one, and follow the same rules, as the transformants of degree one that are inverse to $\sum e_m$.

## § 5. – "Mutators," special transformants of degree two.

Along with $K$, $W$, what other mutators exist that permute the indices and change the sign?

Take the set:

| $t = \alpha e_{11} + \beta e_{12} + \gamma e_{21} + \delta e_{22}$ | | Definitions |
|---|---|---|
| 1) | $Wt = \alpha e_{11} + \beta e_{21} + \gamma e_{12} + \delta e_{22}$ | Transpose the two indices in each term. |
| 2) | $Et = \alpha e_{22} - \beta e_{21} - \gamma e_{12} + \delta e_{11}$ | $W_{12} Z_2 = Z_2 W_{12}$. |
| 3) | $Kt = \alpha e_{22} - \beta e_{12} - \gamma e_{21} + \delta e_{11}$ | $EW = WE$. |
| 4) | $W_{12} t = \alpha e_{22} + \beta e_{21} + \gamma e_{12} + \delta e_{11}$ | The index 2 is converted into the index 1, and vice versa. |
| 5) | $W_{12} W t = \alpha e_{22} + \beta e_{12} + \gamma e_{21} + \delta e_{11}$ | $= W W_{12}$. |
| 6) | $Z_2 t = \alpha e_{11} - \beta e_{12} - \gamma e_{21} + \delta e_{22}$ | Change the sign whenever the index appears. |
| 7) | $W Z_2 t = \alpha e_{11} - \beta e_{21} - \gamma e_{12} + \delta e_{22}$ | $= Z_2 W$. |

We write these expressions more briefly as:

$M_0 \equiv 1$, $M_1 \equiv W$, $M_2 \equiv E$, $M_2 \equiv K$,
$M_4 \equiv W_{12}$ , $M_5 \equiv W_{12}\, W$, $M_6 \equiv Z_2$ , $M_7 \equiv WZ_2$ .

The symbols form an Abelian group of eight elements; (see [**5**], pp. 91, 52).

*N. B.* – The interior squares of the Cayley square contain only indices.

The $M$ with odd indices are transposing; the $M$ with even indices are non-transposing. The group of $M_n$ is:

| $M$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 |
| 2 | 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 |
| 3 | 3 | 2 | 1 | 0 | 7 | 6 | 5 | 4 |
| 4 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 5 | 5 | 4 | 7 | 6 | 1 | 0 | 3 | 2 |
| 6 | 6 | 7 | 4 | 5 | 2 | 3 | 0 | 1 |
| 7 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

The symbols $M_n$ refer to the two indices of each term. If we employ the same definitions on $\rho = x\, e_1 + y\, e_2$ – i.e., on just one index of each term – then we will write $\mathcal{M}_n(\mathcal{W}, \mathcal{E}, \mathcal{K}, \mathcal{W}_{12}, \mathcal{Z}_2, \ldots)$, instead of $M_n(W, E, K, W_{12}, Z_2, \ldots)$. One sees that $\mathcal{M}_n$ coincides with $\mathcal{M}_{n-1}$ : $\mathcal{W} \equiv 1$, $\mathcal{K} \equiv \mathcal{E}$, $\mathcal{W}_{12}\mathcal{W} \equiv \mathcal{W}_{12}$, $\mathcal{W}\mathcal{Z}_2 \equiv \mathcal{W}\mathcal{Z}_2$ .

The $\mathcal{M}_n$ form a group with a composition that is totally different from that of the $M_n$ ([**5**], pp. 93).

| $\mathcal{M}$ | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| 0 | 0 | 2 | 4 | 6 |
| 2 | 2 | −0 | −6 | 4 |
| 4 | 4 | 6 | 0 | 2 |
| 6 | 6 | −4 | −2 | 0 |

This is only a quarter of the Cayley square. One finds the other operators by multiplying with – 1. Thus, $\mathcal{M}_n\,(-\mathcal{M}_n) = -(\mathcal{M}_n, \mathcal{M}_n)$, etc.

$\mathcal{Z}_2$ and $\mathcal{W}_{12}$ are of order two and $\mathcal{E}$ is of order four, and corresponds exactly to the "complementary feature" (*trait de complément*) of Grassmann's school ([**3**], I, pp. 17).

Since:

$$\begin{aligned} \rho &= x\,e_1 + y\,e_2\,, \\ \mathcal{E}\rho &= -y\,e_1 + x\,e_2\,, \\ \mathcal{W}_{12}\,\rho &= x\,e_2 + y\,e_1\,, \\ \mathcal{Z}_2\,\rho &= x\,e_1 - y\,e_2\,, \end{aligned}$$

one can easily verify that:

$$\mathcal{E} \equiv e_{21} - e_{12}\,, \qquad \mathcal{W}_{12} \equiv e_{12} + e_{21}\,, \qquad \mathcal{Z}_2 \equiv e_{11} - e_{22}\,.$$

One thus has:

| | | |
|---|---|---|
| $(e_{21} - e_{12})\,e_1 = e_2\,,$ | and for Grassmann's school | $\mid e_1 = e_2\,,$ |
| $(e_{21} - e_{12})\,e_2 = -\,e_1\,,$ | " " " | $\mid e_2 = -\,e_1\,,$ |
| $(e_{21} - e_{12})^2\,e_1 = -\,(e_{11} + e_{22}) = -\,e_1$ | " " " | $\Vert\, e_1 = \mid e_2 = -\,e_1\,.$ |

Moreover:

$$\mathcal{E} = \mathcal{W}_{12}\,\mathcal{Z}_2 = -\mathcal{Z}_2\,\mathcal{W}_{12}\,.$$

## § 6. – Scalars, vectors, tensors, versors.

Just as we did for transformants, we distinguish scalars $S$, vectors $V$, tensors $T$, and versors $U$.

Let:

$$\begin{aligned} t &= \alpha\,e_{11} + \beta\,e_{12} + \gamma e_{21} + \delta\,e_{22} = S\,t + Vt, \\ K\,t &= \alpha\,e_{22} - \beta\,e_{12} - \gamma e_{21} + \delta\,e_{11} = S\,t - Vt. \end{aligned}$$

From this, we get:

1) $2S\,t = \alpha\,e_{11} + \alpha\,e_{22} + \delta\,e_{22} + \delta\,e_{22} = (\alpha + \delta)\,(e_{11} + e_{22}) = \alpha + \delta.$

$$\boxed{St = \tfrac{1}{2}(\alpha + \beta).}$$

2) $2Vt = t - K\,t = 2\beta\,e_{12} + 2\gamma e_{21} + \alpha\,e_{11} - \alpha\,e_{22} + \delta\,e_{22} - \delta\,e_{11}$
$= 2\beta\,e_{12} + 2\gamma e_{21} + (\alpha + \delta)\,(e_{11} - e_{22})$.

$$\boxed{Vt = \beta e_{12} + \gamma e_{21} + \tfrac{1}{2}(\alpha - \delta)(e_{11} - e_{22}).}$$

Thus:

$$(T\,t)^2 = t \cdot K\,t = \Delta t = \alpha\delta - \beta\gamma.$$

In order for this to be true, one must have::

$$\boxed{Tt = \sqrt{\alpha\delta - \beta\gamma} \quad \text{and} \quad Ut = \frac{1}{Tt} = \frac{\alpha e_{11} + \beta e_{12} + \gamma e_{21} + \delta e_{22}}{\sqrt{\alpha\delta - \beta\gamma}}.}$$

## § 7. – What complex quantities can be employed as coordinates in two-dimensional space?

There exist two possibilities:

I. The coordinates are composed from $e_{mn}$ ($\sum e_{mn}$, resp.).
II. They are $xe_1$ ($ye_2$, resp.).

I.

| | | | | |
|---|---|---|---|---|
| 1) | Abscissa: | $xe_{11}$, | Ordinate: | $ye_{22}$, |
| 2) | “ | $xe_{11}$, | “ | $ye_{12}$, |
| 3) | “ | $xe_{12}$, | “ | $ye_{21}$, according to Hamilton, |
| 4) | “ | $xi = xi_2\,(e_{12} + e_{21})$, | “ | $yk = yi_2\,(e_{11} - e_{22})$. |

We first examine the four combinations: The additions and subtractions according to the rules of a parallelogram of forces. Consequently, in all four cases, each case will possess its proper multiplication.

The third case possesses a property that it gives points (lines, resp.) of two types according to whether the product is composed of an even or odd number of factors, respectively. Cf., the multiplications:

1) $(x\,e_{11} + y\,e_{22})(x_1\,e_{11} + y_1\,e_{11}) = x\,x_1\,e_{11} + y\,y_1\,e_{22}$ ;

for the Grassmann school:

$(x\,e_1 + y\,e_2) \mid (x_1\,e_1 ,\; y_1\,e_2) = x\,x_1 ,\, y\,y_1$ , regarding the coefficients.

2) $(x\,e_{11} + y\,e_{12})(x_1\,e_{11} + y_1\,e_{12}) = x\,x_1\,e_{11} + x\,y_1\,e_{12}$ ;
3) $(x\,e_{12} + y\,e_{21})(x_1\,e_{12} + y_1\,e_{21}) = x_1\,y\,e_{11} + x\,y_1\,e_{11}$ ;

$$(x_1\,y\,e_{22} + xy_1\,e_{11})(x_2\,e_{12} + y_2\,e_{21}) = x\,y_2\,x_1\,e_{12} + \;y\,x_1\,y_2\,e_{21}\,.$$

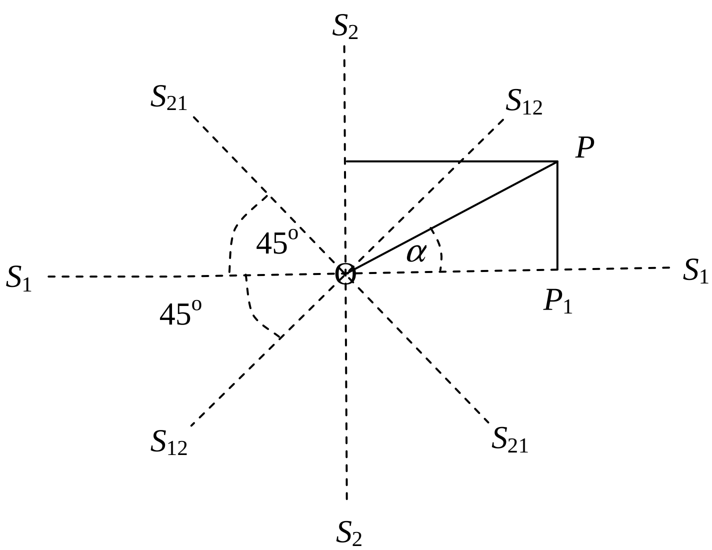
However, we would not like to pursue this line of reasoning further. We direct our attention to case II, in particular: abscissa: $xe_1$, ordinate: $ye_2$ .

## § 8. – The geometric interpretation of the expressions $\sum x_{mn}\,e_{mn}$ and $\sum x_n\,e_n$ .

Let the coordinates of the point $P$ be $xe_1 \equiv OP_1$ and $ye_2 \equiv OP_2$ , in such a way that:

$$(\rho = xe_1 + ye_1) \equiv P \equiv (OP_1 + OP_2) \equiv (OP).$$

This determination of points follows the methods of quaternions; $e_1$ and $e_2$ are vectors in the sense of Grassmann (Ger.: *Strecken*). See (*fig. 1*).




$(\rho = OP) \equiv P,$

$OP = OP_1 + OP_2 = \rho_1 + \rho_2$

$(\rho = \rho_1) \equiv P_1,\;\; (\rho = \rho_2) \equiv P_2.$

The dashed lines represent reflecting lines.

Figure 1.

