Complement 2 in V. P. MASLOV, *Théories des perturbations et méthodes asymptotiques*, Dunod, Gauthier-Villars, Paris, 1972. French translation of the Russian edition,1965, Moscow, by J. Lascoux and R. Seneor.

GENERATING INTEGRAL AND MASLOV'S CANONICAL OPERATOR FOR THE W. K. B. METHOD

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This paper contains a new approach to the results of V. P. Maslov that concern quasi-classical asymptotics (viz., the W. K. B. method).

§ 1. – QUASI-CLASSICAL ASYMPTOTICS

1. – W. K. B. method.

One often resorts to constructing an asymptotic solution to differential equations whose derivative coefficients depend upon a parameter *h* that is supposed to be small:

$$\sum_{k\geq 0}\left(\frac{h}{i}\right)^k u_k \exp\left(\frac{i}{h}S\right), \tag{1.1}$$

in which *S* is a function with real values and the u_k are functions with complex values, and $i^2 = -1$, moreover.

One formally substitutes (1.1) in the equation and upon comparing the coefficients in the development in a series in *h*, one will determine (S, u_k). In other words, one supposes that (1.1) is a formal solution of the differential equation. One such method for constructing an asymptotic solution is ordinarily called the W. K. B. method.

In quantum mechanics, the Schrödinger equation, in its time-dependent form:

$$ih\frac{\partial\psi}{\partial t} = \left[\frac{1}{2}\left(\frac{h}{i}\frac{\partial}{\partial\xi}\right)^2 + v(t,\xi)\right]\psi, \qquad \psi = \psi(h, t, \xi), \tag{1.2}$$

as well as its stationary form:

$$E\ \psi = \left[\frac{1}{2}\left(\frac{h}{i}\frac{\partial}{\partial\xi}\right)^2 + v(\xi)\right]\psi\ , \qquad \psi = \psi(h, E, \xi)\ , \tag{1.3}$$

t, E, R, $\xi \in R^n$, the asymptotic that was described above is also called *quasi-classical*. That is linked with the fact that the functions S and u_k verify some equations that are constructed as functions of the corresponding classical dynamical system in the phase space $M = R^n \oplus R^n$ and are generated by the Hamiltonian function:

$$H = H(t, x) = \tfrac{1}{2}p^2 + v(t, q)\ , \qquad x = \{q, p\} \in M\ . \tag{1.4}$$

For (1.2), the function S verifies the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + H\left(t, \left\{q, \frac{\partial S}{\partial q}\right\}\right) = 0\ , \tag{1.5}$$

and the equation for least action in the case (1.3).

As far as the coefficients u_k are concerned, they obey a recurrent system of ordinary differential equations for total derivatives along the trajectories of the dynamical system. Under certain conditions, one can follow the passage from the quantum dynamical system to the classical dynamical system when $h \to 0$ with the aid of those asymptotic representations. This article would like to introduce and study a much larger class of asymptotic representations than the development (1.1). The necessity of extending the class of developments is dictated by the well-known difficulty that one encounters with the usual approach. For the stationary equation, that difficulty manifests itself in the appearance of regression points, caustics, etc. Their equivalent for the non-stationary equation is the non-invariance of formula solutions of the form (1.1) relative to the dynamics. We shall see that in detail. Suppose that one considers a formal solution of equation (1.2) of the form (1.1) that is equal to the expression:

$$\sum_{k\geq 0}\left(\frac{h}{i}\right)^k u_k^0 \exp\left(\frac{i}{h}S^0\right) \tag{1.6}$$

when $t = 0$. One must the initial condition:

$$S(t, x)\,|_{t=0} = S^0(\xi) \tag{1.7}$$

to the Hamilton-Jacobi equation (1.5). One knows that the Hamilton-Jacobi equation is equivalent (up to components in S that depend upon only time) to the fact that the manifold:

$$\Gamma_t = \left\{ \left\{ q, \frac{\partial S\,(t,q)}{\partial q} \right\} \middle| \, q \in R^n \right\} \tag{1.8}$$

in the space M transforms under the action of a diffeomorphism m_t of that space that is generated by the canonical system:

$$J\,\hat{x} = \frac{\partial H}{\partial x} \;, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} , \qquad I = \mathrm{Id}_{R^n} \,, \tag{1.9}$$

in such a way that:

$$\Gamma_t = m_t \, \Gamma_0 \,. \tag{1.10}$$

Note that the function $S: \; R^n \to R$ is recovered by starting from:

$$\left\{ \left\{ q, \frac{\partial S\,(q)}{\partial q} \right\} \middle| \, q \in R^n \right\}, \tag{1.11}$$

up to a constant.

Consider the manifold:

$$\Gamma^0 = \left\{ \left\{ q, \frac{\partial S\,(q)}{\partial q} \right\} \middle| \, q \in R^n \right\} \,. \tag{1.12}$$

It results from what was said that equation (1.5), with the initial condition (1.7), will have a unique solution for the t ($t_1 < t < t_2$, $t_2 > 0$, $t_1 < 0$) such that the manifolds $m_t\,\Gamma^0$ remain bijectively projectable onto the plane Q, $Q = R^n \oplus 0$, in other words, for all t for which $m_t\,\Gamma^0$ preserves the representation:

$$\{\, \{q, f\,(q)\} \mid q \in R^n \,\} \tag{1.13}$$

under a map $f: \; R^n \to R^n$. In that case, $f = \partial S \,/\, \partial q$. One can verify that for those same values of t, one will have no difficulty in solving the recursive system of equations for the coefficients u_k, when completed by the initial conditions $u_k \,|_{t\,=\,0} = u_k^0$.

Suppose that the development (1.6) is asymptotic for a certain function $\psi^0 = \psi^0(h,\xi)$ when $h \to 0$. Consider the solution to the Cauchy problem that is defined by equation (1.2) and the initial condition:

$$\psi\,(h,\, t,\, x) \,|_{t\,=\,0} = \psi^0(h,\xi) \;. \tag{1.14}$$

Under certain hypotheses, the formal solution that was constructed above will be asymptotic to the exact solution $\psi\,(h, t, \xi)$. But what will the asymptotics of that solution be for $t \notin (t_1, t_2)$?

The class of formal developments that will be introduced later on is invariant under dynamics and can be used for the asymptotic representation of the solution to the Cauchy problem (1.2), (1.4) for all $t \in R$. Invariance under dynamics means that a formal solution that belongs to that class and verifies an initial condition from the same class will exist for all t. Of course, one supposes that the diffeomorphism m_t will exist for all t.

2. – The manifolds in the preceding section of the form (1.11) constitute a subclass of a particular class of n-dimensional manifolds in M that calls *Lagrangian manifolds:* An n-dimensional manifold Γ in M will be called *Lagrangian* if the restriction of the differential form $\omega = \frac{1}{2}(p\,dq - q\,dp)$ to Γ is closed. A general Lagrangian manifold has the form (1.11) if an only if it projects bijectively onto Q.

