"Traiettorie dei raggi luminosi e dei punti materiali nel campo gravitazionale," Nuovo Cim. (5) **5** (1913), 267-300.

Trajectories of light rays and material points in a gravitational field

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Summary

According to the new viewpoint of modern physics, the gravitational action between two masses is not transmitted instantaneously, but with the velocity of light, which is what also happens with the electromagnetic action.

Recently, M. Abraham formulated a new theory of the gravitational field according to which the actions are transmitted with precisely the speed of light (c), *which depends upon the gravitational potential*, which is consistent with the hypothesis that Einstein proposed quite recently ([1]).

Abraham first assumed the Laplace equation for c^2 for a field that was homogeneous, static, and massless ([2]). However, Einstein assumed the Laplace equation for the velocity itself c for the same field in order to make his equivalence principle valid for the propagation of light ([3]). As a result, Einstein ([4]), as well as Abraham ([5]), were led to drop their initial hypotheses in order to assume the Laplace equation for $\sqrt{c}$.

If one assumes a reference system whose z-axis is normal to the equipotential planes of a homogeneous field that is static and massless then with an opportune choice of origin and unit of measurement, the three hypotheses, which are recalled here in the order in which they followed chronologically, would give for c :

(I) $c = \sqrt{z}$ (Abraham),

(II) $c = z$ (Einstein),

(III) $c = z^2$ (Einstein and Abraham) .

([1]) A. Einstein, Ann. Phys. (Leipzig) **35** (1911), pp. 898.
([2]) M. Abraham, Rend. Reale Accad. dei Lincei **22**, 2nd sem. (1911), pp. 679.
([3]) A. Einstein, Ann. Phys. (Leipzig) **38** (1912), pp. 355.
([4]) A. Einstein, Ann. Phys. (Leipzig) **38** (1912), pp. 456.
([5]) M. Abraham, Nuovo Cim. **4** (1912), pp. 459.

Under any of those hypotheses, it would follow that the speed of light would no longer be constant and that its propagation would no longer be rectilinear. The light ray would then describe a curve whose nature would depend upon the particular relation that one establishes between c and z.

In the new theory of gravitation, the equations of motion of material points are also radically modified, in such a way that the trajectory of the point will generally depend upon the way that the speed of light varies, and in that way, the trajectory will be coupled with that of light rays.

In the present article, I propose to establish precisely those relations for passing between the trajectories in the general case of an *arbitrary massless static field* for which c will then be an arbitrary function of the coordinates.

The form that the equations of the light ray will take by virtue of Fermat's principle is stated. From a comparison of those equations with those of the trajectory of a material point that is valid in the new theory, one will arrive at the interesting result that *the trajectories of a material point that emanate from a point in a given direction constitute a manifold of curves that has the light ray that emanates from that point in the same direction as a limit curve.* In other words, if the velocity of a material point approaches that of light then its trajectory will become identical to the light ray (§ **2**).

In the case of a field whose equipotential surfaces are (horizontal) parallel planes, the speed of light will become a maximum whether the ray is horizontal. Furthermore (§ **3**), *the light ray and the material point that emanates horizontally and at the same instant from the same point will simultaneously approach points of the same horizontal, and the horizontal projections of the motion will have the same relationship that the initial velocities did.*

It will then follow that one can determine the trajectory of the material point when one is given that of the light ray.

Therefore, if c is proportional to a positive power of the height z then light will have its maximum velocity at the highest point of the trajectory and will arrive normally to the horizontal plane; in addition, *two arbitrary rays will always describe similar curves.* The trajectories of two material points that have velocities at the points where they are horizontal that are proportional to that of light at those same points (§ **5**).

For the three cases that were posed above, one will then have:

(I) $c = \sqrt{z}$ The ray describes an *ordinary cycloid*, while the material point describes a *Fermat cycloid.*

(II) $c = z$ The ray describes a *circle*, while the material point describes an *ellipse*.

(III) $c = z^2$ The ray describes an *elastic curve*, while the material point describes a *curve that is affine to the elastic curve.*

The time that it takes for either the ray or the material point to reach the point at which it motion is horizontal to the plane for which $c = 0$ is finite and proportional to the height in the first case, but infinite in the other two.

We add that the case of the circle, which was studied before by Garbasso ([1]), is the only case other than that of rectilinear propagation in which the wave surfaces are spheres, with the difference that they are eccentric spheres. (It is precisely because the wave surfaces are spheres that Einstein could assume his equivalence principle in that case.)

Results are obtained in the cases in which the field has equipotential surfaces that are planar and parallel, and in particular, in the three special cases that are of interest to the theory of gravitation from a purely theoretical standpoint, one is given that in the field considered the speed of light must annulled on a certain plane. That cannot be neglected in ordinary optics, in which one considers optical media in which the index of refraction varies with the distance from a given plane.

I must warmly thank prof. Abraham, who suggested this article to me and whose advice has been of valuable assistance to me.

§ 1. – General equations of the trajectories of light rays.

The speed (c) of a light ray at any point in the field in the field considered is a function of only position [$c = c\,(x, y, z)$]. When one assumes *Fermat's principle*, the determination of the trajectory (s) will reduce to a problem of finding a minimum, and more precisely, the ray that passes through one point and another will follow the path that takes the least time. Therefore, the curve travelled must be such that the time required:

$$t = \int \frac{ds}{c}$$

must be a minimum. The calculus of variations then teaches that one must then annul the first variation of that integral:

$$\delta t = \delta \int \frac{ds}{c} = 0,$$

which implies the three equations:

$$\text{(I)} \qquad \frac{d}{ds}\left(\frac{1}{c}\frac{dx}{ds}\right) = \frac{\partial}{\partial x}\frac{1}{c}, \ldots$$

Those are the *general equations* of the trajectories of light rays. It is clear that they are just the equations of the *brachistochrone* for an arbitrary point that moves with velocity c.

In order to integrate (I) when one knows the form of the function $c\,(x, y, z)$, it is enough to find the integral for two of those equations, since the remaining one will then be satisfied.

Indeed, (I) are not independent, but when one is given two of them, the third one will be a consequence of the latter. That is easily verified by multiplying (I) by dx / ds, dy / ds, dz / ds and summing, which will give an identity.

([1]) A. Garbasso, Archiv für Optik I **5** (1908), pp. 251.

§ 2. – General equations of motion for a material point in a gravitational field.

As was said in the introduction, starting from the hypothesis that the speed of light depends upon the gravitational potential, M. Abraham developed a new theory of gravitation, on the basis of which, the theorem of impulse and energy for a material point that is subject to only the force of gravity would be expressed by the equations ([1]):

$$\text{(II)} \qquad \begin{aligned} \frac{d\mathbf{G}}{dt} &= -\frac{E}{c}\,\mathrm{grad}\,c, \\ \frac{dE}{dt} &= \frac{E}{c}\frac{\partial c}{\partial t} \end{aligned}$$

(**G** = impulse vector, E = energy of the point).

