# Carathéodory's approach to the calculus of variations 

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As great and multifaceted as Carathéodory's achievement in other fields might be, the coupling of "Carathéodory and the calculus of variations" is proverbial. There are several grounds for that. First of all, in our century, there are well-defined schools of the calculus of variations only in Italy and the U.S.A. In central Europe, Carathéodory was almost alone in that field, at least in his later years. Secondly, there are traces in all of his work (not the least of which, his book) of the very special love that he had for that classical subject, and in particular, its history, as well, that has already lasted for more than two and a half centuries and encompasses so many great names.

Carathéodory's results in the calculus of variations are varied. However, the direction of his work in that subject was entirely consistent and is characterized completely by the term "field theory." One might call it outmoded from the standpoint of, say, Tonelli's approach, and in fact, that might be connected with his historical inclinations, in that he consciously continued the main currents of the Eighteenth and Nineteenth Centuries and addressed the derivation of existence theorems with the aid of the most modern ideas about integrals (which he was likewise a master of) where others had failed. The objection breaks down completely when one considers the fruits of his labors: In particular, his trailblazing forays into the two realms of "Lagrangian problems" (viz., variational problems with differential conditions) and multiple integrals.

Along with addressing some urgent problems and the history of the topic, throughout the decades, Carathéodory always returned to the question: How can one explain in a simple and intuitive way why fields play such a central role in the calculus of variations? One introduces them, and one then sees how one can begin. However, it is not easy to visualize the very basis for the mechanism. Carathéodory instinctively felt that such a simple visualization must be possible and that it would suddenly illuminate the entire school of thought and open up a common and concise way of accessing all of the important formulas and theorems of the calculus of variations.

One can follow his train of thought in various treatises. "Die Methode der geodätischen Äquidistanten" was the fruit of his twentieth year, which was presented in the article "Variationsrechnung" in the recent "Riemann-Weber" $\left({ }^{1}\right)$ and applied in his great work on the Lagrange problem in the Acta $\left(^{2}\right)$. However, Carathéodory was not content since the mechanism

[^0]was still not entirely clear. In his thirtieth year, his book on the calculus of variations was already in print $\left({ }^{3}\right)$, in which he developed the crucial idea that was incorporated by the editor just in time.

It is known that there is no "royal road" to geometry, but here there is one to the calculus of variations! Nevertheless, in recent years, I must confess my astonishment that he is so little known amongst the broader community of his professional colleagues. Perhaps that is based upon the lastminute revision that was just described, with the consequence that his idea has still not permeated the entire presentation in the book as much as it should have. However, it might also lie in the overall structure of the book, which affords so much space to first-order partial differential equations before the calculus of variations. Indeed, it is precisely the new way of thinking that exhibits the connection between the calculus of variations and first-order partial differential equations most strikingly. However, in order to be able to understand it, one must by no means read through the 163 pages that the first part of the book consists of. Nonetheless, a reader that begins at the correct place (namely, pp. 189) in the lectures will lose their ambition after some time due to their need for many references to the prior material. Only someone that understands both theories and the careful reader of the entire book can fully enjoy the beauty of its conceptual structure.

I therefore believe that I am performing a worthy service when I present that royal road to the calculus of variations once and for all. One will see how the basic idea can be understood with no further assumptions and how everything in the inventory of the calculus of variations will come to light in a few pages: viz., the Euler-Lagrange differential equations, the Legendre condition, the Weierstrass $\mathcal{E}$-function, transversality, Hilbert's independent integral, Kneser's transversality theorem, Erdmann's corner conditions, and the Hamilton-Jacobi theory in mechanics.

Naturally, it does not dispense without any theorems on first-order partial differential equations at all. They are summarized at the end of $\S \mathbf{8}$ with brief outlines of the proofs and bibliography.

In order to show the power of the method when it is applied to difficult problems, in the last sections, I will give a brief depiction of the theory of multiple integrals, which is not treated in Carathéodory's book, while the other most-important application (viz., to the Lagrange problem) is already presented in the book in that way.

## 1. - The problem.

We would like to consider problems with one independent and $n$ dependent variables (i.e., the desired functions): $n=1$ is the case of the simplest variational problem, in which one treats curves in the plane. Like Carathéodory, we would like to denote the integrand in the variational problem by $L$ and let $t$ denote the independent variables, while $x_{1}, \ldots, x_{n}$ denote the desired functions, in order to exhibit the close connection between the calculus of variations and mechanics. In the latter, $t$ is time, $x_{1}, \ldots, x_{n}$ are general coordinates of the problem, and $L$ is the Lagrangian function, which is the integrand of the integral that is to be an extremum (or in any event "stationary")

[^1]according to Hamilton's principle $\left({ }^{4}\right)$. However, insofar as we shall use geometric language here, we will interpret $t$ and $x_{1}, \ldots, x_{n}$ as coordinates in an $(n+1)$-dimensional space $\mathfrak{R}_{n+1}$. In that space, the curves $\mathfrak{C}$ : $x_{i}=x_{i}(t)$ [or more briefly: $x=x(t)$ that will be considered shall generally be "of Bolza class $D_{1}$," i.e., $x_{i}(t)$ is continuous and the derivatives of $x_{i}(t)$ are piecewise continuous (so up to finitely-many jumps). For such curves, whose starting point and endpoint might belong to the values $t^{1}, t^{2}$, resp., one forms the integral:
\[

$$
\begin{equation*}
J_{\mathfrak{C}}=\int_{\mathfrak{C}} L(t, x, p) d t=\int_{t^{1}}^{t^{2}} L(t, x, \dot{x}) d t \tag{1.1}
\end{equation*}
$$

\]

in which the integrand $L\left(t, x_{1}, \ldots, x_{n}, p_{1}, \ldots, p_{n}\right)$, or more briefly $L(t, x, p)$, is a given function of the $2 n+1$ arguments that enter into it that is of class $C_{2}$, i.e., it is continuous, along with its derivatives up to order two $\left(^{5}\right)$. The problem is to minimize that integral [solutions to the maximum problem are derived from what follows by a suitable change of sign (inequality, resp.)], and indeed to make it a relative minimum: One must find a curve $\mathfrak{E}$ (viz., an extremal curve) with the property that $J_{\mathfrak{C}}>J_{\mathfrak{E}}$ for all $\mathfrak{C} \neq \mathfrak{E}$ that lie sufficiently close to $\mathfrak{E}$. If one adds the condition that the directions of $\mathfrak{C}$ and $\mathfrak{E}$ must be sufficiently close then the minimum will be called weak, but otherwise strong. Less is demanded of the comparison curve and more of the solution $\mathfrak{E}$. As far as the "boundary condition" is concerned, we shall first assume that the end points of the desired curve $\mathfrak{E}$ and the comparison curve $\mathfrak{C}$ are two fixed points $1\left(t^{1}, x_{i}^{1}\right)$ and $2\left(t^{2}, x_{i}^{2}\right)$.

## 2. - A field.

The notion of considering a field of curves, instead of a single curve, i.e., a family that simply covers a simply-connected region $\mathfrak{F}$ in $\mathfrak{R}_{n+1}$, will immediately suggest itself in the following argument.

Such a family must depend upon $n$ parameters. We do not need its explicit representation at all, rather we imagine that it is given by:

$$
\begin{equation*}
\dot{x}_{i}=\varphi_{i}(t, x) \quad(i=1, \ldots, n), \tag{2.1}
\end{equation*}
$$

[^2]so by a system of differential equations that specifies the direction of the curve of the family that goes through each point of $\mathfrak{F}$. The functions $\varphi_{i}$ shall then be arranged such that system (2.1) is uniquely soluble everywhere in $\mathfrak{F}$. Naturally, it is sufficient for this that we assume the existence and continuity of its derivatives (in order to be later able to differentiate).

If one replaces the functions $p_{i}$ in $L$ with the functions $\varphi_{i}$ then $(L(t, x, \varphi(t, x))$ will be a spatial function. Now, if:

$$
\begin{equation*}
L(t, x, \varphi)=0 \text { and } \quad L(t, x, p)>0 \quad \text { for } \quad p \neq \varphi \tag{2.2}
\end{equation*}
$$

(i.e., when not all $p_{i}=\varphi_{i}$ ) at every point of $\mathfrak{F}$ then it would be trivial that every segment $\mathfrak{E}$ of a field-curve with endpoints 1 and 2 that lies in $\mathfrak{F}$ will be a solution of the problem in the sense of a strong minimum. That is because one would have $J_{\mathfrak{E}}=0$ for such a curve segment, while any other curve $\mathfrak{C}$ with the same endpoints that lies entirely within the field would certainly contain a line element $(t, x, p)$ that does not belong to the field, and therefore $J_{\mathfrak{C}}>0$ (Fig. $1)$.


Fig. I

## 3. - An equivalent problem.

Obviously, no such family will exist, in general. However, we can perhaps still make use of the argument above when we alter the variational problem in a suitable way. Indeed, there are integrals that are "independent of the path," i.e., whose values over a curve depend upon only its endpoints. For example, if $S(t, x)$ is any spatial function with continuous derivatives $\left({ }^{6}\right)$ then:

$$
\begin{equation*}
J_{\mathfrak{C}}^{*}=\int_{C}\left(S_{t}+S_{x_{i}} p_{i}\right) d t \tag{3.1}
\end{equation*}
$$

is of that kind: If $S_{1}, S_{2}$ denote the values of $S$ at the points 1 and 2 then one will certainly have:

$$
\begin{equation*}
J_{\mathfrak{C}}^{*}=\int_{t^{1}}^{t^{2}}\left(S_{t}+S_{x_{i}} \dot{x}_{i}\right) d t=\int_{t^{1}}^{t^{2}} \frac{d S}{d t} d t=S_{2}-S_{1} \tag{3.2}
\end{equation*}
$$

[^3]That explains the fact that (at least for fixed endpoints) the variational problems with $J_{\mathfrak{E}}$ and $K_{\mathfrak{E}}=J_{\mathscr{E}}-J_{\mathfrak{C}}^{*}$ must possess the same solutions: When $\mathfrak{E}$ and $\mathfrak{C}$ have the same endpoints, $J_{\mathfrak{E}}^{*}=J_{\mathfrak{C}}^{*}$, and therefore $K_{\mathfrak{C}}-K_{\mathscr{E}}=J_{\mathfrak{C}}-J_{\mathfrak{E}}$. Thus, $\mathfrak{E}$ will or will not yield a minimum for $J$ along with $K$.