The function S can be characterized by the data of a Lagrangian manifold Γ that projects bijectively onto Q and the image has the form $\sigma = \omega + \frac{1}{2} d\,(q\,p)$ on Γ.

$$S\,(q) = \Sigma\,(\{q, p\})\,, \qquad \{q, p\} \in \Gamma, \qquad q \in R^n\,. \tag{1.15}$$

The coefficients $u_k : R^n \to C$ of the asymptotic development (1.1) can be considered to be functions on Γ.

Therefore, the representation (1.1) will become a set $\{\Gamma, \Sigma, v\}$, in which v is a formal series of functions on Γ. The generalization of the asymptotic developments that are considered here consists of saying that those sets are constituted by an arbitrary Lagrangian manifold that no longer necessarily projects bijectively onto Q.

We first study the asymptotic development that corresponds to a Lagrangian manifold that projects bijectively onto an arbitrary Lagrangian plane Λ, i.e., onto a linear Lagrangian manifold. The general form of a Lagrangian plane is:

$$\Lambda = g^{-1}Q\,, \tag{1.16}$$

in which g is a transformation of the group G of (inhomogeneous) linear canonical transformations of M. The quantization of the space M generates a unitary representation V of the group of transformations G in $L_2(R^n)$. As far as asymptotic developments that correspond to Lagrangian manifolds that project bijectively onto Λ are concerned, it is natural to take a formal expression:

$$V\,(g)\;\varphi\,, \qquad \varphi = \sum_{k \geq 0} \left(\frac{h}{i}\right)^k u_k \exp\left(\frac{i}{h} S\right)\,. \tag{1.17}$$

One can argue that if one chooses the plane Λ to be the configuration plane in M, instead of Q, then the quantum state that is represented by the element ψ, $\psi \in L_2(R^n)$ will be represented by the element $V^{-1}(g)\psi$. At the next stage in the asymptotic development of a function from $R^n \to C$, one introduces finite or infinite sums of expressions of the form (1.17) that are associated with an arbitrary Lagrangian manifold Γ and an integral Ω (or Σ) of the form ω (or σ) on Γ. One shows that such asymptotic developments already have the property of invariance under dynamics.

We denote those representations by the letter ψ. The representations ψ play a double role in our presentation. On the one hand, independently of asymptotic maps, they occur as formal solutions to equations of type (1.1) and type (1.2). In connection with that, it is necessary to develop a certain formal calculus, and in particular to define some linear operations on ψ, and likewise differentiation $i\,h\,d\,/\,dt$ and the action of an operator of Schrödinger type. On the other hand, the ψ must generate a sequence of functions $\psi^N : R^n \to C$, $N = 0, 1, 2, \ldots$ that are used for the asymptotic development in the same sense as the functions:

$$\sum_{k\geq 0}\left(\frac{h}{i}\right)^k u_k \exp\left(\frac{i}{h}S\right)$$

in the classical W. K. B. method. The result is the application of formal solutions to the asymptotic development of exact solutions. The center of gravity of our presentation is concentrated around its formal construction, and its applications are hardly touched upon.

Different sums of expressions of the form (1.17) can generate one and only one asymptotic development ψ. One shows that ψ can be put into bijective correspondence (up to a natural identification) with the triple $\{\Gamma, \Omega, \mu\}$, where $\mu = \sum_{k\geq 0}\left(\frac{h}{i}\right)^k \mu_k$, in which μ_k are complex differentiable measures on Γ. The correspondence $\psi \leftrightarrow \{\Gamma, \Omega, \mu\}$ is realized with the aid of the symbolic generating integral:

$$\psi(\xi) = \int_\Gamma \mu(dx)\, K_{<\Gamma,\Omega>}(\xi, x)\,, \tag{1.18}$$

in which $K_{<\Gamma,\Omega>}$ is a certain universal kernel. We arrive at that integral by approximating Γ by tangent Lagrangian planes Λ_α to certain points x_α of the manifold Γ and representing ψ by a sum of the form $\sum_\alpha V(g_\alpha)\varphi_\alpha$, in which the support of φ_α is localized to a neighborhood of x_α in a well-defined sense. The integral (1.18) results from a natural passage to the limit for that construction. One can describe the fundamental operations on ψ very simply by means of such an integral.

This article resulted from the work of V. P. Maslov, who was the first to correct the inadequacies in the usual method. Maslov's presentation was constructed on the basis of the expression (1.17), in which g served only to exchange the roles of certain components of the coordinate vector q and that of momentum p. The consideration of an arbitrary g immediately led to the convenient representation of ψ by the generating integral. The essential character of the canonical operator that figured in Maslov's school is certainly equivalent to the generating integral (if one considers only the highest-order terms in the asymptotic development, as Maslov did).

However, the generating integral presents the advantage that its definition is obviously invariant and does not include any concept such as the index of a curve in the Lagrangian manifold (viz., the Maslov index). We also remark that thanks to the passage from the Lagrangian manifolds (on which integrals of the form ω cannot exist) to their coverings (for more details, cf., § **2**), we can consider the asymptotic developments that are generated by arbitrary Lagrangian manifolds and not only the manifolds that satisfy the "quantization conditions," which occupy an important place in Maslov's construction. On the contrary, one immediately obtains those conditions when one considers the applications of the asymptotic development to a stationary equation of the type (1.3) (cf., § **4**).

Let us give the plan of this presentation: The necessary notions from classical and quantum mechanics are summarized in **§ 2**, and a new formula for the Maslov index is also given there. On the latter point, our work overlaps with the articles of Arnol'd and Fuks that are dedicated to the topological interpretation of the Maslov index. **§ 3** is central: In it, we construct the generating integral, and its link with Maslov's canonical operator is explained there.

In **§ 4**, we consider the Cauchy problem for the equation of Schrödinger form:

$$ih\frac{d\boldsymbol{\psi}}{dt} = \mathcal{H}\,\boldsymbol{\psi}, \tag{1.19}$$

and discuss its asymptotic development. A general description of the class of the operator that can play the role of the operator $\mathcal{H}$ in this case is given.

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§ 4. – PHASE SPACE AND QUANTIZATION

1. – Phase space.

One calls the unitary space C^n, which is considered to be real, the *phase space M*. The points of *M* will be denoted by *x*, *a*, … The real and imaginary parts of the scalar product in C^n, (. , .) + [. , .], determine Hermitian and symplectic structures on *M*, resp. The complex structure is given by the operator *J*, which corresponds to multiplication by *i* in C^n. One has [. , .] = (. , *J* .).

Consider the differential form $\omega = \frac{1}{2}[x, dx]$ on *M*. An *n*-dimensional submanifold Γ in *M* is called *Lagrangian* if the form $\omega\,|_\Gamma$ is closed. A linear Lagrangian submanifold is called a *Lagrangian plane*. A subspace Λ is a Lagrangian plane if and only if the form [. , .] is annulled identically. The set of all Lagrangian planes is denoted by Λ, and the set of all Lagrangian subspaces is denoted by Λ_0 . Fix $Q \in \Lambda$. *Q* is considered to be a Euclidian space with the scalar product *qp* = (*q*, *p*), $q, p \in Q$. One can consider the space *M* to be the sum of two exemplars of the space *Q*, and the identification of $x \in M$ with the pair {*q*, *p*}, $q, p \in Q$, is given by the formula $x = q + J\,p$, in addition.