If one now introduces the Lagrange function (L), by means of which and the velocity of the point (v) one can express the values of the impulse and energy as:

$$\text{(1)} \qquad G = \frac{\partial L}{\partial v}, \quad E = v\frac{\partial L}{\partial v} - L,$$

then it will follow that L must be a homogeneous function of degree one in v and c ([2]). In particular, in agreement with the theory of relativity, L will assume the form ([3]):

$$\text{(2)} \qquad L = M\,c\,k\,,$$

in which M is the mass of the point, and:

$$k = \sqrt{1 - \frac{v^2}{c^2}}\,.$$

If one substitutes (2) in (1) then one will have:

$$\text{(1.}a\text{)} \qquad G = \frac{Mv}{ck}, \quad E = \frac{Mc}{k}$$

for the impulse and energy, by means of which, from the first of (II), one will have:

$$\text{(II.}a\text{)} \qquad \frac{d}{dt}\left(\frac{\mathbf{v}}{ck}\right) = -\frac{1}{k}\,\mathrm{grad}\,c,$$

([1]) M. Abraham, Communication to the Intern. Congr. of Math. at Cambridge (22-28 August 1912). Ann. di Mat. pura ed appl., Lagrange Centenary volume (3) tomo XX, pp. 29.

([2]) M. Abraham, Ann. di Mat. pura ed appl., *loc. cit.*

([3]) M. Abraham, Ann. di Mat. pura ed appl., *loc. cit.*, Nuovo Cim. **4** (1912), pp. 459.

which is an equation that coincides with the equation of motion for a point that Einstein gave (second form) ([1]). Since later on we will always confine ourselves to a static gravitational field, for which $\partial c / \partial t = 0$, from the second of (II), we will have the energy theorem:

$$E = \text{const.},$$

and consequently, from the second of (1.*a*):

$$\frac{c}{k} = \text{const.}, \tag{3}$$

which once more expresses the theorem of the conservation of energy in the form that Abraham found in the first article in Nuovo Cim. ([2]).

If one eliminates k from (II.*a*) by means of (3) then one will have:

$$\frac{d}{dt}\left(\frac{\mathbf{v}}{c^2}\right) = -\frac{1}{c}\,\text{grad}\,c. \tag{II.b}$$

In that form, the equations of motion of the material point will coincide with those of Einstein in the first form ([3]).

In terms of components, (II.*b*) can be written:

$$\frac{d}{dt}\left(\frac{v_x}{c}\right) = \frac{d}{ds}\left(\frac{v}{c^2}\frac{dx}{ds}\right)v = -\frac{1}{c}\frac{\partial c}{\partial x}, \ldots,$$

or also:

$$\frac{d}{ds}\left(\frac{v}{c^2}\frac{dx}{ds}\right)v = \frac{c}{v}\frac{\partial \frac{1}{c}}{\partial x}, \ldots, \tag{II.c}$$

We then have the equations of the trajectory of the material point whose form is closely analogous to that of equations (I) for the light ray. We can also prove that *the equations of the trajectory of the material point* (II.*c*) *will become coincident with those of the light ray* (I) *in the limiting case in which the velocity of a given point of the trajectory and that of light are equal at that point.* To that end, we can first prove that if the velocity (v_0) of the material point coincides with that of light (c_0) at a given point P_0 then we will have $v = c$ at any other point P of the trajectory of the material point.

Indeed, it follows from (3) that:

$$\frac{k}{c} = \frac{k_0}{c_0},$$

or

([1]) A. Einstein, Ann. Phys. (Leipzig) **38** (1912), pp. 355.
([2]) Max Abraham, Nuovo Cim. **3** (1912), pp. 211.
([3]) A. Einstein, Ann. Phys. (Leipzig) **38** (1912), pp. 355.

$$1-\frac{v^2}{c^2}=\frac{c^2}{c_0^2}\left(1-\frac{v^2}{c^2}\right),$$

from which:

$$v^2=c^2\left(1-\frac{c^2}{c_0^2}+c^2\frac{v_0^2}{c_0^4}\right), \tag{4}$$

which shows precisely that if $v_0 = c_0$ at a given point P_0 then one will have $v = c$ at all other points of the trajectory of the material point. Now suppose that the velocity of the material point coincides with that of light at a point. At any point of the curve that is represented by (II.*c*), one can set $v = c$, and (II.*c*) will then coincide exactly with (I), which therefore proves the theorem.

Now consider a light ray that emanates from P_0 at a given direction with the velocity c_0 that belongs to that point and a material point that starts from P_0 in the same direction with a velocity $v_0 < c_0$, so it will then describe a trajectory that is distinct from the light ray. Suppose that v_0 now increases while preserving its initial direction, which is equal to that of the ray. The corresponding trajectory will also change, and in the limit of $v_0 = c_0$, it will coincide with the light ray. Therefore: *The trajectories of a material point that originate from that point in a given direction will constitute a manifold of curves that has the light ray that originates from the same point in the same direction as a limiting curve.*

§ 3. – Gravitational field whose equipotential surfaces are parallel planes.

Suppose that the gravitational field has equipotential surfaces that are planes parallel to the *xy*-plane. In that case, the gravitational potentials, as well as *c*, depend upon only *z* :

$$c = c\,(z)\,. \tag{5}$$

Assume that the positive sense of *z* is the one for which $dc / dz > 0$, so that *c* will increase with *z*.

Let us then see what we will have for the trajectories of the light ray and the material point.

a) *Trajectories of the light ray.*

Since, by the hypothesis (5), one will have $\partial c / \partial x = \partial c / \partial y = 0$, and the first two of (I) will give:

$$\frac{dx}{ds}=\pm Ac,\quad \frac{dy}{ds}=\pm Bc \quad (A \text{ and } C \text{ arbitrary positive constants}),$$

from which, it will result that the light rays lie in a vertical plane (that is normal to the *xy*-plane).

Set $B = 0$ and study the trajectories in the vertical *xz*-plane. If one sets $A = 1 / c_0$ then (I) will reduce to just the two equations:

(I.*a*)

$$\frac{dx}{ds} = \pm\frac{v}{c_0},$$

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dz}{ds}\right) = \frac{d}{dz}\left(\frac{1}{c}\right),$$

and once more, we have that one is a consequence of the other. As for the significance of the constant c, it seems obvious from the first of (I.*a*) that since the direction cosine of the trajectory $dx / ds = \pm 1$ for $c = c_0$, c_0 will be the value that velocity assumes at the points where the ray is horizontal (i.e., parallel to the x-axis) and that that value will be the maximum over all points of the ray considered. From the first of (I.*a*), we can also derive the expressions for the horizontal and vertical components, (c_x) and (c_z), resp., of the velocity of light:

(6)

$$c_x = \pm\frac{c^2}{c_0},$$

(7)

$$c_z = \pm c\sqrt{1-\frac{c^2}{c_0^2}}.$$

b) *Trajectories of the material point.*

Under the hypothesis (5), (II.*c*) will also reduce to just two equations, and the curves that they represent will only lie in vertical planes. If we confine ourselves to the xz-plane then we can deduce those equations more simply by starting from (II.*b*):

