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On ray surfaces of constant twist

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The ray surfaces of vanishing twist – viz., the *torses* – can be easily classified and generated in a unified way. It is therefore reasonable to attempt to do something similar with the *ray surfaces of constant twist* ($\neq 0$). As an example of such a surface, in addition to the hyperboloids of revolution and the helical ray surfaces, there are the surfaces that **R. Edlinger** investigated whose osculating hyperboloids are all surfaces of revolution [**1**] and are third-degree algebraic surfaces, namely, **Cayley** surfaces that doubly osculate the absolute conic section [**2**, pp. 241].

The goal of this little note is to give ways of generating certain normal forms for ray surfaces of constant twist, from which all surfaces of that type can be obtained by applying **Minding**'s bends. In addition, a series of properties of those surfaces, as well as their relationships to the theory of curves will be discussed. The analytical treatment of the question that is posed will employ the natural geometry of ray surfaces ([1]) that was founded by **E. Kruppa** [**3**], [**4**], and the formulas from it that are needed in this article are summarized in section 1. With the use of that very convenient calculus, some of the known theorems in the theory of ray surfaces can be proved concisely.

1.

If $\mathfrak{y} = \mathfrak{y}(t)$ is a space curve c and $\mathfrak{e}(t)$ is a unit vector then:

$$\mathfrak{x}(t, v) = \mathfrak{y}(t) + v\,\mathfrak{e}(t) \tag{1}$$

represents a ray surface Φ with the central normal vector:

([1]) A textbook presentation of that topic can be found in [**8**].

$$\mathfrak{n} = \frac{\dot{\mathfrak{e}}}{|\dot{\mathfrak{e}}|} \tag{2}$$

and the central tangent vector:

$$\mathfrak{z} = \frac{\mathfrak{e}\times\dot{\mathfrak{e}}}{|\dot{\mathfrak{e}}|}\,. \tag{3}$$

The *t*-lines then cut out congruent sequences of points from the generators (*v*-lines). For the line of striction *s* of Φ, one has:

$$\mathfrak{s}\,(t) = \mathfrak{y}\,(t) - \frac{\dot{\mathfrak{x}}\;\dot{\mathfrak{e}}}{\dot{\mathfrak{e}}^2}\,\mathfrak{e}\,, \tag{4}$$

and for the *twist d*, one has:

$$d = \frac{(\mathfrak{e}\,\dot{\mathfrak{e}}\,\dot{\mathfrak{y}})}{\dot{\mathfrak{e}}^2}\,. \tag{5}$$

We shall employ the natural geometry of ray surfaces that goes back to **E. Kruppa** and is based upon the right-hand dreibein that is defined by the unit vectors $\mathfrak{e}$, $\mathfrak{y}$, $\mathfrak{z}$, where those vectors can now be regarded as functions of the arc-length *u* along the line of striction *s* . If u_1 (u_3, resp.) is the arc-length of the spherical image of $\mathfrak{e}\,(u)$ [$\mathfrak{z}\,(u)$, resp.] then $\kappa = du_1 : du$ will be called the *curvature* of the ray surface, and $\tau = du_3 : du$ will be called its *torsion*. Furthermore, if $\mathfrak{t}\,(u)$ is the tangent vector of the line of striction then $\sigma(u) = \angle\ \mathfrak{e}\ \mathfrak{t}$ will be the *striction*. The ray surface is defined uniquely (up to motions) when one is given the three functions $\kappa(u)$, $\tau(u)$, $\sigma(u)$. For $\sigma = 0$, one gets the torses. The coordinate representation of Φ (κ, τ, σ) reads:

$$\mathfrak{x}\,(u, v) = \mathfrak{s}\,(u) + v\;\mathfrak{e}\,(u) = \int (\mathfrak{e}\cos\sigma + \mathfrak{z}\sin\sigma)\,du + v\;\mathfrak{e}\,(u)\,. \tag{6}$$