The letters *q* and *p* always give the components of the pair *x* = {*q*, *p*}.

2. – The group *G*.

A diffeomorphism m of the space M is called *canonical* if it leaves the form $d\omega$ invariant. The diffeomorphism m will be canonical if and only if $dm \in \mathrm{Sp}\,(M)$, where dm is the differential of m and Sp (M) is the symplectic group on M, i.e., the group of non-degenerate linear transformations of M that preserve [. , .]. A canonical diffeomorphism transforms a Lagrangian manifold into a Lagrangian manifold. Consider the universal covering group $\mathrm{Sp}(M)$ of the group Sp (M). We denote its elements by A and their canonical projections onto Sp (M) by $\mathring{A}$. The elements of $\mathrm{Sp}(M)$ are naturally parameterized by a triple $\{\theta, \delta, \rho\}$, where θ, δ, ρ are linear transformations of Q, and θ and δ are symmetric. With that terminology:

$$\mathring{A} = \exp J\,\Theta \exp J\,\{0, \delta\} \exp \{\rho, -{}^t\rho\}\ .$$

$\Theta = \{\theta, \theta\}$ and { . , . } are quasi-diagonal 2×2 block-matrices that define a transformation of M and correspond to the decomposition $M = Q + Q$.

One deduces $\exp 2\,J\,\Theta$ from $\mathring{A}$ uniquely. When Θ is fixed, δ and ρ will be determined uniquely. Let G be the semi-direct product of the linear group of the space M and $\mathrm{Sp}(M)$. The elements of G will be denoted by the letter g. They are pairs $g = \{a, A\}$ where $a \in M$. The group G generates a group of transformations G of the space M that operates according to the formula:

$$g\,x = \mathring{g}\,x = a + \mathring{A}\,x\,,$$

and G is the universal covering of $\mathring{G}$. The group $\mathring{G}$ is nothing but the group of linear (inhomogeneous) canonical diffeomorphisms of M.

The general form of a Lagrangian plane is $\Lambda = g\,Q$, where $g \in G$. The set Λ_0 of Lagrangian subspaces can be interpreted as a homogeneous space of Sp (M), and one easily establishes that each $\Lambda \in \Lambda_0$ can be written as $\Lambda = (\exp J\,\Theta)\,Q$, where $\exp 2\,J\,\Theta$ is determined uniquely upon starting from Λ.

3. – Lagrangian pairs.

Let Γ be a connected Lagrangian manifold. In what follows, we denote universal covering space of the manifold Γ by E. We likewise introduce the covering space $E\,(\omega)$, whose characteristic subgroup is a normal divisor $\chi\,(\omega)$ of the group $\pi_1\,(\Gamma)$, which is composed of classes of paths γ that have the property that $\int_\gamma \omega = 0$. The primitive $\Omega : E \to R$ of the form ω exists on E and $E\,(\omega)$, and it will take different values at different points of the fiber of $E\ (\omega)$. Each of the spaces E and $E\ (\omega)$ has its individual advantages.

Those of $E\ (\omega)$ amount to the uniqueness of the correspondences of the type:

$$\psi \leftrightarrow E\ (\omega)$$

(cf., **§ 1**, and for more details, **§ 3**), and the entire presentation can be constructed with $E\ (\omega)$.

However, certain formulations are much simpler if one uses E. In the first part of this discussion, up to section 2, **§ 4**, it will be convenient for us to consider E to be equivalent to $E\ (\omega)$. That convention will then be abandoned, and we will consider only E.

The set $< \Gamma, \Omega >$ will be called a *Lagrangian pair*. Suppose that the map $\tau : E \to G$ has the property that $\tau x = \{x, A\ (x)\}$, in which $A\ (x)\ Q$ is parallel to the tangent plane to E at the point x.

The set $< \Gamma, \Omega, \tau >$ will be called a *Lagrangian triple*. Let $< \Gamma, \Omega, \tau >$ be a Lagrangian triple and let $g = \{a, A\} \in G$. One intends $g < \Gamma, \Omega, \tau >$ to mean the Lagrangian triple $< g\,\Gamma, g\,\Omega, g\,\tau >$, where:

$$\Omega_g\ (x) = \Omega(g^{-1}x) + \tfrac{1}{2}[a, x]\ , \qquad x \in g\,\Gamma.$$

One defines $g < \Gamma, \Omega >$ similarly. An n-dimensional submanifold Γ in M will be called *a bijective projection onto* Q if it has the form:

$$\{\{q, f\ (q)\} \mid q \in D\}, \qquad \text{in which} \qquad f : D \to Q\ ,$$

and D is an open subset of Q. If D is simply connected then the submanifold will be Lagrangian if and only if there exists a function $S : D \to R$ such that $f = \partial S / \partial q$. Hence, $< \Gamma, \Omega >$, where:

$$\Omega\ (x) = S\ (q) - \tfrac{1}{2}\, q\, \frac{\partial S\ (q)}{\partial q} \qquad \text{and} \qquad x = \{q, \ldots\} \in \Gamma$$

is a Lagrangian pair.

4. – Quantization.

One intends that to mean (cf., for example, [**4**]) a map K from the space M to the set of self-adjoint operators on a Hilbert space such that the unitary operators $W\ (x) = \exp i\, h^{-1} K(x)$ form a projective representation of the linear group of the space M in such a way that:

$$W\ (x_1)\ W\ (x_2) = \exp\left\{\frac{i}{2h}[x_1, x_2] W(x_1 + x_2)\right\},$$

in which h is a given constant:

$$h \in \Delta = (0, b)\ .$$

All of the irreducible representation of quantization are unitarily equivalent, and the operators that realize that equivalence are defined up to a phase C, $|C| = 1$.

One refers to the *Schrödinger quantization* of phase space M when one means a quantization such that $\mathfrak{H} = -L_2(Q)$, and the operators $K(x)$ are differential expressions:

$$(K(x)f)(\xi) = \left(q\xi + \frac{h}{i}p\frac{\partial}{\partial\xi}\right)f(\xi) . \qquad \xi \in Q.$$

That quantization is irreducible.

The group G operates naturally on the quantization K :

$$K \to g\,K = A\,K + a\,E ,$$

where

$$(A\,K)(x) = K({}^tA\,x) \quad \text{and} \quad (a\,E)(x) = (a, x)\,E .$$

One easily sees that $g\,K$ is an irreducible quantization when K is. There then exist unitary operators $V(g)$ such that:

$$K\,V(g) = V(g)\,g\,K .$$

They are defined up to a phase C, $|C| = 1$. It is obvious that those operators form a unitary representation of the group G.

5. – Explicit formulas.

Introduce the unitary operators:

$$V(a) = \exp\left\{\frac{-i}{h}K(J\,a)\right\}, \qquad V(A) = \exp\left\{\frac{i}{h}[\ln AK, K]\right\}. \tag{2.1}$$

Here:

$$[B\,K, K] = \sum_{p=1}^{2n}(B\,K)(e_p)(J\,K)(e_p) ,$$

in which $\{e_p\}$ is an orthonormal basis in M and $J\,B = (J\,B)^*$, where $*$ is the Hermitian conjugate in the complexification of M. Normalize $V(A)$ by continuity upon starting from $V(e) = E$.

