# On the calculus of variations for multiple integrals 

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## Introduction.

1.     - The WEIERSTRASS and the related JACOBI-HAMILTON theory of the calculus of variations have proved to be complete in two extreme cases: Namely, when one has a simple integral and $n$ independent functions to be varied or when one has a $\mu$-fold integral and a single function to be varied. By contrast, the general problem of the form:

$$
\begin{equation*}
\int \cdots \int f\left(x_{1}, \ldots, x_{n} ; t_{1}, \ldots, t_{\mu} ; \frac{\partial x_{1}}{\partial t_{1}}, \ldots, \frac{\partial x_{n}}{\partial t_{\mu}}\right) d t_{1} \cdots d t_{\mu} \tag{1.1}
\end{equation*}
$$

has never been actually taken up, if one overlooks some brief remarks that HADAMARD made about the peculiarities of that problem ( ${ }^{1}$ ). In the following pages, the preliminary work that seemed to me essential for the treatment of that problem will be described in detail. My own investigations in regard to it already go back many years and have also been published in fragments $\left({ }^{2}\right)$.

In my study of an important work of HAAR on adjoint variational problems ( ${ }^{3}$ ), I have noticed that my older calculations could be written more symmetrically with a slight modification in the notation. The entire formal system will be communicated once more on that basis. The first chapter, which is dedicated to the derivation of purely formal identities, will then simply include the results of my previous work in a new formulation: However, with the new notations, as well as in response to some advice that Dr. T. RADÓ gave to me, I hope that the presentation will be much clearer. The second chapter is dedicated to the WEIERSTRASS theory for the problem (1.1), which I had previously only touched upon insufficiently. The $E$-function that is associated with that problem will be exhibited here for the first time in ordinary, as well as canonical coordinates. The same thing will be true for the LEGENDRE condition, as well as the differential equation that

[^0]the "geodetic fields" must satisfy. Finally, it will be show that when a geodetic field intersects a surface transversally, it must necessarily be a solution of the EULER-LAGRANGE equations.

By contrast, the problem of constructing "distinguished" geodetic fields, i.e., ones that will generate complete figures of our variational problem, can still not be worked through.

## Chapter I. - Formal identities.

2. Elementary examples of birational involutory contact transformations. - Since we will address a contact transformation in what follows that is birational and involutory, it would be of interest to recall that other transformations that possess those properties have also played a dominant role in the calculus of variation for some time now.

We shall let the symbols:

$$
\begin{equation*}
f, \varphi, p_{i}, \pi_{i} \quad(i=1,2, \ldots, n) \tag{2.1}
\end{equation*}
$$

denote a number of quantities between which the relation should exist:

$$
\begin{equation*}
f+\varphi=p_{i} \pi_{i} \tag{2.2}
\end{equation*}
$$

(with the now conventional omission of the summation sign). We introduce a second series of $2 n$ +2 quantities, $F, \Phi, P_{i}, \Pi_{i}$ by means of the following equations:

$$
\begin{equation*}
F=\varphi, \quad \Phi=f, \quad P_{i}=\pi_{i}, \quad \Pi_{i}=p_{i} \tag{2.3}
\end{equation*}
$$

That transformation is nothing but the LEGENDRE transformation; it possesses the following properties:
a) It is birational and involutory, i.e., one can solve equations (2.3) for the lower-case symbols by simple permuting the upper-case symbols with the lower-case ones. It then following from (2.2) and (2.3) that the relation:

$$
\begin{equation*}
F+\Phi=P_{i} \Pi_{i} \tag{2.4}
\end{equation*}
$$

must be true.
b) It is a contact transformation. In fact, if $f, \varphi, p_{i}, \pi_{i}$ are functions of any sort of the parameters then the relation:

$$
\begin{equation*}
d F-\Pi_{i} d P_{i}=-\left(d f-\pi_{i} d p_{i}\right) \tag{2.5}
\end{equation*}
$$

must always be true.
3. - Naturally, the LEGENDRE transformation is not the only transformation that possesses the properties $a$ ) and $b$ ) of § 2. An entirely-trivial transformation that achieves the same thing is, e.g., the following one:

$$
\begin{equation*}
F=-f, \quad \Phi=-\varphi, \quad P_{i}=-p_{i}, \quad \Pi_{i}=\pi_{i} \tag{3.1}
\end{equation*}
$$

4.     - As a third example, we consider the generalized inversion, which is defined by the following relations:

$$
\begin{equation*}
F=\frac{1}{f}, \quad \Phi=\frac{1}{\varphi}, \quad P_{i}=\frac{p_{i}}{f}, \quad \Pi_{i}=\frac{\pi_{i}}{\varphi} \tag{4.1}
\end{equation*}
$$

The transformation (4.1) is obviously involutory and birational. In addition, one verifies that the relation (2.4) is a consequence of (2.2), as well as the fact that one is dealing with a contact transformation, with the help of the relations:

$$
\begin{align*}
F+\Phi-P_{i} \Pi_{i} & =\frac{1}{f \varphi}\left(f+\varphi-p_{i} \pi_{i}\right)  \tag{4.2}\\
d F-\Pi_{i} d P_{i} & =\frac{1}{f \varphi}\left(d f-\pi_{i} d p_{i}\right), \tag{4.3}
\end{align*}
$$

which are calculated directly. Furthermore, one notes that not only (4.1), but also (2.2), must be employed to exhibit (4.3).
5. - The transformation that A. HAAR employed in his paper that was cited in $\left({ }^{3}\right)$ is a simple combination of the foregoing ones that one will obtain when one sets:

$$
\begin{equation*}
F=-\frac{1}{f}, \quad \Phi=-\frac{1}{\varphi}, \quad P_{i}=-\frac{p_{i}}{f}, \quad \Pi_{i}=\frac{\pi_{i}}{\varphi} . \tag{5.1}
\end{equation*}
$$

6.     - T. LEVI-CIVITA employed a very interesting, but somewhat complicated, birational and involutory contact transformation in order for regularize the three-body problem, and with great success $\left({ }^{4}\right)$. It consists of the following: If one introduces the notation:

$$
\begin{equation*}
a=p_{i} p_{i}, \quad b=p_{i} \pi_{i}, \quad c=\pi_{i} \pi_{i}, \tag{6.1}
\end{equation*}
$$

[^1]\[

$$
\begin{gather*}
F=f, \quad \Phi=\varphi-2 b, \quad P_{i}=\frac{p_{i}}{a}, \quad \Pi_{i}=a \pi_{i}-2 b p_{i},  \tag{6.2}\\
A=P_{i} P_{i}, \quad B=P_{i} \Pi_{i}, \quad C=\Pi_{i} \Pi_{i} \tag{6.3}
\end{gather*}
$$
\]

then after entirely-elementary calculations one will get:

$$
\begin{equation*}
A a=1, \quad B+b=0, \quad A C=a c \tag{6.4}
\end{equation*}
$$

One verifies the properties $a$ ) and $b$ ) of $\S 2$ from those equations very easily.
7. The canonical transformations of the calculus of variations. - The main topic of our investigation is a birational, involutory, contact transformation, that arises from the combination of the generalized inversion of § 4 and the generalized LEGENDRE transformation of my older paper. It has the advantage over the latter that one can exchange the upper-case symbols with the lower-case ones in all formulas, while it has the small disadvantage that it does not go to the usual LEGENDRE transformation in the limiting cases ( $n=1$ or $\mu=1$ ), but to the transformation that HAAR employed.

From now on, in addition to the Latin symbols $i, j, k, \ldots$, which shall run from 1 to $n$, we will also use Greek indices $\alpha, \beta, \gamma, \rho, \sigma, \ldots$, which are taken from 1 to $\mu$. For example, the symbol $p_{i \alpha}$ represents a matrix of $n$ rows and $\mu$ columns.
8. - We consider the variables:

$$
\begin{equation*}
f, \quad \varphi, \quad p_{i \alpha}, \quad \pi_{i \alpha} \tag{8.1}
\end{equation*}
$$

between which the relation:

$$
\begin{equation*}
f+\varphi=p_{i \alpha} \pi_{i \alpha} \tag{8.2}
\end{equation*}
$$

should exist.
We further introduce the symbol:

$$
\begin{equation*}
a_{\alpha \beta}=\delta_{\alpha \beta} f-p_{i \alpha} \pi_{i \alpha} \tag{8.3}
\end{equation*}
$$

in which, as usual, $\delta_{\alpha \beta}$ is supposed to represent the number one or zero according to whether $\alpha=$ $\beta$ or $\alpha \neq \beta$, respectively.

