"Über den Hilbert'schen Unabhängigkeitssatz beim Lagrange'schen Variationsproblem," Rend. Circ. matem. Palermo (1) **31** (1911), 257-272.

On Hilbert's independence theorem for the Lagrange variational problem

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The extension of HILBERT's independence theorem to the case of the LAGRANGE problem has been investigated using various apparently-unconnected methods by A. MAYER (¹), HILBERT (²), and myself (³). A paper by HAHN has recently appeared (⁴) in which he discovered an interesting connection between the theory of the independence theorem and that of the second variation.

The latter work allowed me to return to that question once more in order to give a survey presentation of the theory of the independence theorem in which I shall place special weight upon verifying the intrinsic connection between the various results and methods up to now.

Initially, a purely-analytical presentation of the necessary and sufficient conditions for the validity of the independence theorem in the case of the LAGRANGE problem will be given, while omitting all of the elements that are not directly connected with the problem itself. The starting point will then be the partial differential equations for the slope functions p_i and the multiplier functions μ_β of an extremal field (§ 1), which will be given in a form from which one can immediately read off the fact that of the n (n + 1) / 2 integrability conditions for generalized HILBERT differential expression:

^{(&}lt;sup>1</sup>) A. MAYER, "Über den HILBERTschen Unabhängigssatz in der Theorie des Maximums und Minimums der einfachen Integrale," Ber. Verh. Kgl. Sächs. Ges. Wiss. Leipzig, Math.-phys. Klasse **55** (1903), 131-145; *ibid.*, **57** (1905), 49-67, 313-314.

^{(&}lt;sup>2</sup>) HILBERT, "Zur Variationsrechnung," Nachr. Kgl. Ges. Wiss. Göttingen, Math.-phys. Klasse (1905), 159-180; Math. Ann. **62** (1906), 351-370.

^{(&}lt;sup>3</sup>) BOLZA, "WEIERSTRASS' theorem and KNESER's theorem on transversals for the most general case of an extremum of a simple definite integral," Trans. Am. Math. Soc. **7** (1906), 459-488 and *Vorlesungen über Variationsrechnung*, Leipzig, Teubner, 1909, § 78. I will cite the latter as *Vorlesungen* in what follows and employ the notations that were used there throughout.

^{(&}lt;sup>4</sup>) HAHN, "Über den Zusammenhang zwischen den Theorien der zweiten Variation und der WEIERSTRASS'schen Theorie der Variationsrechnung," Rend. del Circ. Mate. Palermo **29** (1 Semester 1910), 49-78.

$$\{f(x, y, p) - \sum_{i} p_{i} F_{n+i}(x, y, p, \mu)\} dx + \sum_{i} F_{n+i}(x, y, p, \mu) dy_{i}$$

that is constructed from the slope functions and multiplier functions, the *n* conditions that relate to the variable *x* are a consequence of the remaining n(n-1)/2 conditions. The latter can be further reduced in such a way that one will see that they are valid in the entire field as long as they are fulfilled for one special value $x = a^0$. From there on, one will then easily arrive that the various results of A. MAYER and HAHN, and in part, those of HILBERT, as well (§ 2).

The same problem will then be solved by a second method (§ 3) whose starting point will be the formulas for the partial derivatives of the field integral when it is taken over a hypersurface that cuts the field. That method will imply a more geometric formulation of the theorems and lead, in a simple way, on the one hand, to HILBERT's main theorem, and on the other, to the theorems on the transversal hypersurfaces of an extremal field (§ 4).

§ 1. – The partial differential equations of the slope and multiplier functions of an extremal field.

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One addresses the problem of making the integral:

(1)
$$J = \int_{x_0}^{x_1} f(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n') dx \qquad \left(y_i' = \frac{dy_i}{dx} \right)$$

an extremum with the m < n auxiliary conditions (⁵):

(2)
$$\varphi_{\beta}(x, y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n) = 0 \qquad (\beta = 1, 2, \dots, m).$$

We set:

$$F = f + \sum_{eta=1}^m \lambda_eta \, arphi_eta$$
 ,

with undetermined functions λ_{β} of x and then get the differential equations of the problem in the known form (⁶):

(3)
$$F_k - \frac{d}{dx} F_{n+k} = 0, \qquad \varphi_\beta = 0 \qquad (k = 1, 2, ..., n; \beta = 1, 2, ..., m),$$

in which:

$$F_k = \frac{\partial F}{\partial y_k}, \qquad F_{n+k} = \frac{\partial F}{\partial y'_k}$$

^{(&}lt;sup>5</sup>) It will be assumed of the functions *f* and φ_{β} that they have class *C*^{IV} in the region that comes under consideration (in the terminology of my *Vorlesungen*, pp. 13).

^{(&}lt;sup>6</sup>) We will drop the indices in what follows, since we will agree that the indices h, j, j, k will always run through the values 1, 2, ..., n, while the index β runs through the values 1, 2, ..., m.

Now, let $(^7)$:

(4)
$$y_i = Y_i(x, b_1, b_2, ..., b_n), \qquad \lambda_\beta = \Lambda_\beta(x, b_1, b_2, ..., b_n)$$

be an *n*-parameter family of solutions of the differential equations (3). The family of extremals:

(5)
$$y_i = Y_i (x, b_1, b_2, ..., b_n)$$

might define a field \mathfrak{S} in *x*, *y*₁, *y*₂, ..., *y_n* -space in the event that the variables *x*, *b*₁, *b*₂, ..., *b_n* are restricted to a certain region \mathfrak{A} that we will assume to be cylindrical, for the sake of more precision, i.e., it is defined by the conditions:

$$\mathfrak{A}: (b_1, b_2, ..., b_n) \text{ in } \mathfrak{B}; \qquad g(b_1, b_2, ..., b_n) \leq x \leq G,$$

in which \mathfrak{B} denotes a simply-connected (⁸) continuum in the domain of the variables $b_1, b_2, ..., b_n$, and g and G are two functions of $b_1, b_2, ..., b_n$ that are continuous in \mathfrak{B} and satisfy the inequality:

(6)
$$g(b_1, b_2, ..., b_n) < G(b_1, b_2, ..., b_n)$$
.