The operators $V(g) = V(a)\,V(A)$ verify the relation:

$$V(g_1)\,V(g_2) = \exp\left\{\frac{i}{2h}[a_1, A_1\,a_2]V(g_1\,g_2)\right\} .$$

If $A = \{\theta, \delta, \rho\}$ then:

$$(V(a)f)(\xi) = \exp\left(\frac{i}{2h}q\,p\right)\exp\left[\frac{i}{h}p(\xi - q)\right]f(\xi - q) , \tag{2.2}$$

$$a = q + J\,p\,,$$

$$(V^{(3)}(\rho)\,f)(\xi) = |\det{}^{1/2} r\,|\,f\,(r^{-1}\xi)\,, \qquad r = e^{p}\,. \tag{2.3}$$

$V^{(1)}(\theta)V^{(2)}(\delta)$ is an integral operator whose kernel is equal to:

$$\lim_{\varepsilon\to 0}\left[\det\frac{2\pi h}{i}(\cos\theta_\varepsilon\,\delta_\varepsilon+\sin\theta_\varepsilon)\right]^{-1/2}\exp\left\{\left(-\frac{i}{2h}\right)[\xi\,(-\sin\theta_\varepsilon\,\delta_\varepsilon+\cos\theta_\varepsilon)(\cos\theta_\varepsilon\,\delta_\varepsilon+\sin\theta_\varepsilon)^{-1}\xi\right.$$
$$\left.+\ \xi'\,(\cos\,\theta_\varepsilon\,\delta_\varepsilon+\sin\theta_\varepsilon)^{-1}\cos\theta_\varepsilon\,\xi'-2\,\xi'\,(\cos\,\theta_\varepsilon\,\delta_\varepsilon+\sin\theta_\varepsilon)^{-1}\xi]\right\}, \tag{2.4}$$

in which:

$$\theta_\varepsilon = \theta + i\,\varepsilon\,, \quad \delta_x = \delta + i\,\varepsilon\,, \quad V^{(1,2)}(0) = E\,.$$

When cos $\theta\,\delta$ + sin θ degenerates, that expression will define a generalized function. The continuity condition and the normalization will give a unique definition.

One must consider the explicit expressions of this section to be known in quantum mechanics. However, unfortunately, the author has not succeeded in finding any work in which they are given in the form that is necessary for us: We shall then make a few remarks regarding their proofs.

Upon changing the element g into the one-parameter subgroup g_t in the defining relation $K\,V(g) = V\,(g)\,g\,K$ and differentiating with respect to t, we can pass to the following equivalent equation $K\,G = G\,K + g_0'\,K$ for the infinitesimal generator of the group $V\,(g_t)$:

$$V\,(g_t) = \exp G\,t\,.$$

Upon specifying G in the form of a quadratic functional of the operator K and using the definition of K, we will arrive at formula (2.1) for the operator $V\,(g)$. The relation:

$$V\,(g_1)\,V\,(g_2) = \exp\frac{i}{2h}[a_1, A_1\,a_2]V(g_1\,g_2)$$

is then obtained by a simple verification. Formulas (2.2) and (2.3) are obvious. In order to understand (2.4), once more consider the group $V\,(g_t)$ and the equation:

$$\frac{d}{dt}V(g_t) = G\,V\,(g_t)\,, \qquad V(g_0) = E\,.$$

In the Schrödinger representation, G is a second-order differential operator with quadratic coefficients in the independent variables. The kernel of the operator $V\,(g_t)$, i.e., the Green function of the problem, can be found in the form:

$$\exp[\xi A(t)\xi + \xi B(t)\xi' + \xi' C(t)\xi' + D(t)\} \ .$$

Upon substituting that expression in the equation, one will get a system of ordinary differential equations for *B*, *C*, and *D* that is easy to solve explicitly.

6. – Maslov index.

The set θ of symmetric transformations θ of the space Q is the universal covering of the Λ_0-set of Lagrangian subspaces. The projection is given by the formula $\Lambda = (\exp J\,\Theta)\,Q$. Consider the function of θ:

$$v_\varepsilon(\theta) = \det^{-1/2} \cos \theta_\varepsilon \times |\det^{1/2} \cos \theta_\varepsilon| \,,$$

which is defined by continuity and the normalization $v_\varepsilon(0) = 1$.

One easily sees that the product:

$$\lim_{\varepsilon\to 0} v_\varepsilon^{-1}(\theta) V^{(1)}(\theta)$$

is constant on each fiber.

Consider the form $\kappa_\varepsilon = \frac{2}{\pi i} d \ln v_\varepsilon$ on θ. It is a form on Λ_0. Consider the singular form $\kappa = \lim \kappa_\varepsilon$ on Λ_0. Let γ be an oriented curve on Λ_0 with its origin at Λ_1 and its extremity at Λ_2. Suppose that the Lagrangian planes Λ_1 and Λ_2 project bijectively into Q. The *index* ind γ of the curve γ will be the integer ind $\gamma = \int_\gamma \kappa$. The indices of the closed curves obviously define a certain integer cohomology class on Λ_0: It is the *Maslov-Arnol'd characteristic class* [**2**], [**3**]. The Arnol'd formula results directly from our definition: The index of a closed curve γ is equal to the degree of the map $\varepsilon : \gamma \to S^1$, in which ε is the restriction to γ of the map $\Lambda_0 \to S^1$ that is given by the formula det exp $(2i\,\theta)$, and $\Lambda = (\exp J\,\Theta)\,Q$, in addition.

The oriented curve γ on the Lagrangian manifold Γ indices an oriented curve γ' in Λ_0. The index ind γ' is called the *Maslov index* of the curve γ. We also denote it by ind γ. Similarly, a curve γ in the group G induces a curve γ' in Λ_0. We likewise denote the index of the latter curve by ind γ.

7. – Dynamics.

Let m_t, $t \in R$ be the family of canonical diffeomorphisms that are defined by the equation:

$$J\,\dot{x} = \frac{\partial \chi}{\partial x}, \qquad \text{in which} \qquad \chi : R \times M \to R\,.$$

Consider the differential form $\omega - c\, dt$ on $R \times M$. Its restriction to $\bigcup_{-\infty<t<+\infty} E_t$, where $E_t = m_t\, E$, E being the universal covering of the Lagrangian manifold Γ, is a closed form. One denotes a primitive of that form by $\Omega^{(t)}$.

Consider the differential $dm : R \times M \to Sp(M)$, in which dm is defined by the normalization $dm_{0,\,x} = e$ and continuity. Here, $dm_{0,\,x}$ is the value of dm at the point $\{t, x\}$. The trajectory $m_t\, a$, $a \in M$ and $g = \{a, A\} \in G$ correspond to a path in the group $m_t\, g = \{m_t\, a, dm_{t, m_t a} A\}$. Its index coincides with the Morse index of the trajectory $m_t\, a$, in general.