(II.*d*)

$$\frac{d}{dt}\left(\frac{v_x}{c^2}\right) = 0,$$

$$\frac{d}{dt}\left(\frac{v_z}{c^2}\right) = \frac{1}{c}\frac{dc}{dz}.$$

Let the index 0 denote the entities that are referred to the points at which the trajectory is horizontal (so one will then have $v_{0x} = v_0$). From the first of (II.*d*), one will get:

$$\frac{v_x}{c^2} = \pm\frac{v_0}{c_0^2},$$

so

(6.*a*)

$$v_x = \pm v_0\frac{c^2}{c_0^2},$$

and therefore, since $v_z^2 = v^2 - v_x^2$, from (4), one will have:

(7.*a*)
$$v_z = \pm c\sqrt{1-\frac{c^2}{c_0^2}}\,.$$

c) *Comparing the trajectories of the material point and light rays.*

In *a*) we let c_0 denote the maximum value of the speed of light along a ray, which is a value that it is assumed wherever it is horizontal, and we let v_0 denote just the value of the speed of the material point at the points where the motion is horizontal. We must note that v_0 is not the maximum value of v along the trajectory considered, because in fact, from (4) it will have its maximum value v_m for:

$$\frac{d}{dc^2}\left(c^2-\frac{c^4}{c_0^2}+c^4\frac{v_0^2}{c_0^4}\right)=1-2c^2\left(\frac{1}{c_0^2}-\frac{v_0^2}{c_0^4}\right)=0,$$

i.e., for:

$$c=\frac{c_0^2}{\sqrt{2(c_0^2-v_0^2)}},$$

which differs from c_0, and for which we will have:

$$v_m=\frac{c_0^2}{\sqrt{2(c_0^2-v_0^2)}}\,.$$

In the case of free fall, $v_0 = 0$, and one will have $v_m = c_0 / 2$, which is what Abraham found before ([1]).

When one compares (6) with (6.*a*) and chooses the sense of propagation of light and the motion of the point conveniently, it will result that:

(6.*b*)
$$v_x=\frac{v_0}{c_0}c_x\,,$$

so v_x will coincide with c_x only for $v_0 = c_0$, while it will result from (7) and (7.*a*) that one always has:

(7.*b*)
$$v_z = c_z\,.$$

Therefore:

([1]) M. Abraham, Ann. di Mat. pura ed appl., *loc. cit.*

The vertical components of the velocity of a material point and a light ray that are initially both horizontal at the same point P_0 will always be equal.

That means that v_z is independent of the initial velocity v_0 and that:

At any later instant, the light ray and the material point that start from the same instant horizontally at the point P_0 will reach points with equal ordinates – i.e., the same horizontal.

The abscissas of those points (x, z for the light ray, ξ, ζ for the material point) are, if one supposes that P_0 is on the z-axis:

$$x = \int_0^t c_x \, dt, \quad \xi = \int_0^t v_x \, dt.$$

From (6.*b*), the second integral will become:

$$\xi = \frac{v_0}{c_0} \int_0^t c_x \, dt = \frac{v_0}{c_0} x,$$

so

(8) $$\frac{x}{\xi} = \frac{c_0}{v_0}, \quad \text{with} \quad z = \zeta,$$

i.e., *the abscissas of the points of the trajectories of the light ray and the material point that originate horizontally at the same instant from the same point on the z-axis will both have the same ratio as the initial velocities.*

(We shall establish, once and for all, the convention that we shall use Latin or Greek symbols according to whether we are referring to the light ray or the trajectory of the material point, resp., while still using c and v for the respective velocities.)

Based upon (8), the trajectory s of the light ray and all of the possible trajectories σ of the material points that are horizontal at P_0 will constitute a congruence of *affine* curves.

As for the radius of curvature r of a light ray, one will have the simple expression:

(9) $$r = \frac{c_0}{\left| \dfrac{dc}{dz} \right|}.$$

In fact, from (6) and (7), we will have:

(10) $$\frac{dx}{dz} = \frac{c_x}{c_z} = \pm \frac{c}{\sqrt{c_0^2 - c^2}},$$

so

$$\frac{d^2x}{dz^2} = \pm \frac{c_0^2}{(c_0^2 - c^2)^{3/2}} \frac{dc}{dz} .$$

If one substitutes that in the expression for the radius of curvature:

$$r = \left| \frac{\left[1 + \left(\frac{dx}{dz} \right)^2 \right]}{\frac{d^2x}{dz^2}} \right|^{3/2}$$

then one will get (9) precisely.

Analogously, (6.*a*) and (7.*a*) will give:

(10.*a*) $$\frac{d\xi}{d\zeta} = \pm \frac{v_0}{c_0} \frac{c}{\sqrt{c_0^2 - c^2}} ,$$

and therefore:

$$\frac{d^2\xi}{d\zeta^2} = \pm \frac{v_0}{c_0} \frac{c}{\sqrt{c_0^2 - c^2}} \frac{dc}{d\zeta},$$

from which the radius of curvature of the trajectory *s* at the point *Q* (ξ, ζ) will become:

$$\rho = \frac{c_0}{\left| \frac{dc}{d\zeta} \right|} \frac{c_0}{v_0} \left(1 - \frac{c^2}{c_0^2} + c^2 \frac{v_0^2}{c_0^4} \right)^{3/2} ,$$

and from (4), one will have:

(9.*a*) $$\rho = \frac{c_0}{v_0} \frac{v^3}{c^3} r$$

at the points where $\zeta = z$.

In particular, one will have:

(9.*b*) $$\rho_0 = \frac{v_0^2}{c_0^2} r_0 \leq r_0$$

at the point P_0 .

One sees (as would only be natural) that (9) and (9.*a*) will coincide for $v_0 = c_0$, and therefore for $v = c$; i.e., the light rays represent the limiting trajectory that is described by the material point.

Now suppose that:

(11) $$c = a\, z^{\lambda},$$

in particular, in which a and λ are constants that satisfy the condition that $dc / dz > 0$ uniquely, which is equivalent to assuming that the positive sense of z is the one in which c increases.

Therefore:

$$\frac{dx}{dz} = \pm \frac{z^{\lambda}}{\sqrt{z_0^{2\lambda} - z^{2\lambda}}}$$

in that case, which is zero only along the x-axis, and it results from this that if one takes the point P_0 to be on the positive z-axis and supposes that light propagates from P_0 then x, and therefore c_x, will be always increasing or always decreasing.

