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BOUNDARY-VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Bibliography

One should confer the relevant chapters in the following books:

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1. The fundamental problem statement and its origins in mathematical physics. – In what follows, we shall address real, second-order ([1]) ordinary differential equations. The general solution to such an equation will include two integration constants (II A 4.a, no. **6**), that we would like to regard as parameters. In addition, the differential equation itself might include k parameters. If we then let $y_1, y_2, \ldots, y_n$ denote arbitrary solutions of our differential equation then $k + 2n$ parameters will enter into them. If we focus on n segments along the x-axis $a_1 b_1$, $a_2 b_2$, …, $a_n b_n$ then the boundary-value problem is subsumed by the question of *whether and in how many ways those* $k + 2n$ *parameters can be determined such that the solution* y_i ($i = 1, 2, \ldots, n$) *inside of the segment* $a_i b_i$ *is continuous, along with its first derivative, and fulfills certain boundary conditions at the endpoint* a_i *and* b_i .

Up to now, the very extensive class of problems that was just referred to has been developed only to a lesser degree. There are mainly two cases that have already been treated, which we shall refer to as a) and c). Due to its greater importance, however, we shall regard case b) as a special case of c).

a) $n = 1, k = 0$. The boundary conditions consist of saying that the solution to the differential equation should assume certain arbitrarily-prescribed values at the endpoints of the segment.

b) $n = k = 1$. The differential equation is then assumed to be linear and homogeneous. The boundary conditions consist of saying that the quotient y_1' / y_1 approaches arbitrarily-prescribed (perhaps infinite) values at the endpoints of the segment ([2]).

c) $n = k > 1$. The differential equation is assumed to be linear and homogeneous, and the boundary conditions are the same as in b).

Those problem statements have mathematical physics to thank for their existence.

The relationship of case a) to physics is merely indirect, since the problem was taken up by **E. Picard** as an analogue of similar problems in the partial differential equations of mathematical physics ([3]), and indeed only in recent years. We will discuss **Picard**'s investigations, as well as the applications of them that he made to the boundary-value problem b), in no. **7**.

The problem b) goes back to the Eighteenth Century. One must frequently start with a boundary-value problem for a partial differential equation in mathematical physics. In order to solve it, one next forms products that satisfy the partial differential equation and whose factors each include only one variable. Each of those factors will then satisfy an ordinary linear differential equation, and in that way, one will deal with precisely the simplest cases of the boundary-value problems b)

([1]) Higher-order equations will occur only in no. **4**.

([2]) The boundary conditions will be discussed in a somewhat more general, as well as modified, form in no. **3**.

([3]) The fundamental (first) boundary-value problem of potential theory might be mentioned as the original starting point. Cf., II A 7.b, no. **17**.

([4]). The boundary-value problem c) occurs in the same way in more complicated cases. However, **G. Lamé** (1839) ([5]) was the first to treat one such case.

That method of treating physical problems was already known to **L. Euler** in some rather general cases ([6]). However, neither **Euler** nor his contemporaries, with the exception of **J. le R. d'Alembert** (cf., *infra*), knew of the necessity of treating the questions that were raised by it. Rather, the possibility of determining the parameters that were considered in b) was tacitly assumed on physical grounds or merely by analogy with simpler cases. **J. Fourier** (who is regarded as **Euler**'s successor in that domain) did not go much further in that direction, although he did seek an analytical treatment in some individual cases ([7]).

By contrast, as has been often remarked, **D'Alembert** ([8]) knew of the necessity of an analytical treatment. He started from the problem of an oscillating non-uniform string, which led to the equation:

$$\frac{d^2y}{dx^2} = \lambda\, \varphi(x) \cdot y\,,$$

in which $\varphi(x)$ means a given positive function. **D'Alembert** asked whether one could determine the parameter λ such that this equation would admit a solution that would vanish at a and b without vanishing identically. In order to answer that question, he converted the equation above into a **Ricatti** equation (II A 4.b, no. **8**, **27**) by introducing the new dependent variable $z = y'/y$. When he investigated that equation, he arrived at the result ([9]) that the desired parameter determination was always possible, and indeed in such a way that the solution in question y did not vanish between a and b. However, the fact that there was only one such parameter value was not mentioned, any more than he spoke of the infinitely-many parameter values that one obtained by allowing zeroes of y between a and b.

Except for that exception, the boundary-value problem b) [c), resp.] was not only not addressed, but up to the year 1833 [1881, resp.], it was not even formulated, which is when the work of **C. Sturm** [**F. Klein**, resp.], which shall be discussed in nos. **2** and **5**, came out. However, whereas the motivation for **Klein**'s work lay in mathematical physics ([10]), it is not as clear that this

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([4]) It should be remarked that these boundary-value problems sometimes also appear directly (i.e., without the intermediary of a partial differential equation). As an example, one might cite the determination of the equilibrium figures of an inhomogeneous string when one neglects gravity, and one assumes that it lies in a plane that rotates uniformly around the line that connects the two fixed endpoints of the string. In order to arrive at a linear equation, one must assume that the length of the string is only a little larger than the distance between the endpoints.

([5]) J. de math. **4** (1839), pp. 126.

([6]) Cf., Petrop. N. Comm. **10** (1764) [66], pp. 243.

([7]) Thus, e.g., in **§ 284** of *Théorie de la chaleur* (Paris, 1822), he spoke of a differential equation that could be solved by circular functions, and the possibility of determining the parameters followed directly from its known properties. Moreover, in **§ 307**, **Fourier** unnecessarily considered, on the one hand, the determination of imaginary parameters that will not come under consideration here, even when they exist. On the other hand, it would seem that he concluded that a real parameter determination was possible from the assumption that a transcendental equation always has infinitely many real or imaginary roots!

([8]) Berl. Hist. (1763) [70], pp. 244.

([9]) His treatment of it includes a gap that is easily filled.

([10]) And indeed with the ambition of shedding new light on Appendix B of **Thomson** and **Tait**'s *Natural Philosophy* by considering a more general case.

was the case for **Sturm** ([11]), although he obviously recognized the relationship between his results and mathematical physics, and he very soon after made some applications of it in that spirit in conjunction with **J. Liouville**. He originally ([12]) started from finite *difference* equations (I E), and only later adapted the results that he obtained (but did not publish) to *differential* equations.

The developments of the previous century have been prefatory to our boundary-value problems b) and c) in yet another direction. One might consider an article of **P. S. Laplace** ([13]), who started from a problem in the theory of attraction and treated a certain triple integral that was a function of the three polar coordinates (r, ϖ, ϑ) (II A, 7.b, no. **21**). He developed that integral into a series, each term of which included a polynomial in sin ϑ and cos ϑ as a factor. However, instead of evaluating that polynomial directly, as **A. M. Legendre** ([14]) did in a simpler case, **Laplace** remarked that it would satisfy differential equations that would have the form:

$$\frac{d\left[(1-\mu^2)\dfrac{d\beta}{d\mu}\right]}{d\mu}-\left[\frac{n^2}{1-\mu^2}-i(i+1)\right]\beta = 0 .$$

when one sets $\mu = \cos\vartheta$. In that way, **Laplace** knew from the outset that the two parameters i and n were whole numbers, and for him the only purpose of the differential equation was to simplify the calculation of those entire rational functions of μ and $\sqrt{1-\mu^2}$. Nothing was said about a boundary-value problem. However, he was close to addressing the Laplace, and other similar, problems ([15]) in such a way that he assumed that n was a whole number, but sought to determine i in such a way that the differential equation would possess a solution that remained finite for the values $\mu = \pm 1$ (i.e., for the endpoints of the interval that comes under consideration for μ), and in that way, one would be dealing with a boundary-value problem that is only a slight modification of b). In that way, he proposed that the necessary and sufficient condition for such a solution to exist consists of the statement that it should be an entire rational function of μ and $\sqrt{1-\mu^2}$. The boundary-value problem will then be solved here in such a way that we determine the values of i for which the differential equation admits an entire rational function of μ and $\sqrt{1-\mu^2}$ as a solution. As one confirms by an easy calculation, they are the whole-number values of i.