Let a dot or d / dt denote the total derivative along the trajectory of the dynamical system m_t . One has the relation:

$$i\, h \frac{d}{dt}\left[\exp\left(\frac{i}{h}\,\Omega^{(t)} V(m_t\, g)\right)\right]$$

$$= \left\{\chi + \left(K - x, \frac{\partial \chi}{\partial x}\right) + \frac{1}{2}\left(K - x, \frac{\partial^2 \chi}{\partial x^2}(K - x)\right)\right\} \times \exp\left[\frac{i}{h}\,\Omega^{(t)} V(m_t g)\right] . \qquad (2.5)$$

The operations on K are defined in the same way that they were defined on page 9. The proof is obtained by direct calculation. If $< \Gamma, \Omega, \tau >$ is a Lagrangian triple then one intends $m_t < \Gamma, \Omega, \tau >$ to mean the triple $< m_t\, \Gamma, \Omega^{(t)}, m_t\, \tau >$, in which $\Omega^{(t)}\,|_{t\,=\,0} = \Omega$ and $(m_t\, \tau)\, x = m_t\, (\tau\, x)$. Consider the function on E that is given by the formula:

$$K = K_{<\Gamma,\Omega,x>}(x) = \exp\left(\frac{i}{h}\,\Omega^{(t)} V\,(\tau\, x)\,\delta\right) , \qquad (2.6)$$

in which δ is the delta function on Q, and V are the operators that are linked with the Schrödinger quantization. Note that $V\,(g)\;\delta$, for g fixed, defines a generalized function on Q, in general. One can show that one has the relation:

$$K_{<\Gamma,\Omega,x>}(x) = V(g^{-1})\, K_{g<\Gamma,\Omega,x>}(g\, x) \qquad (2.7)$$

§ 3. – GENERATING INTEGRAL

In this section and the following one, the series of the form:

$$\sum_{k \geq 0} (h / i)^k u_k$$

are considered to be formal series in h / i. The expressions of the form:

$$\sum_{k\geq 0}\left(\frac{h}{i}\right)^k u_k, \qquad \text{in which} \qquad u_k = \sum_{l\geq 0}(h/i)^l u_{k,l}$$

must be understood to mean:

$$\sum_{k\geq 0}\left(\frac{h}{i}\right)^k \sum_{l=0}^{k} u_{k-l,l} \;.$$

An expression of the type $D(\xi \mid f, \Gamma)$ will denote a linear differential operator that acts upon the variable ξ whose coefficients depend upon the function f and the geometric object Γ in a finite-order neighborhood of ξ.

1. – Expression for $V\varphi$ **.**

Introduce the formal expression:

$$V(g)\,u\exp\left(\frac{i}{h}S\right), \tag{3.1}$$

i.e., a set $\{g, u, S\}$ such that:

(1). $g \in G$.

(2) $u = \sum_{k\geq 0}\left(\frac{h}{i}\right)^k u_k,$

in which the $u_k : Q \to C$ are such that $\operatorname{supp} u = \bigcup_{k\geq 0} \operatorname{supp} u_k$ is compact.

(3) $S \in C^\infty(\operatorname{supp} u)$, i.e., $S : \operatorname{supp} u \to R$, and S prolongs to an open subset U such that:

$$\operatorname{supp} u \subset U \,.$$

The expression (3.1) is written $V\varphi$, for brevity, where φ denotes $u \exp i\, h^{-1} S$ and is denoted by the symbol $\{u, S\}$. Let S_U be the prolongation of S to U. One can associate S_U with the Lagrangian manifold Γ_{S_U} that projects bijectively onto Q. The subset Γ_{S_U}, which is above supp u, will be denoted by Γ_S . If the Lagrangian manifold Γ_{S_U} projects bijectively onto Q for any prolongation S_U then one says that $g\,\Gamma_S$ projects bijectively onto Q. One understands the symbol:

$$\text{S.P.}\int_Q (2\pi h\, e^{-i\pi/2})^{-\pi/2}\, u \exp\left(\frac{i}{h} f\right) d\xi \;, \tag{3.2}$$

in which u was described above and $f : U \to R$, with supp $u \subset U$ open, has only one non-degenerate critical point ξ_S on U, to mean the formal expression:

$$\left[\sum_{k\geq 0}\left(\frac{h}{i}\right)^k D_k(\xi_S \mid f)\,u\right]\exp\left[\frac{i}{h}\,f(\xi_S)\right] \tag{3.3}$$

that one obtains when one applies the method of stationary phase ([**5**]) to the integral (3.2). One utilizes the explicit formula for the operator $V(g)$ in the Schrödinger representation. One can then make the expression $V\,\varphi$ correspond to a symbolic integral of the form (3.2). The function f will have a unique non-degenerate critical point there if and only if $g\,\Gamma_S$ projects bijectively onto Q. Under that condition, the symbol S.P. $V\,\varphi$ will be meaningful and will define an expression of the form:

$$\varphi_1 = u_1 \exp\left(\frac{i}{h} S_1\right) .$$

The equivalence relation $V_1\,\varphi_1 \sim V_2\,\varphi_2$ is defined by the formula:

$$\varphi_1 = \text{S.P.}(V_1^{-1} V_2)\,\varphi_2 .$$

2. – Class ψ.

We agree to denote the equivalence classes thus-introduced by the letter ψ. Each class ψ can be associated with a pair $< \Gamma_\psi , \Omega_\psi >$. Here, Ω_ψ is the primitive of the form ω on the compactum Γ_ψ. Suppose that the class ψ contains the expression $V(g)\,u\exp(i\,h^{-1}\,S)$. Consider the Lagrangian manifold $<\Gamma_{S_U} , \Omega_{S_U} >$. The pair $< \Gamma_\psi , \Omega_\psi >$ is the restriction of the Lagrangian pair $<\Gamma_{S_U} , \Omega_{S_U} >$ above $g\,\Gamma_S$.

That restriction does not depend upon the choice of representative in the class ψ.

Consider the map $P_{V\varphi} : \Gamma_\psi \to$ supp u that is defined by the formula $P_{V\varphi} = \pi_S \circ g_S^{-1}$, in which π_S is the orthogonal projection of Γ_S onto Q. One has a locality property: $V_1\,\varphi_1 = V_2\,\varphi_2$, so:

$$u_1 \circ P_{V_1\varphi_1} = \left(\sum_{k\geq 0}\left(\frac{h}{i}\right)^k D_k(\bullet \mid P_{V_2\varphi_2}\,\Gamma_\psi)\,u_2\right)\circ P_{V_2\varphi_2} . \tag{3.4}$$

3. – The linear space $\mathcal{L}$ (Γ, Ω).

One says that the class $\boldsymbol{\psi}$ is *dominated* by the Lagrangian pair < Γ, Ω > if the pair < Γ_ψ, Ω_ψ > is a restriction of the Lagrangian pair < Γ, Ω > to Γ_ψ. In order to achieve that, one can consider Γ_ψ to be a subset of E.

