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On a theorem of Jacobi that relates to the integration of first-order partial difference equations

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This year, in his lectures at the Collège de France, Bertrand point to a gap that exists in the method that Jacobi used to integrate first-order partial difference equations. Without recalling that method here, which is known to all geometers, I shall immediately write down the equation upon which the proof of that fundamental theorem is based (*see* Journal de M. Liouville, tome III, page 176):

$$dx - (p_1\, dx_1 + p_2\, dx_2 + \ldots + p_n\, dx_n) = -\, M\,(p_1^0\, dx_1^0 + p_2^0\, dx_2^0 + \cdots + p_n^0\, dx_n^0)\,.$$

Jacobi concluded that $dx - (p_1\, dx_1 + p_2\, dx_2 + \ldots + p_n\, dx_n)$ is zero from that equation and the fact that $dx_1^0,\ dx_2^0,\ \ldots,\ dx_n^0$ are zero. Now that will be permissible only after one has shown that M cannot become infinite.

I would like to give a geometric proof of that for the case of three variables that seems free of any objection to me.

Let:

$$f(x, y, z, p, q) = 0 \tag{1}$$

be a first-order partial difference equation that is to be integrated. First, let us establish the meaning of some terms.

Integral surface: We regard x, y, z as rectangular coordinates. Having said that, any integral of equation (1) represents a surface, to which we give the name of *integral surface.*

Enveloped cone: Consider x, y, z in equation (1) to be the coordinates of a well-defined point, and consider p and q to be the partial derivatives of a function Z of two variables X and Y. If X, Y, Z are the running coordinates then equation (1) will represent an infinitude of developable surfaces, and in particular, a cone that has the point x, y, z for its summit. We shall call that cone the *enveloped cone* at the point x, y, z. It is obvious that each point x, y, z in space corresponds to one and only one enveloped cone and that its equation will result from eliminating p, q, dq / dp from:

$$(a)\qquad \begin{cases} f(x,y,z,p,q)=0, & \dfrac{df}{dp}+\dfrac{df}{dq}\dfrac{dq}{dp}=0, \\ Z-z=p(X-x)+q(Y-y), & X-x+\dfrac{dq}{dp}(Y-y)=0. \end{cases}$$

One sees, in addition, that if one imagines an enveloped cone at a point on an integral surface then the tangent plane to the surface considered will also be tangent to the cone.

Characteristic: Suppose that an integral surface is given. I shall say "characteristic" to mean any line that is traced on the surface such that its tangent at an arbitrary point m is a generator of the enveloped cone, and along which the tangent plane to the cone coincides with the tangent plane to the integral surface at the point m. It is obvious that any integral surface corresponds to a unique system of characteristics.

Let us return to the integration of equation (1) and let us look for its characteristics in order to get an integral surface. x, y, z, p, q will vary according to a well-defined law along one of those lines. We must then establish four relations between those five variables. That question then immediately produces two:

$$(1)\qquad f(x, y, z, p, q) = 0,$$

$$(2)\qquad dz = p\,dx + q\,dy.$$

If one observes that the tangent to the characteristic is a generator of the enveloped cone then one will have:

$$(3)\qquad \frac{dx}{\dfrac{df}{dp}} = \frac{dy}{\dfrac{df}{dq}}$$

from equations (a). On the other hand, if one lets δx, δy, δz, δp, δq denote the variations of x, y, z, p, q, resp., then for an infinitely-small displacement that is performed along the tangent that is conjugate to the tangent to the characteristic, one will have:

$$(5)\qquad dx\,\delta p + dy\,\delta q = 0,$$

$$(6)\qquad \delta x\,dp + \delta y\,dq = 0,$$

and

$$\frac{df}{dx}\,\delta x+\frac{df}{dy}\,\delta y+\frac{df}{dz}\,\delta z+\frac{df}{dp}\,\delta p+\frac{df}{dq}\,\delta q = 0 ,$$

or

$$\left(\frac{df}{dx}+p\,\frac{df}{dz}\right)\delta x+\left(\frac{df}{dy}+q\,\frac{df}{dz}\right)\delta y+\frac{df}{dp}\,\delta p+\frac{df}{dq}\,\delta q = 0 .$$

Due to equations (3) and (5), the last equation will give:

$$\left(\frac{df}{dx}+p\,\frac{df}{dz}\right)\delta x+\left(\frac{df}{dy}+q\,\frac{df}{dz}\right)\delta y = 0 ,$$

and due to equation (6):

(4) $$\frac{dp}{\dfrac{df}{dx}+p\,\dfrac{df}{dz}} = \frac{dq}{\dfrac{df}{dy}+q\,\dfrac{df}{dz}} .$$

That is the desired fourth equation. Now, if one integrates (2), (3), (4), while appealing to (1), moreover, then one will get four relations between x, y, z, p, q, and three arbitrary constants α, β, γ, and then upon eliminating p and q, one will get the two equations:

(b) $$\varphi(x, y, z, \alpha, \beta, \gamma) = 0 , \qquad \varphi_1(x, y, z, \alpha, \beta, \gamma) = 0$$

for an infinitude of curves, among which one will find the characteristics of any integral surface.

