"Ueber $n$ simultane Differentialgleichungen der Form $\sum^{n+m} X_{\mu} d x_{\mu}=0$," Zeit. Math. Phys. 30 (1885), 234-244.

# On $n$ simultaneous differential equations of the form: 

$$
\sum_{\mu=1}^{n+m} X_{\mu} d x_{\mu}=0
$$<br>By

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The Pfaff problem consists of giving a given differential expression:

$$
\sum_{\kappa=1}^{2 k} X_{\kappa} d x_{\kappa} \quad \text { or } \quad \sum_{\kappa=1}^{2 k+1} X_{\kappa} d x_{\kappa}
$$

in which $X_{\kappa}$ mean any functions of the variables $x_{\kappa}$, the form:

$$
\sum_{\rho=1}^{k} U_{\rho} d u_{\rho} \quad \text { or } \quad U d u+\sum_{\rho=1}^{k} U_{\rho} d u_{\rho}
$$

in which the quantities $U_{\rho}, U, u_{\rho}$ once more mean functions of the $x_{\kappa}$, and $u$ is an entirely-arbitrary function of the variables. The integrals of the differential equations:

$$
\sum_{\kappa=1}^{2 k} X_{\kappa} d x_{\kappa}=0 \quad \text { or } \quad \sum_{\kappa=1}^{2 k+1} X_{\kappa} d x_{\kappa}=0
$$

are:

$$
u_{1}=c_{1}, \quad u_{2}=c_{2}, \quad \ldots, \quad u_{k}=c_{k}
$$

or

$$
u=c, \quad u_{1}=c_{1}, \quad u_{2}=c_{2}, \quad \ldots, \quad u_{k}=c_{k}
$$

respectively, when the $c$ are arbitrary constants.
We would like to pose the corresponding problem for a system of $n$ differential expressions with $n+m$ variables:

$$
\begin{aligned}
& X_{1}^{(1)} d x_{1}+X_{2}^{(1)} d x_{2}+\cdots+X_{n}^{(1)} d x_{n}+X_{n+1}^{(1)} d x_{n+1}+\cdots+X_{n+m}^{(1)} d x_{n+m}, \\
& X_{1}^{(2)} d x_{1}+X_{2}^{(2)} d x_{2}+\cdots+X_{n}^{(2)} d x_{n}+X_{n+1}^{(2)} d x_{n+1}+\cdots+X_{n+m}^{(2)} d x_{n+m}, \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
& X_{1}^{(n)} d x_{1}+X_{2}^{(n)} d x_{2}+\cdots+X_{n}^{(n)} d x_{n}+X_{n+1}^{(n)} d x_{n+1}+\cdots+X_{n+m}^{(n)} d x_{n+m} .
\end{aligned}
$$

In that way, we will follow the approach to treating the Pfaff problem that Natani used. (See Borchardt's Journ., Bd. LVIII.) Above all, we will ask what the definition of an integral of the system of equations:

$$
\begin{equation*}
\sum_{\mu=1}^{n+m} X_{\mu}^{(\nu)} d x_{\mu}=0 \tag{1}
\end{equation*}
$$

would be and how many integrals those differential equations possess, in general, when no condition equations exist between those $m(n+m)$ functions $X_{\mu}^{(\nu)}$ of the variables $x_{\mu}$.

We will then see that we have posed a problem that is analogous to the Pfaff problem. Furthermore, we shall address the question of ascertaining the integrals.

In place of the $n+m$ quantities $x_{\mu}$, we imagine that we introduce just as many new variables $v_{1}, v_{2}, \ldots, v_{p} ; u_{1}, u_{2}, \ldots, u_{p}$, which are functions of the $x_{\mu}$. The arbitrary variations $\delta x_{\mu}$, for which the $n$ expressions:

$$
\sum_{\mu} X_{\mu}^{(v)} \delta x_{\mu}
$$

do not need to be zero, are then representable in the form:

$$
\delta x_{\mu}=\sum_{\pi=1}^{p} \frac{\partial x_{\mu}}{\partial v_{\pi}} \delta v_{\pi}+\sum_{\rho=1}^{r} \frac{\partial x_{\mu}}{\partial u_{\rho}} \delta u_{\rho} .
$$

If one expresses the functions $X_{\mu}^{(\nu)}$ in terms of the new variables $v_{\pi}$ and $u_{\rho}$, as well, then one will get $n$ identities:

$$
\sum_{\mu} X_{\mu}^{(\nu)} \delta x_{\mu}=\sum_{\pi} V_{\pi}^{(\nu)} \delta v_{\pi}+\sum_{\rho} U_{\rho}^{(v)} \delta u_{\rho}
$$

in which the functions $V_{\pi}^{(\nu)}$ and $U_{\rho}^{(\nu)}$ are determined in the following way:

$$
V_{\pi}^{(\nu)}=\sum X_{\mu}^{(v)} \frac{\partial x_{\mu}}{\partial v_{\pi}}, \quad U_{\rho}^{(\nu)}=\sum X_{\mu}^{(v)} \frac{\partial x_{\mu}}{\partial u_{\rho}}
$$

If one again replaces the general variations $\delta x_{\mu}$ with the particular ones $d x_{\mu}$ then along with the given system of equations (1), the following one will result:

$$
\sum_{\pi} V_{\pi}^{(v)} d v_{\pi}+\sum_{\rho} U_{\rho}^{(\nu)} d u_{\rho}=0 \quad(n=1,2, \ldots, n)
$$

and that will be fulfilled when either all differential $d v_{\pi}$ and $d u_{\rho}$ are zero, i.e., the $v_{\pi}$ and $u_{\rho}$ are constant or the coefficients of the non-vanishing differentials are zero.