The derived equations:

$$\mathfrak{e}' = \kappa\,\mathfrak{n}, \qquad \mathfrak{n}' = -\kappa\,\mathfrak{e} + \tau\,\mathfrak{z}\,, \qquad \mathfrak{z}' = -\tau\,\mathfrak{n} \tag{7}$$

are constructed in a manner that is analogous to the **Frenet** formulas. For the twist *d*, one has:

$$d = \frac{\sin\sigma}{\kappa}\,. \tag{8}$$

One calculates the normal curvature κ_n , geodetic curvature κ_g , and geodetic torsion τ_g for the line of striction *s* from (1):

$$\kappa_n = -\tau\sin\sigma + \kappa\cos\sigma\,, \qquad \kappa_g = -\sigma', \qquad \tau_g = \kappa\sin\sigma + \tau\cos\sigma\,, \tag{9$_{1,2,3}$}$$

and for its curvature *K* and torsion *T*, one has:

$$K^2 = \kappa_n^2 + \kappa_g^2 , \qquad T = \tau_g + \frac{\kappa_g^2}{\kappa_n^2 + \kappa_g^2}\left(\frac{\kappa_n}{\kappa_g}\right)' . \tag{$10_{1,2}$}$$

The ***Minding*** *bending surfaces* Φ^* of a ray surface Φ (κ, τ, σ) split into two families with the natural equations $\Phi_1^*(\kappa, \tau^*, +\sigma)$ and $\Phi_2^*(\kappa, \tau^*, -\sigma)$, in which τ^* is an arbitrary function of u. The surfaces Φ_1^* possess equal twists along corresponding generators (Minding bending "in the same sense"), and the twist will change sign (Minding bending "in the opposite sense") when one goes to the surfaces Φ_2^*. The surfaces Φ_2^* are Minding bending surfaces in the same sense to the surface Φ_2 (κ, $-\tau$, $-\sigma$), which is congruent in the opposite sense. We would like to understand "bending" to always mean a Minding bending in the same sense, as Minding did. In particular, we have:

Any ray surface can be bent in such a way that its line of striction will be a line of osculation [from (91): $\tau^* = \kappa \cot \sigma$] *or a line of curvature. Every ray surface can be bent into a conoidal ray surface* ($\tau^* = 0$) *and a ray surface with a cone of revolution as its direction cone* (τ^* : κ = const.).

In addition, from (9_2), we have:

The line of striction of a ray surface of constant striction is geodetic.

2.

We assume that Φ (κ, τ, σ) is a ray surface of constant twist d ($\neq 0$), i.e., from (8), we will have:

$$\kappa\,\sigma' \cos \sigma - \kappa' \sin \sigma = 0 . \tag{11}$$

The surface curve $v = v(u)$ will be a *line of osculation* (which differs from the generator) precisely when we have:

$$2\kappa \sin \sigma \cdot v' = \kappa^2 \tau v^2 + (\kappa\,\sigma' \cos \sigma - \kappa' \sin \sigma)\, v + (\tau \sin \sigma - \kappa \cos \sigma) \sin \sigma , \tag{12}$$

as we calculate from (6). The tangent vectors:

$$\mathfrak{x} = \mathfrak{x}_u + \mathfrak{x}_v\, v' = \mathfrak{e}\,(\cos \sigma + v') + \mathfrak{n}\, \kappa v + \mathfrak{z} \sin \sigma \tag{13}$$

to the osculating line at the points of a generator e_0 ($u = u_0$) determine the *osculating hyperboloid* Ω_0 along e_0, and from (12) and (13) its direction cone will read:

$$\kappa\, \tau \sin \sigma\, [x^2 + y^2 + z^2 - (x^2 + 2\,x\,z\frac{\kappa}{\tau} - z^2 \frac{\kappa}{\tau}\cos\sigma)] + (\kappa\,\sigma' \cos \sigma - \kappa' \sin \sigma)\, y\, z = 0 \tag{14}$$

in a coordinates system (x, y, z) in which $\mathfrak{e}_0 = (1, 0, 0)$, $\mathfrak{n}_0 = (1, 0, 0)$, $\mathfrak{z}_0 = (1, 0, 0)$. If the *line of striction of* Φ *is a line of curvature* ($\tau = -\kappa \tan \sigma$) and if Φ has constant twist then when one employs (11), one will have that the direction cone is:

$$x^2 + y^2 + z^2 - (x - z \cot \sigma)^2 = 0 , \tag{15}$$

i.e., Ω_0 is a *hyperboloid of revolution*. One can conclude from this that:

Theorem 1:

If one bends a ray surface of constant twist in such that its line of striction becomes a line of curvature then all of the osculating hyperboloids will be hyperboloids of revolution.