Definitions:

| | | |
|---|---|---|
| $p_1\,\rho$ | denotes the projection of $\rho$ | onto the abscissa, |
| $p_2\,\rho$ | " " | onto the ordinate, |
| $p_3\,\rho$ | " " | onto the origin, |
| $s_1\,\rho$ | denotes the reflection of $\rho$ | onto the abscissa, |

| | | | |
|---|---|---|---|
| $s_2\,\rho$ | “ | “ | onto the ordinate, |
| $s_{12}\,\rho$ | “ | “ | onto the bisector of the first quadrant, |
| $s_{21}\,\rho$ | “ | “ | onto the bisector of the second quadrant, |
| $s_0\,\rho$ | “ | “ | onto the origin. |

Finally, we let:

| | | | | |
|---|---|---|---|---|
| $d_1\rho$ | denote the rotation of $\rho$ | by $90^{\circ}$ | around the | abscissa, |
| $d_2\rho$ | “ | “ | “ | ordinate, |
| $d_3\rho$ | “ | “ | “ | third coordinate $\perp d_1$ and $d_2$, |

$d_1$ is therefore $\equiv k$, $d_2 \equiv i$, $d_3 \equiv j$.

One then obviously has:

1) $\rho' = e_{11}\,\rho = p_1\,\rho$,
2) $\rho' = e_{22}\,\rho = p_2\,\rho$,
3) $\rho' = e_{22}\,(e_{11}\,\rho) = p_2\,(p_1\,\rho) = p_2\,\rho_1 = \rho_0 = 0$.

On the other hand:

$$\rho' = (e_{22}\,e_{11})\rho = 0 \cdot \rho = 0,$$

and similarly:

$$\rho' = e_{11}\,(e_{22}\,\rho) = p_1\,(p_2\,\rho) = p_1\,\rho_2 = \rho_0 = 0.$$

However, one also has:

$$\rho' = p_0\,\rho = 0, \quad \text{from which,} \quad p_0 = p_1\,p_2 = p_2\,p_1 .$$

4) $\rho' = e_{11}\,(e_{11}\,\rho) = e_{11}\,\rho = \rho_1 = (e_{11}\,e_{11})\,\rho = e_{11}\,\rho = \rho_1$.

Similarly:

$$\rho' = e_{21}\,(e_{22}\,\rho), \qquad p_1^2 = p_1, \qquad p_2^2 = p_2 .$$

*N. B.* – One must distinguish between the origin as a reflecting dyname $s_0$ and the origin as a purely geometric form $\rho_0$ : $s_0 = -1$, $\rho_0 = 0$.

5) $\rho' = e_{12}\,\rho = p_1\,s_{12}\,\rho = -\,p_1\,d_3\,\rho = -\,d_3\,p_2\,\rho = s_{12}\,p_2\,\rho$,

6) $\rho' = e_{21}\,\rho = p_2\,s_{12}\,\rho = \quad p_2\,d_3\,\rho = \quad d_3\,p_1\,\rho = s_{12}\,p_1\,\rho$.

Sums and geometric differences (from the parallelogram rule for forces).

1) $\rho' = (e_{11} + e_{22})\,\rho = (p_1 + p_2)\,\rho = \rho$,

2) $\rho' = (e_{11} - e_{22})\,\rho = (p_1 - p_2)\,\rho = s_1\,\rho$,

3) $\rho' = (e_{21} + e_{12})\,\rho = (p_2 + p_1)\,s_{12}\,\rho = s_{12}\,\rho = (p_2 - p_1)\,d_3\,\rho = s_2\,d_3\,\rho = s_{12}\,\rho$,

4) $\rho' = (e_{21} - e_{12})\,\rho = (p_2 - p_1)\,s_{12}\,\rho = s_2\,s_{12}\,\rho = d_3\,\rho$ ,

because it is obvious that $p_1 + p_2 = 1$, $p_1 - p_2 = s_1$ , $p_2 - p_1 = s_2$ .

Since:

$$e_{11} + e_{22} = 1, \qquad e_{11} - e_{22} = -\,i_2\,k,$$
$$e_{21} + e_{21} = -\,i_2\,i, \qquad e_{21} - e_{12} = \;j,$$

one has the geometric interpretation of these imaginary vectors of Hamilton in connection with $\rho = x\,e_1 + y\,e_2$ . One has:

$$-\,i_2\,k \equiv s_1, \qquad i_2\,k = s_0\,s_1 \;= s_1\,s_0 \;= -\,s_1 = s_2\,,$$
$$-\,i_2\,i \equiv s_{12}, \qquad i_2\,i \;= s_0\,s_{12} = s_{12}\,s_0 = -\,s_{12} = s_{21}\,.$$

Now, one can also represent the $e_{mn}$ as geometric sums:

1) $e_{11}\,\rho = \frac{1}{2}(1 - i_2\,k)\,\rho = \frac{1}{2}\rho + \frac{1}{2}s_1 \cdot \rho$ ,

2) $e_{22}\,\rho = \frac{1}{2}(1 + i_2\,k)\,\rho = \frac{1}{2}\rho + \frac{1}{2}s_2 \cdot \rho$ ,

3) $e_{21}\,\rho = \frac{1}{2}(j - i_2\,i)\,\rho = d_3 \cdot \frac{1}{2}\rho + s_{12} \cdot \frac{1}{2}\rho$ , (see *fig.* 2),

4) $e_{12}\,\rho = \frac{1}{2}(-\,j - i_2\,i)\,\rho = -\;d_3 \cdot \frac{1}{2}\rho + s_{12} \cdot \frac{1}{2}\rho$ .

$OM = \frac{1}{2}OP$,

$M' = s_1\,M$,

$M'' = s_1\,M$.

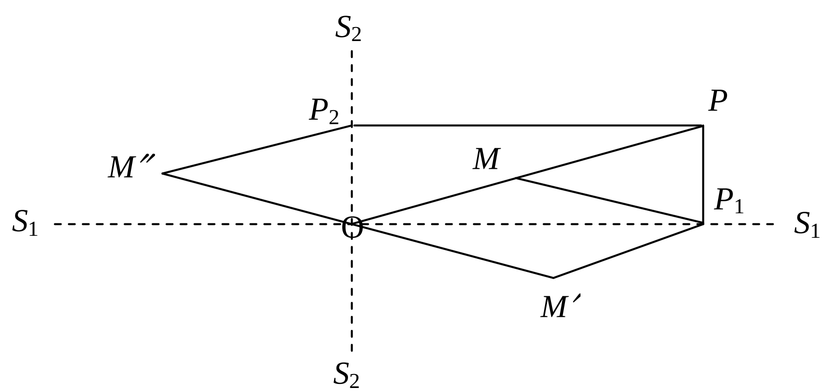



Figure 2. *Characteristic of reflecting lines.*

These lines are completely distinct from ordinary, purely geometric lines; they have no direction, but only a position. A rotation of 180° brings them back to the same position, like multiplying by + 1. One knows that the negative reflecting lines are perpendicular to the positive lines. One must give a double rotational motion of 90° to the reflecting lines in order to bring them to the same position.

*Overview of the symbols that have been employed up to now.*

| | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| 1) | $1$ | $e_{11}+e_{22}$ | $1$ | $1$ |
| 2) | $\mathcal{E}$ | $e_{21}-e_{12}$ | $j$ | $d_3$ |
| 3) | $\mathcal{W}_{12}$ | $e_{12}+e_{21}$ | $-i_2 i$ | $s_{12}$ |
| 4) | $\mathcal{Z}_2$ | $e_{11}-e_{22}$ | $-i_2 k$ | $s_1$ |
| 5) | $-1$ | $-(e_{11}+e_{22})$ | $-1$ | $-1=s_0$ |
| 6) | $-\mathcal{E}$ | $e_{12}-e_{21}$ | $-j$ | $-d_3=s_0 d_3=d_3 s_0$ |
| 7) | $-\mathcal{W}_{12}$ | $-(e_{12}+e_{21})$ | $i_2 i$ | $s_0 s_{12}=s_{12}s_0=s_{21}$ |
| 8) | $-\mathcal{Z}_2$ | $e_{22}-e_{11}$ | $i_2 k$ | $s_0 s_1=s_1 s_0=s_2.$ |

Column headings:

*A*. Change of indices and the sign of $\rho$.
*B*. Transformants with the same effect.
*C*. Corresponding Hamilton symbols.
*D*. Geometric interpretation (geometry of dynames).

The symbols in column *D* form the same group of eight members as the symbols in columns *A* and *B* (cf., § 5).

A quarter of the Cayley square will suffice for this:

| | $1$ | $d_3$ | $s_{12}$ | $s_1$ |
|---|---|---|---|---|
| $1$ | $1$ | $d_3$ | $s_{12}$ | $s_1$ |
| $d_3$ | $d_3$ | $-1$ | $-s_1$ | $s_{12}$ |
| $s_{12}$ | $s_{12}$ | $s_1$ | $1$ | $d_2$ |
| $s_1$ | $s_1$ | $-s_{12}$ | $-d_3$ | $1$ |

Since the complement $\mathcal{E}$ is itself composed of $\mathcal{W}_{12}$ and $\mathcal{Z}_2$ (viz., $\mathcal{E} = \mathcal{W}_{12} \cdot \mathcal{Z}_2$, which is $d_3 = s_{12}\ s_1$), one sees that the other transformants are introduced into the Grassmann system organically, and that will permit an extended amplification of the system.

## § 9. – Geometry in the space of just one dimension.

We content ourselves with the following remark: If one of the two symbols $e_1$, $e_2$, or both of them, represent points then two-dimensional space will change into one-dimensional space, and the results of the first geometry can be easily translated into the geometry of just one dimension.

# CHAPTER II

## Ternary transformants. – three-dimensional spaces.

### § 10. – Reflections.

We have:

$$\rho = x\, e_1 + y\, e_2 + z\, e_3 ;$$

$e_n$ are vectors. One has $t = \sum a_{mn}\, e_{mn}$ , i.e.:

$$t = a_{11}\, e_{11} + a_{12}\, e_{12} + a_{13}\, e_{13} + a_{21}\, e_{21} + a_{22}\, e_{22} + a_{23}\, e_{23} + a_{31}\, e_{31} + a_{32}\, e_{32} + a_{33}\, e_{33} ;$$

$a_{mn}$ are ordinary quantities.

We confine our study to the elementary transformants. They are the ones whose coefficients are ± 1, 0, and each of which appears in the same number between the three indices.

We look for the expressions for the reflections in terms of $\sum e_{mn}$ .

In general, one has:

$$\begin{array}{ll} 1) & t\, e_1 = a_{11}\, e_1 + a_{21}\, e_2 + a_{31}\, e_3 , \\ 2) & t\, e_2 = a_{12}\, e_1 + a_{22}\, e_2 + a_{32}\, e_3 , \\ 3) & t\, e_3 = a_{13}\, e_1 + a_{23}\, e_2 + a_{33}\, e_3 . \end{array}$$

One thus finds, for example, for $s_{12}$, that:

$$\begin{array}{lllll} 1)\ t\, e_1 = e_2 , & \text{i.e.,} & 1)\ a_{21} = 1, & a_{11} = 0, & a_{31} = 0, \\ 2)\ t\, e_2 = e_1 , & & 1)\ a_{12} = 1, & a_{22} = 0, & a_{32} = 0, \\ 3)\ t\, e_3 = -\, e_3 , & & 1)\ a_{33} = -1, & a_{23} = 0, & a_{13} = 0. \end{array}$$

One then has:

$$t = s_{12} = e_{21} + e_{12} - e_{33} .$$

One finds the following expressions for the nine reflecting lines and the nine reflecting planes in the same manner:

$$\begin{array}{lll} s_1 = e_{11} - e_{22} - e_{33} , & \sigma_n = -\, s_n , & n = 1, 2, 3, \\ s_2 = -\, e_{11} + e_{22} - e_{33} , & \sigma_{mn} = -\, s_{mn} , & m, n = 1, 2, 3, \\ s_3 = -\, e_{11} - e_{22} + e_{33} , & & \\ s_{12} = e_{12} + e_{21} - e_{33} , & & \\ s_{21} = -\, e_{12} - e_{21} - e_{33} , & & \\ s_{23} = -\, e_{11} + e_{32} + e_{23} , & & \\ s_{32} = -\, e_{11} - e_{32} - e_{23} , & & \\ s_{31} = e_{31} - e_{22} + e_{13} , & & \end{array}$$

$$s_{13} = -e_{31} - e_{22} - e_{13} .$$

The $\sigma_n$ are the reflecting planes *XY*, *YZ*, *ZX* that are perpendicular to $s_n$ ; one could then say $\sigma_{mn} \perp s_{mn}$ .