If the functions $v_{\pi}$ and $u_{\rho}$ are chosen in such a way that all quantities $V_{\pi}^{(v)}$ vanish and the $u_{\rho}$ are constant then the following $n p$ equations will exist:

$$
\begin{equation*}
\sum X_{\mu}^{(1)} \frac{\partial x_{\mu}}{\partial v_{\pi}}=0, \quad \sum X_{\mu}^{(2)} \frac{\partial x_{\mu}}{\partial v_{\pi}}=0, \quad \ldots, \quad \sum X_{\mu}^{(n)} \frac{\partial x_{\mu}}{\partial v_{\pi}}=0 \quad(\pi=1,2, \ldots, p) \tag{2}
\end{equation*}
$$

and they will replace the given system. Namely, if we regard the $v_{\pi}$ in the latter as the independent variables that must necessarily occur and differentiate with respect to them then that will give the new system (2).

When the $r$ quantities $u_{\rho}$ are set equal to constants, they will fulfill equations (1) and (2), so we call those functions $u_{\rho}$ the integrals of the given system of differential equations. The smaller the number of integrals is, the larger the number of variables that remain arbitrary will be, and the more general that the solution will be. Therefore, the question of the most general solution to the differential equations will coincide with that of the smallest number of integrals. We would now like to look for them.

One has:

$$
\begin{equation*}
\sum_{\mu} X_{\mu}^{(v)} \delta x_{\mu}=\sum_{\rho} U_{\rho}^{(v)} \delta u_{\rho} \quad(v=1,2, \ldots, n), \tag{3}
\end{equation*}
$$

so

$$
\begin{equation*}
X_{\mu}^{(\nu)}=\sum U_{\rho}^{(v)} \frac{\partial u_{\rho}}{\partial x_{\mu}} \quad(\mu=1,2, \ldots, n+m) \tag{4}
\end{equation*}
$$

and the $r(n+1)$ quantities $U_{\rho}^{(\nu)}$ and $u_{\rho}$ are calculated from those $n(n+m)$ equations.
If $n(n+m)$ is divisible by $(n+1) r$, so $n+m$ is also divisible by $n+1$, say:

$$
n+m=k(n+1),
$$

then $r$ cannot be smaller than $k n$, since otherwise that would imply condition equations between the quantities $X_{\mu}^{(v)}$, which might be excluded.

In the case where the number of variables is divisible by number of given equations, plus one, the most general integration problem will consist of a transformation under which the identities:

$$
\begin{equation*}
\sum_{\mu=1}^{k(n+1)} X_{\mu}^{(\nu)} \delta x_{\mu}=\sum_{\rho=1}^{k n} U_{\rho}^{(\nu)} \delta u_{\rho} \quad(v=1,2, \ldots, n) \tag{I}
\end{equation*}
$$

will arise, and the number of integrals $u_{\rho}=c_{\rho}$ of the equations:
(A)

$$
\sum_{\mu=1}^{k(n+1)} X_{\mu}^{(\nu)} d x_{\mu}=0
$$

is the multiple of the number of equations that gives the quotient $\frac{m+n}{n+1}$.
However, if:

$$
m+n=k(n+1)+\kappa,
$$

in which $k$ can assume values from 1 to $n$, then there will be $\kappa$ arbitrary integrals, along with $k n$ determinate ones. In fact, the $n(n+m)$ equations (4) will serve to determine the $n(n k+\kappa)$ quantities $U$. However, since only $n k$ equations will still remain for the $n k+\kappa$ quantities $u_{\rho}, \kappa$ of them will remain arbitrary. We denote them by $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{\kappa}$, and the associated coefficients $U^{(\nu)}$ by $A^{(\nu)}$. Now, the problem of integrating the equations:

$$
\begin{equation*}
\sum_{\mu=1}^{k(n+1)+\kappa} X_{\mu}^{(\nu)} d x_{\mu}=0 \quad(v=1,2, \ldots, n) \tag{B}
\end{equation*}
$$

will be posed as the search for a transformation by which one can exhibit the identities:

$$
\begin{equation*}
\sum_{\mu} X_{\mu}^{(\nu)} \delta x_{\mu}=A_{1}^{(\nu)} \delta \varphi_{1}+A_{2}^{(\nu)} \delta \varphi_{2}+\cdots+A_{\kappa}^{(\nu)} \delta \varphi_{\lambda}+\sum_{\rho=1}^{n k} U_{\rho}^{(\nu)} \delta u_{\rho} \quad(v=1,2, \ldots, n) . \tag{II}
\end{equation*}
$$

The integrals are:

$$
\varphi_{1}=C_{1}, \varphi_{2}=C_{2}, \ldots, \varphi_{\kappa}=C_{k} ; \quad u_{1}=c_{1}, u_{2}=c_{2}, \ldots, u_{n k}=c_{n k}
$$

in which $C$ and $c$ mean arbitrary constants.
If the number of variables is $k(n+1)-\kappa$ then there will be $(k-1) n$ integrals that are determinate and $n+1-\kappa$ of them that are arbitrary.

