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The impulse-energy theorem in Dirac's quantum theory of the electron

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Various expressions for the impulse-energy theorem can be chosen that become identical for a free electron. The most natural choice is a certain asymmetric tensor. The force on a unit volume is the classical Lorentzian one, with no additional force for a magnetic polarization of the electron.

The differential equations of the Dirac theory [*] read:

$$\left\{ i\sum_{\mu=1}^{4}\gamma_\mu\left(p_\mu+\frac{e}{c}A_\mu\right)+mc\right\}\psi = 0. \tag{1}$$

In this equation we have set $p_0 = ip_4$ and $A_0 = iA_4$, and we have:

$$p_0 = \frac{ih}{2\pi c}\frac{\partial}{\partial t}, \qquad p_k = -\frac{ih}{2\pi c}\frac{\partial}{\partial x_k} \qquad (k = 1, 2, 3).$$

If one introduces $x_4 = ict$ then the differential equations above read:

$$\left\{-\gamma_4\left(\frac{h}{2\pi}\frac{\partial}{\partial x_4}-\frac{ie}{c}A_4\right)+\sum_{k=1}^{3}\gamma_k\left(\frac{h}{2\pi}\frac{\partial}{\partial x_4}+\frac{ie}{c}A_4\right)+mc\right\}\psi = 0. \tag{2}$$

In this equation, the γ_μ are four four-rowed matrices that satisfy the equations:

$$\gamma_\mu^2 = 1, \qquad \gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \qquad (\mu\neq\nu); \qquad \mu, \nu = 1, 2, 3, 4. \tag{3}$$

They are understood to multiply with each other in a well-known way; one has, e.g.:

$$(\gamma_1\gamma_2)_r = \sum_p (\gamma_1)_{rp}(\gamma_2)_{ps}.$$

Furthermore, there are four quantities ψ, i.e., the ψ in (1) and (2), with an index understood to be omitted, along with the implied summation over this index. Hence, one has, e.g.:

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† Translated by D. H. Delphenich

* P. A. M. Dirac, Proc. Roy. Soc. (A) **117**, 610, 1928; **118**, 351, 1928.

$$(\gamma_4\,\psi)_r = \sum_p (\gamma_4)_{rp}\,\psi_p\,.$$

We therefore have four differential equations for the four quantities ψ. Moreover, if $-e$ is the charge of the electron and m is its mass then:

$$\varphi_4 = -A_4, \qquad \varphi_k = -A_k \qquad (k = 1, 2, 3)$$

are the components of the four-potential of the electromagnetic field. This field is then regarded as given, i.e., as originating in an arbitrary configuration of given charges and currents (possibly at infinity). The charge of the electron itself is then not to be regarded as a source in the calculations.

With the correct components of the four-potential, (2) reads:

$$\left\{-\gamma_4\left(\frac{h}{2\pi}\frac{\partial}{\partial x_4}+\frac{ie}{c}\varphi_4\right)+\sum_{k=1}^{3}\gamma_k\left(\frac{h}{2\pi}\frac{\partial}{\partial x_4}+\frac{ie}{c}\varphi_4\right)+mc\right\}\psi = 0. \tag{4}$$

We free up the differential quotients with respect to the time coordinate from the matrix γ_4 when we multiply by γ_4:

$$\left\{-\left(\frac{h}{2\pi}\frac{\partial}{\partial x_4}+\frac{ie}{c}\varphi_4\right)+\sum_{k=1}^{3}\gamma_4\gamma_k\left(\frac{h}{2\pi}\frac{\partial}{\partial x_4}+\frac{ie}{c}\varphi_4\right)+\gamma_4 mc\right\}\psi = 0. \tag{5}$$

In this expression, we then replace the matrices γ_4, $\gamma_4\gamma_k$ by their transposes $\tilde{\gamma}_4, \widetilde{\gamma_4\gamma_k}$:

$$(\tilde{\gamma}_\mu)_{rs} = (\gamma_\mu)_{sr}\,,$$

and also the virtual matrices $\partial/\partial x_\mu$ with their transposes $-\partial/\partial x_\mu$, so we obtain four new differential equations:

$$\left\{-\left(-\frac{h}{2\pi}\frac{\partial}{\partial x_4}+\frac{ie}{c}\varphi_4\right)+\sum_{k=1}^{3}\widetilde{\gamma_4\gamma_k}\left(-\frac{h}{2\pi}\frac{\partial}{\partial x_4}+\frac{ie}{c}\varphi_4\right)+\tilde{\gamma}_4 mc\right\}\omega = 0 \tag{6}$$

for four new quantities ω.

