"Sulla teoria delle linee geodetiche," Rendiconti del Reale Istituto Lombardo (2) **1** (1868), 708-718; *Opera matematiche*, v. I, 366-373.

# On the theory of geodetic lines

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It is known that JACOBI's wonderful discovery of the connection that exists between dynamical equations, the isoperimetric equations, and nonlinear first-order partial differential equations has found a useful application in the theory of geodetic lines. Although that application presents no difficulty to those who know of that connection, nonetheless, it does seem as though one can recognize some peculiarities that, lacking any confirmation in the general doctrine, will demand a special proof that would not be inopportune.

One has the integral:

$$\int f(x, y, y')\,dx, \qquad y' = \frac{dy}{dx}, \tag{1}$$

which must render a maximum or a minimum. The indefinite equation of the problem – or as one might say more briefly, the isoperimetric equation – in this case is:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0\,,$$

that is to say:

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x\,\partial y'} - y'\frac{\partial^2 f}{\partial y\,\partial y'} - y''\frac{\partial^2 f}{\partial y'^2} = 0\,.$$

Suppose that one knows a first integral of that equation, and therefore a value for $y'$ as a finite function of $x$, $y$. Imagine that this value has been substituted in the given function:

$$f(x, y, y')\,.$$

One will easily find that the isoperimetric equation assumes the form:

$$\frac{\partial}{\partial y}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right), \tag{2}$$

in which it is intended that one substitutes the value of $y'$ as a function of $x$, $y$ in the expression that is enclosed in parentheses.

One now observes that the element in the integral, i.e., the quantity:

$$f\left(x, y, \frac{dy}{dx}\right) dx \,,$$

is a homogeneous function of degree one in the differentials:

$$dx \,, \qquad dy \,,$$

such that if one calls its derivatives with respect to those differentials $X$, $Y$, resp., then one will have:

$$f\left(x, y, \frac{dy}{dx}\right) dx = X\,dx + Y\,dy$$

identically, in which:

$$X = f - y' \frac{\partial f}{\partial y'} \,, \qquad Y = \frac{\partial f}{\partial y'} \,.$$

Therefore, the proposed integral can assume the form:

$$\int (X\,dx + Y\,dy) \;.$$

It then has the interesting property that when one substitutes the value of $y'$ that is deduced from a first integral of the problem in:

$$X, \qquad Y,$$

the isoperimetric equation will take the form:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} \,,$$

i.e., it will become the usual integrability condition on the differential in two independent variables into which the element $f\,dx$ gets converted.

If $F(x, y)$ is the original function, i.e., one has:

$$dF = \left(f - y' \frac{\partial f}{\partial y'}\right) dx + \left(\frac{\partial f}{\partial y'}\right) dy \tag{3}$$

then one will have:

$$(3') \qquad \frac{\partial F}{\partial x} = f - y' \frac{\partial f}{\partial y'}, \qquad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y'} .$$

As one sees, the determination of the function $F$ depends upon only one quadrature, but one can find a way of determining that function without supposing that it follows from a first integration of the indefinite equation. Indeed, if one eliminates the derivative $y'$ from the preceding two equations then one will obtain a result of the form:

$$(4) \qquad \Phi\left(x, y, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = 0 ,$$

i.e., a nonlinear first-order partial differential equation that the function $F$ must satisfy, and which can then serve to determine it. Since that function does not enter into the equation, but only its derivatives, of the two arbitrary constants that are contained in a complete solution, one of them is simply additive.

If a first integral of the isoperimetric equation is known then when one integrates (3), one will obtain a solution to the partial differential equation (4), which will be a particular solution or a complete solution according to whether that first integral is, in turn, particular or provided by an arbitrary constant, resp. However, conversely, if one knows a solution to equation (4) then one can easily get the value of $y'$ by means of one of the two equations (3′), i.e., a first integral of the isoperimetric equation. Thus, the following consideration is more fruitful in its results.