R. Edlinger [**1**] has examined those surfaces in more detail and gave a kinematic way of generating those surfaces. With the use of his results, one has:

Theorem 2:

If an equilateral paraboloid π *rolls on one of its bending surfaces* ψ *then the vertex generator of* π *that does not contact* ψ *will envelop a ray surface of constant twist. Any surface of constant twist can be bent into such a surface.*

The differential equation of the *lines of curvature* of (6) has the form:

$$\begin{aligned} f(\kappa, \sigma) v'^2 + g(\kappa, \tau, \sigma; v) v' + (\kappa^2 v^2 + \sin^2 \sigma)(\kappa \sin \sigma + \tau \cos \sigma) \\ + v \cos \sigma (\kappa \sigma' \cos \sigma - \kappa' \sin \sigma) = 0 , \end{aligned} \tag{16}$$

in which f, g are functions of the given arguments. If Φ is a ray surface of constant twist and its line of striction s is a line of striction them from (9_3) and (11), (16) will imply that:

$$v'(f v' + g) = 0 , \tag{17}$$

i.e., the v-lines will all be lines of curvature.

Theorem 3:

Any ray surface of constant twist can be bent in such a way that the lines of curvature of one family cut out congruent point-sequences from the generators ([2]).

([2]) The fact that the Edlinger surfaces possess this property is proved with no calculation in [**1**, pp. 346].

3.

If the *line of striction s* of a ray surface is a *line of osculation* then, since $\kappa_n = 0$, from (91), one will have $\tau = \kappa \cot \sigma$, and from (10) [**2**, pp. 147], one will have that its torsion T is:

$$T = \frac{\kappa}{\sin \sigma} = \frac{1}{d} \; . \tag{18}$$

If the line of striction of a ray surface Φ *is a line of osculation of constant torsion then the surface will have constant twist.*

One can infer a theorem of **J. G. Darboux** from (9_2):

The generators of a ray surface are geodetically parallel along the line of striction.

The converse is less known [**2**, pp. 143]:

If the generators are geodetically parallel along a surface curve c then it will be the line of striction.

Namely, if c is given by $u = u\,(t)$, $v = v\,(t)$ then $\mathfrak{e}$ will be geodetically parallel along c when $d\mathfrak{e} / dt$ is parallel to the surface normals at the points of c, so one will have:

$$\mathfrak{e}' \dot{u} \times (\mathfrak{x}_u \times \mathfrak{x}_v) = 0 \; , \tag{19}$$

and from (6), that will yield:

$$v \kappa^2 \, \mathfrak{e} = 0 \; , \tag{20}$$

so $v = 0$.

With that, one gets the following way of generating the ray surfaces of constant twist:

Theorem 4:

If one displaces a line of the osculating strip of a curve of constant torsion in a geodetically parallel way then it will envelop a surface of constant twist. One obtains all surfaces of constant twist by bending such surfaces.

In that way, it is possible to give a coordinate representation of the *normal forms* for surfaces of constant twist for which the lines of striction are lines of osculation. If $\mathfrak{b}\,(\gamma)$ is a unit vector and γ is the arc-length of its spherical image, so $\left| \frac{d\mathfrak{b}}{d\gamma} \right| = 1$, and one introduces the new parameter:

$$u = \frac{1}{T}\gamma \tag{21}$$

in place of γ (where T means an arbitrary constant) then (as an easy calculation will show):

$$\mathfrak{s}\,(u) = \frac{1}{T}\int \left(\mathfrak{b}(u)\times\frac{d\mathfrak{b}}{d\gamma}\right) \tag{22}$$

will represent a curve c of constant torsion T with arc-length u and binormal $\mathfrak{b}$ ([3]). If c is the line of striction and line of osculation of a ray surface of constant twist $d = 1 / T$ that is generated as in Theorem 4 then when one employs the striction σ, tangent vector $\mathfrak{t}$, and principal normal vector $\mathfrak{h}$ of c, its generators $\mathfrak{e}$ will have the representation:

$$\mathfrak{e} = \mathfrak{t}\cos\sigma + \mathfrak{h}\sin\sigma, \tag{23}$$

which will make:

$$\mathfrak{t} = \frac{1}{T}(\mathfrak{b}\times\mathfrak{b}'), \quad \mathfrak{h} = -\frac{1}{T}\,\mathfrak{b}'. \tag{24}$$