Since:

$$\widetilde{\gamma_4\gamma_k} \equiv \tilde{\gamma}_k\tilde{\gamma}_4 = -\tilde{\gamma}_k\tilde{\gamma}_4\,,$$

we obtain, when we multiply (6) by $-\tilde{\gamma}_4$:

$$\tilde{G}\omega \equiv \left\{\sum_{\mu=1}^{4}\tilde{\gamma}_\mu\left(-\frac{h}{2\pi}\frac{\partial}{\partial x_\mu}+\frac{ie}{c}\varphi_\mu\right)-mc\right\}\omega = 0. \tag{7}$$

If we set $\psi = \gamma_4 \chi$ then we get:

$$G\chi \equiv \left\{ \sum_{\mu=1}^{4} \gamma_\mu \left(\frac{h}{2\pi} \frac{\partial}{\partial x_\mu} + \frac{ie}{c} \varphi_\mu \right) - mc \right\} \chi = 0 . \tag{8}$$

These four-dimensional symmetric equations (7) and (8) for ω and χ go over to each other under transposition, just as (5) and (6) do.

We construct:

$$0 = \omega G \chi - \chi \tilde{G} \omega \equiv \frac{h}{2\pi} \sum_\mu \frac{\partial}{\partial x_\mu} (\omega \gamma_\mu \chi) . \tag{9}$$

In this, one has $\omega\chi = \sum_r \omega_r \chi_r$, etc., and one must take into account the identity:

$$\omega \tilde{\alpha} \chi \equiv \chi \, \alpha \, \omega , \tag{10}$$

for any matrix α in the group of γ_μ , and one also must naturally replace, e.g., χ with $\partial\chi / \partial x_\mu$. The quantities:

$$P_\mu = - \, ie \; \omega \gamma_\mu \chi \tag{11}$$

define, for an appropriate normalization of ω, χ, the components of the four-current, and (9) is the conservation law for electricity. Namely, one can, by a canonical transformation:

$$\gamma_\mu \to S \, \gamma_\mu \, S^{-1}, \qquad \chi \to S \, \chi, \qquad \omega \to \omega S^{-1} \equiv \tilde{S}^{-1} \omega ,$$

arrive at the fact that the γ_μ are Hermitian:

$$(\gamma_\mu)_{rs} = \overline{(\gamma_\mu)_{rs}} ,$$

where the overbar means the complex conjugate *. The same is true for the matrices $i\gamma_\mu\gamma_\nu$ ($\mu \neq \nu$), and therefore equations (5) and (6) for ω and χ are complex conjugates of each other (if x_4 and φ_4 are pure imaginary), so they admit complex-conjugate solutions, as well. Correspondingly, if one chooses $\omega = \bar{\psi}$ (= $\overline{\gamma_4 \chi}$) then ω and ψ refer to the same "state," for which P_4/i becomes real and negative, and P_1, P_2, P_3 become real, as one must have.

The foregoing was essentially already discovered by Dirac, up to the four-dimensional symmetric notation. We now go further and construct from (7), (8), and (9):

$$0 = \frac{c}{2} \left(\omega \frac{\partial}{\partial x_\lambda} - \frac{\partial \omega}{\partial x_\lambda} \right) G \chi + \frac{c}{2} \left(\chi \frac{\partial}{\partial x_\lambda} - \frac{\partial \chi}{\partial x_\lambda} \right) \tilde{G} \omega + ie \varphi_\lambda \sum_\mu \frac{\partial}{\partial x_\mu} (\omega \gamma_\mu \chi) .$$

* Proved by P. Jordan and E. Wigner, ZS. f. Phys. **47**, 631, 1928, appendix at the end.

By a simple computation, this yields, taking (10) into account:

$$-\sum_{\mu}\frac{\partial T_{\lambda\mu}}{\partial x_{\mu}}=\sum_{\mu}P_{\mu}\left(\frac{\partial\varphi_{\lambda}}{\partial x_{\mu}}-\frac{\partial\varphi_{\mu}}{\partial x_{\lambda}}\right), \tag{12}$$

where:

$$T_{\lambda\mu}=\frac{hc}{4\pi}\left(\omega\gamma_{\mu}\frac{\partial\chi}{\partial x_{\lambda}}-\frac{\partial\omega}{\partial x_{\lambda}}\gamma_{\mu}\chi\right)+ie\,\varphi_{\lambda}\,\omega\gamma_{\mu}\chi\,. \tag{13}$$

Since the right-hand side of (12) represents the Lorentz four-force on a unit volume that originates in the electromagnetic field, we can regard the $T_{\lambda\mu}$ as the components of the impulse-energy tensor of the electron; it is then generally asymmetric. Just as we did before for the four-current, we show that if the ω and χ refer to the same "state" then the T_{4k}, T_{k4} (k = 1, 2, 3) are pure imaginary, while the remaining $T_{\lambda\mu}$ are real, as it must be. We further emphasize that the force on the unit volume is the classical Lorentz one, and that there are no arbitrary additional forces, as one might expect for an electron with a magnetic moment.

