“Die Methode der geodätischen Äquidistanten und das Problem von Lagrange,” Acta Math. **47** (1925), 199-236.

The method of geodetic equidistants and the Lagrange problem

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INTRODUCTION

1. – In the following pages, the method of geodetic equidistants, which I have already employed on various occasions, will be applied to the general problem of the calculus of variations in an $(n + 1)$-dimensional space with p ordinary differential equations as auxiliary conditions ([1]).

The method of geodetic equidistants offers various other advantages in that so-called *Lagrange problem* besides greater geometric clarity. For example, exhibiting the WEIERSTRASS *E*-function, which would lead to a difficult construction in its usual treatment, is almost immediate. Moreover, one obtains the HILBERT independent integral with almost no calculation. Finally, one is in a position to exhibit all of the relations that exist in HAMILTON-JACOBI theory without needing to construct an extremal field from the outset. In contrast, one can defer the rather complicated field construction until *after* one introduces canonical coordinates, and all the difficulties that arise from the auxiliary conditions will then be eliminated automatically.

CHAPTER I

GEODETIC EQUIDISTANTS

2. Introductory remarks. – We will now employ the following notations in order to simplify our presentation. We give two positive whole numbers p and n, the second of which is greater. The ordinary indices $i, j, \ldots$ shall always run through all numbers form 1 to n, while the singly-primed indices $i', j', \ldots$ shall run through $1, 2, \ldots, p$, and finally, the doubly-primed indices $i'', j'', \ldots$ shall run through the numbers $(p + 1), (p + 2), \ldots, n$.

3. – With those preliminaries, we consider p functions:

$$G_{k'}(t, x_i, \dot{x}_i)$$

of $(2n + 1)$ variables $t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n$. We assume that those functions are analytic, along with all of the remaining ones that we shall consider. However, the reader can (if that is important to him) convince himself that all of the following conclusions will be valid when one assumes that the functions that we shall consider possess continuous partial differential quotients of the first three orders.

In addition, we assume that the equations:

$$G_{k'}(t, x_i, \dot{x}_i) = 0 \tag{1}$$

([1]) "Über die diskontinuierlichen Lösungen in der Variationsrechnung," Diss. Gött. 1904, pp. 63. "Über das allgemeine Problem der Variationsrechnung," Gött. Nach. (1905), 83-90. "Sur une méthode du Calcul des Variations," Rend. Circ. Matem. di Palermo **25** (1908), 36-49. RIEMANN-WEBER, *Die Differential- und Integralgleichungen der Mechanik und Physik,* 7th ed., Braunschweig, Vieweg, 1925, pp. 170-212.

are all fulfilled at a point P_0^* with the coordinates t^0, x_i^0, $\dot{x}_i^0$, and that the matrix:

(2) $$\left(\frac{\partial G_{k'}}{\partial \dot{x}_i}\right),$$

which consists of *p* rows and *n* columns has rank *p*. We can even assume, with no loss of generality, that the variables are enumerated from the outset in such a way that our assumption about the rank of (2) can be written in the form:

(3) $$\left|\frac{\partial G_{k'}}{\partial \dot{x}_{j'}}\right| \neq 0 .$$

3. – Under those assumptions, the equations:

(4) $$G_{k'}(t, x_i, \dot{x}_i) = \varepsilon_{k'}$$

can be solved for the $\dot{x}_{j'}$ when the quantities $|\,t - t^0\,|$, $|\,x_i - x_i^0\,|$, $|\,\dot{x}_{j''} - \dot{x}_{j''}^0\,|$, and $\varepsilon_{k'}$ are taken to be sufficiently small. In that way, we will get equations of the form:

(5) $$\dot{x}_{j'} = \varphi_{j'}(t, x_i, \dot{x}_{m''}, \varepsilon_{k'}) .$$

We now let $\bar{G}(t, x_i, \dot{x}_i)$ denote a function of our (2*n* + 1) variables that is supposed to vanish for all values of those variables for which the *p* equations (1) are fulfilled. If we replace the $\dot{x}_{j'}$ in $\bar{G}$ with their values in (5) then we will get a relation:

(6) $$\bar{G}(t, x_i, \dot{x}_i) = \Phi(t, x_i, \dot{x}_{m''}, \varepsilon_{k'}) ,$$

and our assumption about $\bar{G}$ states that the function Φ must vanish when all $\varepsilon_{k'}$ are zero.

However, one will be able to determine analytic functions $B_{k'}$ of the (2*n* + 1) variables *t*, x_i, $\dot{x}_{m''}$, $\varepsilon_{k'}$ in this case for which the identity:

(7) $$\Phi = \sum_{k'} B_{k'}\,\varepsilon_{k'}$$

is fulfilled. For *p* > 1, there are even infinitely many different systems of functions $B_{k'}$ that satisfy the last relation. We now set:

$$B_{k'}(t, x_i, \dot{x}_{m''}, G_{j'}) = \beta_{k'}(t, x_i, \dot{x}_i) ,$$

in which we employ equations (4). The identity:

(8) $$\bar{G} \equiv \sum_{k'} \beta_{k'} G_{k'}$$

will then follow from (6) and (7). Now since, conversely, any function $\bar{G}$ that satisfies the condition (8) will vanish at the same time as the $G_{k'}$, we can state the following:

Theorem 1:

A necessary and sufficient condition for a function $\bar{G}(t, x_i, \dot{x}_i)$ to vanish at the same time as all $G_{k'}$ consists of demanding that $\bar{G}$ can be represented as a linear form in the $G_{k'}$ with regular analytic coefficients.

4. – We now consider two systems of p equations:

(9) $$G_{k'} = 0 , \qquad \bar{G}_{m'} = 0 ,$$

for which the matrices:

(10) $$\left(\frac{\partial G_{k'}}{\partial \dot{x}_i}\right) \text{ and } \left(\frac{\partial \bar{G}_{m'}}{\partial \dot{x}_i}\right)$$

both have rank p at the point in question.

We would like to show that for the two systems (9) to be equivalent, it is sufficient that each of the $\bar{G}_{m'}$ should vanish at all points where the equations $G_{k'} = 0$ are all fulfilled.