The integrals will not change with one's choice of the $k$ variables $v_{\pi}$, which are functions of $x_{\mu}$ that one considers to be independent, because if one assumes that one has $\mu=n+m$ equations:

$$
x_{\mu}=f_{\mu}\left(v_{1}, v_{2}, \ldots, v_{k}, u_{1}, u_{2}, \ldots, u_{r}\right),
$$

in which the $v$ are arbitrary, but the $u_{\rho}$ have the meaning that is expressed by the identities:

$$
\sum_{\mu=1}^{n+m} X_{\mu}^{(\nu)} \delta x_{\mu}=\sum_{\rho=1}^{r} U_{\rho}^{(\nu)} \delta u_{\rho},
$$

then one will have:

$$
\sum_{\mu=1}^{n+m} X_{\mu}^{(\nu)} \delta x_{\mu}=\sum_{\rho=1}^{r} U_{\rho}^{\prime(v)} \delta u_{\rho}+\sum_{\pi=1}^{k} V_{\pi}^{\prime(v)} \delta v_{\pi} .
$$

However, since the identities that follow from that, namely:

$$
\sum_{\rho} U_{\rho}^{(\nu)} \delta u_{\rho}=\sum_{\rho} U_{\rho}^{\prime(\nu)} \delta u_{\rho}+\sum_{\pi} V_{\pi}^{\prime(\nu)} \delta v_{\pi},
$$

are fulfilled only when:

$$
U_{\rho}^{\prime(\nu)}=U_{\rho}^{(\nu)}, \quad V_{\pi}^{\prime(\nu)}=0
$$

the $u_{\rho}$ will be integrals no matter what the $v_{\pi}$ might be as functions of $x_{\mu}$.
We recognize that the aforementioned transformation problems are analogous to the Pfaff problem.

With the help of the elimination of $\kappa$ variables and their differentials from the arbitrarilychosen equations:

$$
\varphi_{1}=C_{1}, \varphi_{2}=C_{2}, \ldots, \varphi_{\kappa}=C_{k}
$$

and

$$
\delta \varphi_{1}=0, \delta \varphi_{2}=0, \ldots, \delta \varphi_{\kappa}=0,
$$

the integration of equations (B) will come down to the integration of a system of the form (A) in which only $k(n+1)$ variables $x$ and their derivatives will remain in the identities (II) and the first $\kappa$ terms on the right will drop out.

The integrals of equations (A) are:

$$
u_{1}=c_{1}, u_{2}=c_{2}, \ldots, u_{n k}=c_{n k} .
$$

If one differentiates those equations and adds the differentials $d u$, when multiplied by certain quantities, to them then equations (A) will arise. One can also put $\sum X_{\mu}^{(\nu)} \delta x_{\mu}$ into the form $\sum U_{\rho}^{(\nu)} \delta u_{\rho}$ with the help of the integrals, and indeed, the $n^{2} k$ quantities $U_{\rho}^{(\nu)}$ are defined by the equations:

$$
U_{\rho}^{(\nu)}=\sum X_{\mu}^{(\nu)} \frac{\partial x_{\mu}}{\partial u_{\rho}},
$$

in which $x_{\mu}$ are regarded as functions of the $u_{\rho}$, and the $k v_{\mu}$ are regarded as arbitrary.
There are also integrals that include arbitrary functions in place of the arbitrary constants.
All relations between the $U$ and the $u$ that make the $n$ expressions vanish have equations (A) as a consequence, so they will also give a system of integrals. Now, if say $k n-q$ arbitrary relations exist:

$$
u_{q+1}=f_{1}\left(u_{1}, u_{2}, \ldots, u_{q}\right), \quad u_{q+2}=f_{2}\left(u_{1}, u_{2}, \ldots, u_{q}\right), \quad \ldots, \quad u_{k n}=f_{k n-q}\left(u_{1}, u_{2}, \ldots, u_{q}\right)
$$

then the expression will become:

$$
\sum_{\rho} U_{\rho}^{(\nu)} \delta u_{\rho}=\sum_{i=1}^{q}\left(U_{i}^{(\nu)}+U_{q+1}^{(\nu)} \frac{\partial f_{1}}{\partial u_{i}}+U_{q+2}^{(\nu)} \frac{\partial f_{2}}{\partial u_{i}}+\cdots+U_{k n}^{(\nu)} \frac{\partial f_{k n-q}}{\partial u_{i}}\right) \delta u_{i},
$$

and they will vanish when:

$$
U_{i}^{(\nu)}+U_{q+1}^{(\nu)} \frac{\partial f_{1}}{\partial u_{i}}+U_{q+2}^{(\nu)} \frac{\partial f_{2}}{\partial u_{i}}+\cdots+U_{k n}^{(\nu)} \frac{\partial f_{k n-q}}{\partial u_{i}} \quad(i=1,2, \ldots, q, v=1,2, \ldots, n)
$$

are zero. The $k n+(n-1) q$ new relations are also integrals, but they include $k n-q$ arbitrary functions of $q$ variables instead of constants.