We shall let c denote the curves that are represented by the two equations (b), to abbreviate. A first observation that should be made is that no surface that is composed of the curves c can be an integral surface. Let S be a surface that is composed of the curves $c_1, c_2, c_3, \ldots$, which were chosen at random from among the curves c. Take an arbitrary point m on c_p and draw the tangent mt through that point. From equation (3), that tangent will be a generator of the enveloped cone. Draw the tangent plane P to the enveloped cone along mt. If one moves m along c_p then the plane P will move and envelop a developable surface that I shall call Σ_p . Having said that, in order for S to be an integral surface, it is necessary and sufficient that S should be the envelope of the developable surfaces $\Sigma_1, \Sigma_2, \Sigma_3, \ldots$ that are constructed from $c_1, c_2, c_3, \ldots$ in the same way that Σ_p was constructed from c_p . In other words, it is necessary and sufficient that when one supposes that the curves $c_1, c_2, c_3, \ldots$ are infinitely close, the intersection of Σ_p and Σ_{p+1} will be either c_p or c_{p+1} .

Jacobi's theorem consists of saying that S will be an integral surface when the curves $c_1, c_2, c_3, \ldots$ start from the same point. Hence, in order to establish that theorem, it will suffice for us to show that if c_p and c_{p+1} have a common point then c_{p+1} will be found completely on Σ_p . In order to do that, we first state the characteristic properties of the curves c. Now, any of those curves c_p :

1. Has a tangent at each point that is a generator of the enveloped cone that relates to the point. That results from equation (3).

2. Is such that the developable surface Σ_p that one deduces in the manner above has each of its generators tangent to the curve that is the locus of the summits of the enveloped cones that have a tangent plane that is the tangent plane to the developable surface along the generator in question. That results from equation (4).

Now recall the curves c_p, c_{p+1}, and the surface Σ_p . Furthermore, regard c_{p+1} as being deduced from c_p by varying the parameters α, β, γ by quantities that are infinitely small of order one. I shall base normals to Σ_p at the various points of c_{p+1} . Let nn_1 be one of those normals, where n is its point on c_p and n_1 is its foot. I shall draw the rectilinear generator $g\ n_1\ m$ to the surface Σ_p through n_1 and suppose that this line meets c_p at m. I now trace out the curve $m\ n_2\ \alpha$ that is the locus of the summits of the enveloped cones whose tangent plane is the tangent plane to Σ_p along mg. I draw the normal $n_1\ n_2$ to that curve, and finally I connect $n\ n_2$. If $n\ n_1$ is infinitely small of order two then since $n_1\ n_2$ is, as well, it will follow that $n\ n_2$ has order two. One easily sees from this that the tangent to c_{p+1} at the point n makes an angle that is infinitely small of order two with the tangent plane to Σ_p along mg. It will suffice to recall that the tangent to c_{p+1} is a generator of the enveloped cone that relates to the point n and that the tangent plane to Σ_p is tangent to the enveloped cone that relates to n_2 . Now, upon neglecting third-order infinitesimals, the angle that the tangent to c_{p+1} at the point n makes with the tangent plane to Σ_p along mg is nothing but the derivative of the distance nn_1 from c_{p+1} to with respect to the arc-length of c_{p+1} . Indeed, one must prove this very simple general theorem: If one is given a surface c and a developable surface Σ then the derivative of the distance from the points of c to Σ with respect to the arc-length of c is nothing but the cosine of the angle that the tangent to c makes with the normal to Σ that measures the distance in question. Thus, when the distance nn_1 from a point n of c_{p+1} to Σ_p is infinitely small of order two, one can conclude that the derivative of that distance with respect to the arc-length of c_{p+1} will have order two. I say that it results from this that if a point of c_{p+1} has a distance from Σ_p that is infinitely small of order two then the same thing will be true for all points of c_{p+1} , which proves Jacobi's theorem.

Consider the skew surface that is the locus of the normals that are drawn from the various points of c_{p+1} to Σ_p , which is a surface that I shall call σ. I trace out an arc of the curve na on σ that starts from the point n on c_{p+1} that has a distance from Σ_p of order two, has length l, and cuts all of the generators of σ at an angle that is equal to one that the tangent to c_{p+1} at the point n makes with the generator that passes that point. Then, while starting from the point a, and always on the surface σ, I draw a second arc ab of length l that cuts all of the generators of σ at an angle that is equal to the one that the generator of the enveloped cone that relates to a, which is located in the tangent to σ, makes with the generator to σ. Then, upon starting from the point b, and always on the surface σ, I draw a third arc bc of length l that cuts all of the generators of σ at an angle that is equal to the one that is the generator of the enveloped cone that relates to b, which is situated in the tangent plane to σ, makes with the generator of σ, and so on. It is clear that the polygon $nabc$, etc., will tend to c_{p+1} when l decreases indefinitely. Now from what was said above, all of the points of that polygon are at distances from Σ_p of order two, no matter what l is. Hence, all points of c_{p+1} are also at distance of order two.

As one knows, once one has proved Jacobi's theorem, one can deduce a complete integral and then the general integral.