$\mathfrak{e}$ lies in the osculating plane of c and will then be displaced along c in a geodetically parallel way when (since $\kappa_n = 0$) one has:

$$\sigma = -\int \kappa_g\,du = -\int K\,du \tag{25}$$

for the angle σ between $\mathfrak{t}$ and $\mathfrak{e}$, in which $K\,(u)$ is the curvature of c. With the use of (22), it will follow that:

$$K = \left|\frac{\mathfrak{b}\times\mathfrak{b}''}{T}\right|, \tag{26}$$

and therefore one will have:

$$\mathfrak{x}\,(u, v) = d\int(\mathfrak{b}\times\mathfrak{b}')\,du + v\,d\,[(\mathfrak{b}\times\mathfrak{b}')\cos(|\,d\,|\int|\,\mathfrak{b}\times\mathfrak{b}''\,|\,du) + \mathfrak{b}'\sin(|\,d\,|\int|\,\mathfrak{b}\times\mathfrak{b}''\,|\,du] \tag{27}$$

for a ray surface of constant twist d. The natural equations of that ray surface read:

$$\kappa = \frac{1}{d}\sin\sigma, \quad \tau = \frac{1}{d}\cos\sigma, \quad \sigma = -\int K(u)\,du. \tag{28}$$

([3]) Cf., say [**5**, pp. 87].

If one replaces τ with an arbitrary function $\tau^*(u)$ then one will obtain all ray surfaces of constant twist.

4.

A *special ray surface of constant twist* shall now be investigated.

If a ray surface of constant twist has a *geodetic line of striction* then, from (9_2), it will have constant striction, and therefore, from (11):

$$\kappa' \sin \sigma = 0 . \tag{29}$$

The yields either the *torses* with $\sigma = 0$ (and therefore $d = 0$) or κ = const. The line of striction of a ray surface Φ (κ = const., σ = const. $\neq 0, \pi/2$) is, however, a *Bertram curve*, which would follow from (10). Such ray surfaces are *bending surfaces of a hyperboloid of revolution*, since one likewise has κ = const., σ = const. $\neq 0, \pi/2$ for them.

Theorem 5 (4):

The ray surface of constant twist with geodetic lines of striction are (*for* $\sigma \neq 0, \pi/2$) *bending surfaces of a hyperboloid of revolution.*

If a ray surface is an *orthoid* ($\sigma = \pi/2$) then it will follow from (10) [**6**, pp. 78] that:

$$T = \kappa = \frac{1}{d} , \tag{30}$$

i.e., *the binormal surface of a curve of constant torsion has constant twist.* In a manner that is analogous to (27), that will imply a coordinate representation of that surface in the form:

$$\mathfrak{x}(u, v) = d\int (\mathfrak{b} \times \mathfrak{b}')\, du + v\, \mathfrak{b} , \tag{31}$$

which is significant because those surfaces certainly include *algebraic* surfaces. (The order of the binormal surface of an algebraic curve of order n and class m is equal to $n + m$.)(5)

For an orthoid surface of constant twist, from (30), one will also have κ = constant. If one bends one into a *conoidal surface* ($\tau = 0$) then the new surface will have the natural coordinates Φ^* (κ = const., 0, $\pi/2$) and will then be a *helicoid*.

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(4) Cf., also [**2**, pp. 146] and [**6**, pp. 90 and pp. 105].

(5) **Darboux** posed the question of algebraic curves of constant torsion, and they are treated in [**9**] and [**10**], among other places.

Theorem 6 ([4]):

Any orthoid ray surface of constant twist is a bending surface of a helicoid.