The investigations of **Lamé** ([16]) into the differential equation that is named after him were connected with that treatise of Laplace. **Lamé** sought to determine the parameters that entered into it in such a way that the equation would admit a solution that was a polynomial, except for certain simple irrational factors. As will be explained in detail in no. **5**, the problem that is thus defined

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([11]) **Sturm**'s algebraic, as well as the mathematical-physical, tendencies go back to **Fourier**. One sees how intrinsically intermingled those two tendencies were in **Sturm** in his earlier papers in Bull. de Férussac **11**, **12**.

([12]) Cf., J. de math. **1** (1836), pp. 186.

([13]) Paris. Hist. (1782) [85], pp. 113.

([14]) Paris sav. [étr.] **10** (1785), pp. 411.

([15]) So, e.g., in the problem of heat conduction in a ball that was likewise treated by **Laplace** [*Connaissance des temps pour l'an 1823* (1820), pp. 249, *et seq.* and *Méc. cél.*, livre XI (*Oeuvres* 5, pp. 82)].

([16]) J. de math. **4** (1839), pp. 126

will have a relationship to the boundary-value problem c) that is entirely similar to the relationship between the Laplace work that was just discussed and problem b). Those special investigation have then led to the more general ones that will be discussed in no. **6**.

2. The fundamental treatise of Sturm ([17]) and the oscillation theorem in the case of *one* parameter. – **Sturm** started from the general second-order homogeneous linear differential equation, which he wrote in the form:

$$\frac{d\left(K\dfrac{dy}{dx}\right)}{dx}+G\,y\;=0\;. \tag{1}$$

The K and G in that mean arbitrary functions of x and a parameter λ that cannot, however, become infinite in the interval $a \le x \le b$ under consideration. K cannot vanish that interval either ([18]), and will then be assumed to be positive, for simplicity.

A well-defined solution of (1) is now selected in such a way that one requires y and dy / dx to be equal to certain arbitrarily-given functions of λ at a ([19]). **Sturm** then addressed the variation of the function y with λ. In particular, he posed the following two questions:

1) How do the roots of y vary with λ ?

2) How does the ratio $K\dfrac{dy}{dx}\Big/y$ vary with λ when one regards x as constant?

In order to answer those question, he derived the following otherwise-useful formula from the differential equation ([20]):

$\Phi\,(x) - \Phi\,(a)$

$$= \int_a^x y(x,\lambda_1)\cdot y(x,\lambda_2)[G(y,y_2)-G(y,y_1)]\,dx \;-\; \int_a^x \frac{\partial y(x,\lambda_1)}{\partial x}\cdot\frac{\partial y(x,\lambda_2)}{\partial x}[K(y,y_2)-K(y,y_1)]\,dx\,,$$

in which

$$\Phi\,(x) = \; y(x,\lambda_2)\,K(x,\lambda_1)\frac{\partial y(x,\lambda_1)}{\partial x} - y(x,\lambda_1)\,K(x,\lambda_2)\frac{\partial y(x,\lambda_2)}{\partial x}\;.$$

([17]) Read to the Paris Academy in September 1833. J. de math. **1** (1836), 106-186.

([18]) Cf., however, what happens at the endpoints a and b of the interval on pp. 108 of Sturm's treatise.

([19]) In the vast majority of the physical applications, those functions of λ are merely constants.

([20]) Cf., e.g., II A 8, in which the formulas for determining the coefficients in series developments that represent solutions of linear differential equations are subsumed by the formula that was given here as a special case. Such special cases of that formula appear frequently in Sturm's treatise.

Sturm now considered only the case in which the function $K\frac{dy}{dx}\Big/y$ increased monotonically with increasing λ at the point $x = a$ ([21]), and the function G decreased monotonically for every value of x that came under consideration, but K increased monotonically. When he assumed that the difference $\lambda_2 - \lambda_1$ was infinite, he then concluded that ([22]):

1) *The roots of y increase monotonically with increasing* λ.

2) $K\frac{dy}{dx}\Big/y$ *increases monotonically with every value of x that comes under consideration* ([23]).

With those theorems, one is in a position to compare the solutions of two differential equations of the form (1) that include no parameters λ to each other in only the event that the function G in the first equation is nowhere greater and the function K in the first one is nowhere smaller than the corresponding functions in the second equation. That is because obviously one can indeed write down (and in infinitely-many ways) an equation of the form (1) that includes a parameter λ and reduces to two given equations for two values of λ while the functions G and K vary monotonically between those values of λ.

One will get simple theorems when one compares the given equation in that way with an equation in which K and G are constant, so one that can be solved by elementary functions. On the basis of those theorems, **Sturm** ultimately came to the following so-called ([24]):

Oscillation theorem ([25]):

We let λ *vary between two (finite or infinite) values* λ' *and* λ'' ($\lambda' < \lambda''$). *In that way, the function G might decrease monotonically with increasing* λ, *while K increases monotonically, and indeed the ratio G / K might decrease from* $+\infty$ *to* $-\infty$. *Moreover, let* $f_1(\lambda)$ *and* $f_2(\lambda)$ *be two arbitrarily-given functions that increase monotonically with* λ. *There is then one and only one value* $\lambda = \bar{\lambda}$ *for which equation* (1) *possesses a solution y that vanishes an arbitrarily-prescribed number of times in the interval* $a < x < b$ *and for which the function* $K\frac{dy}{dx}\Big/y$ *assumes the values* $f_1(\bar{\lambda})$ *and* $-f_2(\bar{\lambda})$ *at the points a and b, resp.*

([21]) The expressions "monotonically increasing" and "monotonically decreasing" (II A 1, no. **11**) do not exclude the possibility of constancy.

([22]) *Loc. cit.*, pp. 116, 117.

([23]) **Klein** derived that theorem, if only in some special cases, from mechanical considerations. Cf., his *Ueber lineare Differentialgleichungen der zweiten Ordnung*, pp. 268, *et seq.*, 1894.

([24]) That name goes back to **Klein**. Cf., Gött. Nachr. March 1890.

([25]) We are stating this theorem in a somewhat-more-general way than **Sturm** did.