To abbreviate, we shall set the determinant $\left|a_{\alpha \beta}\right|$ equal to $a$ and denote the algebraic complement of $a_{\alpha \beta}$ in that determinant by $\bar{a}_{\alpha \beta}$. We will then have:

$$
\begin{gather*}
a=\left|a_{\alpha \beta}\right|,  \tag{8.4}\\
\delta_{\alpha \beta} a=a_{\alpha \rho} \bar{a}_{\beta \rho}=a_{\sigma \alpha} \bar{a}_{\sigma \beta} . \tag{8.5}
\end{gather*}
$$

9.     - We now introduce a new series of $2(n \mu+1)$ variables:

$$
\begin{equation*}
F, \quad \Phi, \quad P_{i \alpha}, \quad \Pi_{i \alpha} \tag{9.1}
\end{equation*}
$$

which are defined by the following equations:

$$
\begin{align*}
& \frac{F}{f}=\frac{\Phi}{\varphi}=\frac{f^{\mu-2}}{a},  \tag{9.2}\\
& P_{i \alpha}=\frac{1}{a} \pi_{i \rho} \bar{a}_{\alpha \rho}  \tag{9.3}\\
& \Pi_{i \alpha}=\frac{f^{\mu-2}}{a} p_{i \sigma} \bar{a}_{\alpha \sigma} . \tag{9.4}
\end{align*}
$$

It is very remarkable that one can use the relations (9.2) to (9.4) to represent the original variables (8.1) as rational functions of the quantities (9.1) by means of successive solutions of a linear system of equations.
10. - We shall first calculate some identities that follow from the previous relations. First of all, if we replace the summation indices $\alpha$ with $\sigma$ in (9.3) and contract that equation with $a_{\sigma \alpha}$ then when we take (8.5) into consideration, it will follow that:

$$
\begin{equation*}
\pi_{i \alpha}=P_{i \sigma} a_{\sigma \alpha} \tag{10.1}
\end{equation*}
$$

In a completely similar way, we will get from (9.4) that:

$$
\begin{equation*}
p_{i \alpha}=f^{2-\mu} \Pi_{i \rho} \bar{a}_{\rho \alpha} . \tag{10.2}
\end{equation*}
$$

Thirdly, when we consider (8.3), it will follow from (9.3) that:

$$
\begin{aligned}
P_{i \alpha} p_{i \beta} & =\frac{1}{a} \pi_{i \rho} p_{i \beta} \bar{a}_{\alpha \rho} \\
& =\frac{1}{a} \bar{a}_{\alpha \rho}\left(\delta_{\beta \rho} f-a_{\beta \rho}\right),
\end{aligned}
$$

and it follows from this, using (8.5) and (9.2), that:

$$
\begin{equation*}
\bar{a}_{\alpha \beta}=\frac{f^{\mu-2}}{F}\left(\delta_{\alpha \beta}+P_{i \alpha} p_{i \beta}\right) . \tag{10.3}
\end{equation*}
$$

Finally, in order to also exhibit the last of the relations that come under consideration here, with the help of (9.3) and (9.4), we define the equation:

$$
P_{i \alpha} \Pi_{i \beta}=\frac{f^{\mu-2}}{a^{2}} \pi_{i \rho} p_{i \sigma} \bar{a}_{\alpha \rho} a_{\beta \sigma}
$$

It will then follow from (8.3) and (8.5):

$$
P_{i \alpha} \Pi_{i \beta}=\frac{f^{\mu-2}}{a^{2}}\left(\delta_{\sigma \rho} f-a_{\sigma \rho}\right) \bar{a}_{\alpha \rho} a_{\beta \sigma}=\delta_{\sigma \rho} \frac{f^{\mu-1}}{a}-\frac{f^{\mu-2}}{a} a_{\beta \alpha},
$$

or, from (9.2):

$$
\begin{equation*}
P_{i \alpha} \Pi_{i \beta}=\frac{1}{f} p_{i \beta} \pi_{i \alpha} \tag{10.4}
\end{equation*}
$$

From (8.3), the relation (10.4) can also be written symmetrically as:

$$
\begin{equation*}
\frac{1}{F} P_{i \alpha} \Pi_{i \beta}=\frac{1}{f} p_{i \beta} \pi_{i \alpha} \tag{10.5}
\end{equation*}
$$

11.     - We find from (9.2) and (8.2) that:

$$
F+\Phi=\frac{F}{f}(f+\varphi)=\frac{F}{f} p_{i \alpha} \pi_{i \alpha}
$$

or, from (10.5):

$$
\begin{equation*}
F+\Phi=P_{i \alpha} \prod_{i \alpha} \tag{11.1}
\end{equation*}
$$

12.     - We now introduce the notation:

$$
\begin{equation*}
A_{\alpha \beta}=\delta_{\alpha \beta} F-P_{i \alpha} \prod_{i \beta} \tag{12.1}
\end{equation*}
$$

which is analogous to (8.3). It will then follow from (10.4) that:

$$
\begin{equation*}
\frac{A_{\alpha \beta}}{F}=\frac{a_{\alpha \beta}}{f} \tag{12.2}
\end{equation*}
$$

and when we also adapt our previous notations to the upper-case symbols, it will follow from this that:

$$
\begin{align*}
& \frac{A}{F^{\mu}}=\frac{a}{f^{\mu}},  \tag{12.3}\\
& \frac{\bar{A}_{\alpha \beta}}{F^{\mu-1}}=\frac{\bar{a}_{\alpha \beta}}{f^{\mu-1}} . \tag{12.4}
\end{align*}
$$

A comparison of (9.2) and (12.3) now implies that:

$$
\begin{equation*}
\frac{f}{F}=\frac{\varphi}{\Phi}=\frac{F^{\mu-2}}{A} \tag{12.5}
\end{equation*}
$$

Furthermore, when one employs (12.4) and then (12.5), it will follow from (10.2) that:

$$
\begin{equation*}
p_{i \alpha}=\frac{1}{A} \Pi_{i \rho} \bar{A}_{\alpha \rho}, \tag{12.6}
\end{equation*}
$$

and one likewise obtains from (10.1), (12.2), and (12.3) that:

$$
\begin{equation*}
\pi_{i \alpha}=\frac{F^{\mu-2}}{A} P_{i \sigma} A_{\alpha \sigma} . \tag{12.7}
\end{equation*}
$$

If one now compares (8.2) with (11.1), as well as (9.2), (9.3), and (9.4) with (12.5), (12.6), and (12.7), resp., then one will see that one can exchange the upper-case and lower-case symbols in them, and as a result, in all of the remaining equations, as well.

Our transformation is therefore birational and involutory.
13. Introduction of $f, F, p_{i \alpha}, P_{i \alpha}$ as variables. - Up to now, we have alternately used the systems of quantities (8.1) and (9.1) as the basis for our calculations. For many purposes, it is more convenient to develop formulas in which the quantities:

$$
\begin{equation*}
f, \quad F, \quad p_{i \alpha}, \quad P_{i \alpha} \tag{13.1}
\end{equation*}
$$

appear as the basic variables.
In order to do that, we set:

$$
\begin{equation*}
g_{\alpha \beta}=\delta_{\alpha \beta}+P_{i \alpha} p_{i \beta}, \tag{13.2}
\end{equation*}
$$

such that, from (10.3):

$$
\begin{equation*}
g_{\alpha \beta}=\frac{F}{f^{\mu-2}} \bar{a}_{\alpha \beta} . \tag{13.3}
\end{equation*}
$$

In order to calculate the value $g$ of the determinant $\left|g_{\alpha \beta}\right|$, one notes that one has $\left|\bar{a}_{\alpha \beta}\right|=a^{\mu-1}$. It then follows from (13.3) and (9.2) that:

$$
\begin{equation*}
g=F f \tag{13.4}
\end{equation*}
$$

and (13.3) can be written as:

$$
\begin{equation*}
g_{\alpha \beta}=\frac{g}{f^{\mu-1}} \bar{a}_{\alpha \beta} . \tag{13.5}
\end{equation*}
$$

We infer from the last equation that:

$$
g_{\rho \sigma} \bar{g}_{\rho \beta} a_{\alpha \sigma}=\frac{g}{f^{\mu-1}} \bar{a}_{\rho \sigma} \bar{g}_{\rho \beta} a_{\alpha \sigma}
$$

or

$$
\begin{equation*}
a_{\alpha \beta}=\frac{a}{f^{\mu-1}} g_{\alpha \beta}, \tag{13.6}
\end{equation*}
$$

so from (9.2):

$$
\begin{equation*}
\bar{g}_{\alpha \beta}=F a_{\alpha \beta} . \tag{13.7}
\end{equation*}
$$