In what follows, we will confine ourselves to the positive quadrant of x and z and let $c_0 = a\, z_0^{\lambda}$ be the maximum value of c, from which it will result that z_0 is just the maximum value that z assumes along the trajectory – i.e., P_0 is the vertex of the curve s. Therefore, x will increase as z decreases in the quadrant considered. It will then follow that $dx / dz < 0$, so:

(12) $$\frac{dx}{dz} = -\frac{z^{\lambda}}{\sqrt{z_0^{2\lambda} - z^{2\lambda}}} \ .$$

Assuming that the tangent to the ray has the positive sense that corresponds to the sense of propagation of light, the angle φ that the tangent makes with the x-axis will vary between 0 at P_0, where one has $dz / dx = 0$, and $\pi / 2$ along the x-axis, where $dz / dx = \infty$. Therefore, *the light ray will meet* both the axes *normally*, and the angle φ between the tangent and the horizontal axis will be given by:

(12.*a*) $$\tan\varphi = \frac{\sqrt{z_0^{2\lambda} - z^{2\lambda}}}{z^{\lambda}}, \quad 0 \le \varphi \le \frac{\pi}{2} \ .$$

Analogously, one will have:

(12.*b*) $$\frac{d\xi}{d\zeta} = -\frac{v_0}{c_0}\frac{\zeta^{\lambda}}{\sqrt{z_0^{2\lambda} - \zeta^{2\lambda}}} = -\cot\psi$$

for the trajectory of the material point that originates horizontally at P_0, in which ψ is the angle between the tangent to the curve σ at the point Q (ξ, ζ) and the horizontal.

It will follow that:

$$\tan\psi = \frac{c_0}{v_0}\tan\varphi$$

for points of equal height, in which ψ varies between only 0 and $\pi / 2$, and only the curve σ will meet the axes orthogonally.

§ 4. – Correspondence between the trajectories of the material point and those of the luminous point.

Once more under the hypothesis (11), consider two distinct light rays s and s' whose vertices are at P_0 and P_0' along the z-axis. Let P (x, z), P' (x', z') be two points on s and s' that have parallel tangents (Fig. 1).

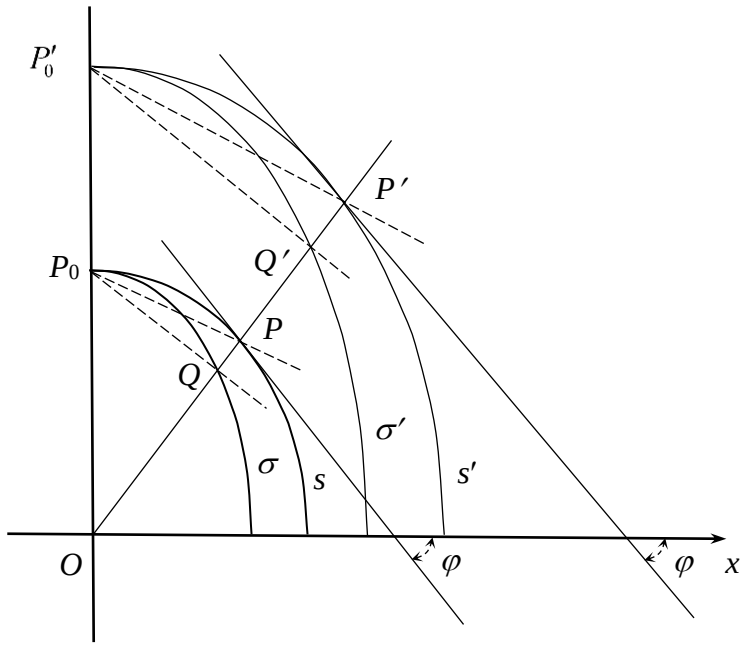

Figure 1.

It will then follow from (12.a) that:

$$\frac{z_0^{2\lambda} - z^{2\lambda}}{z^{2\lambda}} = \frac{z_0'^{2\lambda} - z'^{2\lambda}}{z'^{2\lambda}},$$

so

$$\frac{z}{z'} = \frac{z_0}{z_0'} \,. \tag{13}$$

From (12), one will then have:

$$x = -\int_{z_0}^{z} \frac{dz}{\sqrt{\frac{c_0^2}{c^2} - 1}}, \quad x' = -\int_{z_0'}^{z'} \frac{dz'}{\sqrt{\frac{c_0'^2}{c'^2} - 1}}$$

for the abscissas of P and P'.

Replace the variable z' in the second integral with z, so by (12), one will have $dz' = (z_0' / z_0)$ dz, and therefore:

$$x' = -\frac{z_0'}{z_0}\int_{z_0}^{z}\frac{dz}{\sqrt{\frac{c_0^2}{c^2}-1}} = -\frac{z_0'}{z_0}x,$$

so

$$\frac{x}{x'} = \frac{z_0}{z_0'},$$

or, when one also takes (13) into account:

(13.*a*) $$\frac{x}{x'} = \frac{z}{z'} = \frac{z_0}{z_0'} = \text{const.}$$

That relation is valid for the coordinates of any pair of points P and P' on s and s' that have parallel tangents. It will then follow that P and P' are aligned with the origin O and therefore that the ratio of their distances OP, OP' from O is just a constant that is equal to the ratios in (13.*a*).

Therefore:

The two curves s and s′ are similar and have the origin for their center of similitude.

On the basis of that, when one is given s, it is easy to construct the points of the ray s' that has its vertex at an arbitrary point (P_0') of the z-axis. Indeed, if P is a generic point of the s then the corresponding point P' of s' will be the intersection of the ray OP with the parallel through P_0' to the line P_0P (Fig. 1), as one will easily see from the similar triangles P_0PO, $P_0'P'O$.

Under the hypothesis (11), *the trajectories s and s′ of two material points will also have their vertices at* P_0 *and* P_0' *with initial velocities* v_0 *and* v_0' *such that:*

(14) $$\frac{v_0}{v_0'} = \frac{c_0}{c_0'}$$

are similar.

Indeed, take the point $Q'(\sigma', \zeta')$ on σ' to be the one that corresponds to the point $Q(\sigma, \zeta)$ on σ, in such a way that the tangents to those points are parallel.

From (10.*a*), one will then have:

$$\frac{c_0}{v_0}\frac{\sqrt{c_0^2-c^2}}{c^2} = \frac{c_0'}{v_0'}\frac{\sqrt{c_0'^2-c^2}}{c'^2},$$

so from (14):

$$\frac{c_0^2-c^2}{c^2} = \frac{c_0'^2-c^2}{c'^2},$$

or:

$$\frac{c}{c'} = \frac{c_0}{c'_0},$$

at the points Q and Q', and from (11):

(13.*b*) $$\frac{\zeta}{\zeta'} = \frac{z_0}{z'_0}.$$

As for the abscissas of the points Q and Q, from (12.*b*), one will have:

$$\xi = -\frac{v_0}{c_0}\int_{z_0}^{\zeta}\frac{d\zeta}{\sqrt{\frac{c_0^2}{c^2}-1}}, \quad \xi' = -\frac{v'_0}{c'_0}\int_{z'_0}^{\zeta'}\frac{d\zeta'}{\sqrt{\frac{c_0'^2}{c'^2}-1}},$$

so, analogous to what was done in the preceding case for s and s', from (14) and (13.*b*), one will have:

(13.*c*) $$\frac{\xi}{\xi'} = \frac{\zeta}{\zeta'} = \frac{z_0}{z'_0} = \text{const.}$$

That shows precisely that σ and σ' are similar with their centers of similitude at the origin, and that they will have the same ratio of similitude that was found for s and s'.