Due to the theorem in the previous section, there are, in fact, functions $\beta_{m'k'}$ that are regular and analytic at the point considered and for which the equations:

(11) $$\bar{G}_{m'} = \sum_{k'} \beta_{m'k'} G_{k'}$$

are satisfied identically, and for the equivalence of the two systems of equations (9), it must be further shown that the relation:

(12) $$|\, \beta_{m'k'} \,| \neq 0$$

exists at our point P_0^*. By assumption, the rank of the second matrix in (10) is equal to p. There will then be p indices $n_1, n_2, \ldots, n_p$ such that when one has the relation:

$$S_{i'j'} = \frac{\partial \bar{G}_{i'}}{\partial x_{n_{j'}}},$$

the determinant will be:

$$(13) \qquad | S_{i'j'} | \neq 0 .$$

Now, when one considers that all $G_{k'} = 0$ at P_0^*, it will follow from (11) that:

$$S_{i'j'} = \sum_{k'} \beta_{m'k'} \frac{\partial \bar{G}_{k'}}{\partial x_{n_{j'}}},$$

and one infers from this and the multiplication rule for determinants that:

$$(14) \qquad | S_{i'j'} | = \left| \beta_{m'k'} \right| \cdot \left| \frac{\partial \bar{G}_{k'}}{\partial x_{n_{j'}}} \right| .$$

This latter relation shows us that the relation (12) is a consequence of (13), and we have the theorem:

Theorem 2:

In order for the system of equations (9), *whose matrices* (10) *both have rank p, to be equivalent to each other, it is necessary and sufficient that equations* (11) *must exist, which will make the determinant* $\left| \beta_{m'k'} \right| \neq 0$ *by itself.*

5. – If one starts from equations (5) and sets, in particular:

$$\psi_{j'}(t, x_i, \dot{x}_{m''}) = \varphi_{j'}(t, x_i, \dot{x}_{m''}, 0)$$

and

$$(15) \qquad \Gamma_{j'}(t, x_i, \dot{x}_i) = \dot{x}_{j'} - \psi_{j'}(t, x_i, \dot{x}_{m''})$$

then what was done in § **3** will show that the systems of equations $G_{k'} = 0$ and $\Gamma_{j'} = 0$ are equivalent to each other. One can then write the $\Gamma_{j'}$ as homogeneous linear forms in the $G_{k'}$, and conversely.

6. Statement of the problem. – We shall consider an arbitrary curve C in an $(n + 1)$-dimensional space of (t, x_i) that is defined by the equations:

(16) $$x_i = x_i(t)$$

and set:

$$\dot{x}_i = \frac{dx_i}{dt}$$

along that curve. The equations $G_{k'} = 0$ then represent a system of ordinary differential equations that must be satisfied along all curve that we would like to consider in what follows.

It is clear that we can replace the system of differential equations (1) with any equivalent system $\bar{G}_{m'} = 0$ in our conditions, and in particular, with the system of equations:

(17) $$\Gamma_{j'} = \dot{x}_{j'} - \psi_{j'} = 0 .$$

7. – Now let a function $L(t, x_i, \dot{x}_i)$ be given. We consider the curve integral:

(18) $$I = \int_{t_1}^{t_2} L(t, x_i, \dot{x}_i)\, dt ,$$

which we take along a curve segment C for which the equations of condition (1) are satisfied and whose endpoints are denoted by P_1 and P_2 .

The set of all curves that begin at P_1 and satisfy the equations $G_{k'} = 0$ cover a point-set A in the space of (t, x_i) whose boundary we will define and determine in § **43**. In general, that point-set will be contained in an $(n + 1)$-dimensional region, but the number of dimensions can also be smaller, as we shall see (§ **45**).

If the point P_2, which is contained in A, by assumption, does not lie on the boundary of that point-set then there will be infinitely many curves that connect P_1 to P_2 and satisfy the equations (1). One can then ask whether individual representatives of those curves can be found for which the integral (18) possesses a smaller value than it does for the neighboring ones.

One can further vary the problem in the calculus of variation in many ways by varying the boundary conditions. For example, instead of demanding that the points P_1 and P_2 must be fixed, one might demand that they must be found on given curves or hypersurfaces.

If one would like to treat all of those problems separately and consider singular cases of each of them then a complicated formalism would arise such that one would find their way through it only with great effort. Fortunately, one can, however, preface the treatment of those questions with a much simpler theory in which the aforementioned point-set A plays no role and whose results will provide a universal instrument that will allow one to treat all of the problems that were mentioned, along with many others.

8. Equivalent problems. – There are functions $\bar{L}(t, x_i, \dot{x}_i)$ that are different from L and for which the equation exists:

$$\int_{t_i}^{t_2} \bar{L}\, dt = \int_{t_i}^{t_2} L\, dt$$

when those curve integrals are taken along curves that satisfy the auxiliary conditions (1).

In order for that to be true, it is necessary and sufficient that the difference $(\bar{L} - L)$ must vanish on any curve-element for which the equations $G_{k'} = 0$ are fulfilled. From Theorem 1 of § **3**, for that to be true, it is, in turn, necessary and sufficient that one can write:

$$\bar{L} = L + \sum_{k'} \alpha_{k'} G_{k'} , \tag{19}$$

in which the $\alpha_{k'}$ mean arbitrary analytic functions of the $(2n + 1)$ variables $(t, x_i, \dot{x}_i)$. In particular, one can choose those functions with the help of equations (17) such that $\bar{L}$ depends upon only the $(2n + 1 - p)$ variables $(t, x_i, \dot{x}_{m''})$. (Cf., §§ **31**-**33**).

9. Geodetic gradient. – We consider a one-parameter family of n-dimensional surfaces in the space of (t, x_i) that simply cover a certain region V in that $(n + 1)$-dimensional space and are represented by the equation:

$$S(t, x_i) = \lambda . \tag{20}$$

We now assume that our curve (16) goes through the family of surfaces (20) and contacts none of those surfaces. If we set:

$$\Delta(t, x_i, \dot{x}_i) = S_t + \sum_i S_{x_i} \dot{x}_i , \tag{21}$$

to abbreviate, then the latter fact will be expressed by the relation:

$$\frac{d\lambda}{dt} = \Delta \neq 0 . \tag{22}$$

When we replace the quantity λ in equation (20) with $-\lambda$ in the given case, we can even assume that the inequality:

$$\Delta > 0 \tag{23}$$

is valid along our curve, with no loss of generality.

10. – We now consider the integral (18) between a fixed point on our curve segment that corresponds to the parameter value t_1 and a variable point with the parameter value t. Since we have:

$$\frac{d\lambda}{dt} > 0$$

along the entire curve, due to (22) and (23), we can choose λ to be the independent variable and write equation (18) in the form:

$$I(\lambda) = \int_{\lambda_1}^{\lambda_2} \frac{L}{\Delta} d\lambda .$$

At each point in the region V, the derivative of the latter function with respect to λ, i.e., the expression:

$$\frac{dI}{d\lambda} = \frac{L}{\Delta} ,$$

depends upon only the *direction* of the curve (16) at that point.

*We now define the **geodetic gradient** of our family of surfaces $S = \lambda$ at the point in question to be the direction for which the expression $L : \Delta$ is as small as possible when one considers the auxiliary conditions $G_{k'} = 0$. That minimal value of $L : \Delta$ shall be called the **magnitude** of the geodetic gradient.*

11. – Hence, the rank of the matrix:

$$\left(\frac{\partial}{\partial \dot{x}_i} \frac{L}{\Delta}, \frac{\partial G_{k'}}{\partial \dot{x}_i} \right)$$

which consists of n rows and $(p + 1)$ columns, must be equal to at most p for the geodetic gradient. Namely, if that rank were equal to $(p + 1)$ then one would solve the $(p + 1)$ equations:

$$\frac{L}{\Delta} = u , \qquad G_{k'} = 0$$

for $(p + 1)$ of the variables $\dot{x}_i$, and in that way determine the ones that one considers to be arbitrarily close to them, and for which, on the one hand, the condition equations $G_{k'} = 0$ are fulfilled, and on the other, $L : \Delta$ would possess a smaller value than that of the smallest-possible one.