One can also consider other quantities to represent the impulse-energy tensor, as we would now like to show. Instead of (13), one can also write:

$$T_{\lambda\mu}=\frac{hc}{4\pi}\left\{\omega\gamma_{\mu}\left(\frac{\partial}{\partial x_{\lambda}}+\frac{2\pi ie}{hc}\varphi_{\lambda}\right)\chi+\chi\bar{\gamma}_{\mu}\left(-\frac{\partial}{\partial x_{\lambda}}+\frac{2\pi ie}{hc}\varphi_{\lambda}\right)\omega\right\}. \tag{14}$$

It now follows from (8), upon multiplication by γ_{λ}, that:

$$\frac{hc}{2\pi}\left(\frac{\partial}{\partial x_{\lambda}}+\frac{2\pi ie}{hc}\varphi_{\lambda}\right)\chi=-\sum_{\nu\neq\lambda}\gamma_{\lambda}\gamma_{\nu}\left(\frac{h}{2\pi}\frac{\partial}{\partial x_{\nu}}+\frac{ie}{c}\varphi_{\nu}\right)\chi+mc\gamma_{\lambda}\chi\,,$$

and from (7), upon multiplication by $\tilde{\gamma}_{\lambda}$:

$$\frac{hc}{2\pi}\left(-\frac{\partial}{\partial x_{\lambda}}+\frac{2\pi ie}{hc}\varphi_{\lambda}\right)\omega=-\sum_{\nu\neq\lambda}\tilde{\gamma}_{\lambda}\tilde{\gamma}_{\nu}\left(-\frac{h}{2\pi}\frac{\partial}{\partial x_{\nu}}+\frac{ie}{c}\varphi_{\nu}\right)\omega+mc\tilde{\gamma}_{\lambda}\chi\,,$$

Substitution of these equations into (14) gives:

$$\begin{aligned}\frac{2}{c}T_{\lambda\mu}\ (\lambda\neq\mu)=-\sum_{\mu\neq\nu\neq\lambda}&\left\{\omega\gamma_{\mu}\gamma_{\lambda}\gamma_{\nu}\left(\frac{h}{2\pi}\frac{\partial}{\partial x_{\nu}}+\frac{ie}{c}\varphi_{\nu}\right)\chi+\chi\tilde{\gamma}_{\mu}\tilde{\gamma}_{\lambda}\tilde{\gamma}_{\nu}\left(-\frac{h}{2\pi}\frac{\partial}{\partial x_{\nu}}+\frac{ie}{c}\varphi_{\nu}\right)\omega\right\}\\ &-\omega\gamma_{\mu}\gamma_{\lambda}\gamma_{\nu}\left(\frac{h}{2\pi}\frac{\partial}{\partial x_{\mu}}+\frac{ie}{c}\varphi_{\mu}\right)\chi-\chi\tilde{\gamma}_{\mu}\tilde{\gamma}_{\lambda}\tilde{\gamma}_{\nu}\left(-\frac{h}{2\pi}\frac{\partial}{\partial x_{\mu}}+\frac{ie}{c}\varphi_{\mu}\right)\omega\\ &+mc\,\omega\gamma_{\mu}\gamma_{\lambda}\chi+mc\,\chi\,\tilde{\gamma}_{\mu}\tilde{\gamma}_{\lambda}\,\omega\,.\end{aligned}$$

