“Frenets Formeln für den Raum von Riemann,” Math. Zeit. **6** (1920), 94-99.

Frenet’s formulas for Riemann’s space

By

Wilhelm Blaschke in Hamburg

Translated by D. H. Delphenich

It will be shown here that: In **Riemann**’s space, in which the arc length is described by a quadratic differential form:

$$ds^2 = \sum_{i,k=1}^{n} g_{ik}\, dx_i\, dx_k\ ,$$

one can introduce a moving n-bein for any curve and linearly combine the invariant derivatives of those n vectors with respect to arc length from the n vectors themselves. The generalization of the **Frenet** formulas that is obtained in that way looks just like the special formulas for **Euclid**’s space, and all of the calculation that goes into **Riemann**’s metric in the general case will involve only slightly more work that it does for the special **Euclidian** case.

In order for me to be brief, I would like to keep to the notations that **H. Weyl** used in his wonderful book *Raum – Zeit – Materie* (Berlin, 1918), and in particular, I shall drop the summation sign, following a suggestion by **A. Einstein**. Let the basic quadratic form be assumed to be, say, positive-definite. Along our spatial curve $x_i = x_i\,(s)$, we next call upon the contacting unit vectors:

$$\xi^i = \frac{dx_i}{ds}, \qquad g_{ik}\,\xi^i\xi^k = 1. \tag{1}$$

For an arbitrary vector field φ^i, one can then derive the tensors [cf., **Weyl**, pp. 104 (I′), called “extension” by **A. Einstein**]:

$$\psi_k^i = \frac{\partial\varphi^i}{\partial x_k} + \begin{Bmatrix} k\ r \\ i \end{Bmatrix}\varphi^r \tag{2}$$

and then derive the vectors:

$$\psi_k^i\,\xi^k = \frac{\partial\varphi^i}{\partial x_k}\xi^k + \begin{Bmatrix} k\ r \\ i \end{Bmatrix}\xi^k\varphi^r = \frac{d\varphi^i}{ds} + \begin{Bmatrix} k\ r \\ i \end{Bmatrix}\xi^k\varphi^r \tag{3}$$

from them by "contracting" along our curve $x_i(s)$. The vectors φ^i along the curve $x_i(s)$ still appear on the right-hand side of this. We have then found an invariant differential process θ that makes it possible to derive a new family of vectors $\theta\varphi^i(s)$ from any given one $\varphi^i(s)$ along $x_i(s)$:

(4) $$\theta\varphi^i = \frac{d\varphi^i}{ds} + \begin{Bmatrix} k\ r \\ i \end{Bmatrix} \xi^k \varphi^r .$$

Moreover, one can also arrive at the invariant derivative (4) when one starts from the "parallel displacement" along a curve $x_i(s)$ in **Riemann** space that was described by **Levi-Civita** ([1]). Namely, if s and $s + ds = s + \delta s$ are two neighboring points along the curve then [cf., **Weyl**, pp. 100 (35)]:

$$\varphi^i + \delta\varphi^i = \varphi^i - \begin{Bmatrix} k\ r \\ i \end{Bmatrix} dx_k\, \varphi^r$$

will be the vector that arises by parallel translation from s to $s + ds$. However, one will then have:

(4*) $$\frac{\varphi^i(s+ds) - \varphi^i(s+\delta s)}{ds} = \frac{d\varphi^i}{ds} + \begin{Bmatrix} k\ r \\ i \end{Bmatrix} \frac{dx_k}{ds} \varphi^r = \theta\varphi^i .$$

That θ-process shall be repeatedly applied to the tangent vectors $\xi^i = \xi^i_{(1)}$:

(5) $$\theta\xi^i_{(1)} = \xi^i_{(2)}, \qquad \ldots, \qquad \theta\xi^i_{(k-1)} = \xi^i_{(k)} .$$

(I put the subscripts in parentheses here, since they have nothing to do with the other superscripts and subscripts.) In that way, one will find a moving n-bein $\xi^i_{(k)}$ along the curve $x_i(s)$ that is invariant under coordinate transformations, but which does not consist of orthogonal unit vectors. However, we would like to assume that our curve is "generic" in the sense that the n vectors $\xi^i_{(k)}$ are not linearly dependent. One convinces oneself that this is immesdiately possible in **Euclidian** space when the x_i define a **Cartesian** axis-cross. In that case, the **Christoffel** symbols will all be zero, and the θ-process will reduce to derivation with respect to arc length.