We then attempt to determine whether we can determine the function $S(t, x)$ for a given family $\varphi_{i}(t, x)$ such that integrand of:

$$
\begin{equation*}
K_{\mathfrak{C}}=J_{\mathfrak{C}}-J_{\mathfrak{C}}^{*}=\int_{\mathfrak{C}}\left\{L(t, x, p)-S_{t}-S_{x_{i}} p_{i}\right\} d t \tag{3.3}
\end{equation*}
$$

possesses the desired property (2.2), so we will have:

$$
\begin{equation*}
L(t, x, \varphi)-S_{t}-S_{x_{i}} \varphi_{i}=0 \quad \text { and } \quad L(t, x, p)-S_{t}-S_{x_{i}} p_{i}>0 \quad(p \neq \varphi)! \tag{3.4}
\end{equation*}
$$

Obviously, if that function fulfills (3.4) when it is considered to be a function of the $p_{i}$ for a fixed space point $(t, x)$ then it will have an ordinary minimum for $p=\varphi$. Therefore, all of its partial derivatives with respect to the $p_{i}$ must also be zero at that location:

$$
\begin{equation*}
S_{x_{i}}=L_{p_{i}}(t, x, \varphi) \tag{3.5}
\end{equation*}
$$

After substituting that value for $S_{x_{i}}$, (3.41) will read:

$$
\begin{equation*}
S_{t}=L(t, x, \varphi)-\varphi_{i} L_{p_{i}}(t, x, \varphi) . \tag{3.6}
\end{equation*}
$$

A necessary condition (which is also sufficient, by our assumption on the connectivity between $\mathfrak{F}$ ) for the existence of the function $S$ with the derivatives (3.5) and (3.6) (which can then be calculated by quadratures, and naturally determined only up to an additive constant) is the fulfillment of the well-known integrability conditions between the right-hand sides of (3.5) and (3.6). In order to solve our problem, we must then first take care that the field is arranged such that those conditions are fulfilled. In addition, we must secondly take care that our integrand, which is then stationary for $p=\varphi$ when $(t, x)$ is fixed, actually possesses a minimum there, such that (3.42) is also still fulfilled. We further remark that the existence and continuity of the second derivatives of $S$ follow from (3.5) and (3.6).

## 4. - Integrability conditions. Euler-Lagrange differential equation.

The integrability conditions express the idea that the mixed second derivatives of the function $S$ to be determined are independent of the sequence of differentiations. Hence, the right-hand sides of (3.5) and (3.6) must satisfy the $n(n-1) / 2$ conditions:

$$
\begin{equation*}
\frac{\partial L_{p_{i}}}{\partial x_{k}}=\frac{\partial L_{p_{k}}}{\partial x_{i}} \quad(i \neq k) \tag{4.1}
\end{equation*}
$$

and the $n$ conditions:

$$
\begin{equation*}
\frac{\partial L_{p_{i}}}{\partial t}=\frac{\partial}{\partial x_{i}}\left(L-\varphi_{j} L_{p_{j}}\right) \tag{4.2}
\end{equation*}
$$

in which the arguments that were omitted read $(t, x, \varphi)$ throughout.
We will speak of the conditions (4.1) later, which drop away for $n=1$, so for the simple variational problem in the plane. For the moment, we will assume we have fulfilled them. The right-hand side of (4.2) is:

$$
L_{x_{i}}-\varphi_{j} \frac{\partial L_{p_{j}}}{\partial x_{i}}
$$

since the remaining terms drop out. However, the left-hand side is:

$$
\frac{d L_{p_{i}}}{d t}-\frac{\partial L_{p_{j}}}{\partial x_{i}} \varphi_{j},
$$

in which $d / d t$ means differentiation along the field-curve $\left({ }^{7}\right)$. Due to (4.1), (4.2) will become simply:

$$
\begin{equation*}
\frac{d L_{p_{i}}}{d t}=L_{x_{i}} . \tag{4.3}
\end{equation*}
$$

Those are the equations that one cares to call Euler's in the calculus of variations (in Germany) and the Lagrange equations of the second kind in mechanics. They are a system of second-order differential equations (for $n=1$, there is one of them). Their solutions are called extremals. In order to perform our construction of a solution, not only the desired curve, but all field-curves, must then be extremals. However, not every field of extremals is useful for that purpose (except for $n=1$ ), since the field must fulfill the equations (4.1), in addition. Following Carathéodory, one calls such fields geodetic $\left({ }^{8}\right)$. The possibility of constructing such fields will be discussed at the end of § 8.

[^4]
## 5. - The Weierstrass condition and the strong minimum.

In order to solve our problem, we must still resolve the second point that was stated at the end of § 3. When we substitute the values (3.5) and (2.6) for the derivatives of $S,\left(3.4_{2}\right)$ will read:

$$
\begin{equation*}
L(t, x, p)-L(t, x, \varphi)-\left(p_{j}-\varphi_{j}\right) L_{p_{j}}(t, x, \varphi)>0 . \tag{5.1}
\end{equation*}
$$

If we set:

$$
\begin{equation*}
\mathcal{E}\left(t, x, p, p^{\prime}\right)=L\left(t, x, p^{\prime}\right)-L(t, x, p)-\left(p_{j}^{\prime}-p_{j}\right) L_{p_{j}}(t, x, p) \tag{5.2}
\end{equation*}
$$

- this is the Weierstrass $\mathcal{E}$-function (i.e., excess function) - then that must demand that:

$$
\begin{equation*}
\mathcal{E}(t, x, \varphi(t, x), p)>0 \quad \text { for } p \neq \varphi . \tag{5.3}
\end{equation*}
$$

That is the Weierstrass sufficient condition. If a geodetic field fulfills that condition then, from §§ $\mathbf{2}$ and $\mathbf{3}$, every segment of a field-curve that connects two points 1 and 2 that lie in the field will actually be a solution to the variational problem with fixed endpoints.

Let us actually convince ourselves of this! When we substitute (3.5) and (3.6), the pathindependent integral (3.1) will become:

$$
\begin{equation*}
J_{\mathfrak{C}}^{*}=\int_{\mathfrak{C}}\left\{L(t, x, \varphi)+\left(p_{j}-\varphi_{j}\right) L_{p_{j}}(t, x, \varphi)\right\} d t . \tag{5.4}
\end{equation*}
$$

That is the Hilbert independent integral. Let $\mathfrak{C}$ be a curve that lies in the field and is different from $\mathfrak{E}$, while connecting 1 and 2 (Fig. 1). We will then have:

$$
\begin{equation*}
J_{\mathfrak{C}}-J_{\mathfrak{E}}=J_{\mathfrak{C}}-J_{\mathfrak{E}}^{*}=J_{\mathfrak{C}}-J_{\mathfrak{C}}^{*}=\int_{\mathfrak{C}} \mathcal{E}(t, x, \varphi, p) d t>0 . \tag{5.5}
\end{equation*}
$$

The first equality is true because (5.4) reduces to (1.1) for $p=\varphi$. The second one is true due to the invariance $J^{*}$. The third one is true because $\mathcal{E}(t, x, \varphi, p)$, which is the left-hand side of (5.1), is in fact the difference between the integrands of (1.1) and (5.4), so it is the integrand of the integral $K_{\mathfrak{c}}$ in $\S \mathbf{3}$, as it must be from our construction. Thus, the curve $\mathfrak{E}$ actually provides a minimum, and indeed a strong one: We demand of the comparison curves that they must lie in the field, but nothing beyond that $\left({ }^{9}\right)$.

[^5]
## 6. - The Legendre condition and the weak minimum.

If one develops the function $L\left(t, x, p^{\prime}\right)$ in the neighborhood of $p^{\prime}=p$ in powers of $p_{j}^{\prime}-p_{j}$, writes the quadratic terms as the remainder term, and cancels the absolute and linear terms on the right-hand of (5.2) then that will give:

$$
\begin{equation*}
\mathcal{E}\left(t, x, p, p^{\prime}\right)=\frac{1}{2} \tilde{L}_{p_{i} p_{k}}\left(p_{i}^{\prime}-p_{i}\right)\left(p_{k}^{\prime}-p_{k}\right), \tag{6.1}
\end{equation*}
$$

in which the tilde suggests that the arguments are taken to be $t, x$, and a certain point along the connecting line from $p$ to $p^{\prime}$ in the space of $p_{j}$. Due to the continuity of the functions $L_{p_{i} p_{k}}$, the following is true: If the coefficient $L_{p_{i} p_{k}}$ is positive-definite for a line element $\left(t^{0}, x^{0}, p^{0}\right)$ then the right-side of (6.1) will be positive in a certain neighborhood of the line element, i.e., when the distance between the points $\left(t^{0}, x^{0}\right)$ and $(t, x)$ and the magnitude of the differences $p_{i}-p_{i}^{0}$ and $p_{i}^{\prime}-p_{i}^{0}$ are sufficiently small. It follows from this that: The field-curve $\mathfrak{E}$ yields a weak minimum when the Legendre sufficient condition:

$$
\begin{equation*}
L_{p_{i} p_{k}} u_{i} u_{k}>0 \tag{6.2}
\end{equation*}
$$

is fulfilled for the line elements of $\mathfrak{E}$ and arbitrary $u$, instead of the Weierstrass condition (5.3). Due to the continuity of the functions $\varphi_{i}$, the integrand in (5.5) will then be positive for every line element $p \neq \varphi$ of any curve $\mathfrak{C}$ that connects 1 and 2 , lies in the field sufficiently close to $\mathfrak{E}$, and whose direction deviates from that of $\mathfrak{E}$ (for the same value of $t$ ) sufficiently little.

## 7. - Transversality.

We shall now consider vectors in $\mathfrak{R}_{n+1}$. The vector with the components $\left(1, \dot{x}_{i}\right)$ is the tangent vector to the curve $x_{i}(t)$ at the point $(t, x)$. The direction of the line element $(t, x, p)$ will then be represented by the vector $\left(1, p_{i}\right)$, which is attached to the point $(t, x)$.

Let us consider the $n$-surfaces $S=$ const. in our geodetic field $\mathfrak{F}$ ! We can then represent their surface elements geometrically by their normals, so by the vector grad $S$, which is the vector ( $L-\varphi_{j} L_{p_{j}} L_{p_{i}}$ ), from (3.5), (3.6). We say that those $n$-surface elements are transverse to the line elements $(t, x, j)$ of the field extremals. In general, any line element $(t, x, p)$ will be assigned the

[^6]transverse $n$-surface element at the same point whose normal is ( $L-p_{j} L_{p_{j}}, L_{p_{i}}$ ). Every lowerdimensional element that is contained in it is also said to be transverse to $(t, x, p)$.