Let the inverse functions $(^9)$ of the field be:

(7)
$$b_i = b_i (x, y_1, y_2, ..., y_n),$$

such that one will have:

(8)
$$Y_i(x, \mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_n) = y_i, \qquad \mathfrak{b}_i(x, Y_1, Y_2, \dots, Y_n) = b_i$$

in \mathfrak{S} (\mathfrak{A} , resp.). Let the slope functions of the field be:

(9)
$$p_i(x, y_1, y_2, ..., y_n) = Y'_i(x, \mathfrak{b}_1, \mathfrak{b}_2, ..., \mathfrak{b}_n),$$

while the multiplier functions of the field are:

(10)
$$\mu_{\beta}(x, y_1, y_2, \dots, y_n) = \Lambda_{\beta}(x, \mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_n) .$$

^{(&}lt;sup>7</sup>) The functions Y_i , Y'_i , Λ_β shall have class C'' in the domain \mathfrak{A} . Primes shall always mean derivatives with respect to *x*.

^{(&}lt;sup>8</sup>) Cf., HEFFTER, "Ueber Curvenintegrale im *m*-dimensionale Raun," Trans. Am. Math. Soc. **4** (1903), 142-148, esp. pp. 147.

⁽⁹⁾ Cf., Vorlesungen, loc. cit. (³), pp. 635, 639.

The functions p_i and μ_β will then satisfy a system of partial differential equations that we get as follows: We perform the differentiation with respect to x in the differential equations (3), then replace the y_i , λ_β with the solutions Y_i , Λ_β , and thus obtain n + m identity equations in x, b_1 , b_2 , ..., b_n , which will then remain valid when we replace the b_i in them with the b_i . We shall suggest the substitution of b_i for b_i by the parentheses (). From the above, we will then have:

$$(Y_i) = y_i$$
, $(Y'_i) = p_i$, $(\Lambda_\beta) = \mu_\beta$,

such that the arguments of the derivatives of the functions *F* and φ_{β} will be:

$$x, y_1, \ldots, y_n, p_1, \ldots, p_n, \mu_1, \ldots, \mu_n$$
 $(x, y_1, \ldots, y_n, p_1, \ldots, p_n, \text{resp.})$

or, as we will write more briefly in what follows:

$$x, y, p, \mu$$
 (x, y, p, resp.).

The functions (Y''_i) and (Λ'_{β}) are then expressed in terms of the functions p_i , μ_{β} . From (9) and (8), one has:

(11)
$$\begin{cases} Y'_{i}(x,b_{1},...,b_{n}) = p_{i}(x,Y_{1},...,Y_{n}), \\ \Lambda_{\beta}(x,b_{1},...,b_{n}) = \mu_{\beta}(x,Y_{1},...,Y_{n}). \end{cases}$$

We differentiate those equations with respect to x and then replace b_i with b_i , which gives:

$$(Y_i'') = \frac{\partial p_j}{\partial x} + \sum_i \frac{\partial p_j}{\partial y_i} p_i ,$$

$$(\Lambda_\beta') = \frac{\partial \mu_\beta}{\partial x} + \sum_i \frac{\partial \mu_\beta}{\partial y_i} p_i .$$

Upon substituting those values, we will get the following partial differential equations (¹⁰) for the functions p_i , μ_β as the result of the aforementioned process:

(12)
$$\begin{cases} \left[F_{n+k,0}\right] + \sum_{i} \left[F_{n+k,i}\right] p_{i} + \sum_{j} \left[F_{n+k,n+j}\right] \left(\frac{\partial p_{j}}{\partial x} + \sum_{i} \frac{\partial p_{j}}{\partial y_{i}} p_{i}\right) + \sum_{\beta} \left[\frac{\partial \varphi_{\beta}}{\partial y_{k}'}\right] \left(\frac{\partial \mu_{\beta}}{\partial x} + \sum_{i} \frac{\partial \mu_{\beta}}{\partial y_{i}} p_{i}\right) - [F_{k}] = 0, \\ \left[\varphi_{\beta}\right] = 0. \end{cases}$$

In those equations:

^{(&}lt;sup>10</sup>) Which were already found in that form in A. MAYER, *loc. cit.* (¹), **57** (1905), pp. 54.

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$$F_{n+k,0} = \frac{\partial^2 F}{\partial y'_k \partial x}, \qquad F_{n+k,i} = \frac{\partial^2 F}{\partial y'_k \partial y_i}, \qquad F_{n+k,n+j} = \frac{\partial^2 F}{\partial y'_k \partial y'_j},$$

and the brackets [] shall suggest that the arguments x, y, y', λ have been replaced with x, y, p, μ in the partial derivatives of F and φ_{β} . If one now adds and subtracts the expression:

$$\sum_{i} p_{i} \left\{ [F_{n+i,k}] + \sum_{j} [F_{n+i,n+j}] \frac{\partial p_{j}}{\partial y_{k}} + \sum_{\beta} \left[\frac{\partial \varphi_{\beta}}{\partial y_{i}'} \right] \frac{\partial \mu_{\beta}}{\partial y_{k}} \right\}$$

to the left-hand side of (12) then one will get the following result:

Theorem I:

The slope functions p_i and multiplier functions μ_β of any extremal field satisfy the n + m partial differential equations (equations, resp.):

(13)
$$\begin{cases} \frac{\partial}{\partial x} [F_{n+k}] - \frac{\partial}{\partial y_k} \{ [F] - \sum_i p_i [F_{n+i}] \} + \sum_i p_i \left\{ \frac{\partial}{\partial y_k} [F_{n+k}] - \frac{\partial}{\partial y_k} [F_{n+i}] \right\} = 0, \\ [\varphi_{\beta}] = 0. \end{cases}$$

That is the generalization of BELTRAMI's theorem $(^{11})$ that in the case of the simplest variational problem, the slope function p(x, y) will satisfy the partial differential equation:

$$\frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y'} \right] = \frac{\partial}{\partial y} \left\{ [f] - p \left[\frac{\partial f}{\partial y'} \right] \right\} \,.$$

§ 2. – A purely-analytic formulation of the conditions for the validity of the independence theorem.

We shall now turn to our actual problem, namely, exhibiting the necessary and sufficient conditions for HILBERT's independence theorem to be valid for the field \mathfrak{S} . That means: We would like to examine the conditions under which the differential expression:

(14)
$$\{f(x, y, p) - \sum_{i} p_{i} F_{n+i}(x, y, p, \mu)\} dx + \sum_{i} F_{n+i}(x, y, p, \mu) dy_{i}$$

^{(&}lt;sup>11</sup>) Cf., Vorlesungen, loc. cit. (³), pp. 107.

From the general theory of integrability conditions, we can next write the conditions in question in the form $(^{12})$:

$$\frac{\partial}{\partial x} [F_{n+k}] = \frac{\partial}{\partial y_k} \{ [F] - \sum_i p_i [F_{n+i}] \} ,$$
$$\frac{\partial}{\partial y_h} [F_{n+k}] = \frac{\partial}{\partial y_k} [F_{n+h}] .$$

However, we read off immediately from Theorem I that the first *n* of those equations are a consequence of the remaining n(n-1)/2.

In order for HILBERT's independence theorem to be valid, it is then necessary and sufficient that the slope functions p_i and the multiplier functions μ_β must satisfy the n(n-1)/2 partial differential equations:

(15)
$$\frac{\partial}{\partial y_h} \left[F_{n+k} \right] = \frac{\partial}{\partial y_k} \left[F_{n+h} \right] \,.$$

in the field \mathfrak{S} .

However, that condition can be further reduced in an essential way. Namely, let a^0 be a special value of *x* that satisfies the inequality:

(16)
$$g(b_1, b_2, ..., b_n) < a^0 < G(b_1, b_2, ..., b_n)$$

in \mathfrak{B} , such that intersection of the linear hypersurface $x = a^0$ with the field of extremals (5) lies completely in the field \mathfrak{S} . Due to the assumption (6), there will always be such values of a^0 as long as we restrict the domain \mathfrak{B} sufficiently. We will then have the theorem that *equations* (15) *will be fulfilled in the entire field as long as they are fulfilled for the intersection of the field with the hypersurface* $x = a^0$.

In order to prove that, we first point out that equations (15) are equivalent to the n(n-1)/2 conditions:

(17)
$$\frac{\partial S_k}{\partial b_h} = \frac{\partial S_h}{\partial b_k},$$

in which the function $S_k(x, b_1, ..., b_n)$ is defined by the equation:

^{(&}lt;sup>12</sup>) Cf., footnote (⁸).

(18)
$$S_k(x, b_1, \dots, b_n) = \sum_i F_{n+i}(x, Y, Y', \Lambda) \frac{\partial Y_i}{\partial b_k} .$$

That is because equations (15) say that there is a single-valued and continuously-differentiable function $\Omega(x, y_1, ..., y_n)$ in \mathfrak{S} such that:

(19)
$$F_{n+k}(x, y, p, \mu) = \frac{\partial \Omega}{\partial y_k} .$$

If we now define:

$$S(x, b_1, ..., b_n) = \Omega(x, Y_1, ..., Y_n)$$

then it will follow from (19), when we recall (11), that:

(19.a)
$$\frac{\partial S}{\partial b_k} = S_k \, .$$

Equations (17) are then a consequence of (15), and one likewise easily shows that (15) also follows from (17), conversely, when one goes from the variables x, b_1 , ..., b_n to the variables x, y_1 , ..., y_n using the inverse of the transformation (7).

However, one has, moreover:

Theorem II:

For any n-parameter family of extremals, the n(n-1)/2 differences:

$$\frac{\partial S_k}{\partial b_h} = \frac{\partial S_h}{\partial b_k}$$

are constant along the same extremal, i.e., independent of x.

That is because:

$$\frac{\partial S_k}{\partial x} = \sum_i \left(\frac{\partial Y_i}{\partial b_k} \frac{\partial \overline{F}_{n+i}}{\partial x} + \frac{\partial Y'_i}{\partial b_k} \overline{F}_{n+i} \right),$$

in which we suggest the substitution of Y, Y', Λ for y, y', λ by the overbar. However, from (3):

$$\frac{\partial \overline{F}_{n+i}}{\partial x} = \overline{F}_i \; .$$

When we further consider that $\overline{\varphi}_{\beta} = 0$, we will then get:

$$\frac{\partial S_k}{\partial x} = \frac{\partial F}{\partial b_k}.$$

However, it follows from this that:

$$\frac{\partial}{\partial x} \left(\frac{\partial S_k}{\partial b_h} - \frac{\partial S_h}{\partial b_k} \right) = 0 ,$$

with which Theorem II is proved.