A vector in the linear space $\mathcal{L}$ (Γ, Ω) is a formal sum:

$$\Psi = \sum_{\alpha\in I} \boldsymbol{\psi}_\alpha \tag{3.5}$$

over a set I of classes $\boldsymbol{\psi}_\alpha$ that are dominated by the pair < Γ, Ω > and with the condition that each point $x \in E$ must have a neighborhood that intersects only a finite number of Γ_{ψ_α}. The linear operations in $\mathcal{L}$ (Γ, Ω) are defined in an obvious manner. Define the null vector. Let $\Psi \in \mathcal{L}$ (Γ, Ω). Fix a point x, $x \in E$ and consider one of its neighborhoods $U(x)$. One can suppose that $U(x)$ projects bijectively onto a certain Lagrangian plane Λ. Let U_0 be a neighborhood of x such that $\bar{U}_0 \subset U(x)$. Introduce the corresponding support function η. Set $\eta_\alpha = \eta_0 P^{-1}_{V_\alpha \varphi_\alpha}$. Consider the expressions $V(g_\alpha)(u_\alpha \eta_\alpha)\exp(i h^{-1} S_\alpha)$, in which $V(g_\alpha) u_\alpha \exp(i h^{-1} S_\alpha)$ belongs to the class $\boldsymbol{\psi}_\alpha$. Let $g\,\Lambda = Q$, $g \in G$. Introduce the notation $I(x) = \{\alpha \mid U(x) \cap \Gamma_{\psi_\alpha} \neq \varnothing\}$. For $\alpha \in I(x)$, one has:

$$V(g_\alpha)(u_\alpha \eta_\alpha)\exp\left(\frac{i}{h} S_\alpha\right) = V(g) u^\alpha \exp\left(\frac{i}{h} S\right) .$$

Form the expression:

$$V\varphi = V(g)\left(\sum_{\alpha\in I(x)} u^\alpha\right)\exp\left(\frac{i}{h} S\right) .$$

The vector Ψ will be considered to be zero if:

$$\left(\sum_{\alpha\in I(x)} u^\alpha\right)(P_{V\varphi}\, x) = 0 , \qquad \forall x \in E .$$

4. – Generating integral.

In order to prepare ourselves for certain definitions, we shall begin with some symbolic transformations. We introduce the fundamental notation:

$$T\,\mu = T_{<\Gamma,\Omega,\,\tau>}\,\mu \tag{3.6}$$

for the correspondence that will be described below:

$$T : \{\mu\} \to \mathcal{L}\,(\Gamma, \Omega)\,,$$

in which $<\Gamma, \Omega, \tau>$ is a Lagrangian triple, and:

$$\mu = \sum_{k \geq 0} \left(\frac{h}{i}\right)^k \mu_k\,,$$

with μ_k being differentiable measures with complex values on E.

The fundamental notation is specified more precisely by the symbol:

$$T_{<\Gamma, \Omega, \tau>} = \text{S.P.}\int_E \mu(dx)\, K_{<\Gamma,\Omega,\tau>}(x)\ . \tag{3.7}$$

We shall return to the role of S.P. later on. As for the expression for K, it was described in **§ 2**. The formula (2.7) leads to the symbolic equality:

$$T_{<\Gamma, \Omega, \tau>}\, \mu = V(g^{-1})\, T_{g<\Gamma,\Omega,\tau>}\, \mu_g\,, \tag{3.8}$$

in which $g \in G$ and $\mu_g(\gamma) = \mu(g^{-1}\gamma)$, $\gamma \subset g\,E$.

Theorem 1:

One can establish a bijective correspondence between the vectors Ψ *of the space* $\mathcal{L}\,(\Gamma, \Omega)$ *and the measures* μ *such that the linear operations on* Ψ *correspond to linear operations on* μ .

We can achieve the proof by simultaneously constructing the correspondence T. That construction will always be implicit in what follows. We will show how we can construct the vector Ψ upon starting from the measure μ.

The invertibility of that construction will become obvious.

Take E to be a locally-finite covering $\{E_\alpha\}_{\alpha \in I}$ and a partition of unity $\{\eta_\alpha\}$ that is subordinate to it. We impose the following conditions:

1) Each E_α projects bijectively onto a certain Lagrangian plane Λ_α . Let $g_\alpha \in G$ such that $g_\alpha\, Q = \Lambda_\alpha$.

2) The operator $\delta_\alpha(x_\alpha) + \tan\, \theta_\alpha(x_\alpha)$, in which δ_α and θ_α correspond to the element of the group G :

$$g_\alpha^{-1}\, \tau\, x = \tau_\alpha\, x_\alpha = \{x_\alpha, A_\alpha(x_\alpha)\}, \qquad x \in E_\alpha\,,$$

is not degenerate on E_α .

The symbol $T\ \mu$ naturally corresponds to the symbolic sum $\sum_\alpha T(\eta_\alpha\ \mu)$, which we can represent with the aid of (3.8):

$$\sum_\alpha V(g_\alpha) T_{g_\alpha^{-1}<\Gamma,\Omega,\tau>}(\eta_\alpha\ \mu)_{g_\alpha^{-1}}\,.$$

The integral (3.7), which is associated with $(T_{g_\alpha^{-1}<\Gamma,\Omega,\tau>}(\eta_\alpha\ \mu)_{g_\alpha^{-1}})(\xi))$, has the same form as the integral in (3.2).

The point $x_\alpha = \{\xi, \ldots, \} \in g_\alpha^{-1} E_\alpha$ is the unique non-degenerate critical point of the corresponding function *f*. The symbol S.P. in (3.7) signifies the application of the method of stationary phase that relates to that point:

$$T_{g_\alpha^{-1}<\Gamma,\Omega,\tau>}(\eta_\alpha\ \mu)_{g_\alpha^{-1}} \to \varphi_\alpha = \mu_\alpha \exp\left(\frac{i}{h} S_\alpha\right),$$

and the symbol $T_{<\Gamma,\ \Omega,\ \tau>}\ \mu$ corresponds to the vector (3.5), in which the class $\boldsymbol{\psi}_\alpha$ contains the expression $V(g_\alpha)\ \varphi_\alpha$.

One calls $T\ \mu$ the *generating integral* for the vector Ψ and one writes $\Psi = T\ \mu$.

5. – Relation to the canonical operator.

We remark that $T\ \mu$ will reduce to a class $\boldsymbol{\psi}$ that contains an expression of the form $V(g)\ \varphi$ if and only if the Lagrangian manifold $g^{-1}\Gamma$ projects bijectively onto Q.

In particular, for g equal to unity, the expression for φ is $\varphi = u \exp(i\, h^{-1} S)$, in which:

$$S(\xi) = \Omega(x_\xi) + \tfrac{1}{2}\ \xi\ p\,, \qquad x_\xi = \{\xi, p\} \in \Gamma,$$

$$u_0(\xi) = \frac{d\mu_0}{dx} \left| \det^{-1/2} r^{-1} \cos\theta \right|\Big|_{x=x_\xi} \exp\left(\frac{i}{2}\pi k\right),$$

in which one has, respectively, s = surface element on Γ, $r = e^\rho$, θ and ρ are parameters of $\tau\, x$, and finally k = ind γ, in which γ is the projection onto Λ_0 of a curve on θ that connects $\theta = 0$ to $\theta(x)$ for arbitrary x.

Therefore, the index will enter into the expression for $V(g)\ \varphi$.

Let us show how we can describe Maslov's canonical operator that we can use to describe the higher-order terms asymptotically in terms of those concepts.