A linear dependency then exists between the $(p + 1)$ columns of our matrix. In other words, there are $(p + 1)$ constants $\nu_0, \nu_1, \ldots, \nu_p$ that do not all vanish, in such a way that the n equations:

$$\nu_0 \frac{\partial}{\partial \dot{x}_i}\left(\frac{L}{\Delta}\right) + \sum_{k'} \nu_{k'} \frac{\partial G_{k'}}{\partial \dot{x}_i} = 0 \qquad (i = 1, 2, \ldots, n)$$

will all be satisfied. We have assumed that the matrix (2) has rank p, and it will then follow that $\nu_0 \neq 0$ in the latter equations. If we then perform the differentiation of $L : \Delta$, once we have replaced Δ with its value in (21), then we will get equations that, when multiplied by the non-vanishing quantity $\Delta^2 : \nu_0$ and after introducing the notation:

$$\frac{\nu_{k'} \Delta}{\nu_0} = \mu_{k'} ,$$

will read as follows:

$$\Delta \left(\frac{\partial L}{\partial \dot{x}_i} + \sum_{k'} \mu_{k'} \frac{\partial G_{k'}}{\partial \dot{x}_i}\right) - L\, S_{x_i} = 0 .$$

We now set:

$$M(t, x_i, \dot{x}_i, \mu_{k'}) = L + \sum_{k'} \mu_{k'} G_{k'} , \tag{24}$$

to abbreviate, and remark that the equation $M = L$ is valid for every line element that fulfills the auxiliary conditions (1). Our last equations can then be written:

$$\Delta \frac{\partial M}{\partial \dot{x}_i} - M\, S_{x_i} = 0 \qquad (i = 1, 2, \ldots, n) . \tag{25}$$

That system of equations, when combined with the auxiliary conditions, represents the necessary conditions that the geodetic gradient must satisfy at each point of the region V.

12. The Weierstrass *E*-function. – We imagine that the conditions (25) are fulfilled for the line element $(t, x_i, \dot{x}_i)$ and consider a second line element (t, x_i, x_i') that goes through the same point in the space of (t, x_i) and for which the auxiliary conditions $G_{k'} = 0$ are fulfilled. We now introduce the notations:

$$\left\{\begin{aligned} L' &= L(t, x_i, x_i'), \\ M' &= L + \sum_{k'} \mu_{k'} G_{k'}(t, x_i, x_i'), \\ \Delta' &= S_t + \sum_{k'} S_{x_i} x_i', \end{aligned}\right. \tag{26}$$

for which the $\mu_{k'}$ keep all of their old values.

In order for a line element $(t, x_i, \dot{x}_i)$ to represent a geodetic gradient, it is sufficient that for a sufficiently-small positive ε and for all values x_i' that satisfy the condition $|x_i' - \dot{x}_i| < \varepsilon$, the relation:

$$\frac{L'}{\Delta'} - \frac{L}{\Delta} \geq 0$$

is always fulfilled. Since we have assumed that $\Delta > 0$, we can restrict ourselves to those values of x_i' for which we likewise have $\Delta' > 0$ and write:

$$L' - \frac{\Delta'}{\Delta} L \geq 0 ,$$

instead of the last relation. Furthermore, from our assumptions, the equations $L = M$ and $L' = M'$ also exist. We can also write our condition in the form:

$$M' - \frac{\Delta'}{\Delta} M \geq 0 \tag{27}$$

then. However, we have the identity:

$$M' - \frac{\Delta'}{\Delta} M = M' - M - \frac{\Delta' - \Delta}{\Delta} M , \tag{28}$$

and on the other hand, we infer from (21) and (26) that:

$$\Delta' - \Delta = \sum_i S_{x_i} (x_i' - \dot{x}_i) .$$

If we substitute the latter expression in (28) and replace $M\, S_{x_i} : \Delta$ with its value that follows from (25) then we will ultimately get:

$$M' - \frac{\Delta'}{\Delta} M = M' - M - \sum_i M_{x_i} (x_i' - \dot{x}_i) .$$

If we, with WEIERSTRASS, then introduce the notation:

$$E(t, x_i, \dot{x}_i, x_i', \mu_{k'}) = M' - M - \sum_i M_{x_i} (x_i' - \dot{x}_i) , \tag{29}$$

the condition (27) will ultimately assume the form:

$$E(t, x_i, \dot{x}_i, x_i', \mu_{k'}) \geq 0 . \tag{30}$$

That relation is remarkable due to the fact that the function S (t, x_i) *that determines our family of surfaces no longer entered into it anywhere.*

13. The Legendre condition. – The Weierstrass E-function that was defined by equation (29) can be regarded as the remainder in the Taylor development of the function $M(t, x_i, x_i', \mu_{k'})$ in powers of $(x_i' - \dot{x}_i)$ when one truncates the linear terms in that development. We can then write:

$$W = \tfrac{1}{2}\sum_{i,j} \tilde{M}_{\dot{x}_i \dot{x}_j}(x_i' - \dot{x}_i)(x_j' - \dot{x}_j) \tag{31}$$

when we let $\tilde{M}_{\dot{x}_i \dot{x}_j}$ denote the second partial derivatives of M for an intermediate value $\vartheta x_i'$ + $(1-\vartheta)\,\dot{x}_i$, in which $0 < \vartheta < 1$. On the other hand, by assumption, we must consider the equations $G_{k'}(t, x_i, x_i') = 0$ and $G_{k'}(t, x_i, \dot{x}_i) = 0$, from which, it will similarly follow that:

$$\sum_i \frac{\partial \tilde{G}_{k'}}{\partial \dot{x}_i}(x_i' - \dot{x}_i) = 0\,. \tag{32}$$

The derivatives of the $G_{k'}$ in it are taken for intermediate values $\vartheta_{k'}\, x_i' + (1-\vartheta_{k'})\,\dot{x}_i$ for which the $\vartheta_{k'}$ satisfy the relations $0 < \vartheta_{k'} < 1$.

14. – We now let $\xi_{j''}$ denote a sequence of (n – p) arbitrary real numbers that do not all vanish and set:

$$x_{j''}' = \dot{x}_{j''} + s\,\xi_{j''}\,. \tag{33}$$

Due to the condition (3), for sufficiently-small values of s, we can then calculate the remaining (n – p) functions $x_{j''}'$ as functions of s from the p equations:

$$G_{k'}(t, x_i, x_i') = 0\,,$$

in which we substitute the values (33) of the $x_{j''}'$. Those functions will then have the form:

$$x_{j''}' = \dot{x}_{j'} + s\,\xi_{j'} + ((s^2))\,. \tag{34}$$

If we now substitute the values (33) and (34) in our E-function then that function will satisfy the condition:

(35) $$\lim_{s=0} \frac{2E}{s^2} = \sum_{i,j} M_{\dot{x}_i \dot{x}_j} \xi_i \xi_j ,$$

when it is considered to be a function of s, due to the relation (31). On the other hand, if one substitutes the values (33) and (34) of x_i' in equations (32) then they will be satisfied identically for sufficiently-small values of s, and when one lets s converge to zero, one will get the conditions:

(36) $$\sum_i \frac{\partial G_{k'}}{\partial \dot{x}_i} \xi_i = 0 .$$

15. – We now consider the quadratic form:

(37) $$Q = \sum_{i,j} M_{\dot{x}_i \dot{x}_j} \xi_i \xi_j .$$

If it were negative for certain values ξ_i^0 of the ξ_i that satisfy the linear equations (36) then one must note that not all $\xi_{j'}^0$ can vanish, due to (3). If one substitutes the latter values in (33) and calculates the $x_{j'}'$ using (34), in which one must have $\xi_{j'} = \xi_{j'}^0$, then it would follow from (35) that the E-function must likewise be negative for sufficiently-small values of s.