In three-dimensional space, the negative reflecting lines become reflecting lines that are perpendicular to the lines, and vice versa.

In addition:

$$1 = e_{11} + e_{22} + e_{33} ,$$
$$s_0 = -(e_{11} + e_{22} + e_{33}).$$

*Note*. – The notations are the same as in chapter I, § 8 for the plane *XY*. One obtains notations for the other two planes *YZ*, *ZX* by a circular permutation of the indices 1, 2, 3, and notably the indices 1, 2 of the plane *XY* become 2, 3 in the plane *YZ* and 3, 1 in the plane *ZX*.

## § 11. – Rotations around the coordinate axes.

There are three fourth-order operators that perform a 90° rotation in the left-hand sense (i.e., *levogyrous*):

$$d_1 = e_{32} - e_{23} + e_{11} , \qquad d_1^{-1} = -e_{32} + e_{23} + e_{11} ,$$
$$d_2 = e_{13} - e_{31} + e_{22} , \qquad d_2^{-1} = -e_{13} + e_{31} + e_{22} ,$$
$$d_3 = e_{21} - e_{12} + e_{33} , \qquad d_3^{-1} = -e_{21} + e_{12} + e_{33} .$$

$$d_1^2 = -e_{22} - e_{33} + e_{11} = s_1 , \qquad d_n^3 = d_n^{-1}$$
$$d_2^2 = -e_{33} - e_{11} + e_{22} = s_2 , \qquad d_n^4 = 1, \qquad n = 1, 2, 3,$$
$$d_3^2 = -e_{11} - e_{22} + e_{33} = s_3 .$$

One can easily confirm that:

$$s_{12}\, s_1 = d_3 , \qquad (s_{12}\, s_1)^{-1} = d^{-1} = s_1^{-1} s_{12}^{-1} = s_1\, s_{12} .$$

Moreover:

$$d_1\, d_2\, d_3 = e_{31} - e_{22} + e_{13} = s_{31} .$$

The operators 1, $s_1$, $s_2$, $s_3$, $s_{12}$, $s_{21}$, $d_3$, $d_3^{-1}$ form a group of eight members of the following type 8 = 1(1) + 5(2) + 2(4); i.e., among the eight members, 1 is of degree 1, 5 are of degree 2, and 2 are of degree 4.

The subgroup 1, $s_1$, $s_2$, $s_3$ has type 4 = 1(1) + 3(2).

One will obtain two other groups of eight members of the same type by circular permutations of the indices 1, 2, 3. (See the group of 24 members in § 13.)

*Note.* – It is remarkable that in three-dimensional space, one cannot generally identify $k$, $i$, $j$ with $d_1$, $d_2$, $d_3$ .

$d_1 \equiv k$ only with regard to the points of the plane $YZ$ as operands.

Similarly:

$d_2 \equiv i$ with regard to the points of the plane $ZX$,
and
$d_3 \equiv j$ " " " " $XY$

as operands.

By contrast, $d_1$ ($d_2$, $d_3$, resp.) effect the rotation of all points of tri-dimensional space around the corresponding axes.

Cf., $d_2\, e_3 = e_3$ ; as opposed to $i(j) = -1$.

## § 12. – Operators $\mathcal{D}_n$, $\mathrm{D}_n$ .

$$s_{12} \cdot s_{21} = (e_{12} + e_{21} - e_{33}) \cdot (-e_{11} + e_{32} + e_{23}) = -e_{32} + e_{13} - e_{24} ,$$

which then gives a new operator.

One obtains eight operators in this manner. They effect a counterclockwise (i.e., levogyrous) rotation through 120° around the axes, which forms equal angles with the three coordinates and pass through the origin. They are of third order, and one can call them *regular* since they transpose all three indices of $\rho$, and each of them once.

They are:

| | |
|---|---|
| 1) $\mathcal{D}_0 = \quad e_{21} + e_{32} + e_{12}$ , | 5) $\mathrm{D}_0 = \quad e_{12} + e_{23} + e_{31}$ , |
| 2) $\mathcal{D}_1 = -e_{12} - e_{31} + e_{23}$ , | 6) $\mathrm{D}_1 = -e_{21} - e_{13} + e_{32}$ , |
| 3) $\mathcal{D}_2 = \quad e_{31} - e_{23} - e_{12}$ , | 7) $\mathrm{D}_2 = \quad e_{13} - e_{32} + e_{21}$ , |
| 4) $\mathcal{D}_3 = -e_{32} - e_{13} + e_{21}$ , | 8) $\mathrm{D}_3 = -e_{23} - e_{31} + e_{12}$ . |

$\mathcal{D}_0$ forms equal angles with the $OX$, $OY$, $OZ$ axes,
$\mathcal{D}_1$ " " " $OY$, $OX'$, $OZ$ axes,
$\mathcal{D}_2$ " " " $OX'$, $OY'$, $OZ$ axes,
$\mathcal{D}_3$ " " " $OY'$, $OX$, $OZ$ axes.

Direction $\mathrm{D}_n = -$ direction $\mathcal{D}_n$ .

(see Fig. 3)

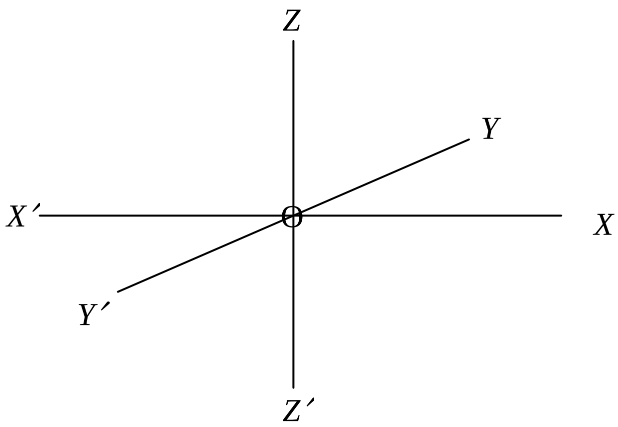



Figure 3.

One sees that:

$$\mathrm{D}_n = W\,\mathcal{D}_n\,,\; n = 0, 1, 2, 3,$$

and

$$\mathrm{D}_n = \mathcal{D}_n^{-1} = \mathcal{D}_n^{2}\,.$$

The operators $\mathcal{D}_n$, $\mathrm{D}_n$, $s_m$, 1; $n$ = 0, 1, 2, 3, $m$ = 1, 2, 3, form a group of twelve members of type 12 = 1(1) + 3(2) + 8(3). (See, § 13.)

## § 13. – The composed group.

The combination of the groups that were studied up to now contains a group of 24 members of type: 24 = 1(1) + 9(2) + 6(4) + 8(3).

(See the table on pp. 124 of the original article.)

# CHAPTER III

# Quaternary transformants.

### § 14. – Fundamental explanations.

$$t = \sum x_{mn}\, e_{mn}\,,\;\; m, n = 0, 1, 2, 3.$$

Above all, we are interested in the transformants that have the same composition as the Hamilton symbols $i$, $j$, $k$.

There are two series of such transformants:

$$\text{I)}\quad \mathcal{T}_1 = e_{32} - e_{23} + e_{01} - e_{10}\,, \qquad \text{II)}\quad \mathcal{T}_1' = e_{32} - e_{23} - e_{01} + e_{10}\,,$$

$$\mathcal{T}_2 = e_{12} - e_{31} + e_{02} - e_{20}\,, \qquad \mathcal{T}_2' = e_{13} - e_{31} - e_{02} + e_{20}\,,$$

$$\mathcal{T}_3 = e_{21} - e_{12} + e_{03} - e_{30}\,, \qquad \mathcal{T}_3' = e_{21} - e_{12} - e_{03} + e_{30}\,,$$

$$\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_3\,,\text{ etc.} \qquad \mathcal{T}_1'\mathcal{T}_2' = \mathcal{T}_3'\,,\text{ etc.}$$

$$\mathcal{T}_n^2 = -(e_{00} + e_{11} + e_{22} + e_{33}) = -1, \quad \text{and likewise} \quad \mathcal{T}_n'^2 = -1, \qquad n = 1, 2, 3.$$

As for the operand $r = x_0\, e_0 + x_1\, e_1 + x_2\, e_2 + x_3\, e_3$, we can interpret it in two ways: First, all four $e_n$ are vectors. Then, ($r = \dots$) is a point in four-dimensional space and ($r = 0$) is the origin.

On the contrary, if only one of the four quantities $e_n$ symbolizes a point then $r$ will be a point in tri-dimensional space.

We choose the simpler option and choose $e_0$ to be the origin, while $e_1$, $e_2$, $e_3$ are vectors in the coordinate directions.

In that way, a point with coordinates $x_1 / x_0$, $x_2 / x_0$, $x_3 / x_0$ will be represented by:

$$m = x_0\, e_0 + x_1\, e_1 + x_2\, e_2 + x_3\, e_3 = x_0\, e_0 + \rho.$$

### § 15. – The representation of the equations given by Olinde Rodrigues and Combebiac ([2], pp. 8, 11) for quaternary transformants.

We write $x'$, $y'$, $z'$ in place of $x$, $y$, $z$ ($x_1'$, $x_2'$, $x_3'$ in place of $x_1$, $y_1$, $z_1$, resp.), while appending $x_0$ and $x_0'$.

One will then have:

$$m' = tm,$$

$$m = \sum x_n\, e_n\,, \qquad m' = \sum x'_n \cdot e_n\,, \qquad n = 0, 1, 2, 3.$$

$$\begin{aligned} t \quad &= 2(\alpha_2\,\beta_3 - \alpha_3\,\beta_2 + \alpha_0\,\beta_1 - \alpha_1\,\beta_0)\,e_{10} + (\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2)\,e_{11} \\ &+ 2(\alpha_1\,\alpha_2 - \alpha_0\,\alpha_3)\,e_{12} + 2(\alpha_1\,\alpha_3 + \alpha_0\,\alpha_2)\,e_{13} + 2(\alpha_3\,\beta_1 - \alpha_1\,\beta_3 + \alpha_0\,\beta_2 - \alpha_2\,\beta_0)\,e_{20} \\ &+ 2(\alpha_2\,\alpha_1 + \alpha_0\,\alpha_3)\,e_{21} + (\alpha_0^2 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2)\,e_{22} + 2(\alpha_2\,\alpha_3 - \alpha_0\,\alpha_1)\,e_{23} \\ &+ 2(\alpha_1\,\beta_2 - \alpha_2\,\beta_1 + \alpha_0\,\beta_3 - \alpha_3\,\beta_0)\,e_{30} + 2(\alpha_3\,\alpha_1 - \alpha_0\,\alpha_2)\,e_{31} \\ &+ 2(\alpha_3\,\alpha_2 + \alpha_0\,\alpha_1)\,e_{32} + (\alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2)\,e_{33} + (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)\,e_{00}\,. \end{aligned}$$

The transformant $t$ represents a displacement whose axis has the coordinates:

$$\frac{p_{01}}{\alpha_1} : \frac{p_{02}}{\alpha_2} : \frac{p_{03}}{\alpha_3} : \frac{p_{23}}{\beta_1 + \dfrac{\alpha_0\beta_0\alpha_1}{\alpha_1^2+\alpha_2^2+\alpha_3^2}} : \frac{p_{31}}{\beta_2 + \dfrac{\alpha_0\beta_0\alpha_2}{\alpha_1^2+\alpha_2^2+\alpha_3^2}} : \frac{p_{12}}{\beta_3 + \dfrac{\alpha_0\beta_0\alpha_3}{\alpha_1^2+\alpha_2^2+\alpha_3^2}}\,,$$

where the rotation angle $2\vartheta$ and the shift along the axis $2\eta$ are given by the relations:

$$\cot\,\vartheta = \frac{\alpha_0}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}\,, \qquad \eta = -\,\frac{\beta_0}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}\,.$$

Upon annulling the $\beta_n$, $n = 0, 1, 2, 3$ in $t$, one will obtain the formulas of Olinde Rodrigues ([**2**], pp. 8), when transformed into transformants.