Can it happen that the surface $S=$ const. will contact the field extremal at a point in the field, or more generally, that a line element $(t, x, p)$ is included in the surface that is transverse to it? In order for that to be true, the vector $\left(1, p_{i}\right)$ must be perpendicular to $\left(L-p_{j} L_{p_{j},} L_{p_{i}}\right)$. That is equivalent to $L=0$. That eventuality will be excluded when we assume that $L>0$ from now on $\left({ }^{10}\right)$.

The one-parameter family of surfaces $S=$ const. belongs to the geodetic field as an essential component, in addition to the $n$-parameter family of field extremals. Carathéodory called the two of them together a complete figure of the variational problem. We would also like to the call the surfaces the transverse trajectories of the family of extremals.

For $n=1$, any field of line elements $p=j(t, x)$, along with the 1 -surface elements (i.e., line elements) that are transverse to them, will be a complete figure, since one can always lay the curves $S=$ const through the latter. For $n>1$, the integrability conditions (4.1) must be fulfilled, in addition, in order for the surface $S=$ const.

An important property of the complete figure should be mentioned. Since the Hilbert integral (5.4), which has the value $S_{2}-S_{1}$ over a curve that extends from 1 to 2 , reduces to the integral $J$ in (1.1) when the curve is a field extremal, the integral $J$ will always have the same value $S_{2}-S_{1}$ when it is extended over any extremal of the field from the surface $S=S_{1}$ to the surface $S=S_{2}$. Any two transverse trajectories then cut out segments of equal "geodetic length" $J$ from all field extremals, so they are "geodetically equidistant" $\left({ }^{8}\right)$. That fact is known as the transversal theorem of A. Kneser.

Example: Fermat's principle. - A particle whose velocity is a function $v(x, y, z)>0$ in ordinary space will need a length of time:

$$
\begin{equation*}
T=\int_{t^{1}}^{t^{2}} d t=\int_{\mathfrak{C}} \frac{d s}{v}=\int_{x^{1}}^{x^{2}} \frac{\sqrt{1+y^{\prime 2}+z^{\prime 2}}}{v(x, y, z)} d x \tag{7.1}
\end{equation*}
$$

to traverse a curve $\mathfrak{C}$ that connects two points 1 and 2 when one takes $x$ to be the independent variable. According to Fermat, light particles move in such a way that this integral is a minimum. In order to apply our theory to that, we must replace $t$ with $x$ and the $x_{i}$ with $y$ and $z$ in our formulas. $L$ is now the integral of (7.1). We will find that the vector ( $L-y^{\prime} L_{y^{\prime}}-z^{\prime} L_{z^{\prime}}, L_{y}, L_{z^{\prime}}$ ) has the same direction as ( $1, y^{\prime}, z^{\prime}$ ). That means that transverse is the same thing as orthogonal here. In the simple case of $v=$ const., the extremals are straight lines, and the Kneser transversal theorem will go to a known theorem on ray congruences (which are the geodetic fields in this case).

[^7]
## 8. - The Legendre transformation and Hamilton-Jacobi theory.

New determining coordinates for the line elements $(t, x, p)$ of $\Re_{n+1}$ are introduced by the Legendre transformation, and the foregoing formulas can be written more simply by means of it. If one can solve the equations:

$$
\begin{equation*}
L_{p_{i}}=y_{i} \tag{8.1}
\end{equation*}
$$

for the $p_{i}$ then that solution might be denoted by:

$$
\begin{equation*}
p_{i}=\psi_{i}(t, x, y) . \tag{8.2}
\end{equation*}
$$

Since the elements of the functional determinant of (8.1) with respect to the $p_{i}$ are coefficients of the quadratic form (6.2), the solution is always possible, e.g., in the neighborhood of the line element of a geodetic field that fulfills the Legendre condition.
$(t, x, y)$ are the new coordinates of the line element. The Lagrangian function $L$, in which one has replaced the $p_{i}$ with the functions (8.2), will be replaced with the Hamiltonian function:

$$
\begin{equation*}
H(t, x, y)=-L(t, x, y)+\psi_{i} y_{i} \tag{8.3}
\end{equation*}
$$

which is suggested by (3.6). For the derivatives of that function, due to (8.1), one gets:

$$
\begin{equation*}
H_{t}=-L_{t}, \quad H_{x_{i}}=-L_{x_{i}}, \quad H_{y_{i}}=\psi_{i} \tag{8.4}
\end{equation*}
$$

directly.
Equations (3.5) and (3.6) are valid for the line elements of a geodetic field; they are now written:

$$
\begin{equation*}
S_{x_{i}}=y_{i}, \quad S_{t}=-H(t, x, y), \tag{8.5}
\end{equation*}
$$

or more simply:

$$
\begin{equation*}
S_{t}+H\left(t, x, S_{x}\right)=0 . \tag{8.6}
\end{equation*}
$$

That is the Hamilton-Jacobi partial differential equation. We see that the function $S(t, x)$ that we started from is the "action function" of analytical mechanics from Hamilton's principle.

One can start from an arbitrary solution of (8.6) in order to obtain a geodetic field. Due to (8.2), (8.51), and (8.43), one will have:

$$
\varphi_{i}(t, x)=\varphi_{i}\left(t, x, S_{x}\right)=H_{y_{i}}\left(t, x, S_{x}\right)
$$

for the direction $\varphi_{i}(t, x)$ of the field curves. One will then obtain the field-curves by integrating the system of $n$ first-order ordinary differential equation:

$$
\dot{x}_{i}=H_{y_{i}}\left(t, x, S_{x}\right) .
$$

If one sets $y_{i}=S_{x_{i}}$ and differentiates that along the curves thus-obtained then that will give $\dot{y}_{i}=$ $S_{x_{i}, t}+S_{x_{i} x_{j}} \dot{x}_{j}$. On the other hand, differentiating (8.6) with respect to $x_{i}$ will then give $S_{x_{i}, t}+H_{x_{i}}+$ $H_{y_{i}} S_{x_{i} x_{j}}=0$. Thus, the curves simultaneously fulfill the equations:

$$
\dot{y}_{i}=-H_{x_{i}}\left(t, x, S_{x}\right),
$$

which are the Euler-Lagrange equations (4.3), when recalculated in terms of the Hamiltonian function. Naturally, the integrability conditions are fulfilled by themselves.

One can also start from the curves, instead of the function $S$. In order to do that, one must replace the system (4.3) of $n$ second-order differential equations with the system of $2 n$ first-order differential equations:

$$
\begin{equation*}
\dot{x}_{i}=H_{y_{i}}\left(t, x, S_{x}\right), \quad \dot{y}_{i}=-H_{x_{i}}\left(t, x, S_{x}\right) \tag{8.7}
\end{equation*}
$$

for the $2 n$ functions $x_{i}(t), y_{i}(t)$, which one calls the canonical equations of the problem. The general solution to that system depends upon $2 n$ parameters. Constructing a field amounts to finding an $n$-parameter family of solutions that simply cover the region in $\mathfrak{R}_{n+1}$ and satisfy the integrability conditions (4.1). If one writes $y_{i}=\chi_{i}(t, x)$ for the line element of the family then (4.1) will read simply:

$$
\begin{equation*}
\frac{\partial \chi_{i}}{\partial x_{k}}=\frac{\partial \chi_{k}}{\partial x_{i}} \tag{8.8}
\end{equation*}
$$

Example from geometrical optics: We write the Legendre transformation of the integrand $L\left(x, y, z, y^{\prime}, z^{\prime}\right)$ in (7.1) thus:

$$
L_{y^{\prime}}=p, \quad L_{z^{\prime}}=q
$$

We will then have:

$$
H(x, y, z, p, q)=-\frac{\sqrt{1-v^{2}\left(p^{2}+q^{2}\right)}}{v}
$$

as the Hamiltonian and:

$$
\begin{equation*}
S_{x}^{2}+S_{y}^{2}+S_{z}^{2}=\frac{1}{v^{2}} . \tag{8.9}
\end{equation*}
$$

The asymmetry that the formulas were endowed with up to now by the choice of the $x$ as independent variables vanishes here.

Example from mechanics: Under the assumptions in footnote 5, one has:

$$
\psi_{i} y_{i}=p_{i} L_{p_{i}}=p_{i} T_{p_{i}}=2 T
$$

(Euler's formula for quadratic forms) and therefore:

$$
H=-L+2 T=T+U
$$

is the total energy. Along an extremal (trajectory), one has:

$$
\frac{d H}{d t}=H_{t}+H_{x_{i}} \dot{x}_{i}+H_{y_{i}} \dot{y}_{i}=H_{t},
$$

due to the canonical equations (8.7). The law of energy $d H / d t=0$ is then valid in the case considered, where $L$, and therefore $H$, as well, do not depend upon $t$ explicitly.

The action function, which is the eikonal in the case of geometrical optics, will mostly be introduced as a function $W\left(P_{1}, P_{2}\right)$ of a starting point and an endpoint, and indeed, as the integral $J$, when it is extended over the extremal that connects them in a region in which any two points can be connected by one and only extremal. When one fixes one endpoint, such a function will then satisfy the partial differential equation (8.6) [(8.9), resp.] as a function of the other endpoint.

In the theory of first-order partial differential equations, equations (8.7) are called the characteristic differential equations of the partial differential equation (8.6). In that theory, one proves the following three theorems, from which everything that one will need for the construction of geodetic fields will follow.

1. Let an $n$-parameter family of extremals [= solutions of (8.7)] that simply covers a region in $\mathfrak{R}_{n+1}$ be given. If the equations (8.8) are fulfilled at a point then they will be true all along the entire extremal of the family that goes through that point.
2. If a line element is given at a point $P$, along with a surface element that goes through it, as well as an $n$ surface patch [perhaps by some twice-continuously-differentiable functions $t\left(u_{1}, \ldots, u_{n}\right), x_{i}\left(u_{1}, \ldots, u_{n}\right)$ ] that includes the point $P$ and the surface element, then one can determine a line element that is transverse to the surface at each point of the surface patch in the neighborhood of $P$ in a continuous manner. The extremals that go through that line element then define a geodetic field that belongs to the given surface as a surface $S=$ const. The function $S$ can be determined with the help of Kneser's transversality theorem.