We can then state the following theorem:

In order for there to exist no line elements $(t, x_i, \dot{x}_i)$ *in the neighborhood of the initial element* (t, x_i, x_i') *for which the E-function is negative, it is necessary that the quadratic form must satisfy* $Q \geq 0$ *for all* ξ_i *that satisfy the linear conditions* (36).

16. – It is known that when one adds the relation $\sum \xi_i^2 = 1$ to the linear conditions (36), the minimum of the quadratic form (37) will be equal to the smallest root ρ_0 of the equation in ρ [1]:

(38) $$D(\rho) = \begin{vmatrix} M_{\dot{x}_i \dot{x}_j} - \delta_{ij}\,\rho & \dfrac{\partial G_{k'}}{\partial \dot{x}_i} \\ \dfrac{\partial G_{m'}}{\partial \dot{x}_j} & 0 \end{vmatrix} = 0 .$$

The theorem in the previous section implies the necessary condition $\rho_0 \geq 0$ for us. However, if $\rho_0 = 0$ then one can say nothing at all in advance about the sign of E. One can easily construct

[1] The δ_{ij} in this means the number zero or one according to whether $i \neq j$ or $i = j$.

examples for which the sign of E changes and other ones for which $E \geq 0$. One would then be dealing with a limiting case for which no general theory is possible, and which we will mainly pass over.

By contrast, if $\rho_0 > 0$ then (since the roots of the algebraic function are continuous functions of its coefficients) there will be a positive quantity η such that for:

$$\left| \tilde{M}_{\dot{x}_i \dot{x}_j} - M_{\dot{x}_i \dot{x}_j} \right| < \eta , \qquad \left| \frac{\partial \tilde{G}_{k'}}{\partial \dot{x}_i} - \frac{\partial G_{k'}}{\partial \dot{x}_i} \right| < \eta ,$$

the smallest roof of $D(\rho) = 0$ will likewise be positive when one replaces the $M_{\dot{x}_i \dot{x}_j}$ in (38) with $\tilde{M}_{\dot{x}_i \dot{x}_j}$ and the $\frac{\partial G_{k'}}{\partial \dot{x}_i}$ with $\frac{\partial \tilde{G}_{k'}}{\partial \dot{x}_i}$.

If one now remarks that the E will appear to be a quadratic form in the $(x_i' - \dot{x}_i)$ when it is written in the form (31) and considers that the conditions $G_{k'}(t, x_i, x_i') = 0$ are equivalent to (32) then it will follow that there is a quantity ε such that the relation $E \geq 0$ will exist for all x_i' that satisfy the auxiliary conditions and for which $| x_i' - \dot{x}_i | < \varepsilon$. In addition, E will vanish only when the line elements x_i' and $\dot{x}_i$ coincide.

The fact that all roots of $D(\rho) = 0$ are positive implies the relation:

$$D(0) = \begin{vmatrix} M_{\dot{x}_i \dot{x}_j} - \delta_{ij}\,\rho & \frac{\partial G_{k'}}{\partial \dot{x}_i} \\ \frac{\partial G_{m'}}{\partial \dot{x}_j} & 0 \end{vmatrix} \neq 0 , \tag{39}$$

which will play an important role in what follows.

17. Remark on the equivalent problems. – The geodetic gradient of a family of surfaces $S(t, x_i) = \lambda$, the Weierstrass E-function, and the quantity ρ_0 that was considered in the last section must, by definition, remain unchanged when our problem is replaced with an equivalent one. That can also be confirmed by calculation.

For example, if we set:

$$\left\{ \begin{aligned} \bar{L} &= L + \sum_{k'} \alpha_{k'} G_{k'} , \\ \bar{G}_{m'} &= \sum_{k'} \beta_{m'k'} G_{k'} , \qquad \left| \beta_{m'k'} \right| \neq 0, \\ \bar{M} &= \bar{L} + \sum_{m'} \bar{\mu}_{m'} \bar{G}_{m'} , \end{aligned} \right. \tag{40}$$

as in §§ **4** and **8**, and determine the $\bar{\mu}_{m'}$ by the relations:

$$\mu_{k'} = \alpha_{k'} + \sum_{m'} \beta_{m'k'}\, \bar{\mu}_{m'} \tag{41}$$

then we will get not only:

$$\bar{M} = M$$

for every line element for which the $G_{k'} = 0$, but also:

$$\bar{M}_{\dot{x}_i} = M_{\dot{x}_i} ,$$

$$\bar{M}_{\dot{x}_i \dot{x}_j} = M_{\dot{x}_i \dot{x}_j} + \sum_{k',n'} \left(\frac{\partial \alpha_{k'}}{\partial \dot{x}_j} + \bar{\mu}_{n'} \frac{\partial \beta_{n'k'}}{\partial \dot{x}_j} \right) \frac{\partial G_{k'}}{\partial \dot{x}_j} + \sum_{m',l'} \left(\frac{\partial \alpha_{m'}}{\partial \dot{x}_i} + \bar{\mu}_{l'} \frac{\partial \beta_{l'm'}}{\partial \dot{x}_i} \right) \frac{\partial G_{m'}}{\partial \dot{x}_j} .$$

Equations (25) for the geodetic gradient will then be satisfied for the same line element only when the multipliers $\mu_{k'}$ (which play an entirely subordinate role, as well will see in the next chapter) are transformed linearly with the help of (41).

The E-function likewise remains invariant, from (29). Finally, if one considers that, due to $G_{k'} = 0$, the equations:

$$\frac{\partial G_{m'}}{\partial \dot{x}_i} = \sum_{k'} \beta_{m'k'} \frac{\partial G_{k'}}{\partial \dot{x}_i}$$

will exist, and one sets:

$$\bar{D}(\rho) = \begin{vmatrix} \bar{M}_{\dot{x}_i \dot{x}_j} - \delta_{ij}\, \rho & \dfrac{\partial \bar{G}_{k'}}{\partial \dot{x}_i} \\ \dfrac{\partial \bar{G}_{n'}}{\partial \dot{x}_j} & 0 \end{vmatrix}$$

then one will infer from our relations that:

$$\bar{D}(\rho) = \left| \beta_{m'k'} \right|^2 D(\rho) ,$$

from which the invariance of ρ_0 will likewise follow.