For given boundary conditions, i.e., for given functions f_1 and f_2, there is then an infinite series of parameter values λ_0, λ_1, λ_2, … (the so-called *distinguished parameter values* [**F. Pockels**, *loc. cit.*]) that one determines by means of the oscillation theorem according to whether one demands that the solution of the differential equation in question should vanish never, once, twice, etc., in the interval. Those parameter values obviously define the set of real roots in the interval (λ', λ'') of a transcendental equation that is easy to write down. For each of those distinguished parameter values, as a result of the oscillation theorem, the differential equation will possess one (and obviously only one, except for a constant factor that remains arbitrary) solution that satisfies the desired boundary conditions at a and b. They are called (**F. Pockels**) *distinguished solutions*, and might be denoted by y_0, y_1, y_2, …, in which the index gives the number of zeroes in each case. **Sturm** then proved the theorem:

The roots of two successive distinguished solutions y_n and y_{n+1} that belong to the same boundary conditions are separate from each other.

The theorems of **Sturm** that were cited here do not, by any means, exhaust the results that were obtained in that treatise. For example, we might refer to his investigations in regard to the roots of the expression $K\,dy\,/\,dx + p\,y$, where p means a function of x that satisfies the inequality $G\,K + p^2 - K\,dp\,/\,dx > 0$ ([26]).

3. Details regarding the case of *one* parameter ([27]). – Soon after Sturm's treatise appeared, a series of works by **Sturm** and **J. Liouville** ([28]) appeared that were devoted to the same topic. However, due to its greater physical interest, the only case that was treated in them was the one in which K was independent of λ, while G had the form $\lambda\,g - l$, in which g and l mean positive functions of x alone. f_1 and f_2 were also assumed to be independent of λ and positive. Naturally, one can give Sturm's original theorems more specific forms under such special assumptions. One then finds that, e.g., by applying a method of **S. D. Poisson** ([29]), that the aforementioned transcendental equation for determining the distinguished parameter values (which is now analytic in λ) possesses no imaginary roots. Therefore, since the interval $\lambda'\lambda''$ now coincides with the entire λ-axis, the distinguished parameter values now define the set of all roots of that equation.

([26]) Even more results of that treatise, as well as the work of **Sturm** and **Liouville** that will be cited directly, will be discussed in II A 8 and II B 4.

([27]) For the method that **Picard** recently applied in order to derive some of **Sturm**'s results, cf., no. **7**.

([28]) **C. Sturm**, J. de math. **1** (1836), pp. 373. **C. Sturm** and **J. Liouville**, *ibid.* **2** (1837), pp. 220. **J. Liouville**, *ibid.* **1** (1836), pp. 253, 269. *ibid.* **2** (1837), pp. 16, 418, 439. Those treatises are meaningful mainly because the series developments in the distinguished solutions were treated for the first time in them. Cf., II A 8.

It might be mentioned here that the Liouville article **2**, pp. 439 treated a special second-order *inhomogeneous* linear differential equation that can, of course, be solved by trigonometric functions.

([29]) Bull. Soc. Philomath. (1826), pp. 145; cf., also **Sturm**, J. de math. **1** (1836), pp. 384, *et seq.* Poisson's proof consists of a simple application of the integral formula that was given in no. **2**.

As one easily proves, they are all positive; they are also simple roots ([30]). One ultimately finds ([31]) that for large n, $\sqrt{\lambda_n}$ can be represented asymptotically by $n\pi / Z$ ([32]), where:

$$Z = \int_a^b \sqrt{g / K}\, dx .$$

We would like to add the following remark that **Liouville** generally did not make. It deals with the question of how the distinguished parameter values will change when the interval ab becomes unbounded. In the simple case when one deals with a differential equation with constant coefficients, those parameter values will cluster along the entire positive half of the λ-axis in such a way that arbitrarily-many parameter values will lie along an arbitrary piece of that axis. One would like to suspect that similar things might be true in the general case. However, **W. Wirtinger** ([33]) was the first to emphasize the fact that this was not the case. Perhaps one can see that most easily in a case that **Wirtinger** did not discuss, namely, the one in which the ratio g / K converges to zero with increasing x so strongly that $\int_a^\infty \sqrt{g / K}\, dx$ has a finite value. Here, one concludes from the asymptotic representation that was written down above that the parameter values do not accumulate at all when the interval ab increases ([34]). However, the case that **Wirtinger** treated is even more interesting: $K = 1$, $l = 0$, $f_1 = f_2 = \infty$, $g =$ an even periodic function. Here, **Wirtinger** proved that the distinguished parameter values naturally accumulate for an increase interval ab, but that they generally fill up only certain pieces of the positive half of the λ-axis increasingly but can leave the remaining pieces of the λ-axis completely empty ([35]).

We now return to **Sturm**'s ground-breaking treatise.

As was the custom at the time, **Sturm** did not say anything more detailed about the nature of the functions that entered into his proofs. However, they must be continuous functions of their arguments in any event, and it emerges from the line of reasoning in the proof that they can also be regarded as differentiable, in part. However, the latter assumption is not necessary for the

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([30]) One might confer, e.g., the **Sturm** article that was cited last.

([31]) **J. Liouville**, J. de math. **2** (1837), pp. 30.

([32]) The case in which the distinguished solutions vanish at one endpoint of the interval a b, but not the other, defines an exception. In that case, $\sqrt{\lambda_n}$ will be represented asymptotically by the formula $(n + \frac{1}{2})\, \pi / Z$.

M. Radaković gave an approximate formula of a different type for the distinguished parameters [Monatsh. für Math. u. Phys. **5** (1894), pp. 228]. It also relates to all values of n, but only for the case $K = 1$, $l = 0$, $f_1 = f_2 = \infty$, and g is a function that varies only slightly between a and b.

([33]) Math. Ann. **48** (1896), pp. 387.

([34]) In order to make that conclusion rigorous, one requires an even-more-specialized argument that can nonetheless be implemented rigorously with ease under certain restricting assumptions in every case.

([35]) Those investigations are connected with the problem of the oscillation of an infinitely-long string, and **Wirtinger** expressed his result by saying that the oscillation would generally correspond to a "band spectrum," in the terminology of optics.

Obviously, this question has the closest relationship to the case that will be discussed much later in which the interval ab extends up to a singular point of the differential equation, because the point $x = \infty$ is indeed a singular one, in general, and in fact an irregular point of the differential equation (II B 4).

theorems that are obtained to be valid. On the other hand, the mathematically-rigorous modern development **Sturm**'s proof cannot be regarded as free from objections at each of its steps. More recently ([36]), the author undertook the task of putting **Sturm**'s main result on solid foundations, and that yielded the result that **Sturm**'s methods could be preserved essentially. In his treatment, the author had restricted himself to equations of the form:

$$\frac{d^2 y}{dx^2} = \varphi \cdot y$$

(which is a case that also received special consideration by **Sturm**), because any equation (1) can be put into that form by changing the independent, as well as the dependent variables ([37]).

The question of whether Sturm's theorems will remain true in the case when one endpoint (or both of them) ([38]) of the interval in question is a singular point of the differential equation has been investigated in several cases and by various methods ([39]). We shall cite only the following two main theorems here, in which one deals with differential equations with *analytic* coefficients that have a *regular singular point* (cf., II B 4) with real exponents at the endpoint a of the interval in question. For the sake of simplicity, we assume that the other endpoint b is a non-singular point.

If one assumes that the boundary condition at the point a is that the solution to the differential equation belongs to large exponents then Sturm's theorems will remain true ([40]).