Now, due to (10.1), it follows that:

$$
\begin{equation*}
F \pi_{i \alpha}=P_{i \sigma} \bar{g}_{\sigma \alpha}, \tag{13.8}
\end{equation*}
$$

and due to (9.4):

$$
\begin{equation*}
f \Pi_{i \alpha}=p_{i \sigma} \bar{g}_{\alpha \sigma} . \tag{13.9}
\end{equation*}
$$

Finally, when the last two relations are solved for $p_{i \alpha}$ and $P_{i \alpha}$, that will yield:

$$
\begin{align*}
& F p_{i \alpha}=\prod_{i \rho} g_{\rho \alpha},  \tag{13.10}\\
& f P_{i \alpha}=\pi_{i \rho} g_{\alpha \rho} . \tag{13.11}
\end{align*}
$$

14. The property of being a contact transformation. - We now assume that the quantities that we consider depend upon any sort of parameters and form the total differential of (13.4) in those parameters. In that way, we will get the relation:

$$
\begin{equation*}
F d f+f d F=d g . \tag{14.1}
\end{equation*}
$$

However, it is known that:

$$
d g=\bar{g}_{\alpha \beta} d g_{\alpha \beta},
$$

and from (13.2):

$$
d g_{\alpha \beta}=P_{i \alpha} d p_{i \beta}+p_{i \alpha} d P_{i \beta} .
$$

Thus, when we consider (13.8) and (13.9), the last two equations will give:

$$
\begin{equation*}
d g=F \pi_{i \beta} d p_{i \beta}+f \prod_{i \alpha} d P_{i \alpha} . \tag{14.2}
\end{equation*}
$$

A comparison of (14.2) with (14.1) will ultimately lead to the relation:

$$
\begin{equation*}
F\left(d f-\pi_{i \beta} d p_{i \beta}\right)+f\left(d F-\Pi_{i \alpha} d P_{i \alpha}\right)=0 \tag{14.3}
\end{equation*}
$$

from which it will follow that our transformation is a contact transformation.
15. Reciprocity. - In a previous article $\left({ }^{5}\right)$, I made the remark (which can be confirmed immediately, moreover) that the determinant $a$ that we introduced in § $\mathbf{8}$ can also be written as a $(\mu+n)$-rowed determinant as follows:

$$
a=\left|\begin{array}{cc}
\delta_{i j} & \pi_{i \beta}  \tag{15.1}\\
p_{j \alpha} & \delta_{\alpha \beta} f
\end{array}\right| .
$$

In that formula, the rows are denoted by $i$ and $\alpha$, while the columns are denoted by $j$ and $\beta$. In precisely the same way, one sees that when one introduces a new system of variables by means of the equations:

$$
\begin{equation*}
b_{i j}=\delta_{i j} f-p_{i \rho} \pi_{j \rho}, \tag{15.2}
\end{equation*}
$$

the determinant $b$ of the $b_{i j}$ can be written:

$$
b=\left|\begin{array}{cc}
\delta_{i j} f & \pi_{i \beta}  \tag{15.3}\\
p_{j \alpha} & \delta_{\alpha \beta}
\end{array}\right| .
$$

[^2]A comparison of (15.1) with (15.3) leads to the relation:

$$
\begin{equation*}
f^{n} a=f^{\mu} b \tag{15.4}
\end{equation*}
$$

from which we infer, with the help of (9.2), that:

$$
\begin{equation*}
\frac{F}{f}=\frac{\Phi}{f}=\frac{f^{n-2}}{b} \tag{15.5}
\end{equation*}
$$

Moreover, it follows from (15.2) that:

$$
b_{s i} P_{s \alpha}=f P_{i \alpha}-p_{s \rho} \pi_{i \rho} P_{s \alpha}
$$

From (13.2), one can write that:

$$
b_{s i} P_{s \alpha}=f P_{i \alpha}-\pi_{i \rho}\left(g_{\alpha \rho}-\delta_{\alpha \rho}\right),
$$

or when one recalls (13.11):

$$
\begin{equation*}
\pi_{i \alpha}=b_{s i} P_{s \alpha} . \tag{15.6}
\end{equation*}
$$

Similarly, we infer from (15.2) that:

$$
\begin{aligned}
b_{i t} p_{t \alpha} & =f p_{i \alpha}-p_{i \rho} \pi_{t \rho} p_{t \alpha} \\
& =p_{i \rho}\left(\delta_{\alpha \rho} f-p_{i \alpha} \pi_{t \rho}\right),
\end{aligned}
$$

or, from (8.5):

$$
\begin{equation*}
b_{i t} p_{t \alpha}=p_{i \rho} a_{\alpha \rho} . \tag{15.7}
\end{equation*}
$$

When one observes (15.4), one will then get from (9.4) that:

$$
\begin{equation*}
\Pi_{i \alpha}=\frac{f^{n-2}}{b} p_{t \alpha} b_{i t} \tag{15.8}
\end{equation*}
$$

Finally, it will follow upon solving (15.6) that:

$$
\begin{equation*}
P_{i \alpha}=\frac{1}{b} \pi_{r \alpha} \bar{b}_{i r} . \tag{15.9}
\end{equation*}
$$

16.     - The similarity of formulas (15.2), (15.5), (15.9) and (15.8) with (8.3), (9.2), (9.3), and (9.4), resp., shows that one can exchange the Latin indices with the Greek ones in all of our equations as long as one also replaces the $a_{\alpha \beta}$ with the $b_{i j}$.
17. Introduction of the parameters $A_{\alpha \beta}, S_{\alpha l}, S_{\alpha \beta}$, and $c_{\alpha \beta}$. - For the treatment of our variational problem, it will be necessary to introduce new parameters and examine their connection with the previous symbols.

To that end, we consider three matrices:

$$
\begin{equation*}
S_{\alpha \beta}, \quad S_{\alpha i}, \quad c_{\alpha \beta}, \tag{17.1}
\end{equation*}
$$

which are coupled with the previous quantities by the relations:

$$
\begin{align*}
& c_{\alpha \beta}=S_{\alpha \beta}+S_{\alpha i} p_{i \beta},  \tag{17.2}\\
& S_{\alpha i}=P_{i \rho} S_{\alpha \rho},  \tag{17.3}\\
& \frac{1}{F}=\left|S_{\alpha \rho}\right| . \tag{17.4}
\end{align*}
$$

Upon substituting (17.3) in (17.2), one will now get:

$$
\begin{aligned}
c_{\alpha \beta} & =S_{\alpha \beta}+S_{\alpha \rho} P_{i \rho} p_{i \beta} \\
& =S_{\alpha \rho}\left(\delta_{\rho \beta}+P_{i \rho} p_{i \beta}\right),
\end{aligned}
$$

or, from (13.2):

$$
\begin{equation*}
c_{\alpha \beta}=S_{\alpha \beta} g_{\rho \beta} . \tag{17.5}
\end{equation*}
$$

Now, when we observe (13.4), it will follow from the multiplication law for determinants that $c=$ $F f\left|S_{\alpha \beta}\right|$, or from (17.4):

$$
\begin{equation*}
c=f . \tag{17.6}
\end{equation*}
$$

Furthermore, one infers from (17.5) that:

$$
c_{\lambda \sigma} \bar{c}_{\lambda \beta} \bar{g}_{\alpha \sigma}=S_{\lambda \sigma} g_{\rho \sigma} \bar{g}_{\alpha \sigma} \bar{c}_{\lambda \beta},
$$

and from (13.4) and (17.6), it will follow from this that:

$$
\begin{equation*}
\bar{g}_{\alpha \beta}=F S_{\lambda \alpha} \bar{c}_{\lambda \beta} . \tag{17.7}
\end{equation*}
$$

Due to (13.8), it will follow from this that:

$$
\pi_{i \alpha}=F S_{\lambda \alpha} \bar{c}_{\lambda \beta}
$$

or from (17.3):

$$
\begin{equation*}
\pi_{i \alpha}=S_{\lambda i} \bar{c}_{\lambda \alpha} \tag{17.8}
\end{equation*}
$$

18.     - We would now like to show that when one assumes (17.2), equations (17.6) and (17.8) will be equivalent to equations (17.3) and (17.4). We now start from the equations:

$$
\begin{align*}
c_{\alpha \beta} & =S_{\alpha \beta}+S_{\alpha i} p_{i \beta},  \tag{18.1}\\
\pi_{i \alpha} & =S_{\rho i} \bar{c}_{\rho \alpha}  \tag{18.2}\\
c & =f, \tag{18.3}
\end{align*}
$$

and we would like to derive (17.3) and (17.4). First of all, it follows from (18.2) that:

$$
\pi_{i} \sigma c_{\alpha \sigma}=S_{\rho i} \bar{c}_{\rho \sigma} c_{\alpha \sigma},
$$

so when we recall (18.3):

$$
\begin{equation*}
f S_{\alpha i}=\pi_{i} \sigma c_{\alpha \sigma} \tag{18.4}
\end{equation*}
$$