Namely, it will be true on the given surface (8.8).
3. All of the extremals that go through a point define a geodetic field to the left and right of that point.

Carathéodory gave proofs of the first two theorems [loc. cit. $\left(^{1}\right.$ ), pp. 192-196] that were independent of the theory of partial differential equations. Later, he abandoned that representation in favor of the elegant method of "Lagrange brackets" that was employed in the chapter on first-order partial differential equations in loc. cit. ${ }^{3}$ ). Cf., §§ 35-48 there. Theorem 3 is almost trivial with the use of that method, cf., § 330.

For a family of solutions, $x_{i}=\xi_{i}\left(t, u_{1}, \ldots, u_{n}\right), y_{i}=\eta_{i}\left(t, u_{1}, \ldots, u_{n}\right)$ of (8.7) with:

$$
\begin{equation*}
\operatorname{det}\left(\frac{\partial \xi_{i}}{\partial u_{\alpha}}\right) \neq 0 \tag{8.9}
\end{equation*}
$$

the Lagrange brackets are defined by:

$$
\left[u_{\alpha}, u_{\beta}\right]=\frac{\partial \xi_{i}}{\partial u_{\alpha}} \frac{\partial \eta_{i}}{\partial u_{\beta}}-\frac{\partial \xi_{i}}{\partial u_{\beta}} \frac{\partial \eta_{i}}{\partial u_{\alpha}}
$$

With the notation that was employed in (8.8), one has $\eta_{i}(t, u)=\eta_{i}(t, u)$, so:

$$
\frac{\partial \eta_{i}}{\partial u_{\beta}}=\frac{\partial \chi_{i}}{\partial x_{j}} \frac{\partial \xi_{j}}{\partial u_{\beta}} \quad \text { and therefore } \quad\left[u_{\alpha}, u_{\beta}\right]=\left(\frac{\partial \chi_{i}}{\partial x_{j}}-\frac{\partial \chi_{i}}{\partial x_{j}}\right) \frac{\partial \xi_{i}}{\partial u_{\alpha}} \frac{\partial \xi_{j}}{\partial u_{\beta}}
$$

It then follows from (8.8) that $\left[u_{\alpha}, u_{\beta}\right]=0$, but due to (8.9), the converse is also true. Upon differentiating the canonical equations with respect to $u_{\beta}$ ( $u_{\alpha}$, resp.), one finds that:

$$
\frac{\partial}{\partial t}\left(\frac{\partial \xi_{i}}{\partial u_{\alpha}} \frac{\partial \xi_{j}}{\partial u_{\beta}}\right)=H_{y_{j} y_{k}} \frac{\partial \eta_{j}}{\partial u_{\alpha}} \frac{\partial \eta_{k}}{\partial u_{\beta}}-H_{x_{j} x_{k}} \frac{\partial \xi_{j}}{\partial u_{\alpha}} \frac{\partial \xi_{k}}{\partial u_{\beta}}
$$

Theorem 1 then follows since the latter equation is symmetric in $\alpha$ and $\beta$.
If all curves of the family go through a fixed point $\left(t^{0}, x^{0}\right)$ then all $\partial \xi_{i} / \partial u_{\alpha}=0$ there, and also all $\left[u_{\alpha}, u_{\beta}\right]=0$, as well. In order to prove Theorem 3, one must only convince oneself that (8.9) is fulfilled for $t \neq t^{0}$ in the neighborhood of $t^{0}$, which poses no difficulty. The proof of Theorem 2 requires somewhat more effort, so I will pass over it.

In order to "embed" a segment of a given extremal in a field, one can perhaps proceed in such that way that one lays an arbitrary $n$-surface patch (e.g., an $n$-plane) through any point of the extremal and carries out the construction in Theorem 2. Theorem 3 can also be employed when one chooses the cusp of the field along the extremal to be outside of the embedded segment.

That raises a further question: How far can a geodetic field that is constructed in that way reach without losing the essential property that it simply covers the neighborhood of the given extremal? Indeed, when two points 1 and 2 along the extremal are given, the segment that connects them must be embedded completely within the field in order for the theory to be applicable. The answer to that question is the subject of the theory of the "second variation," the Jacobi differential equation, conjugate points, and focal points, which is treated thoroughly in all textbooks on the calculus of variations, and which I would not like to go into here.

## 9. - Moving endpoints.

Our considerations can easily be extended to the case in which one endpoint or both of them are not fixed but move on manifolds of dimensions $v \leq$ $n$ (on curves in the planar case) with the help of the concept of transversality and complete figures (§ 7). That is based upon the fact that the equation $J_{\mathfrak{C}}^{*}=J_{\mathfrak{E}}^{*}$ (§ 3) is indeed not only true for the Hilbert integral when both curves possess the same endpoints, but also when the two starting points and the two endpoints lie on the same surface $S=$ const.

Next, let the initial point 1 be fixed, while the endpoint 2 moves on the surface $S=S_{2}$ or on a lowerdimensional manifold $\mathfrak{M}_{2}$ that contains it. (5.5) will then be true, as before, when one understands $\mathfrak{C}$ to mean a


Fig. 2
curve that connects 1 with any point of $\mathfrak{M}_{2}$ and lies in the field. The extremal $\mathfrak{E}$ is also a solution of that problem then (Fig. 2).

Starting with the point 1 and the manifold $\mathfrak{M}_{2}$ as given, one proceeds as follows: One connects 1 and $\mathfrak{M}_{2}$ by an extremal $\mathfrak{E}$ that is transverse to $\mathfrak{M}_{2}$. If $\mathfrak{M}_{2}$ is an $n$-surface then one can construct a geodetic field that includes $\mathfrak{E}$ and a trajectory that is transverse to $\mathfrak{M}_{2}$ using the theorems that were given at the end of $\S 8$. If $\mathfrak{M}_{2}$ has a lower dimension then one proceeds similarly with any $n$ surface that includes $\mathfrak{M}_{2}$ and is transverse to $\mathfrak{E}$. If that field includes the point 1 and the Weierstrass (strong minimum) or Legendre (weak minimum) condition is fulfilled then the extremal $\mathfrak{E}$ will solve the problem.

If both endpoints move on manifolds $\mathfrak{M}_{1}\left(\mathfrak{M}_{2}\right.$, resp. $)$ then one must determine an extremal $\mathfrak{E}$ that intersects $\mathfrak{M}_{1}$ at a point $1\left(\mathfrak{M}_{2}\right.$ at a point 2 , resp.)


Fig. 3 transversally. Just as before, one now constructs a field that includes $\mathfrak{M}_{1}$ or any $\mathfrak{M}_{1}$, possesses $n$ surfaces that are transverse to $\mathfrak{E}$ as transverse trajectories, and includes $\mathfrak{E}$. $\mathfrak{E}$ will then solve the problem when:

1. The field includes the point 2 .
2. The Weierstrass or Legendre condition is fulfilled.
3. $\mathfrak{M}_{2}$ contacts the surface $S=$ const. that goes through 2 externally there [the fact that they contact follows from transversality $\left.\left({ }^{(11}\right)\right]$.

To prove that, let $\mathfrak{C}$ be a curve that lies in the field and connects 1 or a point $1^{\prime}$ of $\mathfrak{M}_{1}$ with a point $2^{\prime}$ of $\mathfrak{M}_{2}$. It meets the surface $S=$ const. that goes through 2 at 3 : $\mathfrak{C}=\mathfrak{C}_{1^{\prime} 3}+\mathfrak{C}_{32^{\prime}}$ (Fig. 3). One then has:

$$
\begin{equation*}
J_{\mathfrak{C}_{13}} \geq J_{\mathfrak{E}} \tag{9.1}
\end{equation*}
$$

and

$$
\begin{equation*}
J_{\mathfrak{C}} \geq J_{\mathfrak{C}_{\mathfrak{C}_{3}}}, \tag{9.2}
\end{equation*}
$$

[^8]since the difference $J_{\mathfrak{C}_{32}}$ is non-negative, due to the fact that $L>0$. The equality sign is true in (9.2) only in the case where $2^{\prime}=2$ (so one will also have $3=2$ then). That will be true in (9.1) only in the case where $\mathfrak{C}_{1^{\prime} 3}$ is a field extremal, so on the case of $2^{\prime}=2$ since $\mathfrak{C} \neq \mathfrak{E}$ is certainly not true. In any case, it will then follow from (9.1) and (9.2) that $J_{\mathfrak{C}}>J_{\mathfrak{E}}$.

## 10. - Corner conditions.

We have indeed allowed corners for the curves that admit concurrence, and therefore for the solution of the problem, but up to now we have constructed only solutions that do not possess corners, because the direction functions $\varphi_{i}(t, x)$ were assumed to be continuous. We would now like to investigate solutions with corners and first consider the case in which the variational problem itself exhibits a discontinuity. The region of $\mathfrak{R}_{n+1}$ in which the variational problem is defined will be divided into two subregions by an $n$-surface patch $\mathfrak{D}$ that is met at least once by each parallel to the $t$-axis. Let two integrals $L$ and $\bar{L}$ be given, one of which is take from the left of $\mathfrak{D}$ (i.e., smaller values of $t$ ), while the other is taken from the right. One will then have:

$$
\begin{equation*}
J_{\mathfrak{C}}=\int_{t^{1}}^{t^{0}} L d t+\int_{t^{0}}^{t^{2}} \bar{L} d t \tag{10.1}
\end{equation*}
$$

if $\mathfrak{C}$ meets the surface $\mathfrak{D}$ when $t=t^{0}$.


Fig. 4

One will obtain a complete figure with whose help one can adapt all of the results up to now when one constructs a direction field that might possess a discontinuity along $\mathfrak{D}$, but in such a way that the function $S(t, x)$ is continuous throughout (Fig. 4). That is because (3.2) will still be true with no changes even when $\mathfrak{C}$ meets the surface $\mathfrak{D}$, and therefore all of the remaining equations, as well. The derivatives of $S$ are given by (3.5) and (3.6), in which cross-sections of $\varphi_{i}$ and $L$ are placed to the right of $\mathfrak{D}$; they will then be continuous on $\mathfrak{D}$. However, since the tangential derivatives of $S$ on $\mathfrak{D}$ are also determined by the values of $S$ on $\mathfrak{D}$, the projection of grad $S$ onto the surface will be continuous, along with $S$. Problems with fixed or moving endpoints that are considered in sections can also be solved in that case when one constructs geodetic fields of broken extremals (viz., broken on $\mathfrak{D}$ ) that fulfill the following corner condition:

The projection of the vector $\left(L-\varphi_{j} L_{p_{j}}, L_{p_{i}}\right)$ onto the tangent plane to $\mathfrak{D}$ shall be continuous.
Namely, if one adds arbitrary additive constants to the two functions $S$ and $\bar{S}$, which belong to the fields in the two subregions, in such a way that those functions agree at a point of $\mathfrak{D}$ then they will agree over all of $\mathfrak{D}$.