18. Geodetic slope curves. – We shall assume, for the moment, that the condition that $L > 0$ is satisfied for the line element in question. Since the relation $M > 0$ will exist in that case, from (24), we can write the conditions (25) for the geodetic gradient symmetrically. To that end, we define a quantity σ by the relation:

$$\Delta = \sigma M . \tag{42}$$

Now, since $M \neq 0$, it follows from (25) that:

$$S_{x_i} = \sigma M_{\dot{x}_i} \qquad (i = 1, 2, \ldots, n)\,, \tag{43}$$

and the last two equations, in conjunction with (21), will yield:

$$S_t = \sigma\left(M - \sum_i M_{\dot{x}_i}\,\dot{x}_i\right). \tag{44}$$

Those equations show that the ratios of the quantities S_{x_i}, S_t are already determined by the $(2n + p + 1)$ quantities $(t, x_i, \dot{x}_i, \mu_{k'})$, or when one considers the auxiliary conditions $G_{k'} = 0$, by the $(2n + 1)$ independent quantities $(t, x_i, \dot{x}_{j''}, \mu_{k'})$. Those quantities shall be called the coordinates of a *complete line element* that cuts the family of surfaces $S = \lambda$ *transversally*.

19. – If we know a transverse complete line element to our family of surfaces $S = \lambda$ at any point in space then we can calculate the neighboring transverse complete line element with the help of equations (43) and (44) only when the functional determinant satisfies:

$$\frac{\partial\left(\sigma M_{\dot{x}_j}, G_{k'}, \sigma\left(M - \sum_k \dot{x}_k M_{\dot{x}_k}\right)\right)}{\partial(\dot{x}_i, \mu_{l'}, \sigma)} \neq 0\,. \tag{45}$$

However, an elementary calculation will show that this determinant can be set equal to $M\,\sigma^n D(0)$. The condition (45) is then fulfilled, due to our last assumptions and the relation (39).

It then follows from this that in a certain neighborhood of P_0, the transverse complete line elements to our family of surfaces can be combined into curves that contact each point of the corresponding geodetic gradients of $S = \lambda$. Those curves shall be called the *geodetic slope curves* of the family of surfaces. At the same time, the $\mu_{k'}$ can be calculated as functions of the (t, x_i) in that same neighborhood of P_0 .

20. Geodetically-equidistant surfaces. – If we substitute the values of $\dot{x}_i$ as functions of (t, x_i) that were just calculated in the expression for $L : \Delta$ then we will get a relation of the form:

$$\frac{L}{\Delta} = \chi(t, x_i) \tag{46}$$

in all cases.

We now pose the following definition:

The family of surfaces $S = \lambda$ shall be called ***geodetically-equidistant*** *when the function $\chi(t, x_i)$ that appears in equation* (46) *remains constant along every surface of our family, i.e., when we can write:*

$$\frac{L}{\Delta} = \omega(S(t, x_i)) . \tag{47}$$

Naturally, the property of a family that it consists of geodetically-equidistant surface depends upon the problem in the calculus of variations being considered, but from what was done in § **17**, it will still apply when one replaces it with an equivalent one. It is a *geometric* property of a family of surfaces. One must understand that we mean the following:

If $\varphi(u)$ is a monotone, continuous, differentiable function then we can also represent our family of surfaces by the equation:

$$\bar{S}(t, x_i) = \varphi(S(t, x_i)) = \bar{\lambda} .$$

However, since it will follow from:

$$\bar{S}_t = \varphi'(S) S_t , \qquad \bar{S}_{x_i} = \varphi'(S) S_{x_i}$$

that σ will merely take on a different value when one replaces the quantities S_{x_i}, S_t with $\bar{S}_{x_i}$, $\bar{S}_t$ in equations (43) and (44), *the geodetic slope curves will remain unchanged.*

By contrast, the expression $L : \Delta$ *will not* remain invariant. One must replace it with $L : \bar{\Delta}$, in which:

$$\bar{\Delta} = \bar{S}_t + \sum_i \bar{S}_{x_i} \dot{x}_i = \varphi'(S) \cdot \Delta .$$

However, $L : \bar{\Delta}$ still remains constant on each surface of the family when (47) is fulfilled, since one will have:

$$\frac{L}{\bar{\Delta}} = \frac{1}{\varphi'(S)} \cdot \frac{L}{\Delta} = \frac{\omega(S)}{\varphi'(S)} .$$

21. – The last formula allows one to write the condition for geodetic equidistance in an especially simple way. Namely, if one notes that due to the facts that $L > 0$ and $\Delta > 0$, one also has $\omega(S) > 0$, then it will follow that one can set $\varphi'(u) = \omega(u)$, since the function $\varphi(u)$ that is calculated from that equation is monotonically increasing. However, $L = \bar{\Delta}$ in this case. In other words, one can *define* the families of geodetically-equidistant surface by the fact that one demands that the quantity $\sigma = 1$ in equations (43) and (44).

The condition for geodetic equidistance can then be written:

(48)
$$\begin{cases} S_{x_i} = M_{\dot{x}_i} , \\ S_t = M - \sum_j \dot{x}_j M_{\dot{x}_j} , \\ G_{k'} = 0. \end{cases}$$

We will also speak of geodetically-equidistant surfaces now when equations (48) *are fulfilled, but the condition* $L > 0$, *which we have used in an essential way up to now, does not exist.*

If one eliminates the $(n + p)$ quantities $\dot{x}_i$, $\mu_{k'}$ from equations (48) then one will get the condition for the geodetic equidistance of the family of surfaces $S = \lambda$ in the form of a partial differential equation for S that can, in its own right, serve to calculate all elements of our problem, as we will see in the next chapter.

22. The Hilbert independent integral. – We consider a family of geodetically-equidistant surfaces for which equations (48) are fulfilled, and which simply cover a certain region in the space of (t, x_i). Secondly, we consider an arbitrary curve:

(49)
$$x_i = x_i (t)$$

that goes through the surfaces of our families and satisfies the auxiliary conditions that $G_{k'} = 0$. If one then sets:

$$\frac{dx_i}{dt} = x_i'(t) ,$$

and one lets P_1 and P_2 denote two points on the curve (40) that correspond to the values t_1 and t_2 of the coordinate t, in which one must have $t_2 > t_1$, and if S_1 and S_2 denote the values of the function $S(t, x_i)$ at those two points then the relation will exist:

$$\begin{aligned} S_2 - S_1 &= \int_{t_1}^{t_2} \frac{dS(t, x_i)}{dt} dt \\ &= \int_{t_1}^{t_2} \left\{ S_t + \sum_j S_{x_j} x_j' \right\} dt . \end{aligned}$$

However, the last relation can now be written in the form:

(50)
$$S_2 - S_1 = \int_{t_1}^{t_2} \left\{ M + \sum_j M_{\dot{x}_j} (x_j' - \dot{x}_j) \right\} dt$$

with the help of equations (48). The value ($S_2 - S_1$) of that line integral depends upon only the endpoints of the curve along which it is taken and is nothing but the HILBERT *independent integral*. In formula (50), the quantities $\dot{x}_i$, $\mu_{k'}$, that enter into the function M and its derivatives are the coordinates of the complete line element that goes through our geodetically-equidistant family of surfaces transversally at the point in question.