If the first integral of the isoperimetric equation contains an arbitrary constant then it is clear that when one varies that constant, the value that $y'$ receives from that integral will be merely a variation of it and that, from (3), the function $F$ will also vary correspondingly. If one denotes those simultaneous variations of $y'$ and $F$ by $D$ then (3) will give:

$$(5) \qquad D\,dF = \frac{\partial^2 f}{\partial y'^2}(dy - y' \cdot dx)\,Dy' .$$

It then follows that for:

$$dy - y' \cdot dx = 0 ,$$

one will have:

$$D\,dF = 0 ,$$

or

$$d\,DF = 0 ,$$

and therefore:

$$(6) \qquad D\,F = \text{const.},$$

which is an equation in which it is legitimate to attribute the significance to $D$ that it is a derivation with respect to the arbitrary (non-additive) constant that is contained in $F$, which is a constant that can be indifferent to whether it is provided by the first integral that served to construct equation

(3) or it is contained in a complete solution of (4). Now since equation (6) is finite with respect to $x$, $y$ and contains two arbitrary constants, it will be nothing but the integral of:

$$dy - y' \cdot dx = 0 \, ,$$

or the complete integral of the isoperimetric equation (*).

It therefore follows from the foregoing that the complete integration of the isoperimetric equation (2) depends substantially on either the search for one of its first integrals (with an arbitrary constant) and a quadrature or the search for a complete solution to a nonlinear first-order partial differential equation into which only the derivatives of the unknown function enter.

The form in which that property presents itself, which was exhibited already by JACOBI, makes it more applicable to the theory of surfaces.

Indeed, let:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

be the square of the line element on a surface. The search for the geodetic lines is equivalent to the search for the relation between $u$, $v$ for which the value of the integral:

$$\int \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

is minimized. Let $U$, $V$ denote the derivatives of the element in that integral with respect to $du$, $dv$. One will then have:

$$U = \frac{E\,du + F\,dv}{ds}\,, \qquad V = \frac{F\,du + G\,dv}{ds}\,,$$

and therefore equation (2) will become:

$$\frac{\partial}{\partial v}\left(\frac{E\,\delta u + F\,\delta v}{\delta s}\right) = \frac{\partial}{\partial u}\left(\frac{F\,\delta u + G\,\delta v}{\delta s}\right)$$

in this case, in which $\delta u$, $\delta v$, $\delta s$ denote the values of $du$, $dv$, $ds$ relative to the geodetics of the system under consideration that passes through the point ($u$, $v$). When the first integral that represents that system has been given, it is easy to calculate the values of $\delta u$ / $\delta s$, $\delta v$ / $\delta s$ relative to that curve.

(*) Clearly, the supposition that $DF$ = const., and therefore $d\,DF = 0$, cannot guarantee that $dy - y' \cdot dx = 0$ in equation (5). Indeed, of the two quantities:

$$Dy', \quad \frac{\partial^2 f}{\partial y'^2}\,,$$

the first one obviously cannot be zero, and the second one can become zero only when $f$ is linear with respect to $y'$, which is a case that cannot arise with a true solution to the isoperimetric problem, as is known.

The preceding equation, which is remarkable in its simplicity, has not perhaps been emphasized yet. It can be deduced with no difficulty from a formula by BONNET (*). CHELINI has also given an equivalent statement of it in a paper that was presented to the Accademia di Bologna in this year (1868) (**).

If one denotes the original function that was previously denoted by $F(x, y)$ by $\varphi(x, y)$ then one will have the equations:

$$\frac{\partial\varphi}{\partial u} = \frac{E\,\delta u + F\,\delta v}{\delta s}\,, \qquad \frac{\partial\varphi}{\partial v} = \frac{F\,\delta u + G\,\delta v}{\delta s}\,,$$

in place of (3′), so if one eliminates $\frac{\delta u}{\delta s}, \frac{\delta v}{\delta s}$ in order to form the equation that is analogous to (4) then one will get:

$$\frac{E\left(\frac{\partial\varphi}{\partial v}\right)^2 - 2F\frac{\partial\varphi}{\partial v}\frac{\partial\varphi}{\partial u} + G\left(\frac{\partial\varphi}{\partial u}\right)^2}{EG - F^2} = 1\,,$$

i.e.:

$$\Delta_1\,\varphi = 1\,, \tag{7}$$

with the notation that we have already used many times before.