Taking into account the identities:

$$\tilde{\gamma}_\mu \tilde{\gamma}_\lambda = \widetilde{\gamma_\lambda \gamma_\mu}, \qquad \tilde{\gamma}_\mu \tilde{\gamma}_\lambda \tilde{\gamma}_\nu = \widetilde{\gamma_\nu \gamma_\lambda \gamma_\mu},$$

the relations (3) and the identities (10) simplify this expression to:

$$\frac{2}{c} T_{\lambda\mu} \; (\lambda \neq \mu) = -\frac{h}{2\pi} \sum_{\lambda \neq \nu \neq \mu} \frac{\partial}{\partial x_\nu} (\omega \gamma_\mu \gamma_\lambda \gamma_\nu \chi) \\ + \omega \gamma_\lambda \left(\frac{h}{2\pi} \frac{\partial}{\partial x_\mu} + \frac{ie}{c} \varphi_\mu \right) \chi + \chi \tilde{\gamma}_\lambda \left(-\frac{h}{2\pi} \frac{\partial}{\partial x_\mu} + \frac{ie}{c} \varphi_\mu \right) \omega . \tag{15}$$

However, from (14) one has:

$$\frac{2}{c} T_{\mu\lambda} = +\omega \gamma_\lambda \left(\frac{h}{2\pi} \frac{\partial}{\partial x_\mu} + \frac{ie}{c} \varphi_\mu \right) \chi + \chi \tilde{\gamma}_\lambda \left(-\frac{h}{2\pi} \frac{\partial}{\partial x_\mu} + \frac{ie}{c} \varphi_\mu \right) \omega .$$

Subtracting this from (15) yields:

$$\frac{2}{c} (T_{\mu\lambda} - T_{\lambda\mu}) \quad (\lambda \neq \mu) = -\frac{h}{2\pi} \sum_{\lambda \neq \nu \neq \mu} \frac{\partial}{\partial x_\nu} (\omega \gamma_\mu \gamma_\lambda \gamma_\nu \chi) , \tag{16}$$

and from this it follows that:

$$\frac{2}{c} \sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} (T_{\lambda\mu} - T_{\mu\lambda}) = \sum_{\mu \neq \lambda} \frac{\partial}{\partial x_\mu} (T_{\lambda\mu} - T_{\mu\lambda}) \\ = -\frac{h}{2\pi} \sum_{\lambda \neq \nu \neq \mu \neq \lambda} \frac{\partial^2}{\partial x_\mu \partial x_\nu} (\omega \gamma_\mu \gamma_\lambda \gamma_\nu \chi) \\ = -\frac{h}{4\pi} \sum_{\lambda \neq \nu \neq \mu \neq \lambda} \frac{\partial^2}{\partial x_\mu \partial x_\nu} (\omega \gamma_\mu \gamma_\lambda \gamma_\nu \chi + \chi \gamma_\mu \gamma_\lambda \gamma_\nu \omega) = 0. \tag{17}$$

Therefore, the divergence of the anti-symmetric tensor with the components $T_{\lambda\mu} - T_{\mu\lambda}$ vanishes, and one can, when one multiplies these components by an arbitrary real number, add them to the tensor that is defined by (13) without altering the conservation law (12). In particular, one can choose the impulse-energy tensor to be, not the T in (13), but the transposed tensor T' with the components $T'_{\lambda\mu} = T_{\mu\lambda}$. However, the original tensor (13) seems to be the most natural one to me. With this choice, one then has that, e.g., c/i T_{41} is the x_1-component of the energy current, and this is in agreement with the meaning of the operators that appear in it: Indeed, it means that the operator $-\frac{hc}{2\pi} \frac{\partial}{\partial x_4}$, when applied to χ, gives the energy, and, from (11), $ic\gamma_1$ gives the current velocity (in the sense of statistical mean). Likewise, the impulse density $1/ic\, T_{14}$ also gets the correct form in this way (on this, compare the statement on the energy density at the conclusion of this article). For a free electron ($\varphi_\mu = 0$) with a given magnitude and direction for its velocity the two tensors T and T' become identical. One then sets:

$$\chi_r = a_r e^{\frac{2\pi}{i}\sum_\mu k_\mu x_\mu},$$

in this case, where a_r , k_μ are constants and the products $k_\mu x_\mu$ are all real. Since ω can be taken to be complex conjugate to $\gamma_4\chi$, one has:

$$\omega_r = b_r e^{-\frac{2\pi}{i}\sum_\mu k_\mu x_\mu},$$

with constants b_r . One thus has:

$$\frac{h}{2\pi}\frac{\partial \chi}{\partial x_\mu} = i\, k_\mu\, \chi \qquad \text{and} \qquad \frac{h}{2\pi}\frac{\partial \omega}{\partial x_\mu} = -\, i\, k_\mu\, \omega,$$

and instead of (7), (8), we have:

$$\sum_\mu i\tilde{\gamma}_\mu k_\mu \omega = mc\,\omega, \qquad \sum_\mu i\gamma_\mu k_\mu \chi = mc\,\chi.$$