Due to the assumed equations $G_{k'}(t, x_i, x_i') = 0$, we now have the relations:

$$M' = M(t, x_i, x_i', \mu_{k'}) = L(t, x_i, x_i') ,$$

and the curve integral of L along the curve (49) will read:

$$J = \int_{t_1}^{t_2} M' \, dt .$$

If we subtract equation (50) from this last equation term-by-term and recall the definition (29) of the Weierstrass E-function then we will ultimately get:

$$J - (S_2 - S_1) = \int_{t_1}^{t_2} E(t, x_i, \dot{x}_i, x_i', \mu_{k'}) \, dt \quad . \tag{51}$$

23. Solutions of some special problems. – The relation (51) is the basic equation from which the solutions to all of the problems that we mentioned in § **7** will arise.

For example, assume that P_1 and P_2 both lie on the same geodetic slope curve C of our family of geodetically-equidistant surfaces.

If the curve (40) initially coincides with C then we must set $x_i' = \dot{x}_i$, and as a result $E = 0$ at each point of that curve, and the curve integral along C, which we shall denote by I, will satisfy the relation:

$$I = S_2 - S_1 . \tag{52}$$

For any other curve γ that connects P_1 to P_2 without leaving the region that is covered by the surfaces $S = \lambda$, and which makes a sufficiently-small angle at each of its points with the geodetic slope curve that goes through that point, from our assumption, we will have $E > 0$, such that the relation $J > (S_2 - S_1)$ will be true, from (51). It follows from that fact and (52) that $J > I$, which solves the first problem in § **7**.

24. – However, a certain difficulty will arise for most boundary conditions that ordinarily present themselves, namely, that one is often compelled to choose the initial surface S_1 to be a structure of dimension less than n. The region in which the surfaces $S = \lambda$ are regular will cover only part of each neighborhood of P_1 in that case, and it is usually impossible to establish whether the comparison curve γ does or does not remain inside of the aforementioned region in the neighborhood of P_1 .

It is then simplest for one to appeal to a trick that L. TONELLI recently discovered ([1]). Briefly, it consists of choosing a point Q between P_1 and P_2 on the comparison curve γ such that the part of γ between Q and P_2 runs inside of the region that is covered by the surfaces $S = \lambda$. A slope curve e_Q of that family of surfaces then goes through the point Q and one proves that when Q is chosen to be sufficiently close to P_1, the piece of γ between P_1 and Q can be embedded in a family of geodetically-equidistant surfaces $\bar{S}(t, x_i) = \lambda$ that contains a complete neighborhood of P_1 in which a slope curve is precisely e_Q and for which the function is constant along the singular surface S_1, in addition. If one then treats each of the two pieces of the comparison curve γ separately with the help of formula (51) then one will ultimately obtain the result that is implied.

CHAPTER II

CANONICAL COORDINATES ([2])

25. Definition. – As before, we set:

$$M(t, x_i, \dot{x}_i, \mu_{k'}) = L(t, x_i, \dot{x}_i) + \sum_{k'} \mu_{k'} G_{k'}(t, x_i, \dot{x}_i)$$

and consider a complete line element for our problem, for which the auxiliary conditions $G_{k'} = 0$ are fulfilled, and the determinant is:

$$D(0) = \begin{vmatrix} M_{\dot{x}_i \dot{x}_j} & \dfrac{\partial G_{k'}}{\partial \dot{x}_i} \\ \dfrac{\partial G_{m'}}{\partial \dot{x}_j} & 0 \end{vmatrix} \neq 0 , \tag{53}$$

from which it already follows that the rank of the matrix (2) must be equal to p, moreover.

The determinant (53) is the functional determinant of the $(n + p)$ functions $M_{\dot{x}_i}$, $G_{m'}$ with respect to the $\dot{x}_j$ and $\mu_{k'}$. It then follows from the inequality (53) that the system of equations:

$$y_i = M_{\dot{x}_i} , \qquad z_{m'} = G_{m'} = M_{\mu_{m'}} \tag{54}$$

([1]) "Sul problema isoperimetrico con un punto terminale mobile," Mem. R. Accad. Bologna (7) **10** (1922-23).

([2]) Most of the results in this chapter were developed before by J. HADAMARD in his *Leçons sur le Calcul des Variations*, Paris, Hermann, 1910, pp. 217-280 in a somewhat different context.

can be solved for the $\dot{x}_j$ and $\mu_{k'}$. In that way, we will get the equations:

$$(55) \qquad \begin{cases} \dot{x}_j = \Phi_j(t, x_i, y_i, z_{m'}), \\ \mu_{k'} = X_{k'}(t, x_i, y_i, z_{m'}). \end{cases}$$

In order to express the auxiliary conditions $G_{m'} = 0$, we merely need to set $z_{m'} = 0$, and equations (55) then teach us that we can characterize our complete line element with the help of the $(2n + 1)$ quantities (t, x_i, y_i), which we would like to call the *canonical coordinates* of the line element.

What was done in § **17** shows, in addition, that the value of the canonical coordinates of a complete line element will remain unchanged when we replace our variational problem with an equivalent one.

26. The Hamiltonian function. – We now construct the Legendre transformation with the help of the equation:

$$(56) \qquad \mathsf{H}(t, x_i, y_i, z_{m'}) = -M + \sum_j \dot{x}_j\, y_j + \sum_{k'} \mu_{k'}\, z_{k'} ,$$

in whose right-hand side we replace the $\dot{x}_j$ and $\mu_{k'}$ with their values in (55). It follows from the known formulas for the Legendre transformation that:

$$(57) \qquad \mathsf{H}_t = -M_t , \qquad \mathsf{H}_{x_i} = -M_{x_i} , \quad \mathsf{H}_{y_j} = \dot{x}_j , \qquad \mathsf{H}_{z_{k'}} = \mu_{k'} .$$

Finally, we define a function $H(t, x_i, y_i)$ by the equation:

$$(58) \qquad H(t, x_i, y_i) = \mathsf{H}(t, x_i, y_i, 0) .$$

If we remark that whenever all $z_{k'}$ vanish, we will have:

$$H_t = \mathsf{H}_t , \qquad H_{x_i} = \mathsf{H}_{x_i} , \qquad H_{y_i} = \mathsf{H}_{y_i}$$

then it will follow from equations (57) that for all line elements for which the auxiliary equations $G_{k'} = 0$ exist, we will have:

$$(59) \qquad H_t = -M_t , \qquad H_{x_i} = -M_{x_i} , \quad H_{y_i} = \dot{x}_i .$$

We can calculate the function H directly without needing to exhibit the function H. In order to do that, we need only to calculate $\dot{x}_j$ and $\mu_{k'}$ from the equations:

(60) $$y_i = M_{\dot{x}_i}, \qquad 0 = G_{k'} .$$

When we consider (55), we will get:

(61) $$\begin{cases} \dot{x}_j = \Phi_j(t, x_i, y_i, 0) = \varphi_j(t, x_i, y_i), \\ \mu_{k'} = X_{k'}(t, x_i, y_i, 0) = \chi_{k'}(t, x_i, y_i). \end{cases}$$

We will then have:

(62) $$H(t, x_i, y_i) = -M(t, x_j, \varphi_j, \chi_{k'}) + \sum_j y_j \varphi_j ,$$

and equations (59) say that the formulas for the Legendre transformation will also remain true when the auxiliary conditions $G_{k'} = 0$ are present, which is a result that can also be easily verified directly.