The function $\varphi$ admits an elegant interpretation. Indeed, if:

$$d\varphi = \frac{E\,\delta u + F\,\delta v}{\delta s}\,du + \frac{F\,\delta u + G\,\delta v}{\delta s}\,dv \tag{8}$$

then one will see that the curve that is represented by the equation $\varphi$ = const., of $d\varphi = 0$, will satisfy the relation:

$$E\,du\,\delta u + F\,(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v = 0\,,$$

and as is known, it expresses the orthogonality of the directions that correspond to the two ratios $du : dv$ and $\delta u : \delta v$. Therefore, the aforementioned curves $\varphi$ = const. are orthogonal to the geodetics that are represented by the first-order integral equation (if that integral exists for a well-defined value of the arbitrary constant or if that integral contains surplus ones for a well-defined system of values for the arbitrary constants). Furthermore, consider two infinitely-close points $(u, v)$, $(u + du, v + dv)$, and the corresponding difference $d\varphi$ between the values of the parameter $\varphi$. If those points are situated along the same geodetic of the system then one will have:

$$du = \delta u\,, \qquad dv = \delta v\,,$$

(*) Journal de Mathématiques pures et appliquées (2) **5** (1860), pp. 166.
(**) Memorie dell'Accademia delle Scienze di Bologna (2), t. VIII, pp. 27.

and therefore:

$$d\varphi = \delta s ,$$

so $d\varphi$ is nothing but the constant geodetic distance between the two trajectories $(\varphi)$ and $(\varphi + d\varphi)$ that pass through those points, and therefore $\varphi$ is the geodetic distance to the point $(u, v)$ from a well-defined orthogonal trajectory of the geodetic system considered. One can also say that the curves $(\varphi)$ are the geodetic developments (*sviluppanti*) of the curve to which all of the geodetics of the system are tangent, or that they form a system of geodetically-parallel curves. Hence, if, for example, the first integral represents the geodetic that passes through a fixed point then the curve $(\varphi)$ will be the geodetic circumference that has its center at that point, and its parameter $\varphi$ will differ from its geodetic radius only by a constant.

All of those properties can also be regarded as being contained in the simple equation (7), according to what was said in art. IV of my "Ricerche di Analisi applicate all Geometria" (*). GAUSS already gave equation in § XXII of his *Disquisitiones generales*… If one starts from it in order to determine the function $\varphi$, and one calls the arbitrary (non-additive) constant $k$ that is contained in one of its complete solutions then, from what was proved, it will be enough to set:

$$\frac{\partial \varphi}{\partial k} = \psi ,$$

in which $\psi$ is an arbitrary parameter, in order to get the system of geodetic lines that correspond to the trajectory $(\varphi)$. It will then follow that if one obtains the values of $u, v$ as functions of $\varphi, \psi$ from the two equations:

$$\varphi(u, v) = \varphi , \qquad \frac{\partial \varphi(u,v)}{\partial k} = \psi$$

then those values will give the line element the form:

$$ds^2 = d\varphi^2 + \Theta \cdot d\psi^2 ,$$

in which $\Theta$ is a function of $\varphi, \psi$ (**).

In the special problem that we are concerned with, the significance of the equation $D\, d\varphi = 0$ can be regarded as intuitive. Indeed, consider two points:

$$M(u, v) , \qquad M'(u + du, v + dv) ,$$

and let the trajectories:

$$(\varphi), \qquad (\varphi + d\varphi)$$

pass through them. Let $M_1$ be the intersection of the second trajectory with the orthogonal geodetic that passes through $M$. Those points form a rectangular triangle at $M_1$ whose cathetus $MM_1$ has the

(*) Giornale di Matematiche, vol. II (1864), pp. 277; or these *Opere*, vol. I, pp. 115.
(**) GAUSS, *Disquisitiones generales*…, § XIX.

value $d\varphi$. Varying the constant in the expression (8) is equivalent to varying the direction of the cathetus $MM_1$ while holding the hypotenuse fixed. Now it is clear that the variation of that cathetus will be zero (i.e., of order higher than two) only when its direction coincides with that of the hypotenuse or when the point $M'$ is located on the geodetic that passes through the point $M$. Setting that variation equal to zero is therefore equivalent to establishing the relation that exists between the increments $du$, $dv$ relative to the geodetics of the system, and setting the equivalent finite relation:

$$D\varphi = \text{const.}$$

is equivalent to writing the finite equation of those lines.