If one assumes that the boundary condition at the point a is that the solution to the differential equation is an arbitrarily-prescribed linear combination with constant coefficients of the solutions that belong to the two exponents then Sturm's theorems will persist in the even that:

(α) *the difference between the exponents at a is less than* 1.

(β) *the exponents are independent of the parameter* λ ([41]).

([36]) N. Y. Bull. April, June, and October 1898. The proof that was given here can be simplified in part. Cf., a preliminary communication of the author in N. Y. Bull., Dec. 1899, pp. 100.

([37]) A second restriction in generality consists of saying that the functions that were denoted by $f_1(\lambda)$ and $f_2(\lambda)$ above will be assumed to be constants. However, it is easy to go from that special case to the general one by means of the first two theorems of the second article.

([38]) The possibility of singular points appearing inside of the interval was also considered by the author.

([39]) **L. Schäfli**, Math. Ann. **10** (1876), pp. 137; the book by the author that was cited above, pp. 174, *et seq.*; **F. Klein**, *Lin. Diff.-Gl. d. 2. Ord.*, pp. 429; **M. B. Porter**, N. Y. Bull. May 1897; **E. B. Van Vleck**, N. Y. Bull., June 1898; **M. Bôcher**, N. Y. Bull., Oct. 1898.

([40]) This was probably stated for the first time by **M. B. Porter** in N. Y. Bull. May 1897.

([41]) **M. Bôcher**, N. Y. Bull., Oct. 1898. For a further extension of these theorems, one can confer a preliminary communication of the author, N. Y. Bull. April 1900.

In conclusion, we shall speak of the modified form of the boundary-value problem b), in which instead of prescribing the values of y'/y at a and b, one demands that the following equations must exist ([42]):

$$y(a) = y(b)\,, \quad y'(a) = y'(b)\ .$$

Fourier was led to pose that question when he investigated the problem of heat condition in a thin wire that closes on itself ([43]). However, when he assumed that the wire was uniform, he was treating one of the cases that would be trivial from our standpoint.

The general problem that was indicated here does not seem to have been dealt with. The special case in which the coefficients of the differential equations possess the same values at the points $a + \xi$ and $b - \xi$ can be easily reduced to the ordinary boundary-value problem b) ([44]). In fact, one needs only to focus upon the half segment $a < x < (a + b) / 2$ and prescribe either $y = 0$ or $y' = 0$ at both endpoints as the boundary condition.

By contrast, the possibility of so-called *two-fold distinguished parameter values* (**F. Pockels**, *loc. cit.*) appearing in that problem arose for the first time, i.e., parameter values for which two linearly-independent, and as a result, all solutions, of the differential equation would be distinguished solutions. For example, in the case of constant coefficients, *all* distinguished parameter values are two-fold.

4. Extension of Sturm's result to higher-order differential equations. – Such an extension to certain fourth-order linear differential equations already presents itself in the physical problem of the vibration of elastic rods and plates, but only some special cases of it will be treated here ([45]). If one addresses an inhomogeneous elastic rod then one will be dealing with the differential equation:

$$\frac{d^2 K\, d^2 y}{dx^4} = \lambda\, g\, y\,,$$

in which K and g means positive functions of x that are independent of λ, and one then deals with the determination of the parameter λ in such a way that the differential equations will possess a solution that fulfills one of the following three boundary conditions at each endpoint a and b :

([42]) That boundary-value problem obviously coincides with the following problem: *For what values of the parameter λ will a second-order homogeneous linear differential equation whose coefficients are periodic functions of x possess at least one solution with the same period?* Cf., pp. below.

([43]) *Théorie de la chaleur*, pp. 239.

([44]) Cf., the author's book that was cited above, pp. 181, *et seq.*

([45]) One finds most of those cases in **Lord Rayleigh**'s *Theory of Sound*. Cf., also **G. Kirchhoff**, Berl. Ber. Oct 1879 and **F. Meyer**, "Zur Capellen," Ann. Phys. Chem. (2) **33** (1888), pp. 661.

clamped end: $\begin{cases} y = 0, \\ \dfrac{dy}{dx} = 0, \end{cases}$ held end: $\begin{cases} y = 0, \\ \dfrac{d^2 y}{dx^2} = 0, \end{cases}$ free end: $\begin{cases} \dfrac{d^2 y}{dx^2} = 0, \\ \dfrac{d^3 y}{dx^3} = 0. \end{cases}$

Based upon general mechanical principles, **Lord Rayleigh** ([46]) gave some results that have a close analogy with Sturm's theorems. More recently ([47]), that question was taken up by **A. Davidoglou** for the case of K = const. by applying Picard's methods (cf., no. **7**).

J. Liouville has treated a somewhat-different boundary-value problem ([48]), namely, the differential equations of arbitrary order that take the form:

$$\frac{d \cdot K\, d \cdot L \cdots d \cdot M\, d \cdot N\, dy}{dx^n} + \lambda\, y = 0\,,$$

in which K, L, …, M, N mean $n - 1$ *positive* functions of x that are independent of the parameter λ. He proved the following oscillation theorem:

One can determine the parameter λ in one and only one way such that the differential equation possesses a solution that possesses an arbitrarily-prescribed number n of roots in the interval ab and satisfies the boundary conditions:

$$y = A\,, \qquad N\frac{dy}{dx} = B\,, \qquad \ldots, \qquad \frac{K\, dL \cdots dM\, dN\, dy}{dx^{n-1}} = D$$

at the point a and the boundary condition:

$$\alpha\, y + \beta\, N\frac{dy}{dx} + \cdots + \nu\, \frac{K\, dL \cdots dM\, dN\, dy}{dx^{n-1}} = 0$$

at the point b. In that, A, B, …, D, as well as α, β, …, ν, means positive constants (some of which might vanish, with the exception of D).

If one denotes the parameter values that are determined by that by λ_n then one will have an infinite series of distinguished parameter values λ_0, λ_1, λ_2, … As **Liouville** proved, they are all positive and ordered by magnitude. Ultimately, the roots separate two successive distinguished solutions from each other.

([46]) *Theory of Sound* **1**, § 187.

([47]) C. R. Acad. Sci. Paris, 12 March and 7 May 1900. Cf., also C. R. Acad. Sci. Paris, 12 February 1900, in which certain *third*-order differential equations were treated by the same mathematician.

([48]) J. éc. polyt., cah. **25** (1837), pp. 85. J. de math. **3** (1838), pp. 255 and 561. Here, we shall mainly deal with the third of those treatises, since the second one is only a brief preview, while first one can be regarded as a preliminary work in which only the special equation $dy^3 / dx^3 = \lambda\, x$ was treated, which is soluble by exponential functions.

Although those theorems are entirely analogous to those of **Sturm**, nonetheless, **Liouville** found that they made it necessary to invent a new method of proof because the integral formula of no. **2** breaks down here. In place of it, **Liouville** appealed to a method that is similar to the one that was applied to prove **Fourier**'s theorem on real roots of algebraic equations (I B 3.a, no. **4**). Just as **Sturm**'s is, **Liouville**'s proof was based upon the fact that for large values of λ, the solutions of the differential equation will possess many roots in the interval ab. One sees that this is true in the case where K, L, …, M, N mean constants in the explicit solution of the differential equation, which can be represented by exponential functions in this case. However, Liouville carried out the transition from that special case to the general case in a very unsatisfactory way.