If $\lambda$, $\mu$, $\nu$ are the angles that the rotational axis makes with the coordinates axes, and $2\vartheta$ is the angle of rotation then one will have the following relations:

$$\cot\,\lambda = \frac{\alpha_1}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}\,, \qquad \cot\,\mu = \frac{\alpha_2}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}\,, \qquad \cot\,\nu = \frac{\alpha_3}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}\,,$$

$$\cot\,\vartheta = \frac{\alpha_0}{\sqrt{\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2}}\,.$$

## § 16. – Reflections.

I. – *A reflecting point* with the equation:

$$m = e_0 + a\,e_1 + b\,e_2 + c\,e_3$$

gives the relation:

$$\boxed{{}^{s}m = 2ae_{10} + 2be_{20} + 2ce_{30} - e_{00} - e_{11} - e_{22} - e_{33}.}$$

*Note*. – The “$s$” in “${}^{s}m$” is an upper index on the left side that is given the Latin name *speculum*; we shall use such indices for certain transformants.

It was by this line of intuition that we obtained the relation below.

Conclusions:

$$\begin{aligned}{}^{s}m_2 \cdot {}^{s}m_1 &= (2\, a_2\, e_{10} + \ldots)(2\, a_1\, e_{10} + \ldots) \\ &= 2(a_2 - a_1)\, e_{10} + 2(b_2 - b_1)\, e_{20} + 2(c_2 - c_1)\, e_{30} + 1;\end{aligned}$$

$$1 = e_{00} + e_{11} + e_{22} + e_{33}\,.$$

Moreover, the origin has the equation:

$${}^{s}m_0 = e_{00} - e_{11} - e_{22} - e_{33}\,,$$

as a reflecting point since $a = b = c = 0$.

II. – *The reflecting line.*

It follows from formulas ([**2**], pp. 11, 13) that for $\eta = 0$, $\vartheta = 90^{\circ}$, $2\vartheta = 180^{\circ}$, since cot $\vartheta = 0$ (cos $\vartheta = 0$, resp.):

$$\alpha_0 = 0 \qquad \text{and} \qquad \beta_0 = 0.$$

If one accepts, *a priori*, that:

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \cos^2\lambda + \cos^2\mu + \cos^2\nu = 1$$

then one will have $x'_0 = 1$, just as $x_0 = 1$, and if “$d$” has linear coordinates:

$$\frac{p_{01}}{\alpha_1} : \frac{p_{02}}{\alpha_2} : \frac{p_{03}}{\alpha_3} : \frac{p_{23}}{\beta_1} : \frac{p_{31}}{\beta_2} : \frac{p_{12}}{\beta_3}$$

then one will have:

$$m' = {}^{s}d \cdot m,$$

where:

$$m = e_0 + x_1\, e_1 + x_2\, e_2 + x_3\, e_3\,, \qquad m' = e_0 + x'_1 e_1 + x'_2 e_2 + x'_3 e_3\,,$$

and

$$
\begin{aligned}
&{}^s d = \sum a_{mn} e_{mn}, \quad m,n = 0,1,2,3,\\
&\text{where:}\\
&a_{11} = \alpha_1^2 - \alpha_2^2 - \alpha_3^2,\ a_{12} = 2\alpha_1\alpha_2, \quad a_{13} = 2\alpha_1\alpha_3, \quad a_{10} = 2(\alpha_2\beta_3 - \alpha_3\beta_2),\\
&a_{21} = 2\alpha_2\alpha_1, \quad a_{22} = -\alpha_1^2 + \alpha_2^2 - \alpha_3^2,\ a_{23} = 2\alpha_2\alpha_3, \quad a_{20} = 2(\alpha_3\beta_1 - \alpha_1\beta_3),\\
&a_{31} = 2\alpha_3\alpha_1, \quad a_{22} = 2\alpha_3\alpha_2, \quad a_{33} = -\alpha_1^2 - \alpha_2^2 + \alpha_3^2,\ a_{30} = 2(\alpha_1\beta_2 - \alpha_2\beta_1),\\
&a_{01} = a_{02} = a_{03} = 0, \quad a_{00} = 1.
\end{aligned}
$$

One gets $\beta_1 = \beta_2 = \beta_3 = 0$ for $d_0$, which is a line that passes through the origin, and if one considers that $\cos\lambda = \alpha_1$ , $\cos\mu = \alpha_2$ , $\cos\nu = \alpha_3$ , with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, then one will find that:

$$
\begin{aligned}
{}^s d_0 = {} & (\alpha_1^2 - \alpha_2^2 - \alpha_3^2)\ e_{11} + 2\alpha_1\alpha_2\ e_{12} + 2\alpha_1\alpha_3\ e_{13}\\
& + 2\alpha_2\alpha_1\ e_{21} + (-\alpha_1^2 + \alpha_2^2 - \alpha_3^2)\ e_{22} + 2\alpha_2\alpha_3\ e_{23}\\
& + + 2\alpha_3\alpha_1\ e_{31} + 2\alpha_3\alpha_2\ e_{32} + (-\alpha_1^2 - \alpha_2^2 + \alpha_3^2)\ e_{33} + e_{00}\ .
\end{aligned}
$$

III. – *The reflecting plane.*

It suffices to choose a point $m$ on a line at a distance from the origin:

$$a\ e_1 + b\ e_2 + c\ e_3\ , \qquad a = h\alpha_1\ , \qquad b = h\alpha_2\ , \qquad c = h\alpha_3\ ,$$

because then:

$${}^s p = {}^s d_0 \cdot {}^s m = {}^s m \cdot {}^s d_0\ .$$

$p$ is a plane that is perpendicular to the line $d_0$ that passes through the point $m$.

$$
\begin{aligned}
{}^s m \cdot {}^s d_0 = {}^s p = {} & 2ae_{10} + 2be_{20} + 2ce_{30} + e_{00}\\
& - (\alpha_1^2 - \alpha_2^2 - \alpha_3^2)\, e_{11} - 2\alpha_2\alpha_1 e_{21} - 2\alpha_3\alpha_1 e_{31}\\
& - 2\alpha_1\alpha_2 e_{12} - (-\alpha_1^2 + \alpha_2^2 - \alpha_3^2)\, e_{22} - 2\alpha_3\alpha_2 e_{32}\\
& - 2\alpha_1\alpha_3\, e_{13} - 2\alpha_2\alpha_3 e_{23} - (\alpha_1^2 - \alpha_2^2 + \alpha_3^2)\, e_{33}.
\end{aligned}
$$

${}^s d_0 \cdot {}^s m$ gives the same result, because the coefficient $a_{10}$ of $\sum a_{mn}\, e_{mn}$ , which is equal to:

$$2a\ (\alpha_1^2 - \alpha_2^2 - \alpha_3^2)\ + 2b \cdot 2\alpha_1\ \alpha_2 + 2c \cdot 2\alpha_1\ \alpha_3\ ,$$

will then become:

$$2h\, (\alpha_1^3 - \alpha_1\alpha_2^2 - \alpha_1\alpha_3^2 + 2\alpha_1\alpha_2^2 + 2\alpha_1\alpha_3^2)$$

$$= 2h(\alpha_1^3 + \alpha_1\alpha_2^2 + \alpha_1\alpha_3^2) = 2h\alpha_1(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) = 2h\alpha_1 = 2a.$$

One has similar expressions for $a_{20}$ and $a_{30}$ .

The multiplication of the factors ${}^s m$, ${}^s d$, ${}^s p$, … in arbitrary quantities (the operations are associative) permits one to form various dynames. (See [**6**].)

For example, ${}^s m\, {}^s p$, or ${}^s p\, {}^s m$ (in German: "Quirl," [**6**], pp. 59, 60, 63, 165), represents an ascending rotational motion of $180^{\circ}$. Two parallel lines or two parallel planes or two points express a translatory motion. Two arbitrary planes effect a rotation around the line of intersection, etc.

*Note*. – The multiplication of $(t\, t_1 \ldots t_n)$ does not need to be realized, since

$$(t\, t_1 \ldots t_n)\, m = t\, [t_1 \ldots (t_n\, m)].$$

### § 16 a). – *(Continued)*.

If the reflecting point $e_m$ is found in the plane ${}^s p$ then one knows that ${}^s m \cdot {}^s p = {}^s p \cdot {}^s m = {}^s d$, ${}^s d \perp {}^s p$ passes through ${}^s m$. If three planes are mutually perpendicular: i.e., $p_1 \perp p_2 \perp p_3 \perp p_1$ then the lines of intersection will be, as well:

$$d_1 \perp d_2 \perp d_3 \perp d_1 \qquad \text{and} \qquad d_1 \perp p_2 ,\ d_2 \perp p_3 ,\ d_3 \perp p_3 .$$

"${}^s m$" is the point of intersection.

One then has:

$${}^s m \cdot {}^s d_n = {}^s d_n \cdot {}^s m = {}^s p_n ,$$
$${}^s m \cdot {}^s p_n = {}^s p_n \cdot {}^s m = {}^s d_n , \qquad {}^s p_n \cdot {}^s d_n = {}^s d_n \cdot {}^s p_n = {}^s m , \qquad n = 1, 2, 3.$$

This gives an Abelian group with eight members, because:

$${}^s m^2 = {}^s d_n{}^2 = {}^s p^2 = 1.$$

Take the simplest case: viz., "$m$" is the origin, $d_n$ are the coordinate axes, and $p_n$ are the coordinate planes. It will then follow from the Olinde Rodrigues equations, or direct calculation, that the effect of reflection on $e_0$, $e_1$, $e_2$, $e_3$ is:

$${}^s m = e_{00} - e_{11} - e_{22} - e_{33} ,$$

$$\begin{aligned} {}^s p_1 &= e_{00} - e_{11} + e_{22} + e_{33} , & {}^s d_1 &= e_{00} + e_{11} - e_{22} - e_{33} , \\ {}^s p_2 &= e_{00} + e_{11} - e_{22} + e_{33} , & {}^s d_2 &= e_{00} - e_{11} + e_{22} - e_{33} , \\ {}^s p_3 &= e_{00} + e_{11} + e_{22} - e_{33} , & {}^s d_3 &= e_{00} - e_{11} - e_{22} + e_{33} . \end{aligned}$$

## § 17. – Transformants that are isomorphic with their operands.

Up to now, the only questions to be addressed were those of the geometric addition of the operands $\sum x_n\, e_n$ and the multiplication of them by transformants; there has been no agreement on the composition rule for the operands themselves; i.e., $e_m\, e_n$ . It can obey various rules, such as those of Grassmann or Hamilton. In the latter case, we will write "$h_n$", instead of "$e_n$" – where "$h$" should suggest then name "$h$(amilton)" – or also "$\lambda_n$".

Furthermore, $h_0$, $h_1$, $h_2$, $h_3$ correspond to 1, $i$, $j$, $k$, respectively. Meanwhile, it must be remarked that $h_0$ can be supplied with the unit "1" only in conjunction with $h_n$, and never in conjunction with the transformants $h_{mn}$ , where:

$$\text{“}h_{00} + h_{11} + h_{22} + h_{33} = 1\text{”} \qquad \text{and} \qquad \text{“}h_{n0} \cdot h_0 = h_n\text{”}.$$

We let "${}^0h_n$" denote the isomorphic transformant if it is found before "$h_n$" – i.e., a *pre-positive transformant* – and by "${}^*h_n$" if it is found after "$h_n$" – i.e., a *post-positive transformant.*

For the inversion of the indices $W \cdot {}^0h_n$ ($W \cdot {}^*h_n$, resp.), we write briefly "${}^{\underline{0}}h_m$" ("${}^{\underline{*}}h_m$", resp.).