However, due to the equivalence of equations (15) and (17), the assertion that was made above will follow immediately from Theorem II, and we will then have arrived at the following fundamental theorem:

Theorem III:

In order for HILBERT's independence theorem to be valid in the field S, it is necessary and sufficient that n(n-1)/2 equations:

(20)
$$\frac{\partial F_{n+k}(a^{0}, y, p(a^{0}, y), \mu(y, a^{0}))}{\partial y_{h}} = \frac{\partial F_{n+h}(a^{0}, y, p(a^{0}, y), \mu(y, a^{0}))}{\partial y_{k}}$$

must be fulfilled on the intersection of the field with the hypersurface $x = a^0$, which is equivalent to saying that (¹³):

(21)
$$\frac{\partial S_k(a^0, b_1, \dots, b_n)}{\partial b_h} = \frac{\partial S_k(a^0, b_1, \dots, b_n)}{\partial b_k}$$

in B.

If that condition is fulfilled then we call the extremal field (5) a *MAYER extremal field*, while the field \mathfrak{S} is a *MAYER field*.

From here on, it is easy for us to now go on to the various forms of the condition for the validity of independence theorem that were given by HILBERT, A. MAYER, and HAHN.

First of all, Theorem III can also be expressed as follows:

If the HILBERT line integral J^* , i.e., the integral of the differential expression (14), is independent of the path for all curves that lie completely on the hypersurface $x = a^0$ and simultaneously in the field \mathfrak{S} then that will also be true for **all** curves that lie completely in the field.

^{(&}lt;sup>13</sup>) HAHN *loc. cit.* (4), pp. 55 derived formulas (21) in a geometric way by means of his theorem on extremal tubes, *cf., infra*, § 3, end.

That is a special case of a more general theorem by HILBERT that we shall address in more detail in § 3 (¹⁴).

If one further differentiates S_k and S_h with respect to b_h and b_k , resp. then, as HAHN pointed out, one will get:

$$\begin{split} \frac{\partial S_k}{\partial b_h} - \frac{\partial S_h}{\partial b_k} &= \sum_{i,j} \left\{ \overline{F}_{n+i,j} \left(\frac{\partial Y_i}{\partial b_k} \frac{\partial Y_j}{\partial b_h} - \frac{\partial Y_i}{\partial b_h} \frac{\partial Y_j}{\partial b_k} \right) + \overline{F}_{n+i,n+j} \left(\frac{\partial Y_i}{\partial b_k} \frac{\partial Y_j'}{\partial b_h} - \frac{\partial Y_i}{\partial b_h} \frac{\partial Y_j'}{\partial b_k} \right) \right\} \\ &+ \sum_{i,\beta} \frac{\overline{\partial \varphi_\beta}}{\partial y_i'} \left(\frac{\partial Y_i}{\partial b_k} \frac{\partial \Lambda_\beta}{\partial b_h} - \frac{\partial Y_i}{\partial b_h} \frac{\partial \Lambda_\beta}{\partial b_h} \right), \end{split}$$

or in the notation of v. ESCHERICH $(^{15})$:

(22)
$$\frac{\partial S_k}{\partial b_h} - \frac{\partial S_h}{\partial b_k} = \psi \left(\frac{\partial Y}{\partial b_k}, \frac{\partial \Lambda}{\partial b_k}; \frac{\partial Y}{\partial b_h}, \frac{\partial \Lambda}{\partial b_h} \right).$$

It will then follow that *Theorem II is identical to the theorem of CLEBSCH in the theory of the second variation that says that:*

(23)
$$\psi\left(\frac{\partial Y}{\partial b_k}, \frac{\partial \Lambda}{\partial b_k}; \frac{\partial Y}{\partial b_h}, \frac{\partial \Lambda}{\partial b_h}\right) = \text{const.}$$

That is the connection between the theory of extremal fields and the theory of the second variation that was mentioned before in the introduction and discovered by HAHN.

Therefore, equations (21) can also be written:

(24)
$$\psi\left(\frac{\partial Y}{\partial b_k}, \frac{\partial \Lambda}{\partial b_k}; \frac{\partial Y}{\partial b_h}, \frac{\partial \Lambda}{\partial b_h}\right)\Big|_{x=a^0} = 0$$

and in that form, it expresses the following theorem of HAHN $(^{16})$:

Theorem IV:

In order for the independence theorem to be valid in the field \mathfrak{S} , it is necessary and sufficient that the n systems of functions that are derived from the family (4) by differentiation with respect to the parameters:

 $^(^{14})$ See below, Theorem VI.*a*.

^{(&}lt;sup>15</sup>) Cf., Vorlesungen, loc. cit. (³), pp. 626.

^{(&}lt;sup>16</sup>) *Loc. cit.* (⁴), pp. 58.

$$\frac{\partial Y_1}{\partial b_k}, \quad \frac{\partial Y_2}{\partial b_k}, \dots, \frac{\partial Y_n}{\partial b_k}; \qquad \qquad \frac{\partial \Lambda_1}{\partial b_k}, \quad \frac{\partial \Lambda_2}{\partial b_k}, \dots, \frac{\partial \Lambda_n}{\partial b_k} \qquad (k = 1, 2, \dots, n)$$

must define a conjugate system of solutions to the accessory system of linear differential equation that belongs to the extremals $b_1, b_2, ..., b_n$, in the terminology of ESCHERICH (¹⁷).