5. The oscillation theorem in the case of several parameters ([49]). – The case that was indicated by c) in no. **1** was first treated by **Klein** in 1881 ([50]) in a deliberate and explicit way, if only for a special differential equation. Indeed, many of the differential equations that appeared in the last century include two parameters ([51]). However, one of them is always known ([52]), and one then deals with the problem of determining the second one by means of Sturm's oscillation theorem. On the other hand, as we will see directly, **Lamé**'s work ([53]) has a relationship to the question that lies before us that is entirely similar to the relationship between the **Laplace** treatise that was discussed in no. **1** and Sturm's oscillation theorem, because **Lamé** said nothing about boundary-value problems, and yet his results, along with those of his followers, yield the solution to the boundary-value problem c) in one special case in a most immediate way.

In the article that was just cited, **Klein** started from the Lamé equation, and indeed in essentially the form:

$$\frac{d^2y}{dx^2}+\frac{1}{2}\left(\frac{1}{x-e_1}+\frac{1}{x-e_2}+\frac{1}{x-e_3}\right)\frac{dy}{dx}-\frac{Ax+B}{4(x-e_1)(x-e_1)(x-e_1)}y=0\,,$$

in which $e_1 < e_2 < e_3$, and considered two segments of the x-axis $a_1 b_1$ and $a_2 b_2$, one of which lies in the intervals $e_1 e_2$, $e_2 e_3$, $e_3 \infty$, but indeed not both of them. He posed the following oscillation theorem ([54]):

One can determine the real parameters A and B (and indeed in only one way) such that Lamé's equation possesses two solutions y_1 and y_2, of which, y_1 vanishes at the points a_1 and b_1 and possesses an arbitrarily-prescribed number of zeroes between those points, while y_2 vanishes at the points a_2 and b_2 and possesses an arbitrarily-prescribed number of zeroes between those points.

([49]) For the relationship to the theory of polygons of circular arcs, cf., II B 4.

([50]) Math. Ann. **18**, pp. 410.

([51]) E. g., the equation that **Laplace** addressed. Cf., no. 1.

([52]) And indeed, often by applying Sturm's oscillation theorem to another differential equation that likewise includes that parameter.

([53]) Cf., no. II B 4.b, namely, **33**, **34**, **36**.

([54]) That formulation was already generalized in various directions in the same treatise. In particular, the segments can bend around the singular points.

Klein proved that theorem by geometric considerations that made no claim to complete rigor. Namely, he considered the auxiliary line $y = A\,x + B$, and when he directed his attention to the one segment $a_1\,b_1$, he investigated how that line would have to move in order for the desired solution u_1 to exist. Based upon some of **Sturm**'s results ([55]), he found that in order to do that, the line would have to envelope a certain *enveloping curve* whose form he examined more closely. When he then considered the second segment $a_2\,b_2$ similarly, he found a second enveloping curve, and he concluded from the form and mutual position of those two curves that those two curves will always possess one and only one common tangent. The equation of that tangent then gave the desired values of A and B. The proof that is thus sketched out was posed in analytic form ([56]) and carried out rigorously by the author ([57]).

In order to connect with Lamé's results, **Klein** let the two segments coincide with the intervals e_1e_2 and $e_2\,e_3$. Moreover, if e_1, e_2, e_3 mean 0 or 1, independently of each other, then **Klein** posed the problem of determining the parameters A and B such that Lamé's equation would possess two solutions y_1 and y_2, of which, y_1 belongs to the exponents $\varepsilon_1 / 2$ and $\varepsilon_2 / 2$ at the points e_1 and e_2, respectively, while y_2 belongs to the exponents $\varepsilon_2 / 2$ and $\varepsilon_3 / 2$ at the points e_1 and e_3, respectively. The somewhat-extended oscillation theorem tells us that there are infinitely many such pairs of values A, B that differ by the number of zeroes of the corresponding functions y_1 and y_2 in the intervals e_1e_2 and $e_2\,e_3$. On the other hand, one sees from an easy function-theoretic argument ([58]) that the two functions y_1 and y_2 (which differ by only an imaginary constant factor) have the form:

$$\sqrt{(x-e_1)^{\varepsilon_1}(x-e_2)^{\varepsilon_2}(x-e_3)^{\varepsilon_3}}\;\Phi\,,$$

in which Φ means a polynomial in x. One will then be likewise led to the desired pair of values A, B when one determines those parameters such that the equation will possess a solution of that form, and therein lies precisely **Lamé**'s original statement of the problem. In particular, one observes the relationship between the oscillation theorem and the theorem on the distribution of roots of the Lamé functions that **Klein** found shortly before it ([59]).

Klein extended the oscillation theorem in two directions in one of his lectures (Winter Semester 1889-90) ([60]): He then directed his considerations to a generalized form of the Lamé equation, which raised no new complications. However, he did that in such a way that in so doing he presumably expressed an oscillation theorem for higher-order Lamé equations ([61]), which made a greater number of segments appear, corresponding to a greater number of parameters. A

([55]) **Klein** did not refer to **Sturm**'s treatise but derived the desired result in a geometric way.

([56]) N. Y. Bull., April 1898, pp. 307, *et seq.*

([57]) One can also confer a modified form of the proof in **Pockels**: *Ueber die Differentiagleichung* $\Delta u + k^2 u = 0$, pp. 118. It was rigorously implemented only in part.

Investigations of the asymptotic values of the parameters A, B in the case of a very large number of zeroes of y_1, as well as y_2, can be found in the dissertation of **C. Jaccottet**, Göttingen 1895.

([58]) Cf., the author's book that was cited before, pp. 213.

([59]) Math. Ann. **18**, pp. 237, *et seq.* One can confer II B 4.b, no. **36**.

([60]) Cf., Gött. Nachr. (1890), pp. 91, *et seq.*

([61]) Cf., II B 4.b, nos. **40**, **42**.

geometric proof was given ([62]) in the simplest case where three parameters are present, in which one operates with enveloping surfaces instead of enveloping curves. In the more complicated cases, that method of proof led to higher-dimensional spaces, and as a result they lost their power to convince, since intuition played an essential role (he made no claim to *rigor*).