Example: The law of refraction. - From Fermat's principle, which we already considered in § 7, the vector ( $L-\varphi_{j} L_{p_{j}}, L_{p_{i}}$ ) is precisely $\frac{1}{v} \mathfrak{e}$, in which $\mathfrak{e}$ is the unit vector in the direction of the ray. If the function $v$ has a jump across a 2 -surface $\mathfrak{D}$ then let $\alpha$ be the "angle of incidence," i.e., the angle that $\mathfrak{e}$ defines with the normal to $\mathfrak{D}$. The length of the projection of $\frac{1}{v} \mathfrak{e}$ onto the tangent plane to $\mathfrak{D}$ is then $\frac{1}{v} \sin \alpha$. One must get the same vector when one projects the vector $\frac{1}{\bar{v}} \overline{\mathcal{e}}$, which belongs to the exiting ray, onto the tangent plane. That means that incident and exiting rays must lie in a plane with the normal to $\mathfrak{D}$ and that one must have:

$$
\frac{1}{v} \sin \alpha=\frac{1}{\bar{v}} \sin \bar{\alpha}
$$

As a second example, we shall consider the case of a "free" corner, so the problem itself does not have to already possess a discontinuity. If one would like to embed an extremal $\mathfrak{E}$ that possesses such a corner (which is imagined to be the solution) in a field whose curves likewise possess corners on an $n$-surface $\mathfrak{D}$ that goes through the corner then $\mathfrak{E}$ must fulfill the same condition as before relative to that surface. However, since the surface $\mathfrak{D}$ can be chosen arbitrarily now, one must demand that $\mathfrak{E}$ fulfills that condition for an arbitrary placement of the tangent plane to $\mathfrak{D}$. That means that the vector ( $L-\varphi_{j} L_{p_{j}}, L_{p_{i}}$ ) itself must be continuous:

$$
\left\{\begin{align*}
L-p_{j} L_{p_{j}} & =\bar{L}-\bar{p}_{j} \bar{L}_{p_{i}},  \tag{10.2}\\
L_{p_{j}} & =\bar{L}_{p_{j}} .
\end{align*}\right.
$$

Those are the Erdmann corner conditions. In that way, $\left(1, p_{i}\right)$ is the direction of the extremal to the right of the corner and $\left(1, \bar{p}_{i}\right)$ is the direction of the one to the right of it. $\bar{L}$ means that the $\bar{p}_{i}$ are substituted in $L$.

All of the previous results will, in turn, be applicable to fields that are discontinuous along a surface $\mathfrak{D}$ and fulfill equations (10.2) (with $\varphi$ in place of $p$ ) there.

Here, let us mention the famous result of Carathéodory ( ${ }^{12}$ ) by which one is inevitably led to consider discontinuous extremals and fields of the type that was just described when one would basically like to focus upon only strong extremals. If one continues a strong extremal sufficiently far then it will happen that it will stop strongly somewhere. Carathéodory found that it is precisely the line element $p$ at such a point where the extremal stops strongly, in general such a thing is something that fulfills the Erdmann conditions (10.2), along with another $\bar{p}$, and that in general the other line element $\bar{p}$ belongs to an extremal that begins strongly at that point, such that one can compose the two strong segments into a discontinuous strong extremal. One can proceed similarly with a field instead of one extremal. $\mathfrak{D}$ will then be the locus of points where the field extremals stop strongly. If one embeds a given segment of a strong extremal in a field in various ways then the surface $\mathfrak{D}$ will prove to be different. One can, in fact, prescribe it arbitrarily in the neighborhood of the corner, in general.

## 11. - Necessity.

It is intrinsic to the considerations that were made here that they lead to sufficient, but not necessary conditions. For some of those conditions, necessity can be seen by a simple additional argument. For example, the fact that the Euler-Lagrange equations are necessary can be made plausible. If a solution of the variational problem (no matter what the boundary condition might be) does not satisfy those equations then one would consider two of its points $P, Q$ that lie so closely to each other that one can connect them by an extremal and one can embed it in a field. $J_{P Q}$ will then be smaller along the extremal than it is along the given curve. The broken path that arises when one replaces the segment $P Q$ with the segment of the extremal will then assign a smaller value to the integral than the given curve. It will therefore not yield a minimum, which is contrary to the assumption $\left({ }^{13}\right)$.

In order to see the necessity of the transversality condition for endpoint 2, which is regarded as moving (regardless of what one assumes about the starting point 1 ), one considers the field of extremals through 1 , which is geodetic, from the third of the theorems that were stated at the end of § 8. The point 1 is assumed to be so close to 2 along the curve in question (which is an extremal that fulfills the Legendre condition) that this field will include the point 2. Now, if the manifold $\mathfrak{M}_{2}$ on which 2 moves were not transverse to the extremal $\mathfrak{E}_{12}$ then it would intersect the surface

[^9]$S=S_{2}$, and there would therefore be a field extremal $\mathfrak{E}^{*}$ whose point of intersection 2 with $\mathfrak{M}_{2}$ would come before its point of intersection 3 with $S=S_{2}$ (Fig. 5). Since $L>0$, one would then have $J_{\mathfrak{E}_{12}^{*}}<J_{\mathfrak{E}_{13}^{*}}=S_{2}-S_{1}=J_{\mathfrak{E}_{12}}$, and $\mathfrak{E}_{12}$ would not yield a minimum again.

The Legendre condition for a weak minimum and the Weierstrass condition for a strong one, with the $\geq$ sign, instead of >, are necessary in any case. However, one can prove that, just like the necessity of the Erdmann corner condition, more conveniently in another direct way, cf., the textbooks on the calculus of variations.


Fig. 5

## 12. - Multiple integrals. Divergence method.

Let an integrand $L\left(t_{1}, \ldots, t_{\mu}, x_{1}, \ldots, x_{n}, p_{11}, p_{12}, \ldots, p_{n \mu}\right) \equiv L\left(t_{\alpha}, x_{i}, p_{i \alpha}\right) \equiv L(t, x, p)$ be given. We once more assume that $L>0$. A $\mu$-surface patch $\mathfrak{F}$ in $\mathfrak{R}_{n+\mu}: x_{i}=x_{i}\left(t_{1}, \ldots, t_{\mu}\right) \equiv x_{i}(t)$ shall be determined such that the integral:

$$
\begin{equation*}
J_{\mathfrak{F}}=\int_{\mathfrak{F}} L(t, x, p) d t=\int_{\mathfrak{G}} L\left(t_{\alpha}, x_{i}, \frac{\partial x_{i}}{\partial t_{\alpha}}\right) d t \tag{12.1}
\end{equation*}
$$

is as small as possible. $d t$ is an abbreviation for $d t_{1} \ldots d t_{\mu}$, the symbol $\int$ stands for $\mu$ such symbols, and $\mathfrak{G}$ is the projection of $\mathfrak{F}$ onto the $t_{1}, \ldots, t_{\mu}$-"plane." For a "problem with a fixed boundary," the $(\mu-1)$-dimensional boundary of $\mathfrak{F}$ is assumed to be fixed.

In order to be able to adapt the theory to this case, one must have an "independent integral," so an integral of the kind (12.1) that depends upon only the boundary of the $\mu$-surface patch but still extends over it. Two possibilities suggest themselves for that purpose that will lead to formulas that are partially quite different. The first of them, which is the simpler one, but as we will see, is not as far-reaching, might be called the "divergence method."

Let $\mu$ functions $S_{1}(t, x), \ldots, S_{\mu}(t, x)$ be given. One writes $S_{\alpha \beta}$ for the derivative of $S_{\alpha}$ with respect to $t_{\beta}$, to abbreviate, and for the ones with respect to $x_{i}$, one writes simply $S_{\alpha i}$. Greek indices will always refer to the independent variables $t$ and run from 1 to $\mu$, while Latin ones will refer to the dependent variables $x$ and run from 1 to $n$. If one replaces the $x_{i}$ in $S_{\alpha}$ with the functions $x_{i}(t)$, whose derivatives will be denoted by $p_{i \alpha}$, then $\left({ }^{14}\right)$ :

[^10]\[

$$
\begin{equation*}
\frac{d S_{\alpha}}{d t_{\beta}}=S_{\alpha \beta}+S_{\alpha j} p_{j \beta} \tag{12.2}
\end{equation*}
$$

\]

are the derivatives of the functions of $t$ that then arise, so they are the derivatives of $S_{\alpha}$ with respect $t_{\beta}$ on the surface $\mathfrak{F}$. The integral:

$$
\begin{equation*}
J_{D i v}^{*}=\int_{\mathfrak{G}} \frac{d S_{\alpha}}{d t_{\alpha}} d t \quad(\text { sum over } \alpha!) \tag{12.3}
\end{equation*}
$$

can be converted into an integral over the boundary of $\mathfrak{G}$ using Gauss's theorem, so it is "pathindependent," i.e., it possesses the same value for all surfaces that have the same boundary.

The formulas of §§ 3-6 can be adapted with no further analysis. A "field" is given by:

$$
\begin{equation*}
p_{i \alpha}=\varphi_{i \alpha}(t, x) . \tag{12.4}
\end{equation*}
$$

Should one be then dealing with an $n$-parameter family of $\mu$-surfaces that simply covers the defining region of the $\varphi_{i \alpha}$, then the differential equations (12.4), in which the $p_{i \alpha}$ then stand for $\partial x_{i} / \partial t_{\alpha}$, must be integrable. However, that is not at all necessary, in general. A surface $x_{i}(t)$ is said to be embedded in the field when the $\varphi_{i \alpha}$ coincide with its derivatives on it; equations (12.4) need to be integrable only it.