We can now derive an orthogonal n-bein of unit vectors η from the n-bein ξ by the orthogonalization process of **E. Schmidt** ([2]). To that end, let the "inner product" of the vectors $\xi^i_{(p)}$, $\xi^i_{(q)}$ be denoted by (p, q):

(6) $$g_{ik}\, \xi^i_{(p)}\, \xi^i_{(q)} = (p, q)$$

([1]) **T. Levi-Civita**, "Nozione di parallelism in una varietà quanlunque…," Rendiconti di Palermo **42** (1917).
([2]) One might confer, say, **G. Kowalewski**'s *Determinantentheorie*, Leipzig, 1909, pp. 423-426.

and the determinant by:

(7)
$$\begin{vmatrix} (1,1) & (1,2) & \cdots & (1,p) \\ (2,1) & (2,2) & \cdots & (2,p) \\ \vdots & \vdots & \cdots & \vdots \\ (p,1) & (p,2) & \cdots & (p,p) \end{vmatrix} = D_{(p)} .$$

From our assumption (viz., $ds^2 > 0$, and therefore $| g_{ik} | > 0$), that determinant $D_{(p)}$ will be positive:

(8)
$$D_p = \| g_{ik} \| \cdot \left\| \xi^i_{(r)} \right\|^2 > 0 \qquad (i, k = 1, 2, \ldots, n \,;\, r = 1, 2, \ldots, p).$$

The desired normalized orthogonal system of η will then be implied by the formulas:

(9)
$$\eta^i_{(p)} = \frac{1}{\sqrt{D_{(p-1)}D_{(p)}}} \begin{vmatrix} (1,1) & \cdots & (1,p-1) & \xi^i_{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ (p,1) & \cdots & (p,p-1) & \xi^i_{(p)} \end{vmatrix}, \quad (p = 1, 2, \ldots, n \,;\, D_0 = 1).$$

(One can then take the roots to be, say, all positive in that way.)

In fact, one immediately confirms the orthogonality:

(10)
$$g_{ik}\, \eta^i_{(p)}\, \xi^k_{(q)} = 0, \qquad \text{and therefore} \quad g_{ik}\, \eta^i_{(p)}\, \eta^k_{(q)} = 0$$

for $p > q$. Moreover, on the grounds of orthogonality, one has the normalization:

(11)
$$g_{ik}\, \eta^i_{(p)}\, \eta^k_{(p)} = 1,$$

in which one replaces all $\xi^i_{(q)}$, $q < p$ in the first factor with zeroes when defining the inner product.

The η now define the moving orthogonal n-bein of our curve $x_i(s)$. As a normalized orthogonal system, the η are linearly independent, and we can then express their invariant derivatives $\theta\,\eta$ as linear combinations of the η:

(12)
$$\theta\,\eta^i_{(p)} = \alpha_{(pq)}\, \eta^i_{(q)}, \qquad \alpha_{(pq)} = g_{ik}\, (\theta\,\eta^i_{(p)})\,\eta^k_{(q)} .$$

That is what the desired **Frenet** formulas will look like, and our problem is to calculate the coefficients α, which will be invariants ("curvatures") of our curve $x_i(s)$, since everything was calculated in an invariant way.