When that is substituted in (18.1), that will give:

$$
f c_{\alpha \beta}=f S_{\alpha \beta}+c_{\alpha \sigma} p_{i \beta} p_{i \sigma},
$$

from which, with (8.3), it will follow that:

$$
\begin{equation*}
f S_{\alpha \beta}=c_{\alpha \sigma} a_{\beta \sigma} . \tag{18.5}
\end{equation*}
$$

Now, it follows from (18.5) that:

$$
f P_{i \rho} S_{\alpha \rho}=c_{\alpha \sigma} P_{i \rho} a_{\rho \sigma}
$$

From (10.1), the right-hand side of the last equation is equal to $c_{\alpha \sigma} \pi_{i \sigma}$, and with the help of (18.4), one will ultimately get:

$$
\begin{equation*}
S_{\alpha i}=P_{i \rho} S_{\alpha \rho} \tag{18.6}
\end{equation*}
$$

i.e., the relation (17.3) that we would like to prove. Equation (17.4) is a consequence of (18.5) in any case when we observe (18.3) and (9.2). That is because:

$$
f^{\mu}\left|S_{\alpha \beta}\right|=a c=f \frac{f^{\mu-1}}{F},
$$

or

$$
\begin{equation*}
F\left|S_{\alpha \beta}\right|=1 . \tag{18.7}
\end{equation*}
$$

## Chapter II. The variational problem.

19. Definition of a geodetic field. - We consider a $\mu$-parameter family of $n$-dimensional surfaces in an $(n+\mu)$-dimensional space whose coordinates might be $\left(x_{1}, \ldots, x_{n}, t_{1}, \ldots, t_{\mu}\right)$, or $\left(x_{i}\right.$, $t_{\alpha}$ ), with the previous notation. Such a family can be represented by $\mu$ equations of the form:

$$
\begin{equation*}
S_{\alpha}\left(x_{i} ; t_{\beta}\right)=\lambda_{\alpha} . \tag{19.1}
\end{equation*}
$$

Furthermore, a $\mu$-dimensional manifold that intersects the family (19.1) will be defined by the equations:

$$
\begin{equation*}
x_{i}=\xi_{i}\left(t_{\alpha}\right) \quad(i=1,2, \ldots, n) \tag{19.2}
\end{equation*}
$$

That will be the case if and only if a one-to-one map of a region $G_{t}$ in the $\mu$-dimensional space of $t_{\alpha}$ to a region $G_{\lambda}$ in the $\mu$-dimensional parameter space of the $\lambda_{\alpha}$ is generated by the system of equations:

$$
\begin{equation*}
S_{\alpha}\left(\xi_{i}\left(t_{\gamma}\right) ; t_{\beta}\right)=\lambda_{\alpha} . \tag{19.3}
\end{equation*}
$$

However, in order for that to be true, the functional determinant:

$$
\begin{equation*}
\Delta=\left|\frac{\partial S_{\alpha}\left(\xi_{i} ; t_{\alpha}\right)}{\partial t_{\beta}}\right| \tag{19.4}
\end{equation*}
$$

must be non-zero in $G_{t}$, in particular.
If one sets:

$$
\begin{gather*}
S_{\alpha i}=\frac{\partial S_{\alpha}}{\partial x_{i}}, \quad S_{\alpha \beta}=\frac{\partial S_{\alpha}}{\partial t_{\beta}},  \tag{19.5}\\
p_{i \alpha}=\frac{\partial \xi_{i}}{\partial t_{\alpha}},  \tag{19.6}\\
c_{\alpha \beta}=S_{\alpha \beta}+S_{\alpha i} p_{i \beta}, \tag{19.7}
\end{gather*}
$$

to abbreviate, then (19.4) will take the form:

$$
\begin{equation*}
\Delta=\left|c_{\alpha \beta}\right|=c . \tag{19.8}
\end{equation*}
$$

We next remark that the integral:

$$
\begin{equation*}
\underbrace{\int \cdots \int}_{G_{t}} \Delta d t_{1} \cdots d t_{\mu} \tag{19.9}
\end{equation*}
$$

represents the volume of the region $G_{\lambda}$ in parameter space onto which the region $G_{t}$ was mapped by the relations (19.3).

However, that volume depends upon the form of the boundary of $G \lambda$.
Therefore, when one considers a second $\mu$-dimensional surface:

$$
\begin{equation*}
x_{i}=\bar{\xi}_{i}\left(t_{\alpha}\right), \tag{19.10}
\end{equation*}
$$

and a region $\bar{G}_{t}$ that is mapped to the same region $G \lambda$ that we just considered by that new surface then the integral:

$$
\begin{equation*}
\underbrace{\int \cdots \int}_{\bar{G}_{t}} \bar{\Delta} d t_{1} \cdots d t_{\mu}, \tag{19.11}
\end{equation*}
$$

which is mapped in a manner that is entirely analogous to (19.9), will possess the same value as (19.9).

If a manifold that also lies on (19.10) were cut out of the surface (19.2) by the boundary of the region $G_{t}$, in particular, then one would have to calculate the integrals (19.9) and (19.11) for the same region $G_{t}$, i.e., one would have to set $\bar{G}_{t}=G_{t}$.
20. - The coordinates of a $\mu$-dimensional surface element in the $(n+\mu)$ space shall now be represented by the $n+\mu+n \mu$ quantities:

$$
\begin{equation*}
x_{i}, \quad t_{\alpha}, \quad p_{i \alpha} . \tag{20.1}
\end{equation*}
$$

We now consider a positive function:

$$
\begin{equation*}
f\left(x_{i}, t_{\alpha}, p_{i \alpha}\right) \tag{20.2}
\end{equation*}
$$

of those quantities and form the expression:

$$
\begin{equation*}
\frac{f\left(x_{i}, t_{\alpha}, p_{i \alpha}\right)}{\Delta\left(x_{i}, t_{\alpha}, p_{i \alpha}\right)}, \tag{20.3}
\end{equation*}
$$

in which $\Delta$ has the same meaning that it had in (19.8). We now fix the $\left(x_{i}, t_{\alpha}\right)$ in (20.3) and seek to determine the $p_{i \alpha}$ such that:

$$
\begin{equation*}
\frac{f}{\Delta}=\operatorname{minimum} \tag{20.4}
\end{equation*}
$$

We say of a surface element (20.1) for which the condition (20.4) is true that it will intersect the family of surfaces (19.1) transversally.

We now assume that we can determine functions:

$$
\begin{equation*}
p_{i \alpha}=p_{i \alpha}\left(x_{i}, t_{\alpha}\right) \tag{20.5}
\end{equation*}
$$

in a certain $(n+\mu)$-dimensional region of the space of $\left(x_{i}, t_{\alpha}\right)$ that generate nothing but surface elements that are intersect transversally by our family (19.1).

If we now introduce the values (20.5) for the $p_{i \alpha}$ into $f\left(x_{i}, t_{\alpha}, p_{i \alpha}\right)$ and $\Delta\left(x_{i}, t_{\alpha}, p_{i \alpha}\right)$, and if the equation:

$$
\begin{equation*}
f=\Delta \tag{20.6}
\end{equation*}
$$

is true at each point of the region considered then we would like to say that the family (19.1) (which belongs to $f$ ) defines a geodetic field. A necessary condition for the existence of (20.4) will be given by the equations:

$$
\frac{\partial}{\partial p_{i \alpha}}\left(\frac{f}{\Delta}\right)=0
$$

and due to (20.6), they can also be written as follows:

$$
\begin{equation*}
f_{p_{i \alpha}}=\frac{\partial \Delta}{\partial p_{i \alpha}} \tag{20.7}
\end{equation*}
$$

Equations (20.6) and (20.7) define the fundamental relations by which a geodetic field is defined.
21. Solving the variational problem. - If one has constructed a geodetic field that intersects a manifold (19.2) transversally in any way then it will always define a solution to the variational problem that belongs to the integral:

$$
\begin{equation*}
\int \cdots \int f\left(x_{i}, t_{\alpha}, p_{i \alpha}\right) d t_{1} \cdots d t_{\mu} \tag{21.1}
\end{equation*}
$$