One can now write down all formulas in § $\mathbf{3}$ with no further discussion. For a geodetic field, one gets:

$$
\begin{aligned}
& S_{\alpha i}=L_{p_{i \alpha}} \\
& S_{\beta \beta}=L-p_{j \gamma} L_{p_{j \gamma}} \quad(\beta \text { is summed over }!)
\end{aligned}
$$

with the arguments $t, x, \varphi(t, x)$. It is obviously necessary for the existence of functions $S$ that accomplish this that:

$$
\frac{\partial L_{p_{i \alpha}}}{\partial x_{j}}-\frac{\partial L_{p_{j \alpha}}}{\partial x_{i}}=0
$$

and

$$
\frac{\partial}{\partial x_{j}}\left(L-p_{j \gamma} L_{p_{j \gamma}}\right)=\frac{\partial L_{p_{i \beta}}}{\partial t_{\beta}} .
$$

As before, the Euler-Lagrange equations will follow from this:

$$
\begin{equation*}
\frac{d L_{p_{i \beta}}}{d t_{\beta}}=L_{x_{i}} \quad(i=1, \ldots, n) \tag{12.5}
\end{equation*}
$$

The Weierstrass $\mathcal{E}$-function will become:

$$
\begin{equation*}
\mathcal{E}\left(t, x, p, p^{\prime}\right)=L\left(t, x, p^{\prime}\right)-L(t, x, p)-\left(p_{j \beta}^{\prime}-p_{j \beta}\right) L_{p_{j \beta}}(t, x, p), \tag{12.6}
\end{equation*}
$$

and the Legendre condition will become:

$$
\begin{equation*}
L_{p_{i \alpha} p_{j \beta}} u_{i \alpha} u_{j \beta}>0 \tag{12.7}
\end{equation*}
$$

Moreover, for a given boundary, an extremal (i.e., a solution to the Euler-Lagrange equations) will actually yield a minimum if it can be embedded in a field and the Weierstrass condition (for a strong minimum) or the Legendre condition (for a weak minimum) is fulfilled.

The Legendre transformation can be performed with no further analysis. I could write down the obvious formulas if I were so asked.

However, one will reach the limits of possibility for that procedure when one would like to adapt the concept of transversality (say, in order to be able to treat problems with a moving boundary). The $n$-surfaces $S_{\alpha}=$ const. will not have the meaning of transverse trajectories to the field as they did in the case of $\mu=1$.

## 13. - Multiple integrals. Determinant method.

There are two natural possibilities for defining an independent integral. With the same notations as in the previous section:

$$
\begin{equation*}
J_{\text {Det }}^{*}=\int_{\mathfrak{E}}\left|\frac{d S_{\alpha}}{d t_{\beta}}\right| d t=\int_{\mathfrak{F}}\left|S_{\alpha \beta}+S_{\alpha j} p_{i \beta}\right| d t \tag{13.1}
\end{equation*}
$$

will also depend upon only the boundary of $\mathfrak{F}$, in which we would like to further assume that the determinant $\left|S_{\alpha \beta}+S_{\alpha j} p_{j \beta}\right| \neq 0$, say, it is $>0$. Namely, under that assumption, the region $\mathfrak{G}$ will be mapped invertibly to a certain region $\mathfrak{H}$ in the $\lambda_{\alpha}$ " "plane" by:

$$
\begin{equation*}
S_{\alpha}(t, x(t))=\lambda_{\alpha} \quad(\alpha=1, \ldots, \mu), \tag{13.2}
\end{equation*}
$$

i.e., precisely one of the $n$-surfaces:

$$
\begin{equation*}
S_{\alpha}(t, x)=\lambda_{\alpha} \quad(\alpha=1, \ldots, \mu) \tag{13.3}
\end{equation*}
$$

goes through each point of $\mathfrak{F}$, and $J_{\text {Det }}^{*}$ is simply the volume of the region $\mathfrak{H}$. $(\mathfrak{H}$ does not need to be simple.) Its value will then depend upon only the nature of the boundary of $\mathfrak{H}$, i.e., exclusively upon which of the surfaces (13.3) go through the boundary of $\mathfrak{F}$.

One sees immediately that one can say somewhat more here, and this is precisely what we will need in order to treat the problems with moving boundaries: $J_{\text {Det }}^{*}$ will also not change when one varies the surface together with its boundary as long as one only observes that the boundary is displaced along the surfaces of the family (13.3) that go through it, so it defines an $(n+\mu-1)$ dimensional "tube."

However, we must next establish that we will obtain, in part, formulas that are entirely different and essentially more complicated than before, because the integrand in (13.3) is nonlinear in the $p_{i \alpha}$.

To abbreviate, let $c_{\alpha \beta}=S_{\alpha \beta}+S_{j \alpha} p_{j \beta}$ and $\left|c_{\alpha \beta}\right|=\Delta$, and let $\overline{c_{\alpha \beta}}$ be the algebraic complement of $c_{\alpha \beta}$ in $\Delta$. Let $p_{i \alpha}\left[=\varphi_{i \alpha}(t, x)\right]$ be a surface element of the field that we would like to construct and let $p_{i \alpha}^{\prime}$ be any other surface element. Since $\Delta$ is the integrand of $J_{\text {Det }}^{*}$, we have to determine the functions $S_{\alpha}$, as before, such that:

$$
\begin{equation*}
L=\Delta \quad \text { and } \quad L_{p_{i a}}=\frac{\partial \Delta}{\partial p_{i \alpha}} \tag{13.4}
\end{equation*}
$$

and if we let $L^{\prime}, \Delta^{\prime}$ denote the functions $L, \Delta$ when $p$ has been replaced with $p^{\prime}$ then the $\mathcal{E}$-function will become $L^{\prime}-\Delta^{\prime}$, in which the $S_{\alpha}$ have been eliminated from $\Delta^{\prime}$ by means of (13.4). That happens as follows:

It follows from (13.4) that $\delta_{\alpha \beta} L=c_{\rho \alpha} \overline{c_{\rho \beta}}$, and from (13.42) that:

$$
\begin{equation*}
L_{p_{i a}}=S_{\rho j} \overline{c_{\rho \beta}} . \tag{13.5}
\end{equation*}
$$

Therefore:

$$
\begin{align*}
\left(S_{\rho \alpha}+S_{\rho j} p_{j \alpha}^{\prime}\right) \overline{c_{\rho \beta}} & =\left\{c_{\rho \alpha}+S_{\rho j}\left(p_{j \alpha}^{\prime}-p_{j \alpha}\right)\right\} \overline{c_{\rho \beta}}  \tag{13.6}\\
& =\delta_{\alpha \beta} L+\left(p_{j \alpha}^{\prime}-p_{j \alpha}\right) L_{p_{j \beta}} .
\end{align*}
$$

Since the determinant of $\overline{c_{\alpha \beta}}$ is $L^{\mu-1}$, due to (13.41), it follows that the determinant is:

$$
\begin{equation*}
\Delta^{\prime} L^{\mu-1}=\left|\delta_{\alpha \beta} L+\left(p_{j \alpha}^{\prime}-p_{j \alpha}\right) L_{p_{j \beta}}\right| . \tag{13.7}
\end{equation*}
$$

Thus, the new $\mathcal{E}$-function will be (one can divide by $L$ since $L>0$ ):

$$
\begin{equation*}
\mathcal{E}\left(t, x, p, p^{\prime}\right)=L^{\prime}-\frac{1}{L^{\mu-1}}\left|\delta_{\alpha \beta} L+\left(p_{j \alpha}^{\prime}-p_{j \alpha}\right) L_{p_{j \beta}}\right| . \tag{13.8}
\end{equation*}
$$

When one develops the determinant on the right-hand side in powers of $p_{j \alpha}^{\prime}-p_{j \alpha}$, that will give:

$$
\mathcal{E}=L^{\prime}-L-\left(p_{j \alpha}^{\prime}-p_{j \alpha}\right) L_{p_{j \alpha}}-\frac{1}{2 L}\left(L_{p_{j \alpha}} L_{p_{k \beta}}-L_{p_{j \beta}} L_{p_{k \alpha}}\right)\left(p_{j \alpha}^{\prime}-p_{j \alpha}\right)\left(p_{k \beta}^{\prime}-p_{k \beta}\right)+\ldots
$$

If one develops $L^{\prime}-L$ then one can also start from that $\mathcal{E}$-function with the quadratic terms and formulate a Legendre condition as:

$$
\begin{equation*}
\left\{L_{p_{j \alpha} p_{k \alpha}}-\frac{1}{L}\left(L_{p_{j \alpha}} L_{p_{k \beta}}-L_{p_{j \beta}} L_{p_{k \alpha}}\right)\right\} u_{j \alpha} u_{k \beta}>0 . \tag{13.9}
\end{equation*}
$$

It is important to remark that for values of the variables $u_{j \alpha}$ that make its rectangular matrix have rank 1 (such that one can write $u_{j \alpha}=v_{j} w_{\alpha}$ ). the quadratic form (13.9) will coincide with the one in (12.7) since the additional terms will vanish due to their skew symmetry. Namely, one can prove that:

$$
L_{p_{j \alpha} p_{k \alpha}} v_{j} v_{k} w_{\alpha} w_{\beta} \geq 0
$$

is necessary for a minimum (viz., "Hadamard's necessary Legendre condition"). That is included in (12.7), as well as in (13.9); cf., also § 14.

Fortunately, one will at least obtain the same Euler-Lagrange equations as in § 12. Namely, when one differentiates along a field-surface, one will get $\frac{d L_{p_{i \beta}}}{d t_{\beta}}=\frac{d}{d t_{\beta}}\left(S_{\rho i} \overline{c_{\rho \beta}}\right)$, due to (13.5). However, one has $\frac{d \overline{c_{\rho \beta}}}{d t_{\beta}}=0$, as one will find by differentiating the formula $\delta_{\alpha \beta} \Delta=c_{\rho \alpha} \overline{c_{\rho \beta}}$, since $c_{\alpha \beta}=\frac{d S_{\alpha}}{d t_{\beta}}$. Furthermore:

$$
\frac{d S_{i \alpha}}{d t_{\beta}}=\frac{\partial S_{\rho i}}{\partial t_{\beta}}+\frac{\partial S_{\rho i}}{\partial x_{j}} p_{j \beta}=\frac{\partial S_{\rho \beta}}{\partial x_{i}}+\frac{\partial S_{\rho j}}{\partial x_{i}} p_{j \beta}=\frac{\partial c_{\rho \beta}}{\partial x_{i}}-S_{\rho j} \frac{\partial p_{j \beta}}{\partial x_{i}},
$$

and as a result, since $L=\Delta$ :

$$
\frac{d L_{p_{i \beta}}}{d t_{\beta}}=\frac{\partial \Delta}{\partial x_{i}}-L_{p_{j \beta}} \frac{\partial p_{j \beta}}{\partial x_{i}}=L_{x_{i}} .
$$

## Q.E.D.

Just as the formula for the $\mathcal{E}$-function follows from (13.7), so does the one form the "Hilbert independent integral":

$$
\begin{equation*}
J_{D e t}^{*}=\int_{\overparen{F}}\left|S_{\alpha \beta}+S_{\alpha j} p_{j \beta}\right| d t=\int_{\overparen{F}} \frac{1}{L^{\mu-1}}\left|\delta_{\alpha \beta} L+\left(p_{j \alpha}-\varphi_{j \alpha}\right) L_{p_{j \beta}}\right| d t . \tag{13.10}
\end{equation*}
$$

Here, in order to lighten the comparison with the previous formulas, we shall write $\varphi_{j \alpha}$ for the surface element of the field and $p_{j \alpha}$ for that of the surface over which we integrate.