Finally, Theorem III immediately implies the form for the validity of the independence theorem that was first given by A. MAYER. Namely, if one lets:

$$y_i = \mathfrak{Y}_i (x; a^0, b_1, ..., b_n, c_1, ..., c_n), \qquad \lambda_\beta = \mathfrak{L}_\beta (x; a^0, b_1, ..., b_n, c_1, ..., c_n)$$

denote the "canonical" system $(^{18})$ of solutions to the differential equations (3), which is characterized by the initial conditions:

(25)
$$\begin{cases} \mathfrak{Y}_{i}\left(a^{0};a^{0},b_{1},\ldots,b_{n},c_{1},\ldots,c_{n}\right)=b_{i},\\ F_{n+i}\left(x,\mathfrak{Y},\mathfrak{Y}',\mathfrak{L}\right)\Big|^{x=a^{0}}=c_{i}, \end{cases}$$

then any *n*-parameter family of solutions to the differential equations (3) that produces a field can be brought into the form $(^{19})$:

(26)
$$\begin{cases} y_i = \mathfrak{Y}_i (x; a^0, b_1, \dots, b_n, C_1, \dots, C_n), \\ \lambda_\beta = \mathfrak{L}_\beta (x; a^0, b_1, \dots, b_n, C_1, \dots, C_n), \end{cases}$$

in which the quantities $C_1, C_2, ..., C_n$ are functions of $b_1, b_2, ..., b_n$, by a parameter transformation.

If we assume that our family (4) is given in precisely that normal form such that we will then have:

$$Y_i(x, b_1, ..., b_n) \equiv \mathfrak{Y}_i(x; a^0, b_1, ..., b_n, C_1, ..., C_n),$$

$$\Lambda_\beta(x, b_1, ..., b_n) \equiv \mathfrak{L}_\beta(x; a^0, b_1, ..., b_n, C_1, ..., C_n),$$

then it will follow from (25) that:

$$\frac{\partial Y_i}{\partial b_k} \bigg|_{x=a^0} = \begin{cases} 1 & \text{when } i = k, \\ 0 & \text{when } i \neq k, \end{cases}$$

and

$$F_{n+i}(x,Y,Y',\Lambda)\Big|^{x=a^0}=C_k.$$

One will then have:

^{(&}lt;sup>17</sup>) Cf., Vorlesungen, loc, cit. (³), pp. 626.

⁽¹⁸⁾ Cf., Vorlesungen, loc, cit. (3), pp. 594.

^{(&}lt;sup>19</sup>) Cf., Vorlesungen, loc, cit. (³), pp. 641, footnote.

$$S_k(a^0, b_1, ..., b_n) = C_k(b_1, ..., b_n),$$

and therefore equations (21) imply the theorem of MAYER $(^{20})$:

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Theorem V:

If one writes the n-parameter field of extremals that yield the field in the normal form:

$$y_i = \mathfrak{Y}_i (x; a^0, b_1, ..., b_n, C_1, ..., C_n)$$

then in order for the independence theorem to be valid, it is necessary and sufficient that the functions $C_1, C_2, ..., C_n$ must be the partial derivatives of one and the same function of $b_1, b_2, ..., b_n$:

(27)
$$C_k = \frac{\partial B(b_1, \dots, b_n)}{\partial b_k}.$$

§ 3. – The geometric formulation of the conditions for the validity of the independence theorem.

The main theorem III shall now be derived in a different way, namely, by means of the method that I used in § **78** of my *Vorlesungen* in order to prove MAYER's Theorem V. In that way, the connection with the geometric methods of HILBERT will also be given most simply then.

To that end, we draw a hypersurface \Re across the field \Im that is subject to only one condition apart from continuity conditions, namely, that each extremal of the field should intersect it at one and only one point. Such a hypersurface can be represented in the form:

(28)
$$\hat{\mathfrak{K}}: x = \xi(b_1, ..., b_n), \qquad y_i = Y_i(\xi, b_1, ..., b_n) \equiv \eta_i(b_1, ..., b_n),$$

when ξ (b_1 , ..., b_n) is the abscissa of the point of intersection of the hypersurface \Re with the extremal (²¹) \mathfrak{E}_b of the family (5). In that way, ξ (b_1 , ..., b_n) might be a function of b_1 , b_2 , ..., b_n that is continuously differentiable in the region \mathfrak{B} and which satisfies the inequality:

$$g(b_1, ..., b_n) < \xi(b_1, ..., b_n) < G(b_1, ..., b_n)$$

^{(&}lt;sup>20</sup>) Loc. cit. (¹), **57** (1905), pp. 56, 60. Cf., also Vorlesungen, loc. cit. (³), pp. 643.

^{(&}lt;sup>21</sup>) I.e., that extremal whose parameters are $b_1, ..., b_n$.

in \mathfrak{B} . We then consider the integral *J*, which is taken along the extremal \mathfrak{E}_b from its point of intersection with the hypersurface \mathfrak{K} up to the point *P* with the abscissa *x*, i.e., the integral:

$$U(x, b_1, ..., b_n) = \int_{\xi(b_1,...,b_n)}^x f(x, Y, Y') dx,$$

that we can also write as:

$$U(x, b_1, ..., b_n) = \int_{\xi(b_1,...,b_n)}^x f(x, Y, Y', \Lambda) \, dx \, ,$$

since $\varphi_{\beta}(x, Y, Y') = 0$. We denote that integral as a function of the coordinates *x*, *y*₁, ..., *y_n* of the point *P* by *W*(*x*, *y*₁, ..., *y_n*), such that:

$$W(x, y_1, ..., y_n) = U(x, b_1, ..., b_n).$$

We call the hypersurface \Re the "initial hypersurface" for the "field integral" W.