Namely, one considers a piece of (19.2) that projects onto a region $G_{t}$ in the space of $t_{\alpha}$ and a corresponding piece of (19.10) that projects onto $\bar{G}_{t}$, in such a way that the relations between $G_{t}$ and $\bar{G}_{t}$ that were explained at the end of $\S 19$ should be true. From the results in $\S \mathbf{1 9}$, combined with (20.6), one will then have:

$$
\begin{equation*}
\int_{G_{t}} f d t_{1} \cdots d t_{\mu}=\int_{G_{t}} \Delta d t_{1} \cdots d t_{\mu}=\int_{G_{t}} \bar{\Delta} d t_{1} \cdots d t_{\mu} . \tag{21.2}
\end{equation*}
$$

If one then denotes the value of $f$ in the surface (19.10) by $\bar{f}$ then, from (21.2), one will have:

$$
\begin{equation*}
\int_{\bar{G}_{t}} \bar{f} d t_{1} \cdots d t_{\mu}-\int_{G_{t}} f d t_{1} \cdots d t_{\mu}=\int_{\bar{G}_{t}}(\bar{f}-\bar{\Delta}) d t_{1} \cdots d t_{\mu} . \tag{21.3}
\end{equation*}
$$

We now remark that it will also follow from $f>0$ and (20.6) that we also have $\Delta>0$. For a weak variation of our original surface patch, we will also have $\bar{\Delta}>0$ then. When we recall (20.4) and (20.6), it will then follow from this that:

$$
\begin{equation*}
\bar{f}-\bar{\Delta}=\bar{\Delta}\left(\frac{\bar{f}}{\bar{\Delta}}-\frac{f}{\Delta}\right)>0 \tag{21.4}
\end{equation*}
$$

with which, our assertion is proved.
22. Introducing canonical variables. - The further treatment of our problem will be simplified considerably when we now introduce the canonical quantities $F, P_{i \alpha}, \Pi_{i \alpha}$ that we examined in the first chapter. In fact, from (19.7) and (19.8), we have:

$$
\begin{equation*}
\frac{\partial \Delta}{\partial p_{i \alpha}}=S_{\lambda i} \bar{c}_{\lambda \alpha} \tag{22.1}
\end{equation*}
$$

and a comparison of that formula with (17.8) and (20.7) will show that we have to set:

$$
\begin{equation*}
\pi_{i \alpha}=f_{p_{i \alpha}} \tag{22.2}
\end{equation*}
$$

According to $\S \S 8$ and $\mathbf{9}$, we can now calculate the $a_{\alpha \beta}, F, \Phi, P_{i \alpha}, \Pi_{i \alpha}$ as functions of the $x_{i}, t_{\alpha}$, $p_{i \alpha}$ by rational operations. We can likewise calculate the determinant $a$, and in particular, verify that it does vanish. In the event that it vanishes identically, the function $f$ upon which our variational problem depends would not be useful in our theory.

However, our goal is to take $x_{i}, t_{\alpha}, P_{i \alpha}$ to be independent variables, and we must then exhibit the condition that must be verified in order for us to be able to express the $p_{i \alpha}$ in terms of those quantities. In order to do that, it would be best for us to employ (10.1), and from (8.3), that is an equation that we can write out in detail as follows:

$$
\begin{equation*}
M_{i \alpha}=\pi_{i \alpha}-P_{i \alpha}\left(\delta_{\sigma \alpha} f-p_{k \alpha} \pi_{k \alpha}\right)=0 . \tag{22.3}
\end{equation*}
$$

This last system of equations shall also be soluble for the $p_{i \alpha}$, and in order to do that, we must write down that the functional determinant satisfies:

$$
\begin{equation*}
\left|\frac{\partial M_{i \alpha}}{\partial p_{j \beta}}\right| \neq 0 \tag{22.4}
\end{equation*}
$$

If we set:

$$
\begin{equation*}
\frac{\partial^{2} f}{\partial p_{i \alpha} \partial p_{j \beta}}=\pi_{i \alpha, j \beta} \tag{22.5}
\end{equation*}
$$

to abbreviate, then it will follow from (22.3) that:

$$
\begin{equation*}
\frac{\partial M_{i \alpha}}{\partial p_{j \beta}}=\pi_{i \alpha, j \beta}-P_{i \alpha} \pi_{j \beta}+P_{i \beta} \pi_{j \alpha}+P_{i \sigma} p_{k \sigma} \pi_{k \alpha, j \beta} \tag{22.6}
\end{equation*}
$$

Now, it follows from our assumptions that $b \neq 0$. Therefore, we can replace the condition (22.4) with the non-vanishing of a determinant whose elements are:

$$
\begin{equation*}
N_{i \alpha, j \beta}=b_{r i} \delta_{\sigma \alpha} \frac{\partial M_{r \sigma}}{\partial p_{j \beta}}=b_{r i} \frac{\partial M_{r \alpha}}{\partial p_{j \beta}} \tag{22.7}
\end{equation*}
$$

It follows from (22.6) and (22.7), with the help of (15.6) that:

$$
\begin{equation*}
N_{i \alpha, j \beta}=b_{r i} \pi_{r \alpha, j \beta}-\pi_{i \alpha} \pi_{j \beta}+\pi_{i \beta} \pi_{j \alpha}+\pi_{i \sigma} p_{k \sigma} \pi_{k \alpha, j \beta} \tag{22.8}
\end{equation*}
$$

Now, since $b_{r i} \pi_{r \alpha, j \beta}=b_{k i} \pi_{k \alpha, j \beta}$ and $b_{k i}+\pi_{i \sigma} p_{k \sigma}=\delta_{i k} f$, we can also write (22.8) in the form:

$$
\begin{equation*}
N_{i \alpha, j \beta}=f \pi_{i \alpha, j \beta}-\pi_{i \alpha} \pi_{j \beta}+\pi_{i \beta} \pi_{j \alpha} \tag{22.9}
\end{equation*}
$$

The introduction of the $P_{i \alpha}$ as independent variables will always be possible as long as the determinant satisfies:

$$
\begin{equation*}
\left|f \frac{\partial^{2} f}{\partial p_{i \alpha} \partial p_{j \beta}}+\frac{\partial f}{\partial p_{i \beta}} \frac{\partial f}{\partial p_{j \alpha}}-\frac{\partial f}{\partial p_{i \alpha}} \frac{\partial f}{\partial p_{j \beta}}\right| \neq 0 . \tag{22.10}
\end{equation*}
$$

23.     - Once we have represented the $p_{i \alpha}$ as functions of $x_{i}, t_{\alpha}, P_{i \alpha}$, from Chapter I, we can determine all of the remaining quantities (so $F, \Phi, \Pi_{i \alpha}$, in particular) as functions of those variables.

If one introduces those functions into (14.3) then the relations will follow immediately:

$$
\begin{gather*}
F f_{\alpha_{i}}=-f F_{\alpha_{i}}, \quad F f_{t_{\alpha}}=-f F_{t_{\alpha}},  \tag{23.1}\\
\Pi_{i \alpha}=\frac{\partial F}{\partial P_{i \alpha}}, \tag{23.2}
\end{gather*}
$$

which we will use later.
24. The $E$-function. - We are now in a position to calculate the WEIERSTRASS excess function that belongs to the integral (1.1) for any geodetic field.

Therefore, equations (20.6) and (20.7) must be true. However, if one considers (19.8), (22.2), and (22.1) then one will see that equations (18.2) and (18.3) must also be true, and from § 18, they are equivalent to (17.3) and (17.4).

If we then substitute the right-hand side of (17.3) for $S_{\alpha}$ in:

$$
\begin{equation*}
\bar{\Delta}=\left|S_{\alpha \beta}+S_{\alpha i} \bar{p}_{i \beta}\right| \tag{24.1}
\end{equation*}
$$

then it will follow from the multiplication rule for determinants, when we observe (17.4), that:

$$
\begin{equation*}
\bar{\Delta}=\frac{1}{F}\left|\delta_{\alpha \beta}+P_{i \alpha} \bar{p}_{i \beta}\right| . \tag{24.2}
\end{equation*}
$$

We now define new quantities $h_{i \beta}$ by the equations:

$$
\begin{equation*}
\bar{p}_{i \beta}=p_{i \beta}+f \cdot h_{i \beta} \tag{24.3}
\end{equation*}
$$

and obtain from (24.2), when we recall (13.2), that:

$$
\begin{equation*}
\bar{\Delta}=\frac{1}{F}\left|g_{\alpha \beta}+f P_{i \alpha} h_{i \beta}\right| . \tag{24.4}
\end{equation*}
$$

If we now remark that it will follow from (13.4) and (13.8) that:

$$
\begin{equation*}
\left(g_{\rho \beta}+f P_{i \rho} h_{i \beta}\right) \bar{g}_{\rho \alpha}=F f\left(\delta_{\alpha \beta}+p_{i \alpha} h_{i \beta}\right) \tag{24.5}
\end{equation*}
$$

then we will get from (24.4) that:

$$
\bar{\Delta}\left|\bar{g}_{\alpha \beta}\right|=F^{\mu-1} f^{\mu}\left|\delta_{\alpha \beta}+\pi_{i \alpha} h_{i \beta}\right| .
$$