The larger that $q$ is, the less arbitrary functions there will be, but the more integrals there will be. It is only the case in which equation (A) has $2 k$ variables that the number of integrals will remain constantly $2 k$.

Finally, the expressions $\sum U_{\rho}^{(\nu)} \delta u_{\rho}$ can be made to vanish in such a way that all of the $U_{\rho}^{(\nu)}$ will be zero, and that system of integrals with no arbitrary constants and functions is called the singular one.

When we set $k=1$ in the system of equations (A) and (B), we will arrive, on the one hand, at the system of total differential equations:

$$
\sum_{\mu=1}^{n+1} X_{\mu}^{(\nu)} d x_{\mu}=0
$$

and on the other, at the system:

$$
\sum_{\mu=1}^{k+1+\kappa} X_{\mu}^{(\nu)} d x_{\mu}=0
$$

After calculating the $n$ ratios:

$$
\frac{d x_{\mu}}{d x_{1}}=\frac{Y_{\mu}}{Y_{1}} \quad(m=2,3, \ldots, n+1),
$$

one writes the first system in the form:

$$
d x_{1}: d x_{2}: \ldots: d x_{n+1}=Y_{1}: Y_{2}: \ldots: Y_{n+1}
$$

That system will be integrated when one knows $n$ mutually-independent integrals $u_{1}=c_{1}, u_{2}=c_{2}$, $\ldots, u_{n}=c_{n}$ of the linear partial differential equation:

$$
\frac{\partial u}{\partial x_{1}} Y_{1}+\frac{\partial u}{\partial x_{2}} Y_{2}+\cdots+\frac{\partial u}{\partial x_{n+1}} Y_{n+1}=0 .
$$

However, the same thing will also be integrated when one can take $n$ of the assumed equations:

$$
\frac{\partial u_{v}}{\partial x_{1}} Y_{1}+\frac{\partial u_{v}}{\partial x_{2}} Y_{2}+\cdots+\frac{\partial u_{v}}{\partial x_{n+1}} Y_{n+1}=0 \quad(v=1,2, \ldots, n)
$$

and derive the identities:

$$
Y_{1} \delta x_{\mu}-Y_{\mu} \delta x_{1}=A_{1}^{(\mu)} \delta u_{1}+A_{2}^{(\mu)} \delta u_{2}+\cdots+A_{n}^{(\mu)} \delta u_{n} \quad(\nu=1,2, \ldots, n),
$$

in which $A_{1}^{(\mu)}, A_{2}^{(\mu)}, \ldots, A_{n}^{(\mu)}$ are functions of $x$. After multiplying the last $n$ identities by suitable factors and adding them, that will yield a system of the form (I).

The second of the systems above possesses $k$ arbitrary integrals and is reducible to the system of total differential equations.

If we set $n=1$ in equations (A) and (B) then we will come to the two Pfaff equations. The adaptation of the known method for solving the equation $\sum_{\mu=1}^{2 k} X_{\mu} d x_{\mu}=0$ would require that we transform the $n$ equations of the system:

$$
\sum_{\mu=1}^{(n+1) k} X_{\mu}^{(v)} d x_{\mu}=0 \quad(\nu=1,2, \ldots, n)
$$

into $n$ other with $k(n+1)-1$ new variables $x^{(1)}$ that are functions of the $k(n+1)$ variables. If we demand that then we can set $n$ of those constants equal to constants and seek to once more transform the equations in $(k-1)(n+1)$ variables that arise in that way into a system of $n$ equations with $(k-1)(n+1)-1$ new variables $x^{(2)}$ and once more set $n$ functions equal to constants, since $n$ arbitrary integrals will indeed exist. If we proceed in the same way then we will ultimately obtain $n$ equations with $n+1$ variables that possess $n$ integrals. In total, we will have to set $k n$ functions of the variables $x_{\mu}$ equal to constants, and they will be integrals of the system.

However, one easily convinces oneself that a transformation of the desired kind with no condition equations for the functions $X_{\mu}^{(\nu)}$ is possible only when the number of equations is one. Even when we convert the given system into another one with just as many variables, we cannot generally succeed in making the new system admit the required transformation.