When one makes use of LAGRANGE's partial integration and the differential equations (3) that the function Y_i , Λ_β satisfy, one will then get the values:

(29)
$$\begin{cases} \frac{\partial U}{\partial x} = f(x, Y, Y') \equiv F(x, Y, Y', \Lambda), \\ \frac{\partial U}{\partial b_k} = \sum_i F_{n+i}(x, Y, Y', \Lambda) \frac{\partial Y_i}{\partial b_k} - T_k \end{cases}$$

for the partial derivatives of U in the known way, in which T_i means the function of $b_1, b_2, ..., b_n$ that is defined by:

(30)
$$T_{k} = f(x, Y, Y') \Big|^{\xi} \frac{\partial \xi}{\partial b_{k}} + \sum_{i} F_{n+i}(x, Y, Y', \Lambda) \frac{\partial \xi}{\partial b_{k}} \Big|^{\xi},$$

and since:

$$\frac{\partial \eta_i}{\partial b_k} = Y_i' \Big|^{\xi} \frac{\partial \xi}{\partial b_k} + \frac{\partial Y_i}{\partial b_k} \Big|^{\zeta},$$

it can also be written:

(31)
$$T_{k} = \left\{ f(x, Y, Y') - \sum_{i} Y_{i}' F_{n+i}(x, Y, Y', \Lambda) \right\}^{\xi} \frac{\partial \xi}{\partial b_{k}} + \sum_{i} F_{n+i}(x, Y, Y', \Lambda) \Big|^{\xi} \frac{\partial \eta_{i}}{\partial b_{k}}$$

We will call the *n* functions $T_1, T_2, ..., T_n$ the *n* transversality expressions for the hypersurface \Re , since their simultaneous vanishing expresses the idea that the hypersurface \Re intersects the

family of extremals (5) transversally (²²). When we go from the variables x, b_1 , ..., b_n to the variables x, y_1 , ..., y_n by the transformation (7), we will get the expressions:

(32)
$$\begin{cases} \frac{\partial W}{\partial x} = f(x, y, p) - \sum_{i} p_{i} F_{n+i}(x, y, p, \mu) - \sum_{i} (T_{i}) \frac{\partial \mathfrak{b}_{i}}{\partial x}, \\ \frac{\partial W}{\partial y_{k}} = F_{n+k}(x, y, p, \mu) - \sum_{i} (T_{i}) \frac{\partial \mathfrak{b}_{i}}{\partial y_{k}} \end{cases}$$

for the partial derivatives of the function W, in which the parentheses () have the same meaning that they had in § **2**.

It then follows that: Should the differential expression (14) be a complete differential, then it would be necessary and sufficient that the expression:

$$\sum_{i} (T_i) \left\{ \frac{\partial \mathfrak{b}_i}{\partial x} \, dx + \sum_{k} \frac{\partial \mathfrak{b}_i}{\partial y_k} \, dy_k \right\}$$

should also be a complete differential. However, when we return to the variables $x, b_1, ..., b_n$ by means of the transformation (5), that will be equivalent to saying that the expression in the independent variables $b_1, ..., b_n$:

$$\sum_i T_i \, db_i$$

is a complete differential in its own right. We thus obtain the:

Theorem VI:

In order for the independence theorem to be valid, it is necessary that for EVERY hypersurface \Re that intersects each extremal of the field at a point, the n transversality expressions:

$$T_{k} = \left\{ f(x, Y, Y') - \sum_{i} Y_{i}' F_{n+i}(x, Y, Y', \Lambda) \right\} \left|^{\xi} \frac{\partial \xi}{\partial b_{k}} + \sum_{i} F_{n+i}(x, Y, Y', \Lambda) \right|^{\xi} \frac{\partial \eta_{i}}{\partial b_{k}}$$

should be the partial derivatives with respect to the b_k of one and the same function $T(b_1, ..., b_n)$:

(33)
$$T_k = \frac{\partial T}{\partial b_k}$$

while it will also be, conversely, sufficient when that condition is fulfilled for A SINGLE hypersurface with the given properties.

^{(&}lt;sup>22</sup>) Cf., Vorlesungen, loc. cit. (³), pp. 648.

Before we infer further consequences from that theorem, we would like to show how it can also be derived from Theorem III. If a^0 has the same meaning as in § 2, and we define:

$$U_0(x, b_1, ..., b_n) = \int_{a^0}^x f(x, Y, Y') dx \equiv \int_{a^0}^x F(x, Y, Y', \Lambda) dx$$

then a simple calculation will yield:

(34)
$$T_{k}(b_{1},...,b_{n}) = \frac{\partial U_{0}(\xi,b_{1},...,b_{n})}{\partial b_{k}} + S_{k}(a^{0},b_{1},...,b_{n})$$

That shows that Theorem VI is equivalent to equations (21). We can also derive Theorem III from Theorem VI by means of formula (34) then, and independently of the developments in §§ 1 and 2.

Theorem VI now has a simple geometric meaning. Namely, if one takes the HILBERT line integral J^* , i.e., the integral of the differential expression (14), along a curve \mathfrak{C} that lies completely in the hypersurface K then one will get:

$$J_{\mathfrak{C}}^* = \int \sum_k T_k \, db_k \, ,$$

when one observes that due to (28) and (11):

$$p_{i}(x, \eta_{1}, ..., \eta_{n}) = Y'_{i}(\xi, b_{1}, ..., b_{n}),$$

$$\mu_{\beta}(x, \eta_{1}, ..., \eta_{n}) = \Lambda_{\beta}(\xi, b_{1}, ..., b_{n}).$$

We can also express Theorem VI in the following form, which goes back to HILBERT (²³), then:

Theorem VI.a:

The HILBERT integral J^* is independent of path for every curve that lies completely in the field as long as that is the case for every curve that lies completely on the hypersurface \Re and, at the same time, completely in the field.