Now since, due to (13.4), we have $\left|\bar{g}_{\alpha \beta}\right|=F^{\mu-1} f^{\mu}$, we will ultimately get:

$$
\begin{equation*}
\bar{\Delta}=f\left|\delta_{\alpha \beta}+p_{i \alpha} h_{i \beta}\right| . \tag{24.6}
\end{equation*}
$$

We now calculate $h_{i \beta}$ from (24.3) and remark that, from § 21, we need to take:

$$
E=\bar{f}-\bar{\Delta} .
$$

Therefore, we finally have:

$$
\begin{equation*}
E=\bar{f}-\frac{1}{f^{\mu-1}}\left|\delta_{\alpha \beta} f+\pi_{i \alpha}\left(\bar{p}_{i \alpha}-p_{i \alpha}\right)\right| \tag{24.7}
\end{equation*}
$$

That formula will go to the usual form for the $E$-function when $\mu=1$. It is remarkable that, here as well, the $E$-function depends upon only the surface elements $p_{i \beta}, \bar{p}_{i \beta}$, but not upon the geodetic field.
25. The Legendre condition. - We shall develop the determinant (24.6) in powers of $h_{i \beta}$ and determine the linear and quadratic terms in that development. In order to do that, we introduce the notation:

$$
\begin{equation*}
m_{\alpha \beta}=\delta_{\alpha \beta}+\pi_{i \alpha} h_{i \beta}, \tag{25.1}
\end{equation*}
$$

from which, with abbreviations that are similar to the ones in § 8, it will follow that:

$$
\delta_{\alpha \beta} m=m_{\rho \beta} \bar{m}_{\rho \alpha},
$$

and after differentiating:

$$
\delta_{\alpha \beta} d m=m_{\rho \beta} d \bar{m}_{\rho \alpha}+\bar{m}_{\rho \alpha} d m_{\rho \beta} .
$$

We contract that equation with $\bar{m}_{\alpha \beta}$, and when we replace the summation index $\beta$ with $\lambda$, we will get:

$$
m d \bar{m}_{\sigma \alpha}=\bar{m}_{\sigma \alpha} d m-\bar{m}_{\rho \alpha} \bar{m}_{\sigma \lambda} d m_{\rho \lambda} .
$$

Now, we know that $d m=\bar{m}_{\sigma \lambda} d m_{\rho \lambda}$, and we will ultimately have:

$$
\begin{equation*}
m d \bar{m}_{\sigma \alpha}=\left(\bar{m}_{\sigma \alpha} \bar{m}_{\rho \lambda}-\bar{m}_{\rho \alpha} \bar{m}_{\sigma \lambda}\right) d m_{\rho \lambda} . \tag{25.2}
\end{equation*}
$$

Now, it follows from (24.6) that:

$$
\begin{equation*}
\frac{\partial \bar{\Delta}}{\partial h_{i \alpha}}=f \bar{m}_{\sigma \alpha} \pi_{i \sigma}, \tag{25.3}
\end{equation*}
$$

and from (25.2):

$$
\begin{equation*}
m \frac{\partial \bar{m}_{\sigma \alpha}}{\partial h_{j \beta}}=\left(\bar{m}_{\sigma \alpha} \bar{m}_{\rho \beta}-\bar{m}_{\rho \alpha} \bar{m}_{\sigma \beta}\right) \pi_{j \rho} . \tag{25.4}
\end{equation*}
$$

Hence, from (25.3), one has:

$$
\begin{equation*}
\frac{\partial^{2} \bar{\Delta}}{\partial h_{i \alpha} \partial h_{j \beta}}=\frac{f}{m}\left(\bar{m}_{\sigma \alpha} \bar{m}_{\rho \beta}-\bar{m}_{\rho \alpha} \bar{m}_{\sigma \beta}\right) \pi_{i \sigma} \pi_{j \rho} . \tag{25.5}
\end{equation*}
$$

Now, for $h_{i \beta}=0$, one has $\bar{m}_{\alpha \beta}=\delta_{\alpha \beta}$, and it will then follow from (25.3) and (25.5) that:

$$
\begin{gather*}
\left.\frac{\partial \bar{\Delta}}{\partial h_{i \alpha}}\right|_{0}=f \pi_{i \alpha},  \tag{25.6}\\
\left.\frac{\partial^{2} \bar{\Delta}}{\partial h_{i \alpha} \partial h_{j \beta}}\right|_{0}=f\left(\pi_{i \alpha} \pi_{j \beta}-\pi_{i \beta} \pi_{i \alpha}\right) . \tag{25.7}
\end{gather*}
$$

If we then develop the $E$-function (24.7) in powers of $\left(\bar{p}_{i \beta}-p_{i \beta}\right)$ then the constant, as well as the linear, terms will be missing. The quadratic terms in the development then define a quadratic form that reads as follows:

$$
\begin{equation*}
2 Q=\left(\bar{p}_{i \alpha}-p_{i \alpha}\right)\left(\bar{p}_{j \beta}-p_{j \beta}\right)\left\{\frac{\partial^{2} f}{\partial p_{i \alpha} \partial p_{j \beta}}-\frac{1}{f}\left(\frac{\partial f}{\partial p_{i \alpha}} \frac{\partial f}{\partial p_{j \beta}}-\frac{\partial f}{\partial p_{j \alpha}} \frac{\partial f}{\partial p_{i \beta}}\right)\right\} . \tag{25.8}
\end{equation*}
$$

The LEGENDRE condition for our problem consists of the requirement that the quadratic form (25.8) must be positive-definite. It should be observed that the determinant of that quadratic form coincides with the expression (22.10). Thus, whenever the LEGENDRE condition is fulfilled, it should also be possible to introduce canonical coordinates.

Finally, it should be observed that the first derivatives of $f$ with respect to the $p_{i \alpha}$ also enter into the LEGENDRE condition.
26. The $E$-function in canonical coordinates. - For the case in which one is given the function $F\left(x_{i}, t_{\alpha}, P_{i \alpha}\right)$ for the variational problem in canonical coordinates from the outset, it is useful to have an expression for the $E$-function into which only those coordinates enter. In order to do that, one sets:

$$
\begin{equation*}
P_{i \alpha}=\bar{P}_{i \alpha}+\bar{F} k_{i \alpha} \tag{26.1}
\end{equation*}
$$

in (24.2) and transforms the expression (24.2) in a manner that is entirely similar to what was done in § 24. One finally finds that:

$$
\begin{equation*}
\frac{F}{\bar{f}} E=F-\frac{1}{\bar{F}^{\mu-1}}\left|\delta_{\alpha \beta} \bar{F}+\bar{\Pi}_{i \beta}\left(P_{i \alpha}-\bar{P}_{i \alpha}\right)\right| \tag{26.2}
\end{equation*}
$$

If one now calculates the LEGENDRE condition from (26.2) then one will find a formula that is entirely analogous to the relation (25.8)

Finally, one notes that as a result of reciprocity (§ 15), the $E$-function can be represented by $n$ rowed determinants in the original coordinates, as well as in canonical ones.
27. The differential equations of geodetic fields. - From §§ 20 and 22, one will get a geodetic field when one simultaneously satisfies the equations:

$$
\begin{equation*}
f=\Delta=c, \quad f_{p_{i c}}=p_{i \alpha}=S_{\lambda i} \bar{c}_{\lambda \alpha}, \tag{27.1}
\end{equation*}
$$

with the notations of § 19. However, from § 18, that system of equations is completely equivalent to the following one:

$$
\begin{align*}
& S_{\alpha i}=P_{i \rho} S_{\alpha \rho},  \tag{27.2}\\
& F \cdot\left|S_{\alpha \beta}\right|=1 . \tag{27.3}
\end{align*}
$$

If one has calculated the function $F\left(x_{i}, t_{\alpha}, P_{i \alpha}\right)$ then one can find a geodetic field as follows: One determines the $P_{i \alpha}$ as rational functions of the first partial derivatives of the $S_{\alpha}\left(x_{i}, t_{\beta}\right)$ from equations (27.2) and substitutes the values thus-found in (27.3). One will then get one first-order partial differential equation for $\mu$ functions $S_{\alpha},(\mu-1)$ of which can then be chosen quite arbitrarily.
28. - The $P_{i \alpha}$ and $F$ can be determined as functions of $\left(x_{i}, t_{\alpha}\right)$, i.e., as functions of position in $(n+\mu)$-dimensional space, by means of a given geodetic field with the help of (27.2) and (27.3), and the remaining quantities can then be determined by applying the formulas of Chapter I.