Since the Pfaff method of integration cannot be employed then, and obviously a generalization of the Clebsch method is not possible either, we restrict ourselves to deriving differential equations for the functions $U_{\rho}^{(\nu)}$ and $u_{\rho}$ that include the "first Pfaff system" as a special system from the $n$ equations:

$$
\sum_{\mu=1}^{k(n+1)} X_{\mu}^{(\nu)} d x_{\mu}=\sum_{\rho=1}^{k n} U_{\rho}^{(\nu)} d u_{\rho}
$$

With the help of the $n k(n+1)$ quantities $X_{\mu}^{(v)}$, we can define $[n k(n+1)]^{2}$ quantities

$$
\begin{equation*}
\frac{\partial X_{\lambda}^{(\nu)}}{\partial x_{\mu}}-\frac{\partial X_{\mu}^{(\nu)}}{\partial x_{\lambda}}=a_{\lambda \mu}^{\left(\nu \nu^{\prime}\right)}, \tag{5}
\end{equation*}
$$

and then determine $n k(n+1)$ quantities $z$ from just as many linear equations:

$$
\begin{equation*}
X_{\mu}^{(\nu)}=\sum_{\nu^{\prime}=1}^{n} \sum_{\lambda=1}^{k(n+1)} a_{\lambda \mu}^{\left(\nu \nu^{\prime}\right)} z_{\left(\nu^{\prime}-1\right) k(n+1)+\lambda} . \tag{6}
\end{equation*}
$$

The determinant of that system of equations:

$$
D=\sum \pm a_{11}^{(11)} a_{22}^{(11)} \cdots a_{k(n+1), k(n+1)}^{(11)} a_{11}^{(2)} \cdots a_{k(n+1), k(n+1)}^{(n, n)}
$$

is skew-symmetric, since one has:

$$
a_{\lambda \mu}^{(\nu \nu)}=0, \quad a_{\lambda \mu}^{\left(\nu \nu^{\prime}\right)}=-a_{\mu \lambda}^{\left(\nu^{\prime} \nu\right)},
$$

and with no condition equation in the $X_{\mu}^{(\nu)}$, it will not vanish either since its order $n k(n+1)$ is even in any event.

If one observes the $n k(n+1)$ equations:

$$
X_{\mu}^{(\nu)}=U_{1}^{(\nu)} \frac{\partial u_{1}}{\partial x_{\mu}}+U_{2}^{(\nu)} \frac{\partial u_{2}}{\partial x_{\mu}}+\cdots+U_{k n}^{(\nu)} \frac{\partial u_{k n}}{\partial x_{\mu}}
$$

and the representations:

$$
\begin{aligned}
a_{\lambda \mu}^{\left(\nu v^{\prime}\right)} & =\left(\frac{\partial U_{1}^{(\nu)}}{\partial x_{\mu}} \cdot \frac{\partial u_{1}}{\partial x_{\lambda}}+\frac{\partial U_{2}^{(\nu)}}{\partial x_{\mu}} \cdot \frac{\partial u_{2}}{\partial x_{\lambda}}+\cdots+\frac{\partial U_{k n}^{(\nu)}}{\partial x_{\mu}} \cdot \frac{\partial u_{k n}}{\partial x_{\lambda}}\right) \\
& -\left(\frac{\partial U_{1}^{\left(\nu^{\prime}\right)}}{\partial x_{\mu}} \cdot \frac{\partial u_{1}}{\partial x_{\lambda}}+\frac{\partial U_{2}^{\left(\nu^{\prime}\right)}}{\partial x_{\mu}} \cdot \frac{\partial u_{2}}{\partial x_{\lambda}}+\cdots+\frac{\partial U_{k n}^{\left(\nu^{\prime}\right)}}{\partial x_{\mu}} \cdot \frac{\partial u_{k n}}{\partial x_{\lambda}}\right) \\
& +\left(U_{1}^{(\nu)}-U_{1}^{\left(\nu^{\prime}\right)}\right) \frac{\partial^{2} u_{1}}{\partial x_{\mu} \partial x_{\lambda}}+\left(U_{2}^{(\nu)}-U_{2}^{\left(\nu^{\prime}\right)}\right) \frac{\partial^{2} u_{2}}{\partial x_{\mu} \partial x_{\lambda}}+\cdots+\left(U_{k n}^{(\nu)}-U_{k n}^{\left(v^{\prime}\right)}\right) \frac{\partial^{2} u_{k n}}{\partial x_{\mu} \partial x_{\lambda}},
\end{aligned}
$$

then equations (6) can be put into the form:

$$
\sum_{\rho=1}^{k n} U_{\rho}^{(\nu)} \frac{\partial u_{\rho}}{\partial x_{\mu}}=\sum_{\nu^{\prime}=1}^{n} \sum_{\lambda=1}^{k(n+1)} \sum_{\rho=1}^{k n}\left(\frac{\partial U_{\rho}^{(\nu)}}{\partial x_{\mu}} \frac{\partial u_{\rho}}{\partial x_{\lambda}}-\frac{\partial U_{\rho}^{\left(\nu^{\prime}\right)}}{\partial x_{\mu}} \frac{\partial u_{\rho}}{\partial x_{\lambda}}+\left(U_{\rho}^{(\nu)}-U_{\rho}^{\left(\nu^{\prime}\right)}\right) \frac{\partial^{2} u_{\rho}}{\partial x_{\mu} \partial x_{\lambda}}\right) z_{\left(\nu^{\prime}-1\right) k(n+1)+\lambda},
$$