Here is an overview of the pre-positive and post-positive isomorphic transformants:

$$1 = h_{00} + h_{11} + h_{22} + h_{33} = {}^0h_0 = {}^*h_0 \,,$$

$$\begin{aligned}
{}^0h_1 &= h_{32} - h_{23} + h_{10} - h_{01}\,, & {}^*h_1 &= h_{32} - h_{23} - h_{10} + h_{01}\,,\\
{}^0h_2 &= h_{13} - h_{31} + h_{20} - h_{02}\,, & {}^*h_2 &= h_{13} - h_{31} - h_{20} + h_{02}\,,\\
{}^0h_3 &= h_{21} - h_{12} + h_{30} - h_{03}\,, & {}^*h_3 &= h_{21} - h_{12} - h_{30} + h_{03}\,.
\end{aligned}$$

One sees that ${}^0h_n$ are identical with $\mathcal{T}_n'$, and ${}^*h_n$, with $\mathcal{T}_n$ , § 14.
One confirms the following rules by calculation:

1) $h_m\, h_n = {}^0h_m \cdot h_n = h_m \cdot {}^*h_n = {}^{\underline{*}}h_n \cdot h_m$ ,
2) ${}^0h_m \cdot {}^*h_n = {}^*h_n \cdot {}^0h_m$ , ${}^0(h_m\, h_n) = {}^0h_m \cdot {}^0h_n$ , ${}^*(h_m\, h_n) = {}^*h_m \cdot {}^*h_n$ ,
3) $({}^{\underset{\sim}{0}}h_m \cdot h_n) \cdot {}^*h_s = {}^0h_m\, (h_m \cdot {}^*h_n) = ({}^{\underset{\sim}{0}}h_m \cdot {}^{\underline{*}}h_s) \cdot h_n$ .

By contrast, for the non-isomorphic transformants, one will have $(t\, e_m)\, t_1 \neq t(e_m\, t_1)$, in general.

4) $({}^*h_n)^{-1} = {}^{\underline{*}}h_n$ and $({}^0h_n)^{-1} = {}^{\underline{0}}h_n$ .

In order for this to be true, one must have:

$$\begin{aligned}
{}^0h_1 \cdot {}^*h_1 &= {}^sd_1\,,\\
{}^0h_2 \cdot {}^*h_2 &= {}^sd_2\,,\\
{}^0h_3 \cdot {}^*h_3 &= {}^sd_3 \qquad \text{(see § 16a)},
\end{aligned}$$

since "$({}^0h_1 \cdot {}^*h_1) \cdot h_n$" is derived from "${}^0h_1 \cdot h_n \cdot ({}^*h_1)^{-1}$".

We have thus arrived at two ways of approaching isomorphic transformants: first, based upon the same composition, and second, by concluding from the effect of the transformants on the operands. We then arrive at a third method: developing the transformants from the Cayley square for the group $h_n$, $n = 0, 1, 2, 3$ (properly speaking, the first quadrant of the square, since most of the members are distinguished from the other ones only by their signs).

| | $h_0, 1$ | $h_1, i$ | $h_2, j$ | $h_3, k$ |
|---|---|---|---|---|
| $h_0, 1$ | $h_0$ | $h_1$ | $h_2$ | $h_3$ |
| $h_1, i$ | $h_1$ | $-h_0$ | $h_3$ | $-h_2$ |
| $h_2, j$ | $h_2$ | $-h_3$ | $-h_0$ | $h_1$ |
| $h_3, k$ | $h_3$ | $h_2$ | $-h_1$ | $-h_0$ |

We denote the following the operations with a semi-colon:

$$h_m \,;\, h_n = h_{mn} \,.$$

From this, one then has:

$${}^0h_s = \textstyle\sum' h_m \,;\, h_n = \sum' h_{mn} = \displaystyle\sum_n h_s\, h_n ; h_n \,,$$

where:

$s$ = index of the left column that is exterior to the row,
$m$ = index of the squares interior to the row $s$,
$n$ = index in the upper row, corresponding to $m$, in such a way that:

$$\text{“}h_s\, h_n = h_m\text{”}.$$

One must be careful with the signs in this.

One finds the post-positive transformants in the same manner, except that one must switch the row with the column; namely:

$${}^*h_s = \textstyle\sum'' h_m \,;\, h_n = \sum'' h_{mn} = \displaystyle\sum_m h_m ; h_m h_s \,,$$

where

$s$ = index in the upper row of the column,
$m$ = indices in the exterior left column,
$n$ = indices in the interior squares of the column $s$, in such a manner that:

$$h_m\, h_s = h_n \,.$$

This method is applicable to all of the other groups.

The composition of the quantities ${}^0h_m$ and ${}^*h_n$ will result in a group of 32 members of the type: $32 = 1(1) + 19(2) + 12(4)$, with a subgroup of operators:

$$\pm\,(1,\ {}^0h_1,\ {}^*h_2,\ {}^0h_1 \cdot {}^*h_2,\ \ {}^0h_2 \cdot {}^*h_1,\ \ {}^0h_2 \cdot {}^*h_3,\ \ {}^0h_3 \cdot {}^*h_1,\ {}^0h_3 \cdot {}^*h_3).$$

## § 18. – The transformants of higher degree.

One can deduce "transformants of transformants" – or *second-degree* transformants – from the group of 32 members that was mentioned above in the same manner. To that end, we write the operators ${}^0h_m \cdot {}^*h_n$ in that sequence as $h_{(mn)}$, and ${}^0h_m = {}^0h_m \cdot {}^*h_0 = h_{(m0)}$, ${}^*h_n = {}^0h_0 \cdot {}^*h_n = h_{(0n)}$. Since "${}^*h_n$" and "${}^0h_r$" commute, a product $({}^0h_m \cdot {}^*h_n) \cdot ({}^0h_r \cdot {}^*h_s)$ will be equal to ${}^0h_p \cdot {}^*h_q = h_{(pq)}$. Therefore, the desired transformant will have the form: ${}^0h_{(mn)} = \sum' h_{(pq)(rs)}$, which is a polynomial with 16 terms, etc.

On the other hand, one can derive ${}^0(h_{mn})$ [${}^*(h_{mn})$, resp.] directly from the square:

| | $h_{00}$ | $h_{01}$ | $h_{02}$ | $h_{03}$ | $h_{10}$ | $h_{11}$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $h_{01}$ | $h_{10}$ | $h_{11}$ | $h_{12}$ | $h_{13}$ | 0 | 0 | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

For example:

$${}^0(h_{10}) = h_{10,00} + h_{11,01} + h_{12,02} + h_{13,03}\,,$$

which is a polynomial with four terms.

One must therefore distinguish ${}^0h_{(mn)}$ from ${}^*h_{mn}$ [${}^*h_{(mn)}$ from ${}^*h_{mn}$, resp.], and furthermore, from ${}^0(h_{mn})$ [${}^*(h_{mn})$, resp.].

Meanwhile, since:

$$h_{(10)} = {}^0h_1 = h_{32} - h_{23} + h_{10} - h_{01}\,,$$

${}^0h_{(10)} = {}^{00}h_1$ can also be represented by:

$${}^0h_{32} - {}^0h_{23} + {}^0h_{10} - {}^0h_{01}\,.$$

One likewise gets ${}^{*0}h_n$, ${}^{0*}h_n$, and ${}^{**}h_n$, upon replacing $h$ with ${}^0h$ [${}^*h$, resp.] in the formulas for ${}^0h_n$ [${}^*h_n$, resp.].

Therefore, ${}^0h_{mn} = {}^0h_m\,;\,{}^0h_n$, etc. (see, § 22a).

## § 19. – The elective transformants.

If one has a quaternion:

$$q = a_0\,h_0 + a_1\,h_1 + a_2\,h_2 + a_3\,h_3$$

then "$h_{00}$" will have the same effect on the quaternion as the Hamilton symbol "$S$".

(Scalar):

$$h_{00}\,q \equiv S\,q = a_0\,h_0\,.$$

In the same fashion, "$h_{11} + h_{22} + h_{33}$" is identical with "$V$" (vector).

Moreover:

$$K(\text{conjugate}) \equiv h_{00} - h_{11} - h_{22} - h_{33}$$

and

$$K^2 \equiv h_{00} + h_{11} + h_{22} + h_{33} = 1.$$

One verifies, in turn, that:

$$SS = S, \qquad VV = V, \qquad KS = SK = S, \qquad KV = VK = -V,$$

$$SV = VS = 0. \qquad \text{([}\mathbf{1}\text{], pp. 43)}$$

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# CHAPTER IV

## Bi-quaternary and quadri-quaternary transformants, Polar and axial vectors. Mu-nu-group and mu-lambda-group. The isomorphic transformants that are derived from these two groups.

### § 20. – Polar and axial vectors.

The discussions up to now have given no motive for distinguishing the two kinds of vectors. At present, we would like to denote the former by $\nu_1$, $\nu_2$, $\nu_3$ and the latter by $\lambda_1$, $\lambda_2$, $\lambda_3$, both of which are in the coordinate directions with the tensor 1; $\nu_0$ ($\lambda_0$, resp.) are then first-order operators.

One must enforce the following rule for their composition ([**7**], pp. 23):

*The vectorial product of two polar vectors, as well as two axial vectors is an axial vector.*

*The vectorial product of a polar vector and an axial vector is a polar vector.*

Therefore, $\lambda_n$ relates to $\nu_n$ in the same way that "+1" does to "–1," or as a real number does to an imaginary number.

From this, one has:

$$\begin{aligned} &\lambda_1 \lambda_2 = \lambda_3 = -\lambda_2 \lambda_1, && \nu_1 \nu_2 = \nu_3, && \lambda_1^2 = -1, \\ &\lambda_2 \lambda_3 = \lambda_1 = -\lambda_3 \lambda_2, && \nu_2 \nu_3 = \nu_1, && \lambda_2^2 = -1, \\ &\lambda_3 \lambda_1 = \lambda_2 = -\lambda_1 \lambda_3, && \nu_3 \nu_1 = \nu_2, && \lambda_3^2 = -1. \end{aligned}$$

Moreover ([**7**], pp. 23):

*Under multiplication by a pseudo-scalar, an axial vector will become a polar vector, and vice versa, a polar vector will become an axial vector.*

This "pseudo-scalar" will be denoted by "$l$". We will show that "$l$" is not presumed to be a scalar, nor a pseudo-scalar, but a true second-order complex operator that has some properties in common with scalars. ("$l$" commutes with the two vectors, and its isomorphic transformant has a constitution that is similar to that of the transformant $t$ = 1.)

One has:

$$\begin{aligned} &l\,\nu_1 = \lambda_1, && l\,\lambda_1 = \nu_1, \\ &l\,\nu_2 = \lambda_2, && l\,\lambda_2 = \nu_2, \\ &l\,\nu_3 = \lambda_3, && l\,\lambda_3 = \nu_3. \end{aligned}$$

From this:

1) $l \cdot l \cdot \lambda_1 = l\, \nu_1 = \lambda_1$ , $\boxed{l^2 = 1,}$

2) $\nu_1\, \nu_2 = l\lambda_1 \cdot l\lambda_1 = \lambda_3 = \lambda_1\, \lambda_2 = l(\lambda_1\, l)\, \lambda_2 = l(l\lambda_1)\, \lambda_2$ .