That fact, which seems surprising on first glance, finds its (geometric) explanation in HILBERT's theorem on extremal surfaces. In order to formulate it, we select an arbitrary one-parameter family (a "pencil of extremals") from the *n*-parameter family (5), and leave it open whether it is or is not a MAYER family, by setting:

$$(35) b_i = B_i(c) .$$

In that definition, the functions $B_i(c)$ should be continuously differentiable in the interval:

^{(&}lt;sup>23</sup>) Loc. cit. (²).

$$(36) c_0 \le c \le c_1$$

and furthermore, the curve that is defined by equations (35) with the inequality (36) should lie completely in the domain of the quantities $b_1, ..., b_n$ in the region \mathfrak{B} .

The equations:

(37)
$$y_i = Y_i(x, B_1, ..., B_n) \equiv h_i(x, c),$$

combined with the inequalities:

(38)
$$c_0 \le c \le c_1, \quad g(B_1, ..., B_n) \le x \le G(B_1, ..., B_n),$$

then represent a pencil of extremal arcs that all belong to the field \mathfrak{G} , or otherwise stated, a surface that is defined by extremal arcs that will call an *extremal surface* and denote by \mathfrak{F} .

Any closed continuous curve in the region \mathfrak{B} (38) in the *x*, *c*-plane will likewise correspond to a closed continuous curve on the surface \mathfrak{F} then, which we will call a closed curve "of the first kind." As opposed to them, it is possible that there are also closed curves "of the second kind" on \mathfrak{F} , i.e., ones whose original version in the *x*, *c*-plane is not closed.

We can the express HILBERT's *theorem* (²⁴) *on extremal surfaces* as follows:

Theorem VII:

The HILBERT line integral will always have the value zero when it is taken over any ordinary (²⁵) *closed curve of the first kind on an extremal surface that belongs to an arbitrary field of extremals.*

There is a simpler proof of that theorem that is based upon our formula (32). Namely, if:

$$\mathfrak{C}_0: \quad x = \overline{x}(t) , \quad c = \overline{c}(t) , \quad t_0 \leq t_1$$

is any ordinary closed curve in the region (38) in the *x*, *c*-plane such that:

(39)
$$\overline{x}(t_0) = \overline{x}(t_1)$$
, $\overline{c}(t_0) = \overline{c}(t_1)$

then the image \mathfrak{C} of it in the surface \mathfrak{F} will be given by the equations:

 \mathfrak{C} : $x = \overline{x}(t)$, $y_i = \mathfrak{y}_i(\overline{x}, \overline{c}) \equiv \overline{y}_i(t)$,

^{(&}lt;sup>24</sup>) HILBERT, *loc. cit.* (²). HAHN gave a second proof, *loc. cit.* (⁴), pp. 53.

^{(&}lt;sup>25</sup>) With the definition on pp. 192 of my Vorlesungen, loc. cit. (³).

$$t_0 \le t \le t_1$$

Since the curve \mathfrak{C} is closed, it will follow from equations (32), which are true for an arbitrary field regardless of whether it is or is not a MAYER field, that when the integral J^* is taken along \mathfrak{C} , it will have the value:

$$J_{\mathfrak{C}}^* = \int_{t_0}^{t_1} \sum_i (T_i) \left\{ \frac{\partial \mathfrak{b}_i}{\partial x} dx + \sum_k \frac{\partial \mathfrak{b}_i}{\partial y_k} dy_k \right\} \bigg|_{y_i = \overline{y}_i}^{x = \overline{x}}.$$

However, from the definition of the functions and $y_i(x, c)$ and due to (8), one has:

$$\mathfrak{b}_i(\overline{x},\overline{y}_1,\ldots,\overline{y}_n)=B_i(\overline{c})$$
,

which is why the expression for the integral will reduce to:

$$J_{\mathfrak{C}}^* = \int_{t_0}^{t_1} \sum_i T_i \Big(B_1(\overline{c}), \dots, B_n(\overline{c}) \Big) B_i'(\overline{c}) \frac{d\overline{c}}{dt} dt \, .$$

If we then let $\Phi(c)$ denote an undetermined integral of the function:

$$\sum_{i} T_i (B_1(\overline{c}), \dots, B_n(\overline{c})) B'_i(\overline{c})$$

then it will follow that:

$$J_{\mathfrak{C}}^* = \left[\Phi(\overline{c})\right]_{t_0}^{t_1},$$

and due to (39), that is equal to zero, which proves the theorem.

Theorem VI.*a* can now be proved in the same way that HILBERT did with the help of that theorem: Let *A* be a point of the hypersurface \Re , let *B* be any point in the field, and let \mathfrak{L} be an ordinary curve that goes from *A* to *B* and lies completely in the field. A field extremal goes through each point *P* of \mathfrak{L} . It cuts the hypersurface \Re at a point *Q*. If *P* describes the curve \mathfrak{L} then that extremal will generate an extremal surface \mathfrak{F} , and at the same time, *Q* will describe its intersection \mathfrak{L}' with the hypersurface \mathfrak{K} . In particular, the point of intersection of the extremal that goes through *B* with the hypersurface \mathfrak{K} might be denoted by *C*. From Theorem VII, when the integral J^* is taken along \mathfrak{L} from *A* to *B*, that will be equal to the integral J^* when it is taken from *A* to *C* along \mathfrak{L}' and from *C* to *B* along the field extremal. If we do the same thing for a second curve \mathfrak{L}_1 that goes from *A* to *B* then that will imply that $J_{\mathfrak{L}}^* = J_{\mathfrak{L}_1}^*$, assuming that $J_{\mathfrak{L}'}^* = J_{\mathfrak{L}_1}^*$. With that, Theorem VI is then proved for curves with one endpoint that lies on \Re . The extension to the general case is then obvious.