Every ray surface can be bent into a *conoidal* one ($\tau = 0$). If κ is the curvature and s is the line of striction of a conoidal ray surface of constant twist d then the central tangent vector $\mathfrak{z}$ will be a constant vector that is normal to the direction plane (which is imagined to be horizontal) and (as a calculation will show immediately) the curve:

$$\mathfrak{x}\,(u) = \mathfrak{s}\,(u) - d\int_{u_0}^{u}\kappa(u)\,du\cdot\mathfrak{x} \tag{32}$$

will be a *plane* curve in the horizontal plane that goes through the point $u = u_0$ of the line of striction. Its tangents:

$$\mathfrak{x} = \mathfrak{e}\cos\sigma + \mathfrak{z}\sin\sigma - d\,\kappa\,\mathfrak{z} = \mathfrak{e}\cos\sigma \tag{33}$$

are parallel to the generators of the ray surface. Its curvature has the value:

$$K\,(u) = \frac{\kappa}{\sqrt{1-d^2\kappa^2}}\,, \tag{34}$$

and one will have:

$$K\,dw = \kappa\,du \tag{35}$$

for its arc-length w. Conversely, if one starts with a plane curve $\mathfrak{y} = \mathfrak{y}\,(w)$ (binormal vector $\mathfrak{b}$ = constant) then:

$$\mathfrak{x}\,(w, v) = \mathfrak{y}\,(w) + d\int_{w_0}^{w}K(w)\,dw\cdot\mathfrak{b} + v\,\frac{d\mathfrak{y}}{dw} \tag{36}$$

will be a conoidal ray surface with constant twist, and one will have:

Theorem 7:

If one measures out the line segment $d\int K\,dw$ along the generator of a perpendicular cylinder through a curve c that lies in a horizontal plane and has curvature K and arc-length w from the points of c then one will get the line of striction of a conoidal ray surface of constant twist d whose generators are parallel to the tangents to c. Conversely, any conoidal ray surface of constant twist can be generated in that way by starting from the normal projection of its line of striction onto a direction plane.

That construction will break down only when one has $K = 0$ or $K = \infty$. From (34), for $K = 0$, one will have $\kappa = 0$, i.e., one will have a *cylinder*. $K = \infty$ will lead to $\kappa = 1/d$, so one will have a *helicoid*. *The helicoid is therefore the only straight conoid of constant twist.*

In particular, if one chooses c to be a *circle* then that will imply the known fact:

The twist of an open straight ray helicoid is equal to the screw parameter.

5.

For any space curve c, there are ∞^1 real lines and two isotropic ones that are rigidly coupled with the sliding dreibein of the curve whose motion along c envelops of a *torse* [**7**]. We ask whether there are lines that are rigidly coupled with the sliding dreibein of a curve that envelop a *ray surface of constant twist* ($\neq 0$) *when the dreibein moves along* c.

If c is given by a position vector $\mathfrak{x} = \mathfrak{x}(s)$ (in which s is the arc-length of c), G is a point with the (fixed) coordinates p, q, r, and α, β, γ are the (fixed) coordinates of a vector $\mathfrak{e}$ relative to the dreibein $\mathfrak{t}$, $\mathfrak{h}$, $\mathfrak{b}$ of c then the ray surface that is enveloped by the lines that go through G and are parallel to e under the motion of the dreibein will be:

$$\begin{aligned}\mathfrak{X}(s, v) &= \mathfrak{x}(s) + p\,\mathfrak{t}(s) + q\,\mathfrak{h}(s) + r\,\mathfrak{b}(s) + v\,[\alpha\,\mathfrak{b}(s) + \beta\,\mathfrak{h}(s) + \gamma\,\mathfrak{b}(s)] \\ &= \mathfrak{y}(s) + v\,\mathfrak{e}(s)\,. \end{aligned} \tag{37}$$

If $\mathfrak{e}$ is a unit vector (isotropic vector, resp.) then one will have:

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \qquad (\alpha^2 + \beta^2 + \gamma^2 = 0, \text{ resp.}). \tag{38}$$

If one calculates the twist d of (37) using (5) then if K and T are the curvature and torsion, resp., of c then one will have:

$$AK^2 + BT^2 + CKT + DK + ET = 0\,, \tag{39}$$

in which one sets:

$$\begin{aligned} &\rho A = q\,\alpha\,\gamma - p\,\beta\,\gamma - d\,(\alpha^2 + \beta^2)\,, && \rho B = -q\,\alpha\,\gamma + r\,\alpha\,\beta - d\,(\beta^2 + \gamma^2)\,, \\ &\rho C = d\,(\alpha^2 - \gamma^2) - p\,\alpha\,\gamma + r\,\beta\,\gamma + 2\,d\,\alpha\gamma\,, && \rho D = -\alpha\,\gamma\,, \quad \rho E = \beta^2 + \gamma^2\,. \end{aligned} \tag{40}$$