The function $H(t, x_i, y_i)$ is called the *Hamiltonian function of our problem.* It will remain invariant when one replaces the problem with an equivalent one (§ **17**). *Now, it is very remarkable that one can acquire all of the data for our problem from the function H (and its derivatives) alone.*

27. The Legendre condition in canonical coordinates. – We then begin by expressing the determinant $D(\rho)$ that we considered in § **16** with the help of the derivatives of H. To that end, we remark that from the definition of the functions (55), when the functional determinant (53) is composed with the inverse determinant:

(63) $$\begin{vmatrix} \dfrac{\partial \Phi_j}{\partial y_m} & \dfrac{\partial \Phi_j}{\partial z_{h'}} \\ \dfrac{\partial X_{k'}}{\partial y_m} & \dfrac{\partial X_{k'}}{\partial y_{h'}} \end{vmatrix},$$

the result of that composition will be equal to the identity matrix:

$$\begin{vmatrix} \delta_{im} & 0 \\ 0 & \delta_{m'h'} \end{vmatrix}.$$

If one then composes the determinant (38) with (63) then that will give:

$$\begin{vmatrix} \delta_{im} - \rho \dfrac{\partial \Phi_i}{\partial y_m} & 0 \\ 0 & \delta_{m'h'} \end{vmatrix}.$$

Finally, one notes that the relations $\Phi_i = H_{y_i}$ will exist due to (55) and (59), so one will ultimately get:

$$D(\rho) = D(0) \left| \delta_{ij} - \rho H_{y_i y_j} \right| . \tag{64}$$

The Legendre condition then consists of the statement that all roots of the equation in ρ:

$$\left| \delta_{ij} - \rho H_{y_i y_j} \right| = 0 \tag{65}$$

should be positive.

28. The *E*-function in canonical coordinates. – Let t, x_i, y_i and t, x_i, y_i' be the canonical coordinates of two complete line elements that go through the same point. Due to equations (59), (62), and (54), we can write:

$$M = -H + \sum_i H_{y_i} y_i ,$$

$$M' = -H' + \sum_i H'_{y'_i} y'_i ,$$

$$\sum_i M_{\dot{x}_i} (x'_i - \dot{x}_i) = \sum_i H'_{y'_i} (y_i - y'_i) .$$

Upon comparing that with (29), we will then have:

$$E = H - H' - \sum_i H'_{y'_i} (y_i - y'_i) . \tag{66}$$

That expression is constructed in precisely the same way as the expression in (29), and we will get:

$$E = \tfrac{1}{2} \sum_{i,j} \tilde{H}_{y_i y_j} (y'_i - y_i)(y'_j - y_j) \tag{67}$$

by a line of reasoning that is similar to the one in § **13**.

Just as in §§ **14-15**, one can conclude from the latter expression that the Legendre condition is fulfilled when all non-vanishing roots of the equation in σ:

$$\left| H_{y_i y_j} - \delta_{ij} \sigma \right| = 0 \tag{68}$$

are positive. One sees immediately that when one sets $\sigma = 1 / \rho$, the latter condition will be equivalent to the one that we posed in the previous paragraph.

29. The condition for geodetic equidistance. – Equations (25), which define the geodetic gradient of a family of surfaces, now read:

$$\Delta\, y_i = M\, S_{x_i} \tag{69}$$

in our canonical coordinates. If we multiply those equations by $\dot{x}_i$ and sum over i, while considering (21), then we will get:

$$\Delta \cdot \sum_i y_i\, \dot{x}_i + M \cdot S_t = M \cdot \Delta ,$$

or since (62):

$$\Delta \cdot H = - M \cdot S_t . \tag{70}$$

When we observe that $\Delta \neq 0$, it will now follow from (69) and (70) that:

$$H\, S_{x_i} + y_i\, S_t = 0 . \tag{71}$$

It is even easier to exhibit the equations (48) for the geodetic equidistance of surfaces in the family $S = \lambda$ with the help of the canonical coordinates.

$$S_{x_i} = y_i , \quad S_t + H\,(t, x_i, y_i) = 0 . \tag{72}$$

The auxiliary conditions $G_{k'} = 0$ *no longer enter into the relations* (71) *for determining the geodetic gradient and* (72) *for geodetic equidistance.*

In particular, the integration of the differential equations (72) leads to precisely the same thing as the usual case with no auxiliary conditions. The usual Cauchy method of characteristics also leads to the same objective here ([1]). The main result consists of the fact that our slope curves coincide with the Cauchy characteristics, and they are solutions of the canonical differential equations:

$$\dot{x}_i = H_{y_i} , \qquad \dot{y}_i = - H_{x_i} , \tag{73}$$

which are the ones that plays such a significant role in mechanics. We do not need to linger any longer on those known facts. By contrast, we would like to see how the function H allows us to express the fact that we are starting from a variational problem with p differential equations as auxiliary conditions.

([1]) See, e.g., the presentation of those things that I gave in the 7th edition of *Der Differential- und Integralgleichungen der Mechanik und Physik* by RIEMANN-WEBER (Braunschweig, Vieweg, 1925), pp. 189-198, which was edited by R. v. MISES.

30. Properties of the Hamiltonian function. – We start from the remark that the determinant (63), which will be equal to unity when it is multiplied by D (0) (§ **27**), is certainly non-zero. When we observe that $\Phi_i = H_{y_i}$, we can write that as:

$$\begin{vmatrix} H_{y_i y_m} & \dfrac{\partial \Phi_j}{\partial z_{h'}} \\ \dfrac{\partial X_{k'}}{\partial y_m} & \dfrac{\partial X_{k'}}{\partial z_{h'}} \end{vmatrix} \neq 0 .$$

However, it follows immediately from the last relation that the rank of the determinant:

$$(74) \qquad \left| H_{y_i y_j} \right|$$

is at least (n – p).

On the other hand, it follows from what was done in § **26** that:

$$G_{k'}(t, x_i, H_{y_i}) \equiv 0 .$$

Since the matrix of $\partial G_{k'} / \partial \dot{x}_i$ has rank p, by assumption, the latter equations teach us that p mutually-independent, linear, homogeneous relations exist between the rows of the determinant (74), and that as a result, the rank of that determinant will be equal to *at most* (n – p).

It follows from those considerations that the rank of (74) is equal to *precisely* (n – p), and we would like to show, moreover, that this property is characteristic of the function H as the Hamiltonian function of a variational problem with p differential equations as auxiliary conditions.