Somewhat later ([63]), **Klein** expressed the conjecture that the oscillation theorem could be extended to other everywhere-regular differential equations under certain circumstances. He was led to it mainly by certain results of **Stieltjes** that showed the validity of his conjecture in one special case, although we will first discuss that in no. **6**. In order to explain Klein's theorem, we would like to start from the general second-order homogeneous linear differential equation that is everywhere regular:

$$\frac{d^2y}{dx^2}+\left(\frac{1-\kappa_1'-\kappa_1''}{x-e_1}+\cdots+\frac{1-\kappa_n'-\kappa_n''}{x-e_n}\right)\frac{dy}{dx} \tag{3}$$

$$+\ \frac{1}{f(x)}\left[\frac{\kappa_1'\kappa_1''f'(e_1)}{x-e_1}+\cdots+\frac{\kappa_n'\kappa_n''f'(e_n)}{x-e_n}+C_{n-2}x^{n-2}+\cdots+C_1x+C_0\right]y=0\,,$$

in which

$$f(x)=(x-e_1)(x-e_2)\ldots(x-e_n)\,.$$

We assume that the coefficients of that equation are real for real values of x. We can then summarize Klein's conjecture as the following theorem:

Consider the $k+1$ *intervals* ($k \le n-2$) $a_0\,b_0$, $a_1\,b_1$, …, $a_k\,b_k$, *which satisfy the following three conditions:*

1) $$a_0 < b_0 \le b_1 \le a_2 \ldots < b_{k-1} < a_k < b_k\,.$$

2) *No singular point* e_i *shall lie inside of the interval, but at most at its ends.*

3) *The singular points that lie at the endpoints of that interval shall have exponents whose difference has an absolute value that is less than 1.*

Moreover, m_0, m_1, …, m_k *might mean arbitrary numbers that are positive integers or zero.*

When one considers the other quantities that enter into the differential equation to be given, one can then determine the parameters C_0, C_1, …, C_k *in one and only one way such that the equation* (3) *will possess* $k+1$ *solutions* y_0, y_1, …, y_k *in such a way that* y_i *will vanish precisely* m_i *times in the interval* $a_i < x < b_i$ *, and at each endpoint of that interval it is proportional to an*

([62]) Although it was only published later. Cf., e.g., **Klein**, *Lin. Diff.-Gl.*, pp. 403, *et seq.*

([63]) In his *Lin. Diff.-Gl.*, 1894, pp. 427, *et seq.* One can also confer the autographed volume from 1891 on differential equations, which nonetheless includes many conjectures by **Klein** that were subsequently proved to be false.

arbitrarily-prescribed linear combination of the fundamental solutions that belong to that point ([64]).

If one drops the condition that the absolute values of the difference between the exponents at the endpoints of the segments must be less than 1 for some of the segments then the theorem will remain unchanged in the event that one demands that at each such singular point, the function y_i in question shall be proportional to the fundamental solution that belongs to the larger exponent at the point in question.

That theorem was proved by the author ([65]), and indeed as a special case of a similar theorem that related to the equation:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + \{\chi(x) + \psi(x)[C_k\, x^k + \cdots + C_1\, x + C_0]\}\, y = 0\,,$$

in which the functions p, χ, ψ do not need to be analytic, but must only satisfy certain conditions.

6. Excursion into polynomial solutions. – When **E. Heine** attempted to extend **Lamé**'s result to the higher-order Lamé equations that he introduced, he was led to take up the following, more-general, problem ([66]):

ψ means a polynomial of degree (p + 1) *in x, while χ and ϑ mean polynomials in x whose degrees do not exceed p (p – 1, resp.). ψ and χ are regarded as given, and one asks whether (in how many ways, resp.) the polynomial ϑ can be determined such that the differential equation:*

$$\psi\frac{d^2y}{dx^2} + \chi\frac{dy}{dx} + \vartheta\, y = 0$$

will possess a polynomial solution of degree n.

Heine answered that question by direct algebraic calculation. He found that *in general* (i.e., when the polynomials ψ and χ are not specialized in any way), there are:

$$[n, p] = \frac{(n+1)(n+2)\cdots(n-p-1)}{1\cdot 2\cdots(p-1)}$$

([64]) Cf., II B 4. It should be mentioned that since the exponents are 0, 1 at a non-singular point, that condition will come down to saying that one can prescribe the value of the ratio y'/y arbitrarily at a non-singular point.

([65]) N. Y. Bull., May and Oct. 1898.

([66]) Berl. Ber., Jan. 1864; **Heine**'s *Kugelfunktionen*, 2nd ed., 1, Berlin, 1878, pp. 472, *et seq*. One can also confer **Klein**, *Lin. Diff.-Gl.*, pp. 191, *et seq.*

different ways of determining ϑ. However, in order to apply that theorem to a special case, one needs to prove each time that the number of functions ϑ is not reduced by the specialization that takes place, and **Heine** had, in fact, given such a proof for the case of higher-order Lamé equations that was especially interesting to him. More general questions of that kind, as well as the reality questions that were connected with them, remained undiscussed by **Heine**.

T. J. Stieltjes ([67]) had answered all of those questions under certain restricting assumptions in a surprisingly simple way. Namely, one assumes that the polynomials ψ and χ are real, so the equation $\psi = 0$ will have nothing but distinct real roots. $a_0 < a_1 < \ldots < a_p$, and furthermore that when one decomposes χ / ψ into partial fractions:

$$\frac{\chi}{\psi} = \frac{\alpha_0}{x-a_0} + \frac{\alpha_1}{x-a_1} + \cdots + \frac{\alpha_p}{x-a_p} ,$$

the α will all be positive (and non-zero). **Stieltjes** then proved that *there will always be* [n, p] *different real polynomials* ϑ *for which the differential equation possesses a polynomial of degree n as a solution, that those polynomial solutions of degree n will all have real roots that lie in the interval* $a_0\, a_p$, *and that they will differ precisely by the way that those roots are distributed in the individual intervals* $a_0\, a_1$, $a_1\, a_2$, …, $a_{p-1}\, a_p$. **Klein** ([68]) called those polynomial solutions *Lamé polynomials*, while **E. B. Van Vleck** ([69]) called them *Stieltjes polynomials*.

Stieltjes derived the proof of his theorem from the remark that the root-points of the polynomial solutions would give the equilibrium locations for a system of n moving unit mass-points that are repelled by $p + 1$ fixed masses $\alpha_0 / 2$, $\alpha_1 / 2$, …, $\alpha_p / 2$ such that the force of repulsion is directly proportional to the masses and inversely proportional to the distance. That proof is directly connected with the mechanical picture but is not based upon it. Rather, it was carried out in a rigorous analytical form.

Those considerations were extended by the author ([70]) in such a way that complex quantities were also brought under consideration, which meant that one would be dealing with a system of points in a plane.

An extension of Stieltjes's theorem by **E. B. Van Vleck** ([71]) should be mentioned in which the polynomials ϑ that are determined in that way also have nothing but real roots that lie in the interval $a_0\, a_p$.

Now, in order to explain the relationship between Stieltjes's theorem and the oscillation theorem, it should be pointed out that in each theorem one is dealing with the general everywhere-regular equation whose singular points are all real, while one exponent at each such point is zero, while the others are algebraically less than 1. The Stieltjes polynomials then belong to zero exponents at all singular points, and that exponent is the smaller one at the point in question when

([67]) Acta Math. **6** (1885), pp. 321.
([68]) *Lin. Diff.-Gl.*, pp. 191.
([69]) N. Y. Bull., June 1898.
([70]) Cf., his book: *Reihenentwicklungen*, etc., pp. 214, *et seq.*, as well as N. Y. Bull., March 1898. See also **Klein**, *Lin. Diff.-Gl.*, pp. 201, *et seq.*
([71]) N. Y. Bull., June 1898.

the difference between the exponents at that point is < 1. **Klein**'s oscillation theorem then subsumes the **Stieltjes** theorem as a special case.