Our proof of Theorem VII likewise includes the proof of a theorem that goes back to HAHN. We would like to assume that the extremal surface \mathfrak{F} is closed in such a way that initial and final extremals coincide, i.e., that:

$$B_i(c_1) = B_i(c_0) \ .$$

We would like to call the extremal surface \mathfrak{F} an *extremal tube* in that case.

Therefore, if the curve \mathfrak{C}_0 in the *x*, *c*-plane satisfies the conditions:

(40)
$$\overline{x}(t_0) = \overline{x}(t_1)$$
, $\overline{c}(t_0) = c_0$, $\overline{c}(t_1) = c_1$,

instead of the conditions (39), then its image \mathfrak{C} on the surface \mathfrak{F} will likewise be a closed curve, but this time it will be a closed curve "of the second kind." We would like to say that it encircles the extremal tube once. The process in the proof above will then give the result:

$$J_{\sigma}^{*}=\Phi\left(c_{1}\right)-\Phi\left(c_{0}\right).$$

However, we will get the same result when we take any other curve $\overline{\mathfrak{C}}$ instead of \mathfrak{C} that likewise encircles the extremal tube once, i.e., it likewise satisfies the conditions (40). We will then get *HAHN's theorem on extremal tubes: The HILBERT line integral has the same value for all curves that encircle the extremal tube once.*

HAHN (²⁶) proved CLEBSCH's theorem (23) with the help of that theorem, as well as Theorem IV, by letting the extremal tube contract to a single extremal.

§ 4. – The transversal hypersurfaces.

In conclusion, the solution shall now be given to the following problem $(^{27})$: Construct a hypersurface \Re that cuts all of the extremals of a given external field transversally.

If such a hypersurface exists then we will call it a *transverse hypersurface* for the field.

We would like to make the problem more precise by demanding that the desired hypersurface should go through a given point $P_0(x^0, y_1^0, \dots, y_n^0)$ inside of the field.

If the hypersurface \Re is once more represented by equations (28) then the analytical form of the problem will read: Determine the function $\xi(b_1, b_2, ..., b_n)$ in such a way that it simultaneously satisfies the *n* partial differential equations:

^{(&}lt;sup>26</sup>) *Loc. cit.* (⁴), pp. 53.

^{(&}lt;sup>27</sup>) Cf., Vorlesungen, loc. cit. (³), pp. 650.

(41)
$$T_1 = 0, \qquad T_2 = 0, ..., \qquad T_n = 0$$

and the initial condition:

(42)
$$\xi(b_1^0, b_2^0, \dots, b_n^0) = x^0,$$

in addition, in which:

$$b_i^0 = \mathfrak{b}_i(x^0, y_1^0, \dots, y_n^0)$$
.

Since equations (41) are included in (33) as a special case, it will then follow that the problem can be soluble at all only when the given family of extremals is a MAYER family. We assume that this condition has been fulfilled. Since equations (19.*a*) are true in that case, from (34), we can also write the conditions (41) as:

$$\frac{\partial}{\partial b_k} \{ U_0(\xi, b_1, \dots, b_n) + S(a^0, b_1, \dots, b_n) \} = 0 \quad (k = 1, 2, \dots, n)$$

then, or:

(43)
$$U_0(x, b_1, ..., b_n) + S(a^0, b_1, ..., b_n) = c,$$

in which *c* is a constant that is independent of $b_1, ..., b_n$ and whose value is determined from the initial conditions (42):

$$c = U_0(x^0, b_1^0, \dots, b_n^0) + S(a^0, b_1^0, \dots, b_n^0) .$$

Our problem is then reduced to solving equation (43) for ξ , in which is essential that the function $S(a^0, b_1, \dots, b_n)$ should be independent of the choice of function ξ . If we write:

$$U_0(x, b_1, ..., b_n) + S(a^0, b_1, ..., b_n) = H(x, b_1, ..., b_n)$$

then:

$$\frac{\partial H}{\partial x} = f(x, Y, Y') \; .$$

We now make the *assumption* that:

(44)
$$f(x,Y,Y') \neq 0 \qquad \text{in }\mathfrak{A}.$$

The discussion pertaining to equation (43) will then easily imply the following result: If \mathfrak{B}_c denotes the subset (²⁸) of the set \mathfrak{B} for which the inequality:

^{(&}lt;sup>28</sup>) At least all points in a certain neighborhood of b_1^0, \ldots, b_n^0 belong to the set \mathfrak{B}_c .

$$H(g(b_1, ..., b_n), b_1, ..., b_n) \le c \le H(G(b_1, ..., b_n), b_1, ..., b_n)$$

exists then for every system of values $b_1, ..., b_n$ that belongs to the set \mathfrak{B}_c , equations (43) will have one and only one solution that satisfies the inequality:

$$(g(b_1, ..., b_n) \le \xi \le G(b_1, ..., b_n)$$

namely:

 $\xi(b_1, ..., b_n; c)$,

and the equations:

$$x = \xi (b_1, ..., b_n; c),$$
 $y_i = Y_i (\xi, b_1, ..., b_n)$

will then represent the desired transverse hypersurface of the field that goes through the point P_0 . We have then arrived at the following theorem, which, in turn, expresses a different form of the necessary and sufficient conditions for the validity of the independence theorem:

Theorem VIII:

Under the assumption (44), one and only one transverse hypersurface goes through each point inside of a MAYER field. Conversely: Each family of extremals for which there is a transverse hypersurface is a MAYER family.

Freiburg i. B. on 21 January 1911.

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