However, one can, conversely, give the $P_{i \alpha}$ as such functions of position from the outset and ask what conditions might be necessary and sufficient for one to be able to find the functions $S_{\alpha}\left(x_{i}, t_{\beta}\right)$ for which the relations (27.2) and (27.3) would be true.

We introduce the linear operator:

$$
\begin{equation*}
L_{i}()=\frac{\partial}{\partial x_{i}}()-P_{i \rho} \frac{\partial}{\partial t_{\rho}}() . \tag{28.1}
\end{equation*}
$$

Equations (27.2) then say that the system of differential equations:

$$
\begin{equation*}
L_{i} S_{\alpha}=0 \tag{28.2}
\end{equation*}
$$

should be true for the $\mu$ independent functions $S_{\alpha}$, and therefore must be a JACOBIAN system. The necessary and sufficient condition for that is known to be the identical vanishing of the bracket expression $\left(L_{i} L_{j}-L_{j} L_{i}\right) S$. That is equivalent to the following relation:

$$
\begin{equation*}
L_{j} P_{i \rho}-L_{i} P_{j \rho}=0 . \tag{28.3}
\end{equation*}
$$

If we then set:

$$
\begin{equation*}
[i j \rho]=\frac{\partial P_{i \rho}}{\partial x_{j}}-\frac{\partial P_{j \rho}}{\partial x_{i}}-\left(P_{j \sigma} \frac{\partial P_{i \rho}}{\partial t_{\sigma}}-P_{i \sigma} \frac{\partial P_{j \rho}}{\partial t_{\sigma}}\right), \tag{28.4}
\end{equation*}
$$

to abbreviate, then we have to write:

$$
\begin{equation*}
[i j \rho]=0 . \tag{28.5}
\end{equation*}
$$

29.     - Let all of the conditions (28.5) be verified. The relation:

$$
\begin{equation*}
\left|S_{\alpha \beta}\right| \frac{\partial\left(T_{1}, \ldots . T_{\mu}\right)}{\partial\left(S_{1}, \ldots . S_{\mu}\right)}=\left|T_{\alpha \beta}\right| \tag{29.1}
\end{equation*}
$$

always exists between two systems $S_{\alpha}$ and $T_{\alpha}$ of $\mu$ independent solutions of the JACOBIAN system (28.2), which represents a known property of functional determinants. If one is given the $T_{\beta}$ then equation (27.3) will be soluble if and only if one can determine the $S_{\alpha}$ as functions of the $T_{\beta}$ such that equation:

$$
\begin{equation*}
\frac{\partial\left(T_{1}, \ldots . T_{\mu}\right)}{\partial\left(S_{1}, \ldots . S_{\mu}\right)}=(F)\left|T_{\alpha \beta}\right| \tag{29.2}
\end{equation*}
$$

will be true. The $(F)$ in that means the function of the $(n+\mu)$ variables $\left(x_{i}, t_{\alpha}\right)$ that one will get when one replaces the $P_{i \alpha}$ in $F\left(x_{i}, t_{\alpha}, P_{i \alpha}\right)$ with functions of $\left(x_{i}, t_{\alpha}\right)$. However, the relation (29.2) can be satisfied if and only if the right-hand side of that equation is itself a function of the $T_{\beta}$, i.e., when it satisfies the JACOBIAN system (28.2). Equation (27.3) will then be equivalent to the system:

$$
\begin{equation*}
L_{i}\left((F) \cdot\left|T_{\alpha \beta}\right|\right)=0 \tag{29.3}
\end{equation*}
$$

Upon developing (29.3), one will get:

$$
\begin{equation*}
\left|T_{\alpha \beta}\right| L_{i}(F)+(F) \bar{T}_{\rho \sigma} L_{i} \frac{\partial T_{\rho}}{\partial t_{\sigma}}=0 . \tag{29.4}
\end{equation*}
$$

Now:

$$
\begin{equation*}
L_{i} \frac{\partial T_{\rho}}{\partial t_{\sigma}}=\frac{\partial^{2} T_{\rho}}{\partial x_{i} \partial t_{\sigma}}-P_{i \lambda} \frac{\partial^{2} T_{\rho \sigma}}{\partial t_{\sigma} \partial t_{\lambda}} . \tag{29.5}
\end{equation*}
$$

On the other hand, since $T_{\rho}$ is, by assumption, a solution of (28.2):

$$
\frac{\partial^{2} T_{\rho}}{\partial x_{i} \partial t_{\sigma}}=\frac{\partial}{\partial t_{\sigma}}\left(\frac{\partial T_{\rho}}{\partial x_{i}}\right)=\frac{\partial}{\partial t_{\sigma}}\left(P_{i \lambda} \frac{\partial T_{\rho}}{\partial t_{\lambda}}\right) .
$$

When that is substituted in (29.5), that will give the equation:

$$
\begin{equation*}
L_{i} \frac{\partial T_{\rho}}{\partial t_{\sigma}}=\frac{\partial P_{i \lambda}}{\partial t_{\sigma}} T_{\rho \lambda} \tag{29.6}
\end{equation*}
$$

We substitute that value in (29.4), and after division by $\left|T_{\alpha \beta}\right|$, we get will the condition that we would like to exhibit:

$$
\begin{equation*}
L_{i}(F)+(F) \frac{\partial P_{i \sigma}}{\partial t_{\sigma}}=0 \tag{29.7}
\end{equation*}
$$

30.     - We now set:

$$
\begin{equation*}
[i]=L_{i}(F)+(F) \frac{\partial P_{i \sigma}}{\partial t_{\sigma}}+\Pi_{j \rho}[i j \rho] \tag{30.1}
\end{equation*}
$$

to abbreviate, and develop $L_{i}(F)$, while considering (23.2). We get:

$$
[i]=\frac{\partial F}{\partial x_{i}}+\Pi_{j \rho} \frac{\partial P_{j \rho}}{\partial x_{i}}-P_{i \sigma}\left(\frac{\partial F}{\partial t_{\sigma}}+\Pi_{j \rho} \frac{\partial P_{j \rho}}{\partial t_{\sigma}}\right)+F \frac{\partial P_{j \sigma}}{\partial t_{\sigma}}+\Pi_{j \rho}[i j \rho],
$$

which is an equation that can be written:

$$
\begin{equation*}
[i]=\frac{\partial F}{\partial x_{i}}-P_{i \sigma} \frac{\partial F}{\partial t_{\sigma}}+\Pi_{j \rho} \frac{\partial P_{i \rho}}{\partial x_{j}}+A_{\sigma \rho} \frac{\partial P_{i \rho}}{\partial t_{\sigma}} \tag{30.3}
\end{equation*}
$$

with our previous notations. Finally, the necessary and sufficient conditions for the existence of a geodetic field read:

$$
\begin{equation*}
[i j \rho]=0, \quad[i]=0 \tag{30.4}
\end{equation*}
$$

31. The EULER equations. - Indeed, it is not difficult to prove that any $\mu$-dimensional manifold that is intersected transversally by a geodetic field must satisfy the EULER differential equations:

$$
\begin{equation*}
\frac{d}{d t_{\alpha}} f_{p_{i \alpha}}-f_{x_{i}}=0 \tag{31.1}
\end{equation*}
$$

However, it is more interesting and instructive to exhibit a general identity from which one can infer that as an immediate consequence.

To that end, we give the $p_{i \alpha}$ to be entirely-arbitrary functions of $\left(x_{i}, t_{\alpha}\right)$ and calculate the remaining quantities $f\left(x_{i}, t_{\alpha}, p_{i \alpha}\right), \pi_{i \alpha}=f_{p_{i \alpha}}, P_{i \alpha}$, etc., with the help of our previous formulas as functions of position, in any case.