Finally, we shall speak of the cases that lie beyond the so-called ***Stieltjes*** *limit*, at which one of the quantities α becomes negative. *The Stieltjes theorem will no longer be true then.* Those cases were examined by **E. B. Van Vleck** ([72]), who started from the **Stieltjes** case and let the quantities $\alpha_0, \ldots, \alpha_p$ decrease continuously and then observed how the roots of the polynomial solutions in question would change. There are two types of situations to consider in that way:

1) Two real polynomial solutions can become imaginary, but only when they were previously identical, and therefore only when one has drifted so far from the Stieltjes case that two real polynomial solutions will take on the same distribution of roots.

2) The distribution of roots for polynomial solutions that remain real can change. That can happen when either a number of roots coincide (which must necessarily happen at one of the points $a_0, \ldots, a_p, \infty$) and some of them are imaginary, or in such a way that roots that remain real will go from one interval to another.

One therefore next (i.e., as long as the polynomial solution remains real) needs to investigate only when and how the roots go through the singular points $a_0, \ldots, a_p, \infty$. In that way, one can determine the distribution of roots of the polynomial solutions precisely in the simplest cases.

One notes that these investigations go beyond the scope of **Klein**'s oscillation theorems, and as a result, they can be regarded as preliminary to the further development of that theorem.

7. The methods that were adapted from partial to ordinary differential equations since 1890 ([73]). – Once **E. Picard** had employed the method of successive approximations that **H. A. Schwarz** ([74]) applied for that purpose by starting from the boundary-value problem and treating similar problems for certain other partial differential equations ([75]), he remarked that the same methods could be applied to the *ordinary* differential equation:

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right). \tag{4}$$

([72]) That work of **Van Vleck** was never published. However, one might cf., **Klein**, *Lin. Diff.-Gl. 2 Ord.*, pp. 226, *et seq.*, where one part of **Van Vleck**'s results was derived along a mechanical route. (**Van Vleck** himself used only analytical methods.) In the case $p = 2$, those results were included, in part, implicitly in many general investigations of **Van Vleck** [Am. J. Math. **21** (1899), pp. 126].

([73]) Cf., in particular, the following works of **Picard**: J. de Math. (4) **6** (1890), pp. 197; *ibid.* (4) **9** (1893), pp. 228; C. R. Acad. Sci. Paris, 19 Feb. 1894, 9 April 1894, 23 April 1894, 14 Feb. 1898. One finds a thorough presentation in **Picard**'s *Traité d'analyse* **3** (1896), pp. 94, *et seq.*

([74]) Fenn. Acta **15** (1885), pp. 315 = *Ges. Abh.* **1**, pp. 244.

([75]) Cf., II A 7.c, no. 5.

The function f might be assumed to be a single-valued continuous function of its three independent arguments, as long as $\alpha \leq x \leq \beta$, $|y| \leq L$, $|y'| \leq L'$ ([76]). We next state the following very general theorem:

For a sufficiently-short ([77]) *interval ab that lies inside the interval* $\alpha\beta$, *there is one and (in case one considers only the case of* $|y| < L$, $|y'| \leq L'$) *only one solution of* (4) *that varies continuously inside of that interval along with its first derivatives and satisfies the inequalities* $|A| < L$, $|B| < L$ *at the points a and b, resp., but otherwise assumes arbitrary values A and B.*

In the derivation of that theorem, as well as the ones that follow, the algorithm of successive approximations was applied in the following way ([78]):

One starts from an initial function y_0 (which is chosen in various ways) and calculates the approximations $y_1, y_2, \ldots$ by the equations:

$$\frac{d^2 y_i}{dx^2} = f(x, y_{i-1}, y'_{i-1}) \qquad (i = 1, 2, \ldots).$$

Each of those equations is integrated in such a way that the solution assumes the desired values A and B at the points a and b, resp. Now, one can prove that those functions y_i approach a limit function *uniformly* (II A 1, no. **16**), so it will follow that this limit function is the desired solution to (4) ([79]).

One can apply the same method to systems of differential equations:

$$\frac{d^2 y_i}{dx^2} = f_i(x, y_1, \ldots, y_m; y'_1, \ldots, y'_m) \qquad (i = 1, 2, \ldots, m)$$

and get a theorem that is analogous to the one that was just stated.

([76]) Other conditions were also imposed upon that function, in part, such as the existence and finitude of its partial derivatives with respect to the second and third argument.

([77]) This was made more precise by three inequalities; cf., *Traité d'analyse*, 3, pp. 96, 97.

([78]) From a remark of **P. Painlevé**, Bull. soc. math. **27** (1899), pp. 150, all of Picard's results (naturally, only to the extent that they do not refer to that method explicitly) can were derived in such a way that one employs the Cauchy method for integrating real differential equations, instead of the method of successive approximations (cf., II A 4, no. 5), which is a method that possesses the greatest-possible domain of convergence, from a remark of **Painlevé** (*loc. cit.*) and **Picard** (C. R. Acad. Sci. Paris, 5 June 1899).

The method of successive approximations, which had been applied to other branches of mathematics all along (cf., e.g., **Fourier**, *Théorie de la chaleur*, **§ 286, 287**), was probably first employed by **J. Liouville** [J. de math. **1** (1836), pp. 255] in order to solve ordinary differential equations. However, the way that method was applied to the latter has no special relationship to boundary-value problems, because the approximations were determined in such a way that they would fulfill certain initial conditions *at a single point*. Cf. II A 4.a, no. **9**. In the form that we use (in which the approximations are determined by limiting values at *two points*), the method seems to have been used only once before Picard's work, namely, by **J. Liouville**, J. de math. **5** (1840), pp. 356.

([79]) We refer to a theorem that **Picard** gave (C. R. Acad. Sci. Paris, 9 April 1894), which said that when f is an analytic function of x, y, y', and a parameter λ, the solution that is determined by the method that was discussed here will likewise be an analytic function of λ.

One can further treat some special cases of (4), namely:

$$(\alpha) \qquad \frac{d^2y}{dx^2} = f(x)\,,$$

$$(\beta) \qquad \frac{d^2y}{dx^2} = \varphi(x)\,y\,,$$

$$(\gamma) \qquad \frac{d^2y}{dx^2} = f(x, y)\,.$$

We would like to discuss those equations in turn.

(α) In this case, the boundary-value problem a) of no. **1** can always be solved, and in only one way, and indeed by the formula:

$$y = \int_a^x \frac{(b-x)(\xi-a)}{a-b} f(\xi)\,d\xi + \int_x^b \frac{(b-\xi)(x-a)}{a-b} f(\xi)\,d\xi + \frac{A-B}{a-b}(x-a) + A\ .$$

As **H. Burkhardt** remarked ([80]), in the especially important case of $A = B = 0$, that formula can be linked with the theory of *one-dimensional Green functions*. Namely, **Burkhardt** understood the Green function $G(x, \xi)$ of the region ab to mean the function of x that vanishes at a and b, is single-valued and varies continuously, along with its first derivative, between those points, except at the point ξ, and satisfies the differential equation $d^2y/dx^2 = 0$, while the function G is continuous at the point $x = \xi$, but dG/dx possesses two values that different by -1.

When $f(x)$ is negative between a and b, one sees from the formula that was written down above that the solution of (α) that vanishes at a and b is positive between a and b and that when f is replaced by an algebraically smaller function, that solution will increase at each point.

The case (α) is important because the result that was obtained here can also be employed in the cases (β) and (γ) by applying the method of successive approximations.