or with a different ordering of the summands, into the form:

$$
\begin{aligned}
& \sum_{\rho} \frac{\partial U_{\rho}^{(\nu)}}{\partial x_{\mu}}\left[\frac{\partial u_{\rho}}{\partial x_{1}} z_{1}+\frac{\partial u_{\rho}}{\partial x_{2}} z_{2}+\cdots+\frac{\partial u_{\rho}}{\partial x_{k(n+1)}} z_{k(n+1)}+\cdots+\frac{\partial u_{\rho}}{\partial x_{1}} z_{(n-1) k(n+1)+1}+\cdots+\frac{\partial u_{\rho}}{\partial x_{k(n+1)}} z_{n k(n+1)}\right] \\
- & \sum_{\rho} \frac{\partial u_{\rho}}{\partial x_{\mu}}\left[\frac{\partial U_{\rho}^{(1)}}{\partial x_{1}} z_{1}+\frac{\partial U_{\rho}^{(1)}}{\partial x_{2}} z_{2}+\cdots+\frac{\partial U_{\rho}^{(1)}}{\partial x_{k(n+1)}} z_{k(n+1)}+\cdots+\frac{\partial U_{\rho}^{(n)}}{\partial x_{1}} z_{(n-1) k(n+1)+1}+\cdots+\frac{\partial U_{\rho}^{(n)}}{\partial x_{k(n+1)}} z_{n k(n+1)}+U_{\rho}^{(\nu)}\right]
\end{aligned}
$$

(7)

$$
\begin{gathered}
=\sum_{\rho=1}^{k n} \sum_{v^{\prime}=1}^{n}\left(U_{\rho}^{\left(v^{\prime}\right)}-U_{\rho}^{(v)}\right)\left[\frac{\partial^{2} u_{\rho}}{\partial x_{\mu} \partial x_{1}} z_{\left(v^{\prime}-1\right) k(n+1)+1}+\cdots+\frac{\partial^{2} u_{\rho}}{\partial x_{\mu} \partial x_{2}} z_{\left(v^{\prime}-1\right) k(n+1)+1}+\cdots+\frac{\partial^{2} u_{\rho}}{\partial x_{\mu} \partial x_{k(n+1)}} z_{n k(n+1)}\right] \\
(\mu=1,2, \ldots, k(n+1), v=1,2, \ldots, n) .
\end{gathered}
$$

We regard those equations as linear in the $n k$ quantities:

$$
Y_{\rho}=\sum_{\lambda=1}^{k(n+1)} \frac{\partial u_{\rho}}{\partial x_{\lambda}}\left(z_{\lambda}+z_{k(n+1)+\lambda}+\cdots+z_{(n-1) k(n+1)+\lambda}\right)
$$

and the $n^{2} k$ quantities:

$$
Z_{\rho}^{(\nu)}=\frac{\partial U_{\rho}^{(1)}}{\partial x_{1}} z_{1}+\frac{\partial U_{\rho}^{(1)}}{\partial x_{2}} z_{2}+\cdots+\frac{\partial U_{\rho}^{(n)}}{\partial x_{k(n+1)}} z_{n k(n+1)}+U_{\rho}^{(\nu)}
$$

and solve for those unknowns.
After introducing the symbols:

$$
\frac{\partial U_{\rho}^{(\nu)}}{\partial x_{\mu}}=P_{\rho, \mu}^{(\nu)}, \quad \frac{\partial u_{\rho}}{\partial x_{\mu}}=Q_{\rho, \mu}
$$

the determinant of the system (7) will read:

$$
\begin{aligned}
\Delta=(-1)^{n^{2} k} & \sum \pm\left(P_{11}^{(1)}, P_{22}^{(1)} \cdots P_{k n, k n}^{(1)} ; Q_{1, k n+1}, Q_{2, k n+2} \cdots Q_{k, k(n+1)} ; 0_{1}, 0_{2} \cdots 0_{k(n-1)},\right. \\
& \left.Q_{1, k(n-1)+1}, Q_{2, k(n-1)+2} \cdots Q_{2 k, k(n+1)} ; 0_{1}, 0_{2} \cdots 0_{k}, Q_{1, k+1}, Q_{2, k+2} \cdots Q_{n k, k(n+1)}\right)
\end{aligned}
$$

(in which the zeroes are given indices that indicate their total number) and will not vanish, in general. It is decomposable into a sum of products of $n$ determinants of order $k(n+1)$, and indeed, those products are constructed in such a way that the first determinant will read:

$$
\left|\begin{array}{llllll}
P_{k, 1}^{\left(k_{1}\right)} & \cdots & P_{k_{k}, 1}^{\left(v_{1}\right)} & Q_{11} & \cdots & Q_{k n, 1} \\
\vdots & & \vdots & \vdots & & \vdots \\
P_{\lambda_{1}, k(n+1)}^{(1)} & \cdots & P_{i_{k}, k(n+1)}^{\left(v_{1}\right)} & Q_{1, k(n+1)} & \cdots & Q_{k n, k(n+1)}
\end{array}\right|,
$$

in which $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}$ mean any $k$ numbers from the sequence $1,2, \ldots, k n$, and $v_{1}$ is one of the numbers $1,2, \ldots, n$. The other $n-1$ determinants are constructed likewise, except that $\lambda_{1}, \ldots, \lambda_{k}$ will then mean other combinations of the numbers $1,2, \ldots, k n$, and $v_{1}$ will also have a different value in each determinant.