The form that the variation $D\, d\varphi$ assumes for geodetic lines, in particular (5), is worthy of note, and it is the following:

$$D\, d\varphi = \frac{(EG - F^2)(\delta u \cdot dv - \delta v \cdot du)(\delta u \cdot D\delta v - \delta v \cdot D\delta u)}{\delta\varphi^3}.$$

Let us make two simple applications.

1. When the surface is one of revolution, let $u$ denote the arc-length of a meridian that contacts a well-defined parallel, let $v$ denote the longitude, and let $r$ (which is a function of $u$) denote the radius of the parallel. One has:

$$ds^2 = du^2 + r^2\, dv^2,$$

and the differential equation of the geodetic lines will have the following first integral:

$$dv = \frac{r_0\, du}{r\sqrt{r^2 - r_0^2}},$$

in which $r_0$ is the arbitrary constant. That integral represents all of the geodetics that are tangent to the parallel of radius $r_0$, and it easy to translate it into a known theorem that is due to CLAIRAUT. If one starts from that integral then one will find that:

$$\varphi = \int_{u_0}^{u} \frac{\sqrt{r^2 - r_0^2}\, du}{r} + r_0\, v,$$

and by virtue of the preceding, that is an equation that represents the complete system of the geodetic developments of the parallel of radius $r_0$, and the parameter $\varphi$ is the length of the arc along the parallel $r_0$ (which corresponds to the value $u = u_0$) between the point of zero longitude and the origin of the development that passes through the point $(u, v)$.

2. In a paper that was inserted in the Annali di Matematica pura ed applicate (*), I found that the surfaces whose geodetic lines are represented by equations that are linear in $u$, $v$ are all contained in the formula:

$$ds^2 = \frac{(v^2+a^2)\,du^2 - 2uv\,du\,dv + (u^2+a^2)\,dv^2}{(u^2+v^2+a^2)^2},$$

in which $a$ is a constant. Let us seek the minimum distance between two points $(u, v)$, $(u_0, v_0)$ . The differential equation:

$$du : dv = u - u_0 : v - v_0$$

obviously represents all of the geodetics that emanate from the point $(u_0, v_0)$. If we start from that first integral and set:

$$u^2+v^2+a^2 = w^2, \qquad u_0^2+v_0^2+a^2 = w_0^2, \qquad u\,u_0 + v\,v_0 + a^2 = t$$

then we will easily find that:

$$d\varphi = -\frac{d\,\frac{t}{w}}{\sqrt{w_0^2 - \left(\frac{t}{w}\right)^2}},$$

so

$$\cos\varphi = \frac{u\,u_0 + v\,v_0 + a^2}{\sqrt{(u^2+v^2+a^2)(u_0^2+v_0^2+a^2)}},$$

in which the constant is chosen in such a way that we will have $\varphi = 0$ for:

$$u = u_0, \qquad v = v_0 .$$

The value of $\varphi$ that is given by that formula is the desired minimum distance, and therefore, if we regard $\varphi$ as a parameter then we will have the equation of the geodetic circumference whose center is at the point $(u_0, v_0)$ and whose geodetic radius is equal to $\varphi$.

In the cited paper, it was seen that the surfaces to which those formulas refer are all applicable to the sphere of radius 1, and with the geometric significance that was assigned to the variables $u$, $v$ here, that will explain the result that was obtained immediately. However, the method that was used there does not in the least assume that one knows that property.

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(*) t. VII (1865), pp. 167; or these *Opere*, vol. I, pp. 262.