With a construction that is analogous to the preceding one, given σ, one can construct the material point that has the initial velocity v'_0 and satisfies (14) for the points of the trajectory σ' with its vertex at P'_0.

Therefore, suppose that s is given. On the basis of (8), one can then construct the congruence of curves σ with their vertex at P_0 for all values of $v_0 < c_0$. One can then construct the curve σ' of a material point that is launched horizontally from P'_0 with a given velocity $v'_0 \leq c'_0$. Conversely, if a material point starts horizontally from a point on the z-axis and passes through Q' then one can always determine that point – i.e., the vertex of the trajectory.

One now poses the more general problem: *Pass a material point with a velocity v' that makes an angle of φ with the x-axis through an arbitrary point Q. Construct the trajectory of the point.*

Let v'_0, c'_0 denote the initial velocities of the point and the light at the vertex of the trajectory. From (6.*a*) and (7.*a*), at the point Q they will become:

(15) $$v'_x = v'_0\frac{c'^2}{c_0'^2}, \quad v'_z = -c'\sqrt{1-\frac{c'^2}{c_0'^2}},$$

in which c' is the velocity light at Q, which is known as a function of the ordinate of Q. One can then get v'_0 and c'_0 as functions of $v'_x = v' \cos\varphi$, $v'_z = v' \sin\varphi$, and c' :

$$v_0' = \frac{v_x'^2 \, c'^2}{\sqrt{c'^2 - v_z'^2}} \, ,$$

$$c_0' = \frac{c'^2}{\sqrt{c'^2 - v_z'^2}} \, .$$

When one calculates v_0' and c_0' in that way by means of the givens in the problem, one will know how to construct the corresponding trajectory σ' under the hypothesis that its vertex is on the z-axis. Having done that, one takes the point Q' of σ' to have the same ordinate as Q and performs a translation of the curve σ' that is parallel to the x-axis and equal to $Q'Q$. In that way, the curve σ' will pass through Q and represent the desired trajectory (Fig. 2).

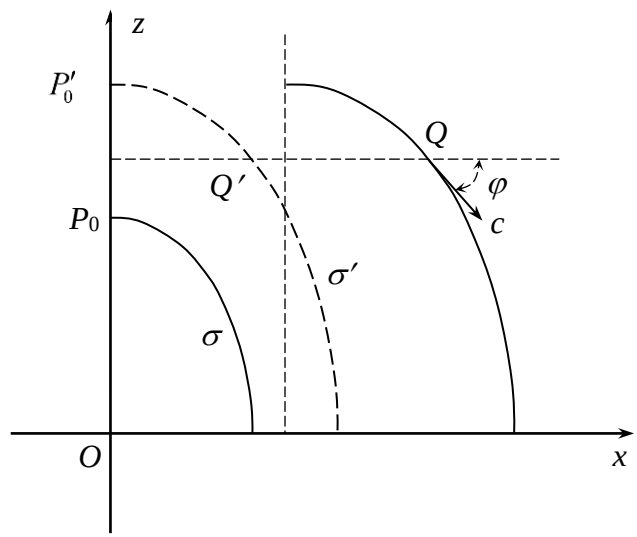

Figure 2.

§ 5. – Determining the trajectories in particular cases.

The relation (8) establishes an affinity between the trajectories σ of material points and the trajectory s of the light ray, on the basis of which, given s, one can construct all of the σ that have the same vertex as s that will be valid in general no matter what the function of z that c might be. However, the relations of similitude between s and σ when they have distinct vertices that were found in the preceding § were valid under the hypothesis (11), namely, that $c = a\, z^{\lambda}$. In order to effectively determine s and σ, it will then be necessary to fix the form of c, and therefore to give definite values to a and λ .

For the sake of simplicity, we begin by choosing the units of measurement in such a way that $a = 1$.

We saw in the introduction how the cases in which c is proportional to $\sqrt{z}$, z, and z^2 present a special interest, or the cases in which $\lambda = 1/2$, $\lambda = 1$, $\lambda = 2$, respectively. For our considerations,

we shall confine ourselves to the positive quadrant of the *xz*-plane (in case 1, it will be indispensable that we have $z > 0$ in order for the curves to be real), so that we will always satisfy the relation:

$$\frac{dc}{dz} > 0.$$

We shall now study the three cases individually.

Case 1:

$$\lambda = \tfrac{1}{2}$$

or

(16) $$c = \sqrt{z}\ .$$

a) *Light rays.*

In order to determine the trajectory *s* of the light ray, it is enough to integrate (I.*a*). However, it is simpler to start from (12), and when one takes (12.*a*) and (16) into account, it can be written:

(17) $$-\frac{dz}{dx} = \sqrt{\frac{z_0}{z} - 1} = \tan\varphi.$$

From (17), one will have:

(18) $$z = z_0 \cos^2 \varphi\,,$$

so:

$$\frac{dz}{dx} = -\,2\, z_0 \sin\varphi \cos\varphi\,,$$

and therefore, also from (17):

$$\frac{dx}{d\varphi} = \frac{dx}{dz}\frac{dz}{d\varphi} = 2\, z_0 \cos^2\varphi\,,$$

from which, integration will give:

(18.*a*) $$x - x_0 = z_0\left(\varphi + \frac{\sin^2\varphi}{2}\right),$$

in which x_0 is the abscissa of the point for which $\varphi = 0$, so its ordinate will then be $z = z_0$, from (18.*b*). That is, one addresses the vertex P_0 of the curve when one assumes, as usual, that it is on the *z*-axis, and one can then write (18.*a*) and (18) as:

$$\begin{cases} x = \dfrac{z_0}{2}(2\varphi + \sin 2\varphi), \\ z = \dfrac{z_0}{2}(1 + \cos 2\varphi). \end{cases} \tag{19}$$

Those equations define a *cycloid* that is generated by a circle of radius $z_0 / 2$ that rolls along the x-axis.

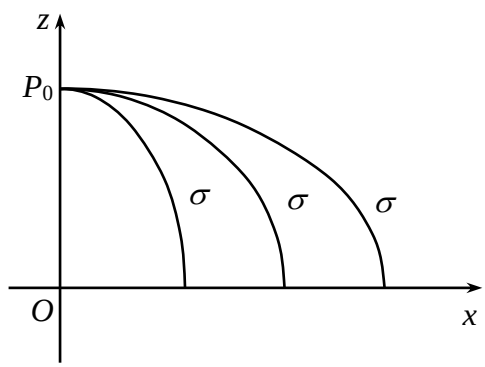

Figure 3.

Furthermore, it already results from the expression for the radius of curvature in (9), and with (16), that:

$$r = 2\sqrt{z_0\, z}\,,$$

so the curve is a cycloid (Fig. 3).