I will next show: Since the η define a normalized orthogonal system, the matrix of the α will be skew-symmetric. Namely, if:

(13) $$\frac{d}{ds} g_{ik}\, \eta^i_{(p)}\, \eta^k_{(q)} = 0\,.$$

The left-hand side expands to:

$$\frac{\partial g_{ik}}{\partial x_r} \xi^r\, \eta^i_{(p)}\, \eta^k_{(q)} + g_{ik} \left(\theta\, \eta^i_{(p)} - \begin{Bmatrix} h\,l \\ i \end{Bmatrix} \xi^r\, \eta^i_{(p)} \right) \eta^k_{(q)} + g_{ik} \left(\theta\, \eta^i_{(q)} - \begin{Bmatrix} h\,l \\ i \end{Bmatrix} \xi^r\, \eta^i_{(q)} \right) \eta^k_{(p)}.$$

Since [**Weyl**, pp. 99 (31), pp. 98 (29)]:

$$\frac{\partial g_{ik}}{\partial x_r} = \begin{bmatrix} i\,r \\ k \end{bmatrix} + \begin{bmatrix} k\,r \\ i \end{bmatrix}, \qquad g_{ik} \begin{Bmatrix} h\,l \\ i \end{Bmatrix} = \begin{bmatrix} h\,l \\ k \end{bmatrix},$$

if further follows that:

(14) $$\frac{d}{ds} g_{ik}\, \eta^i_{(p)}\, \eta^k_{(q)} = g_{ik} \left(\theta\, \eta^i_{(p)} \right) \eta^k_{(q)} + g_{ik}\, \eta^k_{(p)} \left(\theta\, \eta^i_{(q)} \right),$$

and equation (13) will then reduce to our assertion:

(15) $$\alpha_{(pq)} + \alpha_{(qp)} = 0.$$

Moreover: One can easily see from the way that the vectors η were constructed that:

(16) $$\alpha_{(pq)} = 0 \qquad \text{for} \qquad p < q + 1.$$

In fact, $\eta_{(p)}$ is a linear combination of the $\xi_{(1)}, \xi_{(2)}, \dots \xi_{(p)}$, and therefore the invariant derivative $\theta\eta_{(p)}$ will depend upon only $\xi_{(1)}, \xi_{(2)}, \dots \xi_{(p+1)}$, or what amounts to the same thing, $\eta_{(1)}, \dots \eta_{(p+1)}$.

From (15), (16) the matrix of α will then have the form:

(17) $$\| \alpha_{(pq)} \| = \begin{Vmatrix} 0 & +\frac{1}{\rho_{(1)}} & 0 & 0 & \cdots & 0 \\ -\frac{1}{\rho_{(1)}} & 0 & +\frac{1}{\rho_{(2)}} & 0 & \cdots & 0 \\ 0 & -\frac{1}{\rho_{(2)}} & 0 & +\frac{1}{\rho_{(3)}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & +\frac{1}{\rho_{(n-1)}} \\ 0 & 0 & 0 & \cdots & -\frac{1}{\rho_{(n-1)}} & 0 \end{Vmatrix},$$

and the **Frenet** formulas (12) will read:

$$(18)\qquad \begin{cases} \theta\eta^i_{(p)} = -\dfrac{1}{\rho_{(p-1)}}\eta^i_{(p-1)} + \dfrac{1}{\rho_{(p)}}\eta^i_{(p+1)}, \\ \dfrac{1}{\rho_{(p)}} = g_{ik}(\theta\eta^i_{(p)})\eta^k_{(p+1)}, \qquad 0 < p < n; \quad \dfrac{1}{\rho_{(0)}} = \dfrac{1}{\rho_{(n)}} = 0. \end{cases}$$

We only have to calculate the $n - 1$ "curvatures" $1 : \rho$ then.