If one lets $C_{\alpha}^{(\beta)}$ denote the number of combinations of $\alpha$ elements from the class $\beta$ then there will obviously be:

$$
C_{n k}^{(k)} \cdot C_{n k-k}^{(k)} \cdots C_{n k-(n-1) k}^{(k)}=\frac{(n k)!}{(k!)^{n}}
$$

products of the stated kind. The sum of all those terms (each of which is given the associated sign) is equal to $\Delta$, as one easily sees by noting the theorem:

If a system of $n^{2}$ elements in $m$ rows has more than $n-m$ columns then its determinant will be zero.

In order to calculate the numerator in $(-1)^{n^{2} k} Y_{\rho}$, one must replace the columns:

$$
P_{\rho, 1}^{(\nu)} \cdots P_{\rho, k(n+1)}^{(\nu)} \quad(v=1,2, \ldots, n)
$$

with

$$
A_{1}^{(\nu)} \cdots A_{k(n+1)}^{(\nu)}
$$

in the aforementioned $\frac{(n k)!}{(k!)^{n}}$ products, where one understands $A_{\mu}^{(\nu)}$ to mean the double sum on the right-hand side of equation (7). Along with the latter, there are other $n$-term products of determinants of order $k(n+1)$ that are constructed as follows: A first determinant reads:

$$
\left|\begin{array}{llllllllll}
P_{\lambda_{1}, 1}^{(\nu)} & \cdots & P_{\lambda_{k}, 1}^{(\nu)} & Q_{11} & \cdots & Q_{\rho-1,1} & P_{\lambda_{k+1}, 1}^{(\nu)} & Q_{\rho+1,1} & \cdots & Q_{k n, 1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
P_{\lambda_{1}, k(n+1)}^{(\nu)} & \cdots & P_{\lambda_{k}, k(n+1)}^{(\nu)} & Q_{1, k(n+1)} & \cdots & Q_{\rho-1, k(n+1)} & P_{\lambda_{k}+1, k(n+1)}^{(\nu)} & Q_{\rho+1, k(n+1)} & \cdots & Q_{k n, k(n+1)}
\end{array}\right|,
$$

in which $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k+1}$ mean any $k+1$ numbers in the sequence $1,2, \ldots, k n$. The second, third, $\ldots(n-2)^{\text {th }}$ determinants of the product have the form:

$$
\left|\begin{array}{llllll}
P_{\lambda_{1}^{\prime}, 1}^{\left(y^{\prime}\right)} & \cdots & P_{\lambda_{k}^{\prime}, 1}^{\left(\nu^{\prime}\right)} & Q_{11} & \cdots & Q_{k n, 1} \\
\vdots & & \vdots & \vdots & & \vdots \\
P_{\lambda_{1}^{\prime}, k(n+1)}^{\left(v^{\prime}\right)} & \cdots & P_{\lambda_{k}^{\prime}, k(n+1)}^{\left(v^{\prime}\right)} & Q_{1, k(n+1)} & \cdots & Q_{k n, k(n+1)}
\end{array}\right|
$$

in which the $\lambda_{1}^{\prime}, \ldots, \lambda_{k}^{\prime}$ mean other numbers in the sequence $1,2, \ldots, k n$, and $v^{\prime}$ assumes $n-2$ sets of $v$ different values from the sequence $1,2, \ldots, n$ in succession. Among the numbers $1,2, \ldots$, $k n\left(1,2, \ldots, n\right.$, resp.), if the following ones $\lambda_{1}^{\prime \prime}, \ldots, \lambda_{k-1}^{\prime \prime}\left(v^{\prime \prime}\right.$, resp.) remain then the latter determinant will have products of the form:

$$
\left|\begin{array}{lllllll}
P_{\lambda^{\prime \prime}, 1}^{\left(v^{\prime \prime}\right)} & \cdots & P_{\lambda_{k}^{\prime \prime}, 1}^{\left(\nu^{\prime \prime}\right)} & A_{1}^{\left(v^{\prime \prime}\right)} & Q_{11} & \cdots & Q_{k n, 1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
P_{\lambda_{1}^{\prime \prime}, k(n+1)}^{\left(v^{\prime \prime}\right)} & \cdots & P_{\lambda_{k}^{\prime \prime}, k(n+1)}^{\left(v^{\prime \prime}\right)} & A_{k(n+1)}^{\left(\nu^{\prime \prime}\right)} & Q_{1, k(n+1)} & \cdots & Q_{k n, k(n+1)}
\end{array}\right| .
$$