It is obvious that the Maslov construction utilizes the concept of index.

Each class $\boldsymbol{\psi}$ contains an expression of the form $V^{(1)}(\overset{\circ}{\theta})u \exp i/h S$, such that the proper values of $\overset{\circ}{\theta}$ are zero or $\pi/2$. Consider the Lagrangian pair $<\Gamma, \Omega>$ and the function $v : E \to C$.

Introduce the Lagrangian triple $<\Gamma, \Omega, \tau>$ and the measure:

$$\mu(\gamma) = \mu_0(\gamma) = \int_\gamma v \,|\det{}^{1/2} r\,|\, s(dx) \ .$$

Introduce the vector $T_{<\Gamma, \Omega, \tau>}\,\mu$ and represent it in the form (3.5), but while choosing a representative of the form:

$$V^{(1)}(\overset{\circ}{\theta}_\alpha)u_\alpha \exp\left(\frac{i}{h}S_\alpha\right)$$

in each $\boldsymbol{\psi}_\alpha$.

Maslov's canonical operator is the map from $\{<\Gamma, \Omega>, v\}$ to the associated function $Q \to C$, which has the form:

$$\sum_\alpha V^{(1)}(\overset{\circ}{\theta}_\alpha)u_\alpha \exp\left(\frac{i}{h}S_\alpha\right) \ .$$

One supposes that v has compact support and that $\{\alpha \mid \operatorname{supp} v \cap E_\alpha \neq \emptyset\}$ is finite.

§ 4. – APPLICATIONS OF THE GENERATING INTEGRAL

Consider the Cauchy problem for the formal equation:

$$i\,h\frac{d}{dx}\Psi(t) = \mathcal{H}(t)\,\Psi(t) \qquad (4.1)$$

at the points 1, 2, with the initial condition:

$$\Psi(0) = \Psi \in \mathcal{L}(\Gamma, \Omega) \ .$$

In connection with that, one defines a certain class of linear operators on the spaces $\mathcal{L}(\Gamma, \Omega)$ and then defines the expression $i\,h\frac{d}{dx}\Psi(t)$. Section 3 will discuss the applications to asymptotic series.

1. – Quasi-classical operator.

We shall define some linear operators on the space $\mathcal{L}$ (Γ, Ω) that have a special form that we shall call *quasi-classical*.

A quasi-classical operator $\mathcal{H}$ is given by its Hamiltonian function:

$$H = \sum_{k\geq 0}\left(\frac{h}{i}\right)^k H_k\,, \qquad H_k : M \to C\,.$$

If $H = H_0$ and $\Psi = T_{<\Gamma,\Omega,\tau>}\,\mu$ then:

$$\mathcal{H}\,\Psi = T_{<\Gamma,\Omega,\tau>}\hat{H}(\Gamma,\tau)\mu\,, \tag{4.2}$$

in which:

$$\hat{H}(\Gamma,\tau)\mu = \sum_{l\geq 0}\left(\frac{h}{i}\right)^l D_l\,(x\,|\,H_0,\Gamma,\tau)\,\mu\;. \tag{4.3}$$

D_k depends upon H_0 linearly.

For a general H, we must set:

$$\hat{H}\mu = \sum_{k\geq 0}\left(\frac{h}{i}\right)^k \sum_{l\geq 0}\left(\frac{h}{i}\right)^l D_l\,(x\,|\,H_k,\Gamma,\tau)\,\mu \tag{4.4}$$

in (4.3). We shall now describe the construction that leads to the explicit form for the expression of D_l . It will suffice to consider $H = H_0$ and $\mu = \mu_0$. We then perform the sequence of symbolic transformations:

$$\begin{aligned}\mathcal{H}\,\Psi &\sim \text{S.P.}\int_E \mu(dx)\sum_{k\geq 0}\frac{1}{k!}\left(K - x, \frac{\partial}{\partial y}\right)^k H(y)\big|_{y=x}\,K_{<\Gamma,\Omega,\tau>}(x)\\ &\sim \sum_{k\geq 0}\text{S.P.}\int_E \mu(dx)\,K\,F_k\;.\end{aligned} \tag{4.5}$$

The definition of F_k is obvious. It is easy to see that F_k is a polynomial in h and x that depends linearly on H and its derivatives at the point x. Each term can be specified as an expression of the form $T\,\mu$. In order to do that, it is necessary to use the transformation of **§ 2**, which transforms (there it was $T\,\mu$, but here it is) that term into $\Psi = \sum_\alpha \psi_\alpha$, and then represent the result in the form $T\,\mu$. In summary:

$$\sum_{k\geq 0}\text{S.P.}\int_E \mu(dx)\,K\,F_k \sim T_{<\Gamma,\Omega,\tau>}\left(\sum_{j\geq E\left(\frac{k+1}{2}\right)}\left(\frac{h}{i}\right)^j d_{k,j}\,(x\,|\,H,\Gamma,\tau)\,\mu\right). \tag{4.6}$$

Here, $E\,(t)$, $t \in R$ is the integer part of t. In the definition of (4.3), one must set:

$$D_0 = H\,, \qquad D_l = \sum_{k=1}^{2l} d_{k,l}\,, \qquad l \geq 1\,.$$

We now free ourselves from the hypothesis that E is equivalent to E (ω) that was made in **§ 2**. Upon examining the foregoing, we see that this manifests itself only in the annihilation of the measure μ on E in Ψ by the inverse transformation to T. We must give the Lagrangian triple on E. However, the inverse transformation of T is necessary here only for the construction of the differential operator D_1 . Taking into account their local character, it is obvious that the formulas obtained can also be taken to be definitions in the general case. An analogous remark can also be made at the beginning of the following section.

Let us change the notation. We agree to let Ψ denote the symbol $T_{<\Gamma,\,\Omega,\,\tau>}\,\mu$, i.e., the set of Lagrangian triples and measures. We intend $\mathcal{L}$ (Γ, Ω) to mean the linear space of those symbols that are generated by ordinary linear operations on μ for a fixed $<\Gamma,\,\Omega,\,\tau>$.

The space that was introduced in number 3 in **§ 3** will now be distinguished from $\mathcal{L}$ (Γ, Ω). One denotes it by $\mathcal{L}_T$ (Γ, Ω). The element of $\mathcal{L}_T$ (Γ, Ω) that corresponds to Ψ, $\Psi \in \mathcal{L}$ (Γ, Ω), will be denoted by Ψ_T .

2. – The Cauchy problem.