Furthermore, we introduce the symbol $d \psi / d t_{\alpha}$ for any function $\psi\left(x_{i}, t_{\alpha}\right)$ that enters into (31.1), in particular, and is defined by the relation:

$$
\begin{equation*}
\frac{d \psi}{d t_{\alpha}}=\frac{\partial \psi}{\partial t_{\alpha}}+\frac{\partial \psi}{\partial x_{i}} p_{i \alpha} \tag{31.2}
\end{equation*}
$$

With those preliminaries, we consider the relation (13.8):

$$
F \pi_{i \alpha}=P_{i \sigma} \bar{g}_{\sigma \alpha}
$$

and deduce from that equation by differentiation that:

$$
\begin{equation*}
F d \pi_{i \alpha}=-\pi_{i \alpha} d F+\bar{g}_{\sigma \alpha} d P_{i \sigma}+P_{i \sigma} d \bar{g}_{\sigma \alpha} . \tag{31.3}
\end{equation*}
$$

Now, from a formula that is obtained from equation (25.2), we have:

$$
g d \bar{g}_{\sigma \alpha}=\left(\bar{g}_{\sigma \alpha} \bar{g}_{\rho \lambda}-\bar{g}_{\rho \alpha} \bar{g}_{\sigma \lambda}\right) d \bar{g}_{\rho \lambda}
$$

so, from (13.4), (13.8), (13.9), and (13.2):

$$
\begin{gather*}
F f P_{i \sigma} d \bar{g}_{\sigma \alpha}=F\left(\pi_{i \alpha} \bar{g}_{\rho \lambda}-\pi_{i \lambda} \bar{g}_{\rho \alpha}\right)\left(p_{j \lambda} d P_{j \rho}+P_{j \rho} d p_{j \lambda}\right), \\
f P_{i \sigma} d \bar{g}_{\sigma \alpha}=\left(f \pi_{i \alpha} \Pi_{\rho \lambda}-\bar{g}_{\rho \alpha} p_{j \lambda} \pi_{i \lambda}\right) d P_{j \rho}+F\left(\pi_{i \alpha} \pi_{j \lambda}-\pi_{i \lambda} \pi_{j \alpha}\right) d p_{j \lambda} \tag{31.4}
\end{gather*}
$$

Upon substituting that relation in (31.3), we will get:

$$
\begin{equation*}
F f d \pi_{i \alpha}=-f \pi_{i \alpha} d F+\left(f \pi_{i \alpha} \Pi_{\rho \lambda}+\bar{g}_{\rho \alpha} b_{j i}\right) d P_{j \rho}+F\left(\pi_{i \alpha} \pi_{j \lambda}-\pi_{i \lambda} \pi_{j \alpha}\right) d p_{j \lambda} \tag{31.5}
\end{equation*}
$$

We now introduce the notation:

$$
\begin{equation*}
\Omega_{i}=F f\left(\frac{d \pi_{i \alpha}}{d t_{\alpha}}-f_{x_{i}}\right)-F \pi_{i \alpha} \pi_{j \lambda}\left(\frac{d p_{j \lambda}}{d t_{\alpha}}-\frac{d p_{j \alpha}}{d t_{\lambda}}\right) \tag{31.6}
\end{equation*}
$$

and obtain from (31.5):

$$
\begin{equation*}
\Omega_{i}=-F f-f \pi_{i \alpha} \frac{d(F)}{d t_{\alpha}}+\left(f \pi_{i \alpha} \Pi_{j \rho}+\bar{g}_{\rho \alpha} b_{j i}\right) \frac{d P_{j \rho}}{d t_{\alpha}} . \tag{31.7}
\end{equation*}
$$

Now, from (23.1), (23.2), and (31.2), one has:

$$
\begin{gathered}
-F f f_{x_{i}}=f^{2} F_{x_{i}}=f^{2} \frac{\partial(F)}{\partial x_{i}}-f^{2} \Pi_{j \rho} \frac{\partial P_{j \rho}}{\partial x_{i}} \\
\frac{d(F)}{d t_{\alpha}}=\frac{\partial(F)}{\partial x_{k}} p_{k \alpha}+\frac{\partial(F)}{\partial t_{\alpha}} \\
\frac{d P_{j \rho}}{d t_{\alpha}}=\frac{\partial P_{j \rho}}{\partial x_{k}} p_{k \alpha}+\frac{\partial P_{j \rho}}{\partial t_{\alpha}}
\end{gathered}
$$

When everything is substituted in (31.7), after some simplifications, that will imply that:

$$
\begin{equation*}
\Omega_{i}=f b_{k i} \frac{\partial(F)}{\partial x_{k}}+f\left(\Pi_{k \rho} b_{j i}-\Pi_{j \rho} b_{k i}\right) \frac{\partial P_{j \rho}}{\partial x_{k}}-f \pi_{i \alpha} \frac{\partial(F)}{\partial t_{\alpha}}+\left(f \pi_{i \alpha} \Pi_{j \rho}+\bar{g}_{\rho \alpha} b_{j i}\right) \frac{\partial P_{j \rho}}{\partial t_{\alpha}} . \tag{31.8}
\end{equation*}
$$

With the use of (15.6) and (28.1), that can be written:

$$
\begin{equation*}
\Omega_{i}=f b_{k i} L_{k}(F)-f \Pi_{j \rho} b_{k i}\left(\frac{\partial P_{j \rho}}{\partial x_{k}}-\frac{\partial P_{k \rho}}{\partial x_{j}}\right)+\left(f \pi_{i \alpha} \Pi_{j \rho}+\bar{g}_{\rho \alpha} b_{j i}\right) \frac{\partial P_{j \rho}}{\partial t_{\alpha}} \tag{31.9}
\end{equation*}
$$

However, from (28.4) and (30.1), one has:

$$
\begin{gathered}
\frac{\partial P_{j \rho}}{\partial x_{k}}-\frac{\partial P_{k \rho}}{\partial x_{j}}=[j k r]+P_{k \sigma} \frac{\partial P_{j \rho}}{\partial t_{\sigma}}-P_{j \sigma} \frac{\partial P_{k \rho}}{\partial t_{\sigma}}, \\
L_{k}(F)=[k]+\Pi_{j \rho}[j k \rho]-(F) \frac{\partial P_{k \rho}}{\partial t_{\sigma}} .
\end{gathered}
$$

If one introduces those quantities into (31.9) and notes that from (15.6), (13.7), (12.2), and (12.1):

$$
\begin{gathered}
b_{k i} P_{k \sigma}=p_{i \sigma}, \\
\bar{g}_{\rho \sigma}=F a_{\rho \sigma}=f A_{\rho \sigma}=f\left(\delta_{\rho \sigma} F-P_{j \rho} \Pi_{j \sigma}\right),
\end{gathered}
$$

then almost all of the terms will vanish, and what will remain is:

$$
\begin{equation*}
\Omega_{i}=f b_{k i}[k] \tag{31.10}
\end{equation*}
$$

We will get the identity that we would like to exhibit by comparing (31.6) and (31.10); it reads:

$$
\begin{equation*}
\frac{d \pi_{i \alpha}}{d t_{\alpha}}-f_{x_{i}}=\frac{b_{k i}}{F}[k]+\frac{\pi_{i \alpha} \pi_{j \beta}}{f}\left(\frac{d p_{j \alpha}}{d t_{\beta}}-\frac{d p_{j \beta}}{d t_{\alpha}}\right) \tag{31.11}
\end{equation*}
$$

32.     - If the function of position $p_{i \alpha}\left(x_{j}, t_{\beta}\right)$ that have before us, in particular, belongs to a geodetic field that intersects a $\mu$-dimensional manifold transversally then we will have:

$$
[k]=0, \quad \frac{d p_{j \alpha}}{d t_{\beta}}=\frac{d p_{j \beta}}{d t_{\alpha}}
$$

at any point of that manifold. The left-hand side of (31.11) must then vanish on that manifold, and it will be an integral of the EULER equations (31.1).

We would like to call the geodetic field a distinguished field when an extremal can be laid through each point of that field that intersects the field transversally. The extremals in question, which are defined by those extremals and the manifolds $S_{\alpha}=\lambda_{\alpha}$, are called a complete figure of the variational problem.
33. - We consider an arbitrary family of extremals that simply cover a region of $(n+\mu)$ dimensional space. The left-hand side and the last term in the identity (31.11) must vanish then, from which it will follow that all $[k]=0$.

However, in order for the extremals to define a field that generates a complete figure of the variational problem, one must also have that all $[j k \rho]=0$, and it is known that this does not always have to happen, even for $\mu=1$.


[^0]:    ${ }^{(1)}$ J. HADAMARD, "Sur quelques questions de calcul des variations." Bull. Soc. Math. de France 33 (1905), 7380.
    ( ${ }^{2}$ ) C. CARATHÉODORY, "Über die kanonischen Veränderlichen in der Variationsrechnung der mehrfachen Integrale," Math. Ann. 85 (1922), 78-88; "Über ein Reziprozitätsgesetz der verallgemeinerten LEGENDREschen Transformation," Math. Ann. 86 (1922), 272-275,
    $\left({ }^{3}\right)$ A. HAAR, "Über adjungierte Variationsprobleme und adjungierte Extremalflächen," Math. Ann. 100 (1928), 481-502.

[^1]:    $\left({ }^{4}\right)$ T. LEVI-CIVITA, "Sur la régularization du problème des trois corps," Acta Mathematica 42 (1920), 99-144.

[^2]:    $\left({ }^{5}\right)$ Cf., the citation in ( ${ }^{2}$ ).