Therefore:

$\boxed{\lambda_1\, l = l\lambda_1;}$ and, in general: $\boxed{\lambda_n\, l = l\lambda_n, \quad n = 1, 2, 3.}$

3) $l\lambda_1 \cdot l\lambda_1 = \boxed{\nu_1\, \nu_2 = -1,}$ because $l\lambda_1 \cdot l\lambda_1 = ll\lambda_1\lambda_2$ ,

4) $l\lambda_2 \cdot l\lambda_1 = \nu_2\, \nu_1 = ll\, \lambda_2\lambda_1 = -\lambda_3 = -\nu_1\, \nu_2$ , $\boxed{\nu_2\, \nu_1 = -\nu_1\, \nu_2 .}$

*Note.* – According to Hamilton, $\lambda_n^2 = -1$. Abraham ([**7**], pp. 14) writes the scalar product $\lambda_n^2 = +1$, so one must write strictly $S_A \cdot \lambda_n^2 = +1$, while for Hamilton's school, $S_H \cdot \lambda_n^2 = -1$, which is naturally permitted. Then, $S_A \equiv -\lambda_{00}$, and by contrast, $S_H \equiv +\lambda_{00} = -S_A$ .

With that, the composition of the operators ± (1, $l$, $\lambda_1$, $\lambda_2$, $\lambda_3$, $\nu_1$, $\nu_2$, $\nu_3$) is well-defined and gives a group of sixteen members of the type 16 = 1(1) + 3(2) + 12(4).

One must keep in mind that $\lambda_n$ , $n$ = 0, 1, 2, 3 forms a group by itself, while this is not the case with $\pm\, \nu_n$ ; $\pm\, \nu$ will form a group only in conjunction with $\pm\, \lambda_n$ .

The corresponding isomorphic transformants contain eight different indices and can be referred to as *bi-quaternary*.

However, we shall not dwell on this, since we wish to envision quadri-quaternary transformants.

## § 21. – Mu-nu-group and mu-lambda-group.

If we add the origin $\pm\, \mu \equiv \pm\, \nu_4$ to $\pm\, \nu_m$ , $\pm\, \lambda_n$ then we will double the number of operators.

We are in agreement with Grassmann, for whom:

$$e_0\, (1 + |\, )\, e_0 = e_0\, e_0 + e_0\, |\, e_0 = 0 + 1 = 1,$$

as well as with Combebiac, for whom $\mu^2 = 1$, if we set $\mu^2 = \nu_4^2 = 1$.

The other rules of composition follow from a prior comparison with the Grassmann symbols.

Once more, let $e_0$ be the origin, and by contrast, $e_1$, $e_2$, $e_3$ must be the terminal points of the vectors that were denoted by these letters above (§ 14). For now, the latter will be $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ and $\varepsilon_0 \equiv e_0$ .

Then:

$$\varepsilon_1 = e_1 - e_0 ,$$

$$\varepsilon_2 = e_2 - e_0\,,$$
$$\varepsilon_3 = e_3 - e_0\,.$$

*N. B.* – Grassmann denoted the direction of $e_n$ by “$e_0 - e_n$”, which is merely a question of convention.

One then has:

$$\nu_1 \equiv \varepsilon_1 = -\,e_0 + e_1\,,$$
$$\nu_2 \equiv \varepsilon_2 = -\,e_0 + e_2\,,$$
$$\nu_3 \equiv \varepsilon_3 = -\,e_0 + e_3\,.$$

One can then pose the equations:

$$\varepsilon_1\,\varepsilon_2 \equiv (e_1 - e_0)(e_2 - e_0) = e_1\,e_2 - e_0\,e_2 - e_1\,e_0 + e_0\,e_0 = e_0\,e_1 + e_1\,e_2 + e_2\,e_0 \neq (e_3 - e_0)$$

only along with $\nu_1 \nu_2 = \lambda_3 \neq \nu_3$ .

We will use the synonymous expressions “polar vector,” “difference of two points,” “points at infinity” for the symbols $\nu_1,\ \nu_2,\ \nu_3$ . A polar vector permits a displacement parallel to itself in all of tri-dimensional space.

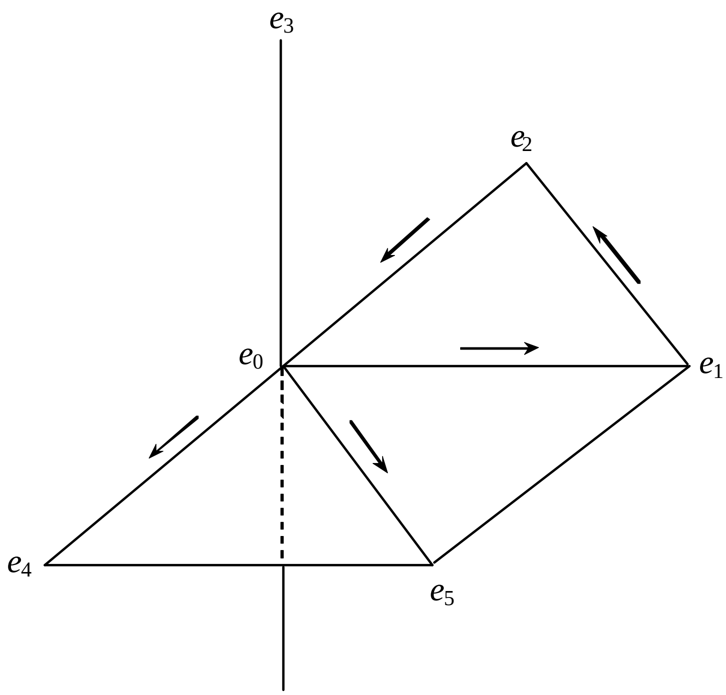


Figure 4.

As for $\lambda_n$ with $n$ = 1, 2, 3, $\lambda_3$, for example, is equal to $e_0\ e_1\ +\ e_1\ e_2\ +\ e_2\ e_0$ , namely, a sum of segments of three lines that form a closed figure that is a triangle that is traversed in the positive sense, as seen from $e_2$ . (See Fig. 4).

The segments of the fixed lines can have a translatory displacement only along their lines. They permit a conversion into a parallelogram:

$$= e_1 e_2 + e_0 e_3 = e_1 e_2 - e_3 e_0,$$

which is a sum of two anti-parallel segments or the difference of two parallel segments that have the same lengths.

Since $e_2\ e_0 = e_0\ e_4$, and according to the rule for geometric addition, one will have $e_0\ e_1 + e_0\ e_4 = e_0 e_5$ . This parallelogram can be converted into another parallelogram in its plane or in a parallel plane under the single condition that its tensor, which is equal to the twice the area of the triangle $e_0\ e_1\ e_2$, remain unaltered.

Ultimately, these parallelograms can be identified with the unique closed line in their planes, with the line of intersection of the planes parallel to the parallelograms; i.e., with the line at infinity.

The Hamilton symbols *i*, *j*, *k* correspond to these parallelograms exactly, as their composition shows; they are thus exclusively axial vectors.

When Hamilton used lines perpendicular to the parallelograms in place of them, with lengths that were equal to the areas of the latter, they were *auxiliary* axial vectors, which are very useful for certain constructions, but ultimately they must be converted into parallelograms again.

We use the expressions “axial vector,” “parallelogram,” difference of two segments,” line at infinity.”

The other rules of composition are easily obtained by starting with the following considerations:

1) $\mu\, v_1 = e_0\,(e_1 - e_0) = e_0\, e_1 = \delta_1$ ,

so a segment that is part of a fixed line will move only along the line that carries it.

Moreover:

$$v_1\, \mu = e_0\,(e_1 - e_0) = e_1\, e_0 = -\, e_0\, e_1\ ,$$

so

$$\boxed{\mu v_1 = -v_1 \mu,}$$

and similarly for the other indices.

2) $\mu\, \lambda_3 = e_0\,(e_0\, e_1 + e_1\, e_2 + e_2\, e_0) = e_0\, e_1\, e_2$ ,

which is part of a fixed plane = $\pi_3$ that moves in the plane that carries it.

$$\lambda_3\, \mu = (e_0\, e_1 + e_1\, e_2 + e_2\, e_0)\, e_0 = e_1\, e_2\, e_0 = e_0\, e_1\, e_2\,;$$

thus:

$$\boxed{\mu \lambda_3 = \lambda_3 \mu,}$$

as with the school of Combebiac.

One will find the corresponding equations by a circular permutation of the indices 1, 2, 3.

3) All that remains to be discussed is “$\mu\, l$.” The geometric significance of “$l$” results from the following reflection:

$$l\, \lambda_3\, \lambda_3 = v_3\, \lambda_3 = -\, l = (e_3 - e_0)(e_0 e_1 + e_1 e_2 + e_2 e_0) = e_3 e_0 e_1 + e_3 e_1 e_2 + e_3 e_2 e_0 - e_0 e_1 e_2\ ,$$
$$l = e_0\, e_1\, e_2 + e_0\, e_2\, e_3 + e_0\, e_3\, e_1 + e_3\, e_2\, e_1\ ,$$

(See Fig. 5)

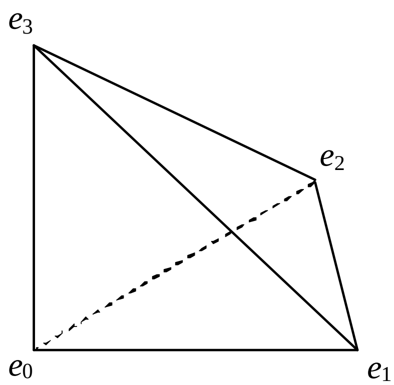

Figure 5

From this, "$l$" will represent the surface of the tetrahedron $e_0e_1e_2e_3$. The figure is displaceable, translatory, and rotatory in all of space, and equatable with the plane at infinity. Since the trinomial $e_0\, e_2\, e_3 + e_0\, e_3\, e_1 + e_3\, e_2\, e_1$ can be transformed in a plane parallel to $e_0e_1e_2$ with the same area and the opposite position. "$l$" will therefore be the difference between two fixed parallel planes that have the same area and can thus be equated with the plane at infinity, which is the unique closed plane that is a surface of the tetrahedron.

The two parallel planes delimit a cube with six (3! = 6) times the volume of the tetrahedron $e_0e_1e_2\, e_3$ . This volume is the tensor of the figure.

One finally has:

$$\boxed{\mu l} = -\,\mu \cdot \nu_1\, \lambda_1 = +\,\nu_1\, \lambda_1 \cdot \mu = \boxed{-\,\mu l} = -\,\Psi.$$

One obtains the same result by the Grassmann algebra, since:

$$l\mu = (e_0\, e_1\, e_2 + e_0\, e_2\, e_3 + e_0\, e_3\, e_1 + e_3\, e_2\, e_1)\; e_0 = e_3\, e_2\, e_1 e_0\,,$$

and

$$\mu\, l = e_0\; (e_0\, e_1\, e_2 + e_0\, e_2\, e_3 + e_0\, e_3\, e_1 + e_3\, e_2\, e_1) = e_0\, e_3\, e_2\, e_1 = -\, e_3\, e_2\, e_1 e_0\,.$$

$l\,\mu = \Psi$ is then a fourth-order, three-dimensional quantity; i.e., a geometric solid.

Unlike Grassmann, we do not set $e_0\, e_1\, e_2\, e_3$ equal to the unity directly, but set it equal to $\Psi$; however, $\Psi^4 = 1$.

At present, one can construct a Cayley square for this group of 32 members. In order to do that, it will naturally suffice to consider just the first quadrant of the square. [See Table I. (*)]

One can denote the operators of the aforementioned group by just one letter that has sixteen different indices – for example, $\pm\, l_n$ , $n$ = 0, 1, …, 15.

In another manner, one can denote these operators by two letters, each of which has four indices – for example, $\mu_m\ \nu_n$ or $\mu_m\ \lambda_n$ ; $m$, $n$ = 0, 1, 2, 3.

We shall appeal to this latter method in order to avoid indices that appear twice.

By analogy with compound words, we call $\mu_m$ a *determinative operator* and $\nu_n$ ($\lambda_n$, resp.) a *primitive operator*.