In order to determine the derivatives $\theta\eta$ from (9), we first note that when one replaces the η in (14) with the ξ, it will then follow that:

$$(19)\qquad \frac{d}{ds}(p, q) = (p + 1, q) + (p, q + 1)\,.$$

Moreover, the definition (4) of θ implies the rule for calculation:

$$(20)\qquad \theta\,\lambda\,\xi^i = \frac{d\lambda}{ds}\cdot\xi^i + \lambda\cdot\theta\,\xi^i\,.$$

Therefore, when the θ-process is applied to (9), that will give:

$$(21)\qquad \begin{cases} \theta\eta^i_{(p)} = \dfrac{d}{ds}\dfrac{1}{\sqrt{D_{(p-1)}D_{(p)}}}\cdot\sqrt{D_{(p-1)}D_{(p)}}\cdot\eta^i_{(p)} \\ + \dfrac{1}{\sqrt{D_{(p-1)}D_{(p)}}}\begin{vmatrix} (1,1) & \cdots & (1,p-1) & \xi^i_{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ (p+1,1) & \cdots & (p+1,p-1) & \xi^i_{(p+1)} \end{vmatrix} + \dfrac{1}{\sqrt{D_{(p-1)}D_{(p)}}}\begin{vmatrix} (1,1) & \cdots & (1,p) & \xi^i_{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ (p,1) & \cdots & (p,p) & \xi^i_{(p)} \end{vmatrix} \end{cases}$$

when one first differentiates the first "factor" with respect to the rows and the differentiates the second one with respect to the columns. Upon taking the inner product with:

$$\eta^k_{(p+1)} = \frac{1}{\sqrt{D_{(p-1)}D_{(p)}}}\begin{vmatrix} (1,1) & \cdots & (1,p-1) & \xi^i_{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ (p+1,1) & \cdots & (p+1,p-1) & \xi^i_{(p+1)} \end{vmatrix},$$

one will observe that one can replace all $\xi_{(q)}$, $q < p + 1$ on the right-hand side of (21) with zeroes, due to their orthogonality to $\eta_{(p+1)}$. One will then find the desired result:

(22) $$\frac{1}{\rho_{(p)}} = \frac{\sqrt{D_{(q-1)}\,D_{(p+1)}}}{D_{(p)}}.$$

One can employ the covariant coordinates φ_i instead of the contravariant coordinates φ^i throughout. The formula:

(23) $$\theta\,\varphi_i = \frac{d\varphi_i}{ds} - \begin{Bmatrix} i\,k \\ r \end{Bmatrix} \xi^k \varphi_r$$

[cf., **Weyl**, pp. 101 (36)] will enter in place of (4), and the **Frenet** formulas (18) will remain valid when one switches the upper symbols *i*, *k* with the lower ones.

The formulas (22) also provide the curvatures of a curve in **Euclid**'s space when one employs an arbitrary curvilinear coordinate system.

In conclusion, I would like to add some references. If one specializes our formulas to a **Cartesian** axis-cross in **Euclid**'s space and one lets $n \to \infty$ then one will obtain the result that **G. Kowalewski** got ["Les formules de **Frenet** dans l'espace fonctionnel," C. R. Acad. Sci. Paris **151** (1910), 1338-1340]. If one specializes our formulas to non-**Euclid**ian space then one will get the **Frenet** formulas for non-**Euclid**ian space geometry that were first given for $n = 3$ by **L. Bianchi** ["Sulle superficie a curvature nulla in geometrica ellittica," Annali di Matematica (2) **24** (1896), 92-129, esp., pp. 101]. **G. Kowalewski** presented the **Frenet** formulas in non-**Euclid**ian space for arbitrary *n* ["Zur Differentialgeometrie…," Sitzungsberichte Wien, Math. Nat. Klasse **120** (1911), Sec. II.a.1, pp. 531-542.] Cf., also the dissertation of **E. Stranski** that was suggested by **G. Pick** ["Zur Infinitesimal geometrie der Kurven in elliptische Räume," *ibidem*, **121** (1912), Sec. II.a.1, pp. 813-827]. At the suggestion of **C. Carathéodory**, **P. Finsler** has investigated the geometry of curves in R_n with the metric that is induced by an arbitrary variational problem ["Über Kurven und Flächen in allgemeinen Räumen," Dissertation, Göttingen, 1918].

Tübingen in May 1919.

(Received on 9 May 1919)