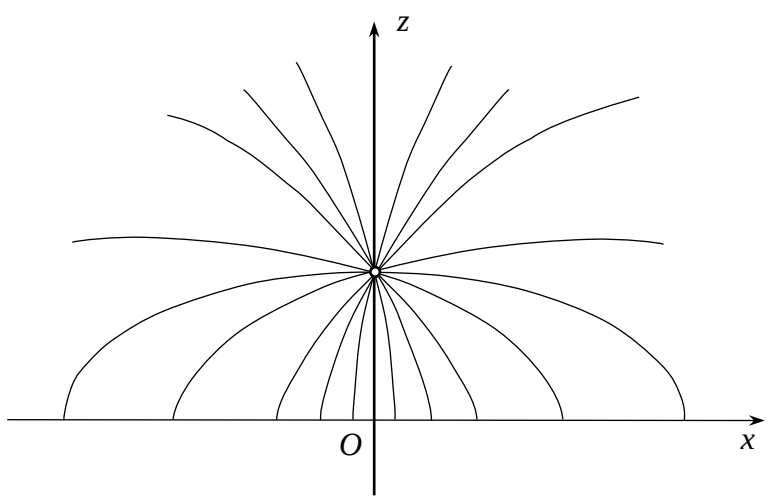

Figure 4.

Therefore, for $c = \sqrt{z}$, *light rays will describe cycloids* that have the horizontal axis for their bases. That was exhibited before by Abraham ([1]), who found, as Giovanni Bernoulli had found

([1]) M. Abraham, Nuovo Cim. **3** (1912), pp. 211.

already ([1]) regard to that, that from Fermat's principle, light rays that propagate in an inhomogeneous medium with a velocity that is proportional to $\sqrt{z}$ will describe cycloids.

Light rays that radiate from a point and all start with the same velocity will generate a congruence of cycloids that all intersect the horizontal axis normally (Fig. 4).

Differentiating the first of (19) with respect to time will give:

$$\dot{x} = c_x = z_0 (1 + \cos 2\varphi)\ \dot{\varphi}\,,$$

and from the second one, along with (6):

$$2\,z\,\dot{\varphi} = \frac{z}{\sqrt{z_0}}\,,$$

so:

$$\dot{\varphi} = \frac{1}{2\sqrt{z_0}} = \frac{1}{2c_0} = \text{const.}\ ;$$

i.e.: *the angular velocity with which the tangent rotates* (and therefore the generating circle) *is constant along the ray.*

When φ is expressed as a function of time and the time is measured from the instant at which the ray passes through the vertex:

$$\varphi = \frac{1}{2c_0}\,t\,.$$

The time that is required for the ray to reach the x-axis will then be:

$$T = \pi\, c_0\,.$$

The distance that is travelled will then be $4 \cdot z_0 / 2 = 2c_0^2$ (viz., the length of a quadrant of the cycloid), so the mean velocity $\sqrt{c}$ will be given by:

$$\overline{c} = \frac{2c_0^2}{T} = \frac{2c_0}{\pi}\,.$$

b) *Trajectories of the material point.*

As for the trajectory σ of a material point that starts horizontally from P_0 with a velocity v_0, one knows that the points of s and σ that have the same ordinate have abscissas that have the ratio c_0 / v_0. One can then write the parametric equations of σ by starting from (19) and multiplying the first one by v_0 / c_0, while leaving the second one unaltered:

([1]) Giov. Bernoulli, *Acta Eruditorum*, Leipzig, May, 1697, pp. 211.

(19.*a*)
$$\xi = \frac{v_0\sqrt{z_0}}{2}(2\varphi + \sin 2\varphi),$$
$$\zeta = \frac{z_0}{2}(1 + \cos 2\varphi).$$

As we have established already, the ξ and ζ denote the coordinates of a generic point of σ, and φ always denotes the angle between the tangent and the curve s at the point whose ordinate is ζ. The curve that is defined by (19.*a*) is affine to the ordinary cycloid and is a *Fermat cycloid* precisely ([1]). (See Fig. 3)

Naturally, the time required is always the same regardless of v_0 and is once more equal to $T = \pi c_0$.

Case 2:

$$\lambda = 1$$

or

(20) $$c = z\ .$$

a) *Light rays.*

In order to determine s, one must recall the expression for the radius of curvature in that case, which is, from (9):

$$r_0 = c_0 = z_0 = \text{const.}$$

Therefore: *The light rays traverse circles whose centers are on the x-axis* ([2]) (Fig. 5), and if one supposes that the vertex of s is on z then its equation will be:

(21) $$x^2 + z^2 = z_0^2\ .$$

When one introduces the angle φ between the tangent and the x-axis, one will have:

$$x = z_0 \sin\varphi, \quad z = z_0 \cos\varphi\ .$$

Differentiating the first one with respect to time will give:

$$\dot{x} = c_x = z_0 \cos\varphi \cdot \dot{\varphi} = z\,\dot{\varphi},$$

([1]) P. Fermat. *Oeuvres*, vol. II, letter to Cercavi, June 1660. Fermat studied that curve independently of any optical questions, and observed that its rectification would depend upon that of the circle of the parabola according to whether $\xi / x < 1$ or $\xi / x > 1$, resp.

([2]) Garbasso, *loc. cit.*

and from (6.*a*), $c_x = z^2 / z_0$, so:

$$\dot{\varphi} = \frac{z}{z_0} = \cos\varphi.$$

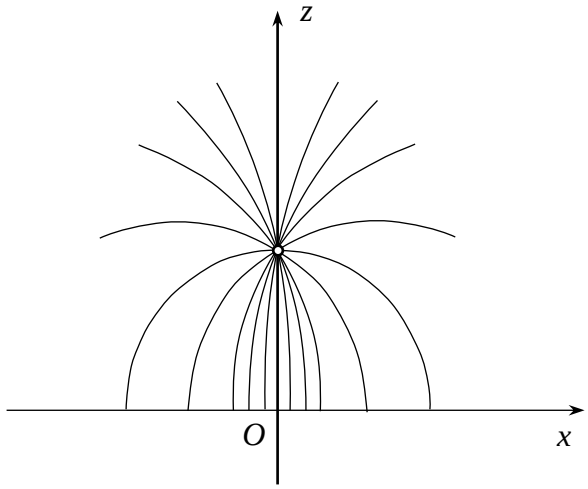

Figure 5.

Therefore, $dt = d\varphi / \cos\varphi$, and when one integrates this, one will get:

$$t = \log\sqrt{\frac{1+\sin\varphi}{1-\sin\varphi}},$$

in which one sets $t = 0$ for $\varphi = 0$.

It will then follow that the time T required by the ray to reach the x-axis when it starts from the vertex will become infinite like $\log\infty$.

b) *Trajectories of the material point.*

By virtue of (8), those trajectories are obtained from (21) by setting $x = (c_0 / v_0)\ \xi_0$ and $z = \zeta$:

$$\frac{c_0^2}{v_0^2}\xi^2 + \zeta^2 = z_0^2,$$

from which, dividing by $c_0^2 = z_0^2$ will give:

(21.*a*)
$$\frac{\xi^2}{v_0^2} + \frac{\zeta^2}{c_0^2} = 1,$$

which is the equation of an ellipse whose axes coincide with the coordinate axes.

Hence, in this case, *the material point will describe ellipses that are all inside the circles* (21). (See Fig. 6.)