One can define:

$$
(n-1)!C_{k n}^{(k+1)} \cdot C_{n k-k-1}^{(k)} \cdots C_{k n-(n-2) k-1}^{(k)} \cdot C_{k n(n-1) k-1}^{(k-1)}=(n-1)!\frac{(k n)!}{(k+1) \cdot(k-1) \cdot(k!)^{n-1}}
$$

such products, so there will be, in total:

$$
\frac{(k n)!}{(k+1)(k!)^{n}(k-1)!}[(k+1) \cdot(k-1)!+(n-1)!k!]
$$

$n$-term products of determinants of order $k(n+1)$ in the numerator of $(-1)^{n^{2} k-1} Z_{\rho}^{(\nu)}$, and no other terms of that order can occur.

One sees from that description of the structure of the values for the $n k(n+1)$ unknowns $Y_{\rho}$ and $Z_{\rho}^{(\nu)}$ that those values will generally prove to be different, except in the case of $n=1$, where all of the quantities $A_{\mu}^{(v)}$ will vanish. The solutions of the system (7) are then:

$$
\begin{align*}
& \frac{\partial u_{\rho}}{\partial x_{1}} z_{1}+\frac{\partial u_{\rho}}{\partial x_{2}} z_{2}+\cdots+\frac{\partial u_{\rho}}{\partial x_{2 k}} z_{2 k}=0 \\
& \frac{\partial U_{\rho}}{\partial x_{1}} z_{1}+\frac{\partial U_{\rho}}{\partial x_{2}} z_{2}+\cdots+\frac{\partial U_{\rho}}{\partial x_{2 k}} z_{2 k}+U_{\rho}=0
\end{align*}
$$

because the determinant:

$$
\Delta=(-1)^{k} \sum \pm \frac{\partial U_{1}}{\partial x_{1}} \frac{\partial U_{2}}{\partial x_{2}} \cdots \frac{\partial U_{k}}{\partial x_{n}} \frac{\partial u_{1}}{\partial x_{k+1}} \frac{\partial u_{2}}{\partial x_{k+2}} \cdots \frac{\partial u_{k}}{\partial x_{2 k}}
$$

does not vanish, in general. ( $U_{\rho}$ is written for $U_{\rho}^{(1)}$ in that.)

The $k$ equations ( $\beta$ ) imply the following $k-1$ equations:

$$
z_{1} \frac{\partial}{\partial x_{1}}\left(\frac{U_{\rho}}{U_{k}}\right)+z_{2} \frac{\partial}{\partial x_{2}}\left(\frac{U_{\rho}}{U_{k}}\right)+\cdots+z_{2 k} \frac{\partial}{\partial x_{12 k}}\left(\frac{U_{\rho}}{U_{k}}\right)=0 \quad(\rho=1,2, \ldots, k-1),
$$

and therefore the $2 k-1$ functions $u_{1}, u_{2}, \ldots, u_{k}, \frac{U_{1}}{U_{k}}, \frac{U_{2}}{U_{k}}, \ldots, \frac{U_{k-\mu}}{U_{k}}$ will all satisfy the same linear partial differential equation:

$$
\frac{\partial \varphi}{\partial x_{1}} z_{1}+\frac{\partial \varphi}{\partial x_{2}} z_{2}+\cdots+\frac{\partial \varphi}{\partial x_{2 k}} z_{2 k}=0,
$$

whose general solution $\varphi$ is an arbitrary function of the latter functions. However, it is also a solution of the first Pfaff problem, and $\varphi=c$ is a first integral of the Pfaff equation $\sum_{\mu=1}^{2 k} X_{\mu} d x_{\mu}=$ 0. Clebsch showed how one can introduce and carry out further integrals with the help of that determination (Borchardt's Journal, Bd. LX).

It becomes clear here why one must successively make a transition from one integral $\varphi=c$ to a second one, from the second one to a third one, etc., in the Pfaff problem. Due to the coincidence of the values for $Y_{\rho}$ and $Z_{\rho}$, or due to the agreement in the differential equations for the functions $u_{\rho}$ and $U_{\rho} / U_{k}$, one cannot, in fact, find a system of associated functions $u_{\rho}$ and $U_{\rho} / U_{k}$ that solves the problem.

The facts that one simultaneously considers $n k(n+1)$ differential equations that one finds in the $n k(n+1)$ solutions of the system (7) in the general case of $n$ equations (A) with $k(n+1)$ variables, and that they must imply a system of associated functions $u_{\rho}$ and $U_{\rho}^{(\nu)}$ that solve the problem naturally makes further investigations difficult, and the complexities of the differential equations (which have order two with respect to the functions $u_{\rho}$ and order one with respect to the functions $U_{\rho}^{(\nu)}$ ) are not easy to discuss, even for low values of $n$ and $k$. That fact must then characterize the relationship between the Pfaff problem and the general one.

Prague, 18 December 1884.