At any moment, one can pass to one letter with sixteen indices by writing:

$$\mu_m \nu_n = \nu_{(mn)}\,, \qquad\qquad [\mu_m\lambda_n = \lambda_{(mn)}, \text{ resp.}],$$

where $(mn) = 4m + n$, $m$, $n$ = 0, 1, 2, 3. In this sequence:

$$\nu_n \cdot \mu_m = \mu_r \nu_s = \nu_{(rs)}\,.$$

(*) DHD: The Table is on page 139 of the original article.

Moreover, $\mu_0$, $\mu_1$, $\mu_2$, $\mu_3$ correspond to 1, $\mu$, $\Psi$, $l$, respectively. For example, one then has $\mu_1 \nu_0 = \nu_{(10)} = \nu_4$, which is the origin, while $\nu_n$ and $\lambda_n$ preserve their previous significance. One will then have:

$$\mu_0 \nu_0 \equiv \mu_0 \lambda_0 \equiv 1, \qquad \mu_0 \nu_n \equiv \nu_n , \qquad \mu_0 \lambda_n \equiv \lambda_n ,$$

$$\mu_m \nu_0 \equiv \mu_m \lambda_0 \equiv \mu_m, \qquad \mu_1 \equiv \mu , \qquad \mu_2 \equiv \Psi, \qquad \mu_3 \equiv l,$$

$$\mu_1 \nu_m \equiv \delta_m , \qquad \mu_2 \nu_m \equiv - \pi_m , \quad \mu_3 \nu_m \equiv \lambda_m , \qquad m = 1, 2, 3.$$

Moreover:

$$\mu_1 \nu_m = \mu_1 l \cdot l\ \nu_m = - \Psi \cdot \lambda_m = - \mu_2 \lambda_m ,$$

$$\mu_2 \nu_m = \mu_2 l \cdot l\ \nu_m = - \mu \cdot \lambda_m = - \mu_2 \lambda_m ,$$

$$\mu_3 \nu_m = \mu_3 l \cdot l\ \nu_m = ll \cdot \lambda_m = \mu_0 \lambda_m ,$$

$$\mu_0 \nu_m = \mu_0 l \cdot l\ \nu_m = l \cdot \lambda_m = \mu_3 \lambda_m .$$

[See Table II and III (*).]

One advantage of the $\mu$-$\lambda$-group, which is opposite to the $\mu$-$\nu$-group, is that all the $\mu_m$ commute with all the $\lambda_m$ .

## § 22. – The pre-positive and post-positive isomorphic transformants that are derived from Tables II and III.

Since the various letters ($\mu$, $\nu$ [$\mu$, $\lambda$, resp.]) commute with their indices, the indices of equal letters preserve their sequence; one can stipulate the rule:

$$\boxed{\mu_m \nu_n ; \mu_r \nu_s = (\mu_m ; \mu_r)(\nu_n ; \nu_s), \qquad [\mu_m \lambda_n ; \mu_r \lambda_s = (\mu_m ; \mu_r)(\lambda_n ; \lambda_s), \text{resp.}].}$$

(*) Tables II and III are on pages 140 and 141, resp. of the original article. The comment at the bottom of both pages translates as: "The interior squares refer to only the indices in the sequence $\mu$, $\nu$, and their signs."

Table IIa). *Pre-positive transformants.*

It results from Table II that:

$$
\begin{aligned}
&{}^0(\mu_0\ \nu_0) = (\ \ \mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\ (\nu_{00} + \nu_{11} + \nu_{22} + \nu_{33}),\\
&{}^0(\mu_1\ \nu_0) = (\ \ \mu_{01} + \mu_{10} - \mu_{23} - \mu_{32})\ (\quad \text{“} \quad ),\\
&{}^0(\mu_2\ \nu_0) = (-\mu_{02} + \mu_{20} + \mu_{31} - \mu_{13})\ (\quad \text{“} \quad ),\\
&{}^0(\mu_3\ \nu_0) = (\ \ \mu_{03} + \mu_{30} + \mu_{12} + \mu_{21})\ (\quad \text{“} \quad );
\end{aligned}
$$

$$
\begin{aligned}
&{}^0(\mu_0\ \nu_1) = (\mu_{00} - \mu_{11} - \mu_{22} + \mu_{33})(\nu_{10} - \nu_{01}) + (\ \ \mu_{30} + \mu_{03} + \mu_{12} + \mu_{21})\ (\nu_{32} - \nu_{23}),\\
&{}^0(\mu_1\ \nu_1) = (\mu_{10} - \mu_{01} - \mu_{23} + \mu_{32})(\quad \text{“} \quad ) + (\ \ \mu_{02} - \mu_{20} - \mu_{31} + \mu_{13})\ (\quad \text{“} \quad ),\\
&{}^0(\mu_2 \nu_1) = (\mu_{20} + \mu_{02} - \mu_{31} - \mu_{13})(\quad \text{“} \quad ) + (-\mu_{01} - \mu_{10} + \mu_{32} + \mu_{23})\ (\quad \text{“} \quad ),\\
&{}^0(\mu_3 \nu_1) = (\mu_{30} + \mu_{03} - \mu_{12} - \mu_{21})(\quad \text{“} \quad ) + (\ \ \mu_{00} - \mu_{11} + \mu_{22} + \mu_{33})\ (\quad \text{“} \quad );
\end{aligned}
$$

$$
\begin{aligned}
&{}^0(\mu_0 \nu_2) = (\mu_{00} - \mu_{11} - \mu_{22} + \mu_{33})(\nu_{20} - \nu_{02}) + (\ \ \mu_{30} + \mu_{03} + \mu_{12} + \mu_{21})\ (\nu_{13} - \nu_{31}),\\
&{}^0(\mu_1 \nu_2) = (\mu_{10} - \mu_{01} - \mu_{23} + \mu_{32})(\quad \text{“} \quad ) + (\ \ \mu_{02} - \mu_{20} - \mu_{31} + \mu_{13})\ (\quad \text{“} \quad ),\\
&{}^0(\mu_2\ \nu_2) = (\mu_{20} + \mu_{02} - \mu_{31} - \mu_{13})(\quad \text{“} \quad ) + (-\mu_{01} - \mu_{10} + \mu_{32} + \mu_{23})\ (\quad \text{“} \quad ),\\
&{}^0(\mu_3 \nu_2) = (\mu_{30} + \mu_{03} - \mu_{12} - \mu_{21})(\quad \text{“} \quad ) + (\ \ \mu_{00} - \mu_{11} + \mu_{22} + \mu_{33})\ (\quad \text{“} \quad );
\end{aligned}
$$

$$
\begin{aligned}
&{}^0(\mu_0 \nu_3) = (\mu_{00} - \mu_{11} - \mu_{22} + \mu_{33})(\nu_{30} - \nu_{03}) + (\ \ \mu_{30} + \mu_{03} + \mu_{12} + \mu_{21})\ (\nu_{21} - \nu_{12}),\\
&{}^0(\mu_1 \nu_3) = (\mu_{10} - \mu_{01} - \mu_{23} + \mu_{32})(\quad \text{“} \quad ) + (\ \ \mu_{02} - \mu_{20} - \mu_{31} + \mu_{13})\ (\quad \text{“} \quad ),\\
&{}^0(\mu_2 \nu_3) = (\mu_{20} + \mu_{02} - \mu_{31} - \mu_{13})(\quad \text{“} \quad ) + (-\mu_{01} - \mu_{10} + \mu_{32} + \mu_{23})\ (\quad \text{“} \quad ),\\
&{}^0(\mu_3 \nu_3) = (\mu_{30} + \mu_{03} - \mu_{12} - \mu_{21})(\quad \text{“} \quad ) + (\ \ \mu_{00} - \mu_{11} + \mu_{22} + \mu_{33})\ (\quad \text{“} \quad ).
\end{aligned}
$$

One easily shows that $\mu_m \nu_n\,;\,\mu_r \nu_s$ gives the same results as $(\mu_m;\,\mu_r)(\nu_n;\nu_s)$.

1) $\mu_m \nu_n\,;\,\mu_r \nu_s = \nu_{(mn)}\,;\,\nu_{(rs)} = \nu_{(mn)\,(rs)}$ .

$\nu_{(mn)\,(rs)} \cdot \nu_{(rs)} = \nu_{(mn)} = \mu_m \nu_n$ ;

2) $(\mu_m;\,\mu_r)(\nu_n;\nu_s) = \mu_{mr} \cdot \nu_{ns}$ ,

$\mu_{mr} \cdot \nu_{ns} \cdot \nu_{(rs)} = \mu_{mr}\ \mu_r \cdot \nu_{ns}\ \nu_s = \mu_m \nu_n = \nu_{(mn)}$ .

Table IIb). *Post-positive transformants.*

$^{*}(\mu_0\ \nu_0) = (\mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\nu_{00} + \nu_{11} + \nu_{22} + \nu_{33}),$
$^{*}(\mu_1\ \nu_0) = (\mu_{01} + \mu_{10} + \mu_{23} + \mu_{32})\,(\nu_{00} - \nu_{11} - \nu_{22} - \nu_{33}),$
$^{*}(\mu_2\ \nu_0) = (\mu_{02} - \mu_{20} + \mu_{31} - \mu_{13})\,(\nu_{00} - \nu_{11} - \nu_{22} - \nu_{33}),$
$^{*}(\mu_3\ \nu_0) = (\mu_{03} + \mu_{30} - \mu_{12} - \mu_{21})\,(\nu_{00} + \nu_{11} + \nu_{22} + \nu_{33});$

$^{*}(\mu_0\ \nu_1) = (\ \ \mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})(\nu_{01} - \nu_{10}) + (\mu_{30} + \mu_{03} - \mu_{12} - \mu_{21})\,(\nu_{32} - \nu_{23}),$
$^{*}(\mu_1\ \nu_1) = (\ \ \mu_{10} + \mu_{01} + \mu_{23} + \mu_{32})(\quad \text{“} \quad) + (\mu_{02} - \mu_{20} + \mu_{31} - \mu_{13})\,(\quad \text{“} \quad),$
$^{*}(\mu_2\ \nu_1) = (-\,\mu_{20} + \mu_{02} + \mu_{31} - \mu_{13})(\quad \text{“} \quad) + (\mu_{01} + \mu_{10} + \mu_{32} + \mu_{23})\,(\quad \text{“} \quad),$
$^{*}(\mu_3\ \nu_1) = (\ \ \mu_{30} + \mu_{03} - \mu_{12} - \mu_{21})(\quad \text{“} \quad) + (\mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\quad \text{“} \quad);$

$^{*}(\mu_0\ \nu_2) = (\ \ \mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})(\nu_{02} - \nu_{20}) + (\mu_{30} + \mu_{03} - \mu_{12} - \mu_{21})\,(\nu_{13} - \nu_{31}),$
$^{*}(\mu_1\ \nu_2) = (\ \ \mu_{10} + \mu_{01} + \mu_{23} + \mu_{32})(\quad \text{“} \quad) + (\mu_{02} - \mu_{20} + \mu_{31} - \mu_{13})\,(\quad \text{“} \quad),$
$^{*}(\mu_2\ \nu_2) = (-\,\mu_{20} + \mu_{02} + \mu_{31} - \mu_{13})(\quad \text{“} \quad) + (\mu_{01} + \mu_{10} + \mu_{32} + \mu_{23})\,(\quad \text{“} \quad),$
$^{*}(\mu_3\ \nu_2) = (\ \ \mu_{30} + \mu_{03} - \mu_{12} - \mu_{21})(\quad \text{“} \quad) + (\mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\quad \text{“} \quad);$

$^{*}(\mu_0\ \nu_3) = (\ \ \mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})(\nu_{03} - \nu_{30}) + (\mu_{30} + \mu_{03} - \mu_{12} - \mu_{21})\,(\nu_{21} - \nu_{12}),$
$^{*}(\mu_1\ \nu_3) = (\ \ \mu_{10} + \mu_{01} + \mu_{23} + \mu_{32})(\quad \text{“} \quad) + (\mu_{02} - \mu_{20} + \mu_{31} - \mu_{13})\,(\quad \text{“} \quad),$
$^{*}(\mu_2\ \nu_3) = (-\,\mu_{20} + \mu_{02} + \mu_{31} - \mu_{13})(\quad \text{“} \quad) + (\mu_{01} + \mu_{10} + \mu_{32} + \mu_{23})\,(\quad \text{“} \quad),$
$^{*}(\mu_3\ \nu_3) = (\ \ \mu_{30} + \mu_{03} - \mu_{12} - \mu_{21})(\quad \text{“} \quad) + (\mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\quad \text{“} \quad);$