The time it takes to fall, and for free fall ($v_0 = 0$), in particular, is only logarithmically infinite, as it is for the light ray.

Case 3:

$$\lambda = 2,$$

or

(22) $$c = z^2 .$$

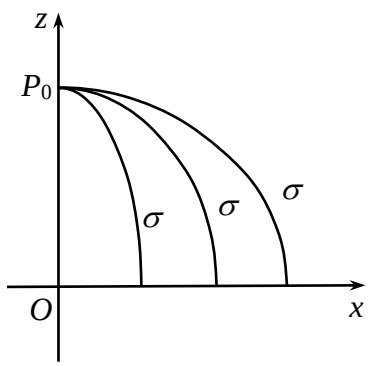

Figure 6.

a) *Light rays:*

From (12), one has:

(23) $$\frac{dz}{dx} = -\frac{z^2}{\sqrt{z_0^4 - z^4}},$$

from which one can get x as a function of elliptic integrals in z .

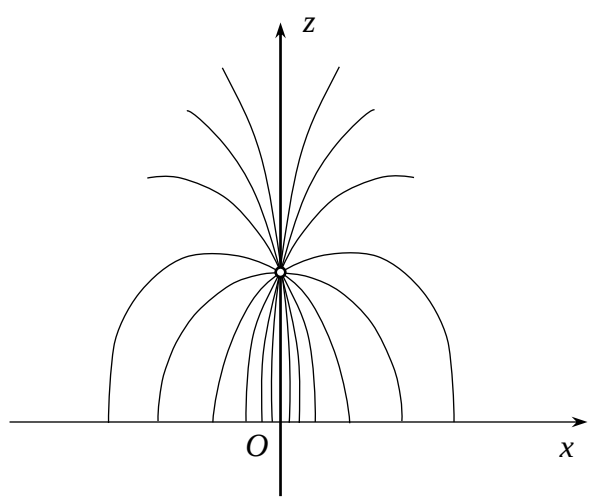

Figure 7.

However, it is not necessary to integrate (23) in order to determine the curve that is described by the light ray because it is known that (23) represents an *elastic curve* that intersects normally the coordinate axes with respect to which it is symmetric.

The curves s and s' in Fig. 1 are elastic curves. We have a congruence of such curves in Fig. 7.

The elastic curve represents the form that a thin elastic band will assume when it is fixed at one extreme (in Fig. 1, the vertex) and the other extreme is subject to a force that is normal to the plane of the curve ([1]). That curve represents just the normal section of a cloth that is supported by two parallel axes and inflated by the wind (i.e., air), or if the supports are horizontal, filled with a ponderous liquid: That is why one also calls it a *linteare curve* (*curva linteare*) ([2]).

Finally, Euler used the calculus of variations to determine that one among the plane curves that have one extreme on x and the other on z and have equal length and enclose an equal are with the axes whose area will generate the maximum or minimum volume when it is rotated around the x-axis, and he arrived at the elastica precisely ([3]).

From (12.*a*), we have:

$$\tan\varphi = \sqrt{\frac{z_0^4}{z^4} - 1}\,,$$

which gives:

$$\cos\varphi = \frac{z^2}{z_0^2}, \quad \sin\varphi = \sqrt{1 - \frac{z_0^4}{z^4}}\,.$$

Differentiating the first of these with respect to time will give:

$$-\sin\varphi \cdot \dot{\varphi} = 2\frac{z}{z_0^2}\dot{z}\,,$$

and from (7), one has $\dot{z} = -\,z^2\sqrt{1 - \frac{z^4}{z_0^4}}$, so one will have:

$$\sin\varphi \cdot \dot{\varphi} = 2\frac{z^3}{z_0^2}\sqrt{1 - \frac{z^4}{z_0^2}} = 2\frac{z^3}{z_0^2}\sin\varphi\,,$$

from which:

$$\dot{\varphi} = 2\frac{z^3}{z_0^2}\,.$$

([1]) Giac. Bernoulli, *Opera* I, page 122, III, pp. 512. Poisson, *Traité de Méc.* (1833) I, pp. 600.
([2]) Giov. Bernoulli, *Opera* I, pp. 451, 576, and 639.
([3]) L. Euler, *Methodus inveniendi lineas*…, Lausanne and Geneva (1977), Chap. V, ex. I.

Therefore, in the third case, the angular velocity with which the tangent to ray rotates is proportional to z^3. Since $z = z_0\sqrt{\cos\varphi}$, one can then write:

$$\dot{\varphi} = 2\,z_0\sqrt{\cos^3\varphi}\,,$$

and when that is integrated, it will give:

$$t = \frac{1}{2\,z_0}\int_0^{\varphi}\frac{d\varphi}{\sqrt{\cos^3\varphi}} = \frac{1}{z_0}\int_u^1\frac{du}{u^2\sqrt{1-u^4}}\ ,$$

in which we have set $\sqrt{\cos\varphi} = z / z_0 = u$. Thus, for $0 < z \le z_0$ or for $0 < u \le 1$, the following development of the integrand will be valid:

$$\frac{1}{u^2}\frac{1}{\sqrt{1-u^4}} = \frac{1}{u^2} + \frac{1}{2}u^2 + \frac{1\cdot 3}{2\cdot 4}u^6 + \cdots,$$

and integrating this series will give:

$$t = \frac{1}{z_0}\left[-\frac{1}{u} + \frac{1}{2}\frac{u^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{u^7}{7} + \cdots\right]_u^1\ .$$

From that, in the limit of $u = 0$, or for $z = 0$, one will have that time required by the ray to reach the x-axis will become infinite:

$$T = \frac{1}{z_0}\lim_{u\to 0}\frac{1}{u} = \lim_{z\to 0}\frac{1}{z}\ .$$

b) *Trajectories of the material point.*

The trajectory of the material point is a *curve that is affine to the elastica*, and from (8) and (23), its equation will be:

(23.*a*)
$$\frac{d\xi}{d\zeta} = -\frac{v_0}{c_0}\frac{\zeta^2}{\sqrt{z_0^4 - \zeta^4}}\ .$$

As for the construction of the curves σ (Fig. 8), once one has constructed s and knows how to proceed, for points of s and σ that lie on the same horizontal, one will have:

$$\frac{x}{\xi} = \frac{c_0}{v_0} \ .$$

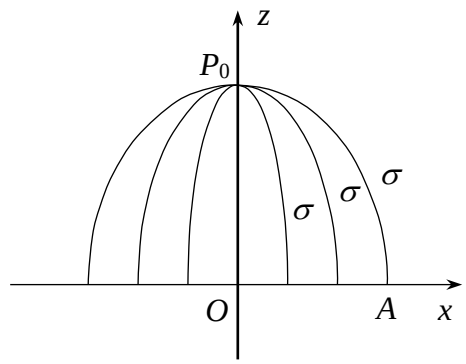

Figure 8.

As was essentially explained before, we shall now briefly discuss the elastic curves that are presented in the figure.

c) *Construction of the elastic curve.*

From (23), we will have (for points in the positive quadrant of the *xy*-plane):

$$x = -\int_1^z \frac{z^2\, dz}{\sqrt{1-z^4}} ,$$

in which we have set $z_0 = 1$, for simplicity.