Consider the Cauchy problem (4.1). Define the operation $ih\dfrac{d}{dt}$. Suppose that the dependency of Ψ (t) on t has the form $\Psi(t) = T_{m_t <\Gamma,\Omega,\tau>}$, in which m_t is the diffeomorphism that was described in **§ 2**. Based upon the formula (2.5), one will arrive at an expression of the type (4.5) for $ih\dfrac{d}{dt}\Psi(t)$. Upon specifying it, one will be naturally led to the following definition:

$$ih\frac{d}{dt}\Psi(t) = T_{m_t <\Gamma,\Omega,\tau>}\left\{ih\frac{d\dot{\mu}_t}{d\mu_t} + \chi + \sum_{k\geq 1}\left(\frac{h}{i}\right)^k \sum_{j=1}^{2} d_{j,k}(\bullet\,|\,\chi,\Gamma,\tau)\right\}\mu_t\,. \tag{4.7}$$

Upon returning to equation (4.1), suppose that $H_0 = \chi$. Equation (4.1) will then be equivalent to the equality:

$$ih\,\dot{\mu}_t + H_0\mu_t + \sum_{k\geq 1}\left(\frac{h}{i}\right)^k\left[\sum_{j=1}^{2} d_{j,k}(\bullet\,|\,H_0,\cdots) - \sum_{l\geq 0}\left(\frac{h}{i}\right)^l D_l(\bullet\,|\,H_0,\cdots\right]\mu_t = 0\,, \tag{4.8}$$

and after simplification, that will come down to the system of recurrence equations:

$$(\dot{\mu}_t)_k + H_0(\mu_t)_k = N_k\,((\mu_t)_i\,,\ i<k), \quad k = 0, 1, 2, \ldots, \tag{4.9}$$

along with $N_0 = 0$. In the Cauchy problem, those equations are completed with initial conditions that will then determine μ_t uniquely. The Cauchy problem is then solved. The choice of t in the Lagrangian triple < Γ, Ω, τ > that defines the correspondence $\mu \to \Psi_T$ is largely arbitrary. In particular, one can always give K in the canonical form:

$$K = K^{(1)} = \exp\frac{i}{h}\Omega^{(t)}V(x)V^{(1)}(\theta)\,\delta\,.$$

The change of K to $K^{(1)}$ corresponds to a local linear transformation of the corresponding measures μ and $\mu^{(1)}$, and $\mu_0 = \mu_0^{(1)}\,|\det^{1/2}\tau\,|$, in addition. If such a change is performed on the solution to the Cauchy problem, while considering the fact that $H_1 = 0$, then one will get:

$$(\mu_t^{(1)})_0(m_t\,dx)/(\mu_0^{(1)})_0(dx) = [s_0\,(dx)/s_t(m_t\,dx)]^{1/2}, \tag{4.10}$$

in which s_t is the surface element of Γ_t .

The generating integral with a kernel of the form $K^{(1)}$ is described in [**6**]. Formula (4.10) establishes a link between the solutions to the Cauchy problem that were given there and the ones that were obtained here.

3. – Asymptotic maps.

In the expression for $V\varphi$, we let $V\varphi^N$ denote the element of L_2 (Q) that is given by the formulas:

$$V\varphi^N = V(g)\,u^N\exp\left(\frac{i}{h}S\right),$$

in which:

$$u^N = \sum_{k=0}^{N}\left(\frac{h}{i}\right)^k u_k \;\;. \tag{4.11}$$

One easily sees that it results from $V_1\,\varphi_1 = V_2\,\varphi_2$ that:

$$V_1\varphi_1^N = V_2\varphi_2^N = O(h^{N+1})\;.$$

We intend $O(h^k)$, $k = 0, 1, 2, \ldots$ to mean an element of L_2 (Q) such that $h^{-k}\,O(h^k)$ has a limit for $h \in \Delta$.

We intend $\boldsymbol{\psi}^N$ to mean the class of functions $V\varphi^N$ for which $V\varphi$ belongs to the class $\boldsymbol{\psi}$.

Let $\overset{\circ}{\mathcal{L}}(\Gamma,\Omega)$ denote the subset of elements of $\mathcal{L}$ (Γ, Ω) whose measures have compact support μ.

Let:

$$\Psi \in \overset{\circ}{\mathcal{L}}(\Gamma,\Omega) \quad \text{and} \qquad \Psi_T = \sum_\alpha \Psi_\alpha \,.$$

We take the set I to be finite (if that is possible). Set:

$$\Psi_T^N = \sum_\alpha \Psi_\alpha^N \,.$$

One says that Ψ^α, $\Psi \in \overset{\circ}{\mathcal{L}}(\Gamma,\Omega)$, is an asymptotic development of the element ψ_h, $\psi_h \in L_2(Q)$ if:

$$\psi_h - \Psi_T^N = O(h^{N+1}), \qquad\qquad N = 0, 1, 2, \dots$$

One writes $\psi_h \sim \Psi$.

We say that the linear operator H on L_2 (Q) that depends upon h generates the quasi-classical operator $\mathcal{H}$ if $\forall N$ and $\forall \Psi \in \overset{\circ}{\mathcal{L}}(\Gamma,\Omega)$:

1) $\Psi_T^N \in \mathcal{D}(H)$ (viz., the domain of H).

2) $H\,\Psi_T^N = (\mathcal{H}\Psi)_T^N + O(h^{N+1})$.

One can give some simple effective criteria for recognizing whether H is an operator that generates a quasi-classical operator. In particular, they are verified under some simple hypotheses on v in the case of the Schrödinger operator, such that the corresponding Hamiltonian function is given by the formula (1.4).

Suppose that $H = H(t)$ depends upon t. Consider the Cauchy problem in L_2 (Q):

$$i\,h\frac{d}{dt}\psi(t) = H(t)\;\psi(t) + f(t)\,, \qquad\qquad \psi(0) = \psi\,. \tag{4.12}$$

We suppose that:

1) The operator $H(t)$ generates the quasi-classical operator $\mathcal{H}(t)$:

2) The problem (4.12) is soluble, and:

$$\|\,y(t)\,\| \le C(t)[\|\psi\| + \sup_{0\le\tau\le t}\|\,f(\tau)\,\|]\,,$$

in addition. $C(t)$ does not depend upon h.

For some efficacious criteria for the validity of 2), cf., [**7-9**].

Theorem 2:

If H verifies the conditions 1), 2), *and* $f = 0$, *but* $\psi \sim \Psi$, *with* $\Psi \in \overset{\circ}{\mathcal{L}}(\Gamma, \Omega)$, *then* $\psi(t) \sim \Psi(t)$ *will be the solution to the problem* (4.1).

The proof will become obvious if one takes into account the fact that:

$$i\,h\,\frac{d}{dt}\Psi_t^N = \left(i\,h\,\frac{d}{dt}\Psi\right)_t^N + O(h^{N+1}) \ .$$

An analogous result that is expressed in terms of the canonical operator is contained in Maslov's book.

In conclusion, here are some remarks about the use of the generating integral in the study of asymptotics of proper elements of the operator that generate the quasi-classical operator.

One shows that one can associate each closed compact Lagrangian manifold Γ that is invariant with respect to the dynamical system m_t and verifies a certain stability condition with an element Ψ, $\Psi \in \mathcal{L}(\Gamma, \Omega)$ that asymptotically approaches a proper function of the operator *H* for a certain sequence h_n ($n = 1, 2, \ldots$), $h_n \to 0$ as $n \to \infty$. As $h_n \to 0$, that proper function will be concentrated on Γ in a certain sense. The sequence h_n is determined by means of the Maslov-Arnol'd characteristic class of the manifold Γ.

A modification of the generating integral will permit one to obtain an analogous result for lower-dimensional manifolds; for example, closed orbits of dimension one. The role of the stability conditions in that set of problems was observed in some particular cases in the articles [**10**], [**11**], etc.

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