Should the twist d of (37) be constant then the coefficients in (39) would be constant. If no quadratic relationship with constant coefficients (viz., the *general* case) exists between curvature and torsion then all of the coefficients of (39) must vanish, and if one observes (38) and (40) then that will yield only two isotropic solutions:

$$\mathfrak{X}(s, v) = \mathfrak{x} \pm i\, d\, \mathfrak{t} + v\,(\mp i\, \mathfrak{h} + \mathfrak{b}) \,. \tag{41}$$

Theorem 8:

The only lines (in general) that are rigidly coupled with the sliding dreibein of a curve and envelop a ray surface of constant twist d ($\neq 0$) *as the dreibein moves along c are two isotropic lines that intersect the tangent normally and are at a distance of* $\pm i\, d$ *from the normal plane and have a slope of* $\pm i$ *with respect to the osculating plane.*

Moreover, those two lines are at a distance of $\pm i\, d$ from those two isotropic lines that cover a torse during their motion and are parallel to them.

With *special space curves* for which a relation of the type (39) exists, there are even more lines that lead to ray surfaces of constant twist. A more detailed discussion of the generalization of the problem of *Cesàro curves* will be presented in another place, but only a special case of it shall be treated here.

We seek those curves for which *lines parallel to the tangent* exist that are fixed in the moving dreibein and envelop ray surfaces of constant twist ($\neq 0$). In order to do that, we must set $\beta = \gamma = 0$ in (37) and thus, from (39) (for $K \neq 0$), we will have:

$$d\, K - q\, T = 0 \,. \tag{42}$$

The curve c is then a *slope line* with constant slope:

$$\tan \omega = \frac{T}{K} = \frac{d}{q} \tag{43}$$

with respect to the reference plane (which is imagined to be horizontal). All of the ∞^2 lines that are parallel to a tangent to c generate ray surfaces of constant twist under the motion of the dreibein of c, and indeed the twists coincide for those ∞^1 surfaces whose generating lines intersect the normal plane in a line ($q = q_0$) that is parallel to the binormal:

$$\mathfrak{X}(s, v) = \mathfrak{x}(s) + q_0\, \mathfrak{h} + r\, \mathfrak{b} + v\, \mathfrak{t} = \mathfrak{y}(s) = v\, \mathfrak{t} \,. \tag{44}$$

When one uses (4) and (43), the lines of striction for those surfaces read:

$$\mathfrak{s}(s) = \mathfrak{y}(s) + r \tan \omega \cdot \mathfrak{t} \,, \tag{45}$$

i.e., the lines of striction will be generated by the points of those of the lines that intersect the principal normals of c that run parallel to the **Darboux** vector of the slope line:

$$\mathfrak{d} = T\, \mathfrak{t} + K\, \mathfrak{b} = K (\tan \omega \cdot \mathfrak{t} + \mathfrak{b}) \,, \tag{46}$$

and are therefore perpendicular. The ray surfaces of constant twist that are generated in that way have the same direction cone as the curve c, so they are cones of revolution (conical curvature τ : $\kappa = \tan\omega =$ const.). **E. Kruppa** called skew ray surfaces of constant conical curvature and constant twist *generalized slope surfaces* [**3**, pp. 170].

Conversely, if a generalized slope surface with line of striction $\mathfrak{s}\,(u)$ is given then the curve:

$$\mathfrak{x}\,(u) = \mathfrak{s}\,(u) - d\,\frac{\kappa}{\tau}\,\mathfrak{n} \tag{47}$$

will be a slope line with a slope of $\tan\omega = \tau / \kappa$, as one easily confirms.

Theorem 9:

Any line that is rigidly coupled to the sliding dreibein of a slope line and parallel to the tangent at an initial location will envelop a generalized slope surface under the motion of the dreibein along the slope line. The lines of striction of that generalized slope surface will then be described by the points of a perpendicular plane through the principal normal to the initial location. Conversely, any generalized slope surface can be generated in that way.

From 1, any ray surface of constant twist can be bent into a generalized slope surface.

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