From this, it follows that:

$$-\, mc\ \gamma_\lambda\, \chi = \sum_\mu \gamma_\lambda \gamma_\mu k_\mu \chi = (2k_\lambda - \sum_\mu \gamma_\mu \gamma_\lambda k_\mu\,)\chi\,,$$

and since $\frac{h}{2\pi}\gamma_\lambda \frac{\partial \chi}{\partial x_\nu} = ik_\nu \gamma_\lambda \chi$, one has:

$$-\frac{h}{2\pi} mc\gamma_\lambda \frac{\partial \chi}{\partial x_\nu} = k_\nu\,(2k_\lambda\ \omega\chi - \sum_\mu \omega\, \gamma_\mu\, \gamma_\lambda\ k_\mu \chi)$$

$$= 2\ k_\nu\, k_\lambda\ \omega\chi - \ k_\nu \sum_\mu \chi\, \tilde{\gamma}_\lambda \tilde{\gamma}_\mu k_\mu \omega\ = 2\ k_\nu\, k_\lambda\ \omega\chi + imc\ k_\nu \chi\ \tilde{\gamma}_\lambda\ \omega$$

$$= 2\ k_\nu\, k_\lambda\ \omega\chi + imc\ k_\nu \chi\ \gamma_\lambda\ \omega = 2\ k_\nu\, k_\lambda\ \omega\chi + \frac{h}{2\pi} mc\omega\gamma_\lambda \frac{\partial \chi}{\partial x_\nu}\,.$$

Thus, one has:

$$\frac{h}{2\pi} mc\omega\gamma_\lambda \frac{\partial \chi}{\partial x_\nu} = -\, k_\nu\, k_\lambda\ \omega\chi,$$

and likewise:

$$\frac{h}{2\pi} mc \frac{\partial \omega}{\partial x_\nu}\gamma_\lambda = k_\nu\, k_\lambda\ \omega\chi,$$

from which the symmetry of the tensor T follows.

In classical electrodynamics, the right-hand side of (12) can be expressed as the divergence of the impulse-energy tensor of the electromagnetic field. As usual, this is, however, impossible here, which was already the case in the earlier relativistic generalization of the Schrödinger theory that was presented by Gordon and others, because the four-current P of our electron is not the source of the field that acts on it, and is therefore not connected with that field through the Maxwell equations. Our system of electron + electromagnetic field is therefore not to be regarded as a closed one since there is no proper tensor whose divergence vanishes (naturally, the tensor $T' - T$ does not come into consideration), which the conservation of energy and momentum would guarantee. One must consider the interaction of the electron with whatever particles generate the given field. However, the presentation of a relativistic quantum mechanics for many electrons (after eliminating the field quantities) encounters difficulties that are connected with the introduction of retarded potentials [*].

Just as in the earlier "relativistic Schrödinger equation," a gradient is physically trivial in the four-potential. If one replaces φ_μ with $\varphi_\mu + \partial S / \partial x_\mu$ (where S is a scalar), χ with $e^{-\frac{2\pi ie}{hc}S}\chi$, and ω with $e^{+\frac{2\pi ie}{hc}S}\omega$ then all expressions and equations remain unchanged.

On the physical meaning of the tensor T, we make the following remark: The operator $-\frac{hc}{2\pi}\frac{\partial}{\partial x_4}$, when applied to χ, represents the energy, just as when $\frac{hc}{2\pi}\frac{\partial}{\partial x_4}$ is applied to ω. For that reason:

$$-\frac{hc}{4\pi}\left(\omega\gamma_4\frac{\partial\chi}{\partial x_4} - \frac{\partial\omega}{\partial x_4}\gamma_4\chi\right) \tag{18}$$

is the real (see above) energy density, since $\omega\gamma_4\chi$ is the probability density. Furthermore, φ_4/i is the electrostatic potential and $-e$ is the charge of the electron. Thus, $ie\varphi_4$ is its potential energy in a static field, and $ie\varphi_4\omega\gamma_4\chi$ is the corresponding energy density. When this is subtracted from (18) it gives the energy density of "matter" $-T_{44}$, which would thus make it the density of "kinetic" energy when one understands this term to mean "rest energy" mc^2, which classically means the expression $\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$. One then observes that the mass of the electron does not enter into this impulse-energy explicitly.

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[*] Cf., H. Tetrode, ZS. f. Phys. **10**, 317, 1922, where this question was discussed on the basis of classical mechanics, and, in principle, by W. Schottky, Die Naturw. **9**, 492 and 506, 1921.