Table IIIa). *Pre-positive transformants.*

$^{0}(\mu_0\ \lambda_0) = (\ \ \mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\lambda_{00} + \lambda_{11} + \lambda_{22} + \lambda_{33}),$
$^{0}(\mu_1\ \lambda_0) = (\ \ \mu_{01} + \mu_{10} - \mu_{23} - \mu_{32})\,(\qquad \text{“} \qquad),$
$^{0}(\mu_2\ \lambda_0) = (-\,\mu_{02} + \mu_{20} + \mu_{31} - \mu_{13})\,(\qquad \text{“} \qquad),$
$^{0}(\mu_3\ \lambda_0) = (\ \ \mu_{03} + \mu_{30} + \mu_{12} + \mu_{21})\,(\qquad \text{“} \qquad);$

$^{0}(\mu_0\ \lambda_1) = (\ \ \mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\lambda_{10} - \lambda_{01} + \lambda_{32} - \lambda_{23}),$
$^{0}(\mu_1\ \lambda_1) = (\ \ \mu_{01} + \mu_{10} - \mu_{23} - \mu_{32})\,(\qquad \text{“} \qquad),$
$^{0}(\mu_2\ \lambda_1) = (-\,\mu_{02} + \mu_{20} + \mu_{31} - \mu_{13})\,(\qquad \text{“} \qquad),$
$^{0}(\mu_3\ \lambda_1) = (\ \ \mu_{03} + \mu_{30} + \mu_{12} + \mu_{21})\,(\qquad \text{“} \qquad);$

$^{0}(\mu_0\ \lambda_2) = (\ \ \mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\lambda_{20} - \lambda_{02} + \lambda_{13} - \lambda_{31}),$
$^{0}(\mu_1\ \lambda_2) = (\ \ \mu_{01} + \mu_{10} - \mu_{23} - \mu_{32})\,(\qquad \text{“} \qquad),$
$^{0}(\mu_2\ \lambda_2) = (-\,\mu_{02} + \mu_{20} + \mu_{31} - \mu_{13})\,(\qquad \text{“} \qquad),$
$^{0}(\mu_3\ \lambda_2) = (\ \ \mu_{03} + \mu_{30} + \mu_{12} + \mu_{21})\,(\qquad \text{“} \qquad);$

$^{0}(\mu_0\ \lambda_3) = (\ \ \mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\lambda_{30} - \lambda_{03} + \lambda_{21} - \lambda_{12}),$
$^{0}(\mu_1\ \lambda_3) = (\ \ \mu_{01} + \mu_{10} - \mu_{23} - \mu_{32})\,(\qquad \text{“} \qquad),$
$^{0}(\mu_2\ \lambda_3) = (-\,\mu_{02} + \mu_{20} + \mu_{31} - \mu_{13})\,(\qquad \text{“} \qquad),$
$^{0}(\mu_3\ \lambda_3) = (\ \ \mu_{03} + \mu_{30} + \mu_{12} + \mu_{21})\,(\qquad \text{“} \qquad).$

Table IIIb). *Post-positive transformants.*

$$
\begin{aligned}
{}^{*}(\mu_0\,\lambda_0) &= (\mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\ \ \lambda_{00} + \lambda_{11} + \lambda_{22} + \lambda_{33}),\\
{}^{*}(\mu_1\,\lambda_0) &= (\mu_{01} + \mu_{10} + \mu_{23} + \mu_{32})\,(\qquad\qquad \text{“} \qquad\qquad),\\
{}^{*}(\mu_2\,\lambda_0) &= (\mu_{02} - \mu_{20} + \mu_{31} - \mu_{13})\,(\qquad\qquad \text{“} \qquad\qquad),\\
{}^{*}(\mu_3\,\lambda_0) &= (\mu_{03} + \mu_{30} - \mu_{12} - \mu_{21})\,(\qquad\qquad \text{“} \qquad\qquad);
\end{aligned}
$$

$$
\begin{aligned}
{}^{*}(\mu_0\,\lambda_1) &= (\mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(-\lambda_{10} + \lambda_{01} + \lambda_{32} - \lambda_{23}),\\
{}^{*}(\mu_1\,\lambda_1) &= (\mu_{01} + \mu_{10} + \mu_{23} + \mu_{32})\,(\qquad\qquad \text{“} \qquad\qquad),\\
{}^{*}(\mu_2\,\lambda_1) &= (\mu_{02} - \mu_{20} + \mu_{31} - \mu_{13})\,(\qquad\qquad \text{“} \qquad\qquad),\\
{}^{*}(\mu_3\,\lambda_1) &= (\mu_{03} + \mu_{30} - \mu_{12} - \mu_{21})\,(\qquad\qquad \text{“} \qquad\qquad);
\end{aligned}
$$

$$
\begin{aligned}
{}^{*}(\mu_0\,\lambda_2) &= (\mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(-\lambda_{20} + \lambda_{02} + \lambda_{13} - \lambda_{31}),\\
{}^{*}(\mu_1\,\lambda_2) &= (\mu_{01} + \mu_{10} + \mu_{23} + \mu_{32})\,(\qquad\qquad \text{“} \qquad\qquad),\\
{}^{*}(\mu_2\,\lambda_2) &= (\mu_{02} - \mu_{20} + \mu_{31} - \mu_{13})\,(\qquad\qquad \text{“} \qquad\qquad),\\
{}^{*}(\mu_3\,\lambda_2) &= (\mu_{03} + \mu_{30} - \mu_{12} - \mu_{21})\,(\qquad\qquad \text{“} \qquad\qquad);
\end{aligned}
$$

$$
\begin{aligned}
{}^{*}(\mu_0\,\lambda_3) &= (\mu_{00} + \mu_{11} + \mu_{22} + \mu_{33})\,(\ \lambda_{30} - \lambda_{03} + \lambda_{21} - \lambda_{12}),\\
{}^{*}(\mu_1\,\lambda_3) &= (\mu_{01} + \mu_{10} + \mu_{23} + \mu_{32})\,(\qquad\qquad \text{“} \qquad\qquad),\\
{}^{*}(\mu_2\,\lambda_3) &= (\mu_{02} - \mu_{20} + \mu_{31} - \mu_{13})\,(\qquad\qquad \text{“} \qquad\qquad),\\
{}^{*}(\mu_3\,\lambda_3) &= (\mu_{03} + \mu_{30} - \mu_{12} - \mu_{21})\,(\qquad\qquad \text{“} \qquad\qquad).
\end{aligned}
$$

The advantage of the operators ${}^{0}(\mu_m\,\lambda_n)$ [${}^{*}(\mu_m\,\lambda_n)$, resp.] over ${}^{0}(\mu_m\,\nu_n)$ [${}^{*}(\mu_m\,\nu_n)$, resp.] pops into view.

If one puts the common factors outside then one will get a monomial in each case.

One does not need to perform the multiplication $\sum \mu_{mn} \sum \lambda_{rs}$ . Observe that:

$$\sum \mu_{mn} \cdot \sum \lambda_{rs} \ \cdot \mu_p\,\lambda_q = [\sum \mu_{mn} \cdot \mu_p\,(\sum\lambda_{rs} \cdot \lambda_q]\ .$$

In addition:

$${}^{0}(\mu_m\,\lambda_n) = {}^{0}\mu_m \cdot {}^{0}\lambda_n \qquad \text{and} \qquad {}^{*}(\mu_m\,\lambda_n) = {}^{*}\mu_m \cdot {}^{*}\lambda_n\ .$$

The $\lambda_n$ correspond exactly to the axial vectors $h_n$ in § 17 and that also the $\mu_n$ form a proper group:

| | $\mu_0$ | $\mu_1$ | $\mu_2$ | $\mu_3$ |
|---|---|---|---|---|
| $\mu_0$ | $\mu_0$ | $\mu_1$ | $\mu_2$ | $\mu_3$ |
| $\mu_1$ | $\mu_1$ | $\mu_0$ | $-\mu_3$ | $-\mu_2$ |
| $\mu_2$ | $\mu_2$ | $\mu_3$ | $-\mu_0$ | $-\mu_1$ |
| $\mu_3$ | $\mu_3$ | $\mu_2$ | $\mu_1$ | $\mu_0$ |

In this group, one can, in turn, develop the isomorphic transformants that are in agreement with the ones in Table III (as well as Table II).

One can also represent them by the axial vectors $h_n$, and in fact:

$$ {}^{0}\mu_0 = {}^{*}\mu_0 = {}^{0}h_0 = {}^{*}h_0 \,, $$

$$ \begin{aligned} {}^{0}\mu_0 &= -\,{}^{0}h_0 \cdot {}^{*}h_3 \,, & {}^{*}\mu_1 &= {}^{0}h_3 \cdot {}^{*}h_2 \,, \\ {}^{0}\mu_2 &= -\,{}^{*}h_2 \,, & {}^{*}\mu_2 &= -\,{}^{0}h_2 \,, \\ {}^{0}\mu_3 &= {}^{0}h_2 \cdot {}^{*}h_1 \,, & {}^{*}\mu_3 &= -\,{}^{0}h_1 . \end{aligned} $$

**§ 22 a)**. *(Continued)*. – The preceding method is also applicable to § 18. Since one must regard ${}^{0}h$ and ${}^{*}h$ as if they were two different letters, one can write:

$$ {}^{0}h_m \cdot {}^{*}h_n \,;\, {}^{0}h_r \cdot {}^{*}h_s = ({}^{0}h_m \,;\, {}^{0}h_r)\,({}^{*}h_n \,;\, {}^{*}h_s) = {}^{0}h_{mr} \cdot {}^{*}h_{ns} \,, $$

and one finds from the group ${}^{0}h_m \cdot {}^{*}h_n$ that:

$$ \begin{aligned} {}^{00}h_1 &= ({}^{0}h_{10} - {}^{0}h_{01} + {}^{0}h_{32} - {}^{0}h_{23})({}^{*}h_{00} + {}^{*}h_{11} + {}^{*}h_{22} + {}^{*}h_{33}), \\ {}^{*0}h_1 &= ({}^{0}h_{01} - {}^{0}h_{10} + {}^{0}h_{32} - {}^{0}h_{23})(\quad \text{“} \quad), \\ {}^{0*}h_1 &= ({}^{0}h_{00} + {}^{0}h_{11} + {}^{0}h_{22} + {}^{0}h_{33})({}^{*}h_{10} - {}^{*}h_{01} + {}^{*}h_{32} - {}^{*}h_{23}), \\ {}^{**}h_1 &= (\quad \text{“} \quad)({}^{*}h_{01} - {}^{*}h_{10} + {}^{*}h_{32} - {}^{*}h_{23}). \end{aligned} $$

One sees that:

$$ \begin{aligned} {}^{00}h_1 &= {}^{0}({}^{0}h_1 \cdot {}^{*}h_0) = {}^{00}h_1 \cdot {}^{0*}h_0 \,, \\ {}^{*0}h_1 &= {}^{*}({}^{0}h_1 \cdot {}^{*}h_0) = {}^{*0}h_1 \cdot {}^{**}h_0 \,, \\ {}^{0*}h_1 &= {}^{0}({}^{0}h_0 \cdot {}^{*}h_1) = {}^{00}h_0 \cdot {}^{0*}h_0 \,, \\ {}^{**}h_1 &= {}^{*}({}^{0}h_0 \cdot {}^{*}h_1) = {}^{*0}h_0 \cdot {}^{**}h_0 \,, \qquad \text{etc.} \end{aligned} $$

(*To be continued…*)

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