We already saw that $J_{\text {Det }}^{*}$ remains unchanged when we displace the boundary of the surface along the "tube" of the surfaces $S_{\alpha}=\lambda \alpha$ that go through it. For $p=\varphi$, it reduces to the integral $J$. In a geodetic field that consists of an $n$-parameter family of extremals, we then have the generalization of the Kneser transversality theorem: An $(n+\mu-1)$-tube of surfaces $S_{\alpha}=\lambda_{\alpha}$ cuts out equal values of the integral $J$ from the segments of field extremals.

That leads to a definition of transversality and sufficient conditions for moving boundaries in a natural way. For a field extremal $\mathfrak{E}$ with the boundary $\mathfrak{R}$, one has $J_{\text {Det }}^{*}=J_{\mathfrak{E}} . J_{\text {Det }}^{*}$ will have the same value $J_{\mathfrak{E}}$ for any other surface $\mathfrak{F}$ that lies in the field and whose boundary lies on the tube that is determined by $\mathfrak{R}$, and therefore $J_{\mathfrak{F}}-J_{\mathscr{E}}=\int_{\mathfrak{F}} \mathcal{E} d t \geq 0$, and the equality sign is valid only when $\mathfrak{F}$ is also a field extremal. If we take a boundary-value problem for which the boundary moves on the tube then $\mathfrak{E}$ will probably yield a minimum, but not an actual one. By contrast, if the boundary moves on a manifold $\mathfrak{M}$ of dimension $v+\mu-1(0 \leq v \leq n ; v=0$ is the case of a fixed boundary) that contacts the tube along $\mathfrak{R}$, but otherwise lies outside of it, then we will have a true minimum. That is because if $\mathfrak{F}$ is now a comparison surface whose boundary lies on $\mathfrak{M}$ then $\mathfrak{F}$ will protrude beyond the tube, and $J_{\mathfrak{F}}$ will generally be larger than before since $L>0$. The equality sign is possible only when the boundary $\mathfrak{F}$ is $\mathfrak{R}$ and it is a field extremal, so only for $\mathfrak{F}=\mathfrak{E}$.

The transversality condition expresses the idea that $\mathfrak{M}$ contacts the tube. We next consider the $\mu$-surface element of $\mathfrak{E}$ that is given by $p_{i \alpha}$ at a point of $\mathfrak{R}$ and the $n$-surface element of the surface $S_{\alpha}=\lambda_{\alpha}$ that goes through it. It is called the $n$-surface element that is transverse to it. We will soon see that it can be calculated with no knowledge of the functions $S$. Together with the $(\mu-1)$ surface element of the boundary $\mathfrak{R}$ (which is contained in the $\mu$-surface element above), it spans the $(n+\mu-1)$-surface element of the tube, and the $(\nu+\mu-1)$-surface element of $\mathfrak{M}$ must be contained in it: That is the transversality condition $\left({ }^{15}\right)$.

We would like to calculate the transversal $n$-surface element. The element $p_{i \alpha}=\partial x_{i} / \partial t_{\alpha}$ of a $\mu$-surface $x_{i}(t)$ will be spanned by the $\mu$ vectors ( $\delta_{\alpha \beta}, p_{i \beta}$ ), $\beta=1, \ldots, \mu$. An $n$-surface can (under suitable assumptions) be described by functions $t_{\alpha}(x)$. If one sets $\partial t_{\alpha} / \partial x_{i}=-P_{i \alpha}$ then its element will be spanned by the $n$ vectors $\left(P_{j \alpha},-\delta_{i j}\right), j=1, \ldots, n$. The minus sign will be justified by the remark that one will now get the $n$-surface element that is perpendicular to the $\mu$-surface element above precisely when $P_{i \alpha}=p_{i \alpha}$. If one is dealing with a surface $S_{\alpha}=\lambda_{\alpha}$ then $S_{\alpha}(t(x), x)$ will be constant, and therefore:

[^11]\[

$$
\begin{equation*}
S_{\alpha \rho} P_{i \rho}=S_{\alpha i} . \tag{13.11}
\end{equation*}
$$

\]

One can calculate the $P_{i \alpha}$ from that when $\left|S_{\alpha \rho}\right| \neq 0$.
In order to calculate those $P_{i \alpha}$ from the $p_{i \alpha}$, it is convenient to introduce an abbreviation: $a_{\alpha \rho}$ $=\delta_{\alpha \rho} L-p_{j \alpha} L_{p_{j \beta}}$. One infers from (13.6) that:

$$
\begin{equation*}
a_{\alpha \rho}=S_{\rho \alpha} \overline{c_{\rho \beta}} \tag{13.12}
\end{equation*}
$$

If one multiplies (13.11) by $\overline{c_{\alpha \beta}}$ then one will get $a_{\rho \beta} P_{i \rho}$ on the left from that; one will get $L_{p_{j \beta}}$ on the right, due to (13.5). Thus:

$$
\begin{equation*}
P_{i \alpha}=\frac{\overline{a_{\alpha \beta}}}{a} L_{p_{j \beta}} \tag{13.13}
\end{equation*}
$$

where $a$ denotes the determinant and $\overline{a_{\alpha \beta}}$ denotes the algebraic complement of $a_{\alpha \rho}$. That is the formula for transversality. Naturally, one must assume $a \neq 0$ in that. However, our assumption that $L>0$ will then suffice to confirm that a surface element and its transversal do not contact, i.e., have no common direction. It is necessary and sufficient for this that the determinant:

$$
\left|\begin{array}{cc}
\delta_{\alpha \beta} & P_{j \alpha} \\
p_{i \beta} & -\delta_{i j}
\end{array}\right|
$$

whose columns are defined by all of the $\mu+n$ spanning vectors, is non-zero. By a simple conversion, it will go to:

$$
\left|\begin{array}{cc}
g_{\alpha \beta} & 0 \\
p_{i \beta} & -\delta_{i j}
\end{array}\right|=(-1)^{n}\left|g_{\alpha \beta}\right|,
$$

with $g_{\alpha \beta}=\delta_{\alpha \beta}+P_{i \alpha} p_{i \beta}$. Since $L_{p_{j \beta}}=P_{i \rho} a_{\rho \beta}$, one will have:

$$
g_{\alpha \beta} a_{\rho \gamma}=a_{\beta \gamma}+p_{i \beta} L_{p_{i y}}=\delta_{\alpha \beta} L,
$$

and as a result, $\left|g_{\alpha \beta}\right| \cdot a=L^{\mu}$, from which the assertion follows.
The condition $\left|S_{\alpha \beta}\right| \neq 0$, which is also necessary for one to be able to take the $x$ to be the independent variables for the transverse surfaces, is easy to express in terms of the $p_{i \alpha}$. It follows immediately from (13.12) that:

$$
\left|S_{\alpha \beta}\right|=\frac{a}{L^{\mu-1}} .
$$

The stated condition is then equivalent to $a \neq 0$.

The reciprocal value:

$$
\begin{equation*}
H=\frac{L^{\mu-1}}{a} \tag{13.14}
\end{equation*}
$$

when expressed in terms of the $P_{i \alpha}$, was what Carathéodory introduced [loc. cit. $\left(^{8}{ }^{8}\right.$ ] as a generalized Hamiltonian function. Its generalized Legendre transformation will then be:

$$
\begin{aligned}
& p_{i \alpha} \rightarrow P_{i \alpha}, \\
& L \rightarrow H
\end{aligned}
$$

It does not reduce to the usual one that was described in $\S \mathbf{8}$ for $\mu=1$ but will represent a somewhat more complicated system of formulas that nonetheless has greater symmetry. For the problem of shortest arc-length (least area, resp.), one has $P_{i \alpha}=p_{i \alpha}$ and $H=L$. E. Hölder $\left({ }^{16}\right)$ had revealed the simple geometric meaning of that Hamiltonian function for $\mu=1$.

One also finds the corner condition of § $\mathbf{1 0}$ quite easily here. If one demands continuity ( $S_{\alpha}^{\prime}=$ $S_{\alpha}$ ) across an $(n+\mu-1)$-dimensional discontinuity surface $\mathfrak{D}$ then the tangential derivatives must also be continuous, i.e., the gradients of the functions $S_{\alpha}^{\prime}=S_{\alpha}$ all have the same directions as the normals to $\mathfrak{D}$. If one now has an extremal $\mathfrak{E}$ with a ( $\mu-1$ )-dimensional kink $\mathfrak{K}$, with the quantities $p_{i \alpha}, P_{i \alpha}$ on the one side of $\mathfrak{K}$ and the quantities $p_{i \alpha}^{\prime}, P_{i \alpha}^{\prime}$ on the other, then one must demand the possibility of such a construction for an arbitrary surface $\mathfrak{D}$ that is laid through $\mathfrak{K}$. Whenever (13.11) is true on the one side of $\mathfrak{K}$, the corresponding relation with primes will be true on the other. One infers the continuity of all derivatives of the $S_{\alpha}$ from that, along with the corner conditions:

$$
\begin{equation*}
P_{i \alpha}^{\prime}=P_{i \alpha}, \quad H^{\prime}=H, \tag{13.15}
\end{equation*}
$$

in which the symbols are abbreviations for the expression (13.13) and (13.14), with and without primes.