(β) Since we are dealing with a linear differential equation, the boundary-value problem that was denoted by a) in no. **1** will generally admit one and only one solution. An exception to that will occur only when the solution to the differential equation that vanishes at a also vanishes at b. That case cannot occur as long as φ is positive in the entire interval ab. *We would now like to assume that φ is negative.* Moreover, we next consider only the case in which the interval ab is short enough that the solution to (β) that vanishes at a does not vanish between a and b or at b, or what amounts to the same thing, we restrict ourselves to the case in which (β) possesses a solution

([80]) Bull. soc. math. **22** (1894), pp. 71. **Burkhardt** appealed to Picard's work only to the extent that he treated a more general boundary-value problem for equation (α), and likewise with the use of a Green function.

that is positive in the entire interval (including the endpoints). As **Picard** proved, *that solution can be represented by the method of successive approximations*.

Moreover, **Picard** gave a criterion for determining whether the case that is considered does or does not occur that was closely linked with the aforementioned work of Schwarz. In order to do that, he considered the constants:

$$W_n = -\int_a^b \varphi(x)\, u_n(x)\, dx\,,$$

in which $u_0 = 1$, and the successive approximations u_1, u_2, … that are defined by the method that was described above vanish at the point a and b. The quotients $c_n = W_n / W_{n-1}$ approach a limiting value c as n increases. *When* $c < 1$, *the solution to* (β) *that vanishes at a will not vanish between a and b or at b. When* $c > 1$, *that solution vanishes between a and b, and when* $c = 1$, *it will vanish at b, but not between a and b*. In that last case, that solution that vanishes at a and b will appear to be the limit of the functions u_n .

The constant c depends, on the one hand, on the values of the function φ, and on the other, on the length of the interval ab. Indeed, one finds that this constant increases, on the one hand, when the function φ is replaced with an algebraically smaller function, and on the other, when the interval ab is replaced with a larger one. It follows from this that the roots of the solution that vanishes at a will also move towards the point a for algebraically degreasing φ. In order to do that, a special case of the Sturm theorem that was discussed in no. **2** will be proved again.

If one replaces equation (β) by the equation:

$$(\beta') \qquad \frac{d^2 y}{dx^2} = \lambda\, \varphi(x)\, y$$

then c will be replaced by $\lambda\, c$. As a result, (β') will have a solution that vanishes at a and b but does not vanish between those points for $\lambda = 1 / c$, and no other value of λ. Therefore, in that way, not only will the existence of the distinguished parameter value that was denoted by λ_0 in no. **2** be proved by an entirely new method ([81]), but one will also arrive at an Ansatz for calculating that parameter value.

If we let y_a and y_b denote the solutions of (β') that vanish at a and b, resp., then they will coincide with $\lambda = 1 / c$. If we let λ grow beyond that value then the root of y_a that originally lay at b will move to the left, while the root of y_b that originally lay at a will move to the right. Those two roots will occur for a certain value of λ, and in that way, we will get the second distinguished parameter value λ_1 . **Picard** proved the existence of the other distinguished parameter value by proceeding in a similar way.

The solution to (β') that assumes arbitrarily-prescribed values A and B at a and b, resp. (so A and B cannot both vanish), is a single-valued analytic function of λ in the entire plane of complex λ, except at the points λ_0, λ_1, λ_2, … As **Picard** proved, it possesses a pole for those distinguished

([81]) Naturally, that is only true in the case when the corresponding distinguished solution vanishes at a and b.

values. Starting from that fact, **Picard** obtained a limit process for calculating those distinguished parameter values ([82]).

(γ) In this case, one next assumes that the function $f(x, y)$ that enters into the differential equation is single-valued and continuous for $a \leq x \leq b$ and for all values of y, and that $f(x, 0) = 0$. Furthermore, $\partial f / \partial y$ should exist and be negative in that region. Finally, the absolute value of that derivative should decrease when y increases through positive values or decreases through negative values. **Picard** then proved the following theorems, among others:

If a solution of (γ) *exists that assumes positive values at a and b and does not vanish between a and b then that solution can be represented by the method of successive approximations.*

If the solutions of the two auxiliary equations:

$$\frac{d^2y}{dx^2} = \left[\frac{\partial f}{\partial y}\right]_{y=0} \cdot y\,, \qquad \frac{d^2y}{dx^2} = \left[\frac{\partial f}{\partial y}\right]_{y=+\infty} \cdot y$$

that vanish at a vanish again for the first time at α *(*β*, resp.)* ([83]), *and* $\alpha < c < \beta$*, then there will be one and only one solution to* (γ) *that vanishes at a and c but remains positive between those points. One can obtain that solution by the method of successive approximations.*

We shall mention some special results that relate to the case in which the desired solution is not everywhere positive. In particular, we refer to the search for *periodic* solutions in the case where f means a periodic function of x ([84]).

For the case in which equation (γ) means an everywhere-positive function that increases along with y, **Picard** proved the theorems:

There never exist two different solutions to (γ) *that assume the same values at a and b.*

If M means a positive constant, and one has the inequality $|\partial f / \partial y| < M$ *for negative y then one can find a solution to* (γ) *that vanishes at a and b* ([85]) *by the method of successive approximations when the length of the interval ab is less than* $\pi / \sqrt{M}$.

If one has a longer interval and one defines the successive approximations $y_1, y_2, y_3, \ldots$ that vanish at a and b then the odd-order approximations will approach a limit u uniformly, while the even-order approximations will approach a limit v uniformly ([86]). However, as **Picard** showed in

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([82]) Those investigations of **Picard** regarding equation (β) were adapted to certain third and fourth-order linear differential equation by **A. Davidoglou** (C. R. Acad. Sci. Paris 1900, 12 Feb., 12 March, 7 May).

([83]) We assume that these points both lie in the interval *ab*.

([84]) *Traité d'analyse* 3, pp. 142. Cf., also **H. Poincaré**, *Les méth. nouv. de la mécanique cél*, 1, Paris 1892, chap. 3.

([85]) The restriction to the limiting value zero at a and b is not essential, since the general case will be reduced to that case by the substitution $z = y + \alpha x + \beta$.

([86]) Cf., *Traité d'analyse* 3, pp. 146, as well as the part about uniform convergence in C. R. Acad. Sci. Paris, 14 Feb. 1898.

an example, those functions do not always coincide. They also will not satisfy equation (γ) then, but the two equations:

$$\frac{d^2u}{dx^2} = f(x, v)\,, \qquad \frac{d^2v}{dx^2} = f(x, v)\,.$$

Thus, the method of successive approximations is not always applicable to the problem of finding a solution of (γ) that vanishes at a and b. However, **Picard** proved that such a solution always exists by giving a method for its calculation that was patterned precisely on Schwarz's method of alternating procedures ([87]).

Picard's investigations into equation (γ) admit a generalization, on the one hand, by allowing more general boundary conditions, as **H. Burkhardt** has remarked ([88]). On the other hand, as **Picard** himself had showed, one can consider systems of differential equations.

We emphasize expressly that singular points of the differential equation *did not* appear in Picard's investigations.

([87]) II A 7.b, no. **28**.
([88]) Bull. soc. math. **22** (1894), pp. 74.