If $z \leq 1$ then the integrand can be developed into the series:

$$\frac{z^2}{\sqrt{1-z^4}} = z^2 + \frac{1}{2} z^6 + \frac{1\cdot 3}{2\cdot 4} z^{10} + \cdots$$

If one integrates this series (which is legitimate) then one will have:

$$x = \left[\frac{z^3}{3} + \frac{1}{2}\frac{z^7}{7} + \frac{1\cdot 3}{2\cdot 4}\frac{z^{11}}{11} + \cdots \right]_z^1 ,$$

or, upon setting:

$$C_0 = \frac{1}{3}, \quad C_n = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{1}{4n+3} \quad (n = 1, 2, \ldots),$$

$$x = C - \sum_{n=0}^{\infty} C_n\, z^{4n+3},$$

in which the constant:

$$C = \sum_{n=0}^{\infty} C_n$$

is found to lie between 0.598 and 0.601 ([1]). If one assumes the value 0.6 then the error that is introduced will be negligible for the graphical construction of the points of the curve. Thus:

$$x = 0.6 - \sum_{n=0}^{\infty} C_n \, z^{4n+3}. \tag{24}$$

For $z = 0$, the value of the horizontal semi-axis of the curve (Fig. 8) will be:

$$OA = 0.6 \, .$$

If the vertical semi-axis is $OP_0 = 1$ then one will see that the semi-axes of an arbitrary elastic curve have the ratio 10 to 6.

The series that appears in (24) is rapidly convergent and permits one to calculate approximately the values of x that correspond to the given values of z when one takes into account only the first terms of the series: In that way, one can construct the points of the elastic curve.

For example, that is how one gets the elastic curve s' in Fig. 1.

The construction of the other elastic curves whose vertical diameter is different from that of the curve that was already illustrated in Figure 1 can be obtained by recalling the similitude that that exists between those curves. Hence, the effective construction will become difficult in that way. It is preferable to appeal directly to an elastic band that is fixed at one extremity and subject at the other to a force that is normal to the band, which will assume the geometric form of the elastic form to a great degree of approximation.

To that end, I shall appeal to a steel band whose thickness is about 0.2 mm and whose width is 2 cm. Fix one end with a clamp in such a way that the free band would be parallel to the x-axis. Then bend the band by pressing down on it normally at the other extremity until the band becomes normal to the x-axis at that extremity. One will then get a quadrant of a curve that coincides with the one that is obtained for points. It would be even more practical to curve the band into an arc until the extremity becomes parallel while keeping it in that position with a string, which will then represent the minor diameter of the elastic curve. In various tests, one will then find that the ratio of the two diameters of the curve that is represented by the band will oscillate in the neighborhood of 10/6 precisely. In order to be more certain, it is better to curve that band in such a way that the ratio of the two diameters is appropriate to the one that is desired by the theory. In that way, the form of the band will coincide reasonably with that of the elastic curve, and the band can serve as a convenient guide to drawing the curve.

([1]) Giac. Bernoulli, *Opera, loc. cit.*

§ 6. – Comparing the trajectories in the three special cases.

In Fig. 9, the three curves s_1, s_2, s_3 with a common vertex are a cycloid, a circle, and an elastic curve, respectively, which are then the curves that are described by light rays that emanate horizontally from P_0 according to whether:

$$c = \sqrt{z}\,, \quad c = z, \quad c = z^2,$$

respectively.

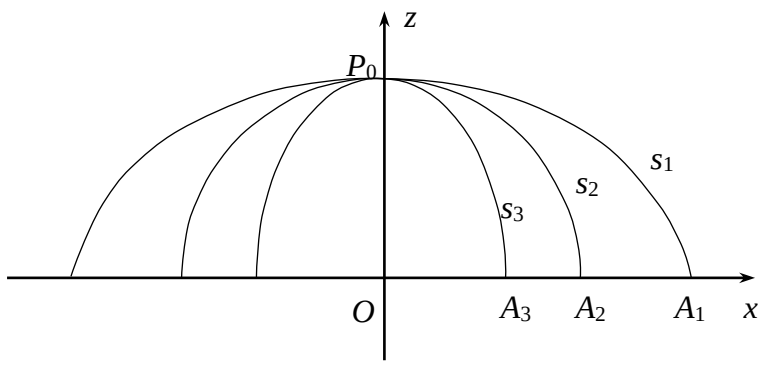

Figure 9.

One then sees that the cycloid is on the outside, the elastic curve in on the inside, and the circle is between those two curves. From (9), their radii of curvature at points of the same horizontal are:

$$2\sqrt{z_0\, z}\,, \quad z_0\,, \quad \frac{z_0^2}{2z},$$

respectively. In particular, P_0 assumes the values:

$$2z_0\,, \quad z_0\,, \quad \tfrac{1}{2}\, z_0\,,$$

respectively, which are in decreasing geometric proportion (viz., each of them is half of the preceding one). When z varies from z_0 to 0, the ray of the cycloid will decrease and become zero the limit of $z = 0$, the second will remain constant, and the third will increase and become infinite for $z = 0$.

The points A_1, A_2, A_3 on the x-axis that are arrived at have the abscissas:

$$\pi\, z_0\,, \quad z_0\,, \quad 0.6\, z_0\,,$$

respectively. A_1 will be reached in the finite time $\pi\sqrt{z_0}$, while the ray will asymptotically approach A_2 and A_3 .

As far as the material point that starts horizontally from P_0 with the velocity v_0 is concerned, in each of the three cases, it will describe trajectories whose radii of curvature at P_0 are, from (9.*b*):

$$2v_0^2\,, \quad \frac{v_0^2}{z_0}\,, \quad \frac{v_0^2}{2z_0^3}\,.$$

In the case of the Fermat cycloid, the initial radius of curvature will not depend upon the height of the vertex then, while in the other two cases, it will be inversely proportional to z_0 and z_0^3, resp. Therefore, in any case, it will be proportional to the square of the initial velocity. The points on the x-axis that are reached in each case then have abscissas that are those of the points A_1, A_2, A_3 multiplied by v_0 / c_0 ; i.e.:

$$\pi v_0 \sqrt{z_0}\,, \quad v_0\,, \quad 0.6\,\frac{v_0}{z_0}\,.$$

The time it takes to fall will then coincide with the time required for the ray to reach the x-axis.

The results obtained might be of interest to ordinary optics, independently of the theory of gravitation. Bernoulli already observed that the case of the cycloid can present itself in an inhomogeneous medium whose density is inversely proportional to $\sqrt{z}$, so our considerations will be valid for inhomogeneous isotropic media whose indices of refraction vary with distance from a given plane ([1]).

Milan, March 1913.

(Received 26 March 1913)

([1]) For the fact that such media exist in reality, see, e.g., A. Barbasso and G. Fubini, Atti della R. Accad. delle Sci. di Torino, vol. XLIV. Announced on 27 December, 1908.