## 14. - The multiplicity of field theories for multiple integrals.

We have just learned about two "field theories" for multiple integrals. The second one proves to be somewhat more effective. Nonetheless, the first one is by no means "false": It leads to reasonable sufficient conditions for a fixed boundary $\left({ }^{17}\right)$. It can happen that an extremal segment fulfills it, so it yields a minimum, without fulfilling the Legendre condition (13.9) of the other theory. Since all of those conditions are sufficient (but not necessary), that is not a contradiction.

[^12]Lepage showed $\left({ }^{18}\right)$ that those two theories are only special cases of a great variety of possible field theories. The presentation of his developments requires some other tools. It shows that the quadratic and higher-order terms in $p_{i \alpha}^{\prime}-p_{i \alpha}$ can be chosen to be completely arbitrary in the $\mathcal{E}$ function, but only when linked with certain conditions of skew symmetry. The quadratic terms will then enter into the Legendre condition. As long as one can embed a sufficiently-small piece of an extremal in a geodetic field of the associated kind $\left({ }^{19}\right)$, one has the remarkable result that an extremal will yield a minimum in the small when only any of the quadratic forms:

$$
\begin{equation*}
L_{A}(u)=\left(L_{p_{i \alpha} p_{j \beta}}+A_{i \alpha, j \beta}\right) u_{i \alpha} u_{j \beta} \tag{14.1}
\end{equation*}
$$

is positive-definite, in which the $A_{i \alpha, j \beta}$ have arbitrary symmetry in the $i, j$ and skew-symmetry in the $\alpha, \beta$. For $u_{i \alpha}=v_{i} w_{\alpha}$, everything reduces to:

$$
\begin{equation*}
H(v, w)=L_{p_{i \alpha} p_{j \beta}} v_{i} v_{j} w_{\alpha} w_{\beta} . \tag{14.2}
\end{equation*}
$$

In 1903, Hadamard showed $\left({ }^{20}\right)$ that $H(v, w) \geq 0$ is necessary for a minimum. $H(v, w)>0$ would be sufficient if one could show that under that assumption, one can always find quantities $A_{i \alpha, j \beta}$ such that $L_{A}(u)$ is positive-definite. As Carathéodory communicated to me in a letter, he had already posed that problem to $P$. Finsler around 1920, but the latter published his investigations into it only much later. Finsler and F. J. Terpstra $\left({ }^{21}\right)$ have proved that the theorem is correct for only $\mu \leq 2$ or $n \leq 2$, while it is false when both of them are $>2$. One cannot get by with the Hadamard condition then.

Among all of the field theories, that of Carathéodory in § $\mathbf{1 3}$ plays a distinguished role. That is because I have shown $\left({ }^{22}\right)$ that one will be led to it inevitably when one seeks a field theory for which the results of $\S 7$ and $\S \mathbf{9}$ on transversality and the problem with moving boundary are true. Only that theory will then imply sufficient conditions for all possible boundary-value problems. Nevertheless, one cannot do without it, either. That is because, as was mentioned before, a great

[^13]many solutions to many problems would be lost if one wished to restrict oneself to only extremals that fulfill the Legendre condition (13.9). That is based upon the sizable yawning gap that always lies between "necessary" and "sufficient."


[^0]:    ( ${ }^{1}$ ) Frank-Mises, Die Differential- und Integralgleichungen der Mechanik und Physik I, Braunschweig, 1925. Chapter 5: "Variationsrechnung," pp. 170-212.
    $\left({ }^{2}\right)$ C. Carathéodory, "Die Methode der geodätischen Äquidistanten und das Problem von Lagrange," Acta Math. 47 (1925), 199-236.

[^1]:    $\left({ }^{3}\right)$ C. Carathéodory, Variationsrechnung und partielle Differentialgleichungen erster Ordnung, Leipzig and Berlin, 1935.

[^2]:    $\left.{ }^{( }{ }^{4}\right)$ In a simple mechanical problem, $L=T-U$, where $T$, namely, the kinetic energy, is a quadratic form in the derivatives $\dot{x}_{i}$ with coefficients that depend upon $x_{i}$, and $U$, namely, the potential energy, depends upon only the $x_{i}$. $L$ does not depend upon $t$ explicitly then.
    $\left({ }^{5}\right)$ In order to apply suitable existence uniqueness theorems to the differential equations that will be derived, we must further assume the existence of some third derivatives. We would not like to concern ourselves with the question of how we might arrive at the fewest-possible assumptions of that kind in this brief presentation.

[^3]:    $\left({ }^{6}\right)$ Doubled indices that appear in a term, such as $i$ here, are always summed from 1 to $n$.

[^4]:    $\left({ }^{7}\right)$ The derivatives of the functions $S(t, x)$ and $L(t, x, p)$ with respect to their arguments will be denoted by indices, while the derivatives of the "spatial function" $L(t, x, \varphi(t, x))$ with respect to $t$ or the $x$ will be denoted by the round $\partial$. However, if one replaces the $x$ by functions of $t$ and the $p$ by their derivatives in that function or in $S$ then what will come about is a function of only $t$ whose derivative (viz., "derivative along the curve") will be denoted by the straight $d$.
    $\left({ }^{8}\right)$ C. Carathéodory, "Über die Variationsrechnung bei mehrfachen Integralen," Acta Szeged 4 (1929), 193-216. The term is not employed in the book [loc. cit., $\left({ }^{3}\right)$ ], but he had introduced it in the theory of multiple integrals, in particular. The surfaces $S=$ const. are called "geodetically equidistant," due to their properties that are discussed in § 7. The usage of the term is connected with the term "geodetic lines" for the shortest lines on curved surfaces: If one introduces (1.1) as a measure of length (viz., a "Finsler space") then the extremals of the field will be the shortest ones. In my use of the word "extremal," I deviate from Carathéodory and appeal to the general usage, since he assumed

[^5]:    that the extremals possessed the "minimum property in the small," which demands that they must necessarily satisfy the Legendre condition additionally. Cf., § 11.
    $\left({ }^{9}\right)$ Carathéodory used the concept of a "strong minimum" only for the problems in the parametric representation, for which all directions would really be allowable then, while in the ones that are presented here, the directions are restricted by the fact that the line elements must pierce the planes $t=$ const. in the sense of increasing $t$. For them,

[^6]:    Carathéodory restricted the domain of the directions even further from the outset and spoke of only a weak minimum. The terminology in the text above is justified by the remark that we have less to do with curve problems than we do with function problems: If one speaks of derivatives, which is better than speaking of directions, then they will actually be subject to no sort of restriction for a strong minimum.

[^7]:    $\left({ }^{10}\right)$ For the case of planar variational problems in their parametric representation, it will follow from a paper by W . Damkähler and E. Hopf [Math. Ann. 120 (1947/49), 12-20] that the restriction to positive-definite problems is no loss of generality. That is not the case for the usual representation, but the restriction might be allowed here since we would like to present only the broad relationships.

[^8]:    $\left({ }^{11}\right)$ That condition goes back to Bliss and is connected with the focal point condition of that author.

[^9]:    ( ${ }^{12}$ ) C. Carathéodory, "Über die starken Maxima und Minima bei einfachen Integralen," Math. Ann. 62 (1906), 449-503.
    $\left({ }^{13}\right)$ The process will break down when the extremal segment does not satisfy the Legendre condition. It is necessary anyway with the sign $\geq$ instead of >, cf., infra. If the given curve fulfills that condition with $>$ then the extremal segment will also satisfy it when one chooses the points $P$ and $Q$ to be sufficiently close to each other. Naturally, the classic proof of necessity that one finds in the textbooks is basically to be preferred, since it is completely independent of the Legendre condition. Equality will be valid for the transversality condition, due to the condition that $L>0$.

[^10]:    $\left({ }^{14}\right)$ One observes that the straight $d$ also means a partial derivative here.

[^11]:    $\left({ }^{15}\right)$ The necessity of that condition was proved by H. Boerner, Math. Zeit. 46 (1940), 720-742.

[^12]:    $\left({ }^{16}\right)$ E. Hölder, Jber. Deutsch. Math. Verein. 49 (1939), 162-178.
    $\left({ }^{17}\right)$ One can find formulas that belong to that (even for much more general cases) in the book by Th. De Donder: Théorie invariantive du calcul des variations, nouv. éd., Paris, 1935.

[^13]:    $\left({ }^{18}\right)$ Th.-H.-J. Lepage, Bull. Acad. Roy. Belg. V., s. 22 (1936), 716-729 and 1036-1046; ibid. 27 (1941), 27-46; ibid. 28 (1942), 73-92 and 247-265.
    $\left({ }^{19}\right)$ This was proved for the geodetic fields in § $\mathbf{1 2}$ (viz., "De Donder-Weyl geodetic fields") by H. Weyl: Ann. Math. 36 (1935), 607-629 and for the Carathéodory ones in § 13 by H. Boerner, Math. Ann. 112 (1936), 187-220. Cf., also E. Hölder, loc. cit $\left({ }^{16}\right)$. L. van Hove [Bull. Acad. Roy. Belg. 31 (1945), 278-285 and 625-638] once more proved and simplified the embedding for $\S \mathbf{1 2}$ and then showed how one can reduce an embedding in $\S \mathbf{1 3}$ to one in $\S \mathbf{1 2}$ with the help of a trick of Hölder's [loc. cit. (16)]. Nothing still seems to be known for arbitrary Lepage fields. However, R. Debever [Bull. Acad. Roy. Belg. 23 (1937), 809-815] proved that for an arbitrary family of extremals, one can additionally determine the $A_{i \alpha, j \beta}$ such that they define a geodetic field for those extremals. There is a paper by L. van Hove on the "second variation" in Mém. Acad. Roy. Belg., v. 24, issue 5.
    $\left({ }^{20}\right)$ J. Hadamard, La Propagation des ondes, Paris, 1903, pp. 253, for $n=\mu=3$. H. Boerner gave a more general proof, loc. cit. $\left({ }^{15}\right)$. L. van Hove [Proc. Kon. Nederl. Akad. Wetensch. 50 (1947), 18-23] showed that the condition, with the $>$ sign, is sufficient for a "minimum in the small," i.e., when one varyies the surface in the neighborhood of a point.
    ${ }^{(21)}$ P. Finsler, Comm. math. Helv. 9 (1937), 182-187 and 187-191. F. J. Terpstra, Math. Ann. 116 (1939), 166-180.
    $\left({ }^{22}\right)$ Loc. cit. $\left({ }^{15}\right)$.