31. – In order to do that, we would like to prove that when H means an arbitrary function of t, x_i, y_i for which the Hessian determinant (74) has rank (n – p), we can construct functions L and auxiliary conditions $G_{k'} = 0$ whose Hamiltonian function coincides with the given one. Since all equivalent problems possess the same Hamiltonian function, it would suffice to find one particular problem among them.

Now, in algebra one proves that a *symmetric* determinant of (n – p) contains at least one non-vanishing principal subdeterminant of order (n – p) ([1]). A principal subdeterminant is known to be a subdeterminant that one obtains by deleting the rows and columns that cross the principal diagonal.

With our assumptions about the rank of (74), we can then choose our notations in such a way that the relation:

$$(75) \qquad \left| H_{y''_i y''_j} \right| \neq 0$$

([1]) See, e.g., MAX. BÔCHER, *Introduction to higher Algebra*, Macmillan, New York, 1907, § 20, pp. 56.

is fulfilled. As a result, we can solve the $(n - p)$ equations:

$$\dot{x}_{j''} = H_{y_{j''}}(t, x_i, y_i) \tag{76}$$

for the $(n - p)$ quantities $y_{i''}$ and in that way obtain the relations:

$$y_{i''} = \omega_{i''}(t, x_i, \dot{x}_{j''}, y_{k'}) . \tag{77}$$

Now, since, on the other hand, all $(n - p + 1)$-rowed subdeterminants of (74) vanish identically, the functions that one obtains by substituting the values $\omega_{i''}$ of the $y_{i''}$ in the $H_{y_{k'}}$ *will be independent of the* $y_{k'}$. If one performs those substitutions in the p equations:

$$\dot{x}_{k'} = H_{y_{k'}}$$

then one will get p relations:

$$\Gamma_{k'}(t, x_i, \dot{x}_i) = \dot{x}_{k'} - \psi_{k'}(t, x_i, \dot{x}_{j''}) = 0 , \tag{78}$$

which represent the auxiliary conditions of the desired problem in the form (15).

32. – We now calculate the total differential of the function:

$$\bar{L} = - H(t, x_i, y_{k'}, \omega_{j''}) + \sum_{k'} \psi_{k'}\, y_{k'} + \sum_{j''} \dot{x}_{j''}\, \omega_{j''} \tag{79}$$

and obtain:

$$d\bar{L} = \left(-H_t + \sum_{k'} y_{k'} \frac{\partial \psi_{k'}}{\partial t}\right) dt + \sum_i \left(-H_{x_i} + \sum_{k'} y_{k'} \frac{\partial \psi_{k'}}{\partial x_i}\right) dx_i + \sum_{j''} \left(\omega_{j''} + \sum_{k'} y_{k'} \frac{\partial \psi_{k'}}{\partial \dot{x}_{j''}}\right) d\dot{x}_{j''} , \tag{80}$$

when we remark that the coefficients of the remaining terms:

$$\sum_{k'} (-H_{y_{k'}} + \psi_{k'})\, dy_{k'} + \sum_{j''} (-H_{y_{j''}} + \dot{x}_{j''})\, d\omega_{j''}$$

must vanish identically, from the foregoing. Equation (80) now teaches us that $\bar{L}$ is a function of the $(2n - p + 1)$ quantities t, x_i, $\dot{x}_{i''}$ alone and is independent of the p quantities $y_{k'}$. In particular, it follows from this that the $\omega_{j''}$ must be *linear* functions of the $y_{k'}$, since one can infer from (80) that:

(81) $$\omega_{j''} = \bar{L}_{\dot{x}_{j''}} - \sum_{k'} y_{k'} \frac{\partial \psi_{k'}}{\partial x_{j''}},$$

and $\bar{L}_{\dot{x}_{j''}}$, $\frac{\partial \psi_{k'}}{\partial x_{j''}}$ are independent of $y_{k'}$. It further follows from (79) that the function $H(t, x_i, y_{k'}, \omega_{j''})$ is likewise a linear function of the $y_{k'}$ ([1]).

33. – We would now like to show that the Hamiltonian function of $\bar{L}$ coincides with the given function H when we consider the auxiliary conditions $\Gamma_{k'} = 0$.

To that end, we set:

(82) $$\bar{M} = \bar{L} + \sum_{k'} \bar{\mu}_{k'} \Gamma_{k'}$$

and

(83) $$\bar{y}_i = \bar{M}_{\dot{x}_i} .$$

When we note that $\bar{L}$ does not depend upon the $\dot{x}_{k'}$ and consider the form (78) of the $\Gamma_{k'}$, it will follow, first of all, that:

(84) $$\bar{y}_{k'} = \bar{\mu}_{k'} .$$

Secondly, it follows with the help of (80) and (78):

$$\bar{y}_{j''} = \bar{M}_{\dot{x}_{j''}} = \omega_{j''}(t, x_i, \dot{x}_{m''}, y_{k'}) + \sum_{k'} (y_{k'} - \bar{\mu}_{k'}) \frac{\partial \psi_{k'}}{\partial \dot{x}_{j''}} .$$

However, $\bar{M}_{\dot{x}_{j''}}$ is independent of $y_{k'}$, and the right-hand side of the latter equations will remain true when we replace $y_{k'}$ with any values. For example, we can write:

$$y_{k'} = \bar{\mu}_{k'} = \bar{y}_{k'}$$

and obtain:

(85) $$\bar{y}_{j''} = \omega_{j''}(t, x_i, \dot{x}_{m''}, \bar{y}_{k'}) .$$

The functional determinant of the $\omega_{j''}$ with respect to the $\dot{x}_{m''}$ is, by the definition of the $\omega_{j''}$, equal to the inverse of (75), and therefore $\neq 0$. One can then add the latter equations to those of (78) and

([1]) That fact says that the figuratrix of our problem is a developable surface. See HADAMARD, *loc. cit.*, pp. 266.

(84) in order to calculate the $\dot{x}_i$ and $\mu_{k'}$ as functions of the canonical coordinates t, x_i, $\bar{y}_i$. The determinant that corresponds to the determinant (53) in our problem is then $\neq 0$, as it should be.

Our Hamiltonian function $\bar{H}$ now reads:

$$\bar{H}(t, x_i, \bar{y}_i) = -\bar{M} + \sum_i x_i \bar{y}_i \ .$$

Due to the auxiliary conditions $\Gamma_{k'} = 0$, we can set $\bar{M} = \bar{L}$, $\dot{x}_{k'} = \psi_{k'}$ in the last equation and obtain:

$$\bar{H} = -\bar{L} + \sum_{k'} \psi_{k'} \bar{y}_{k'} + \sum_{j''} \dot{x}_{j''} \bar{y}_{j''} ,$$

or, with the help of (79):

$$\bar{H} = H(t, x_i, y_{k'}, \omega_{j''}) + \sum_{k'} \psi_{k'} (\bar{y}_{k'} - y_{k'}) + \sum_{j''} \dot{x}_{j''} (\bar{y}_{j''} - \omega_{j''}) .$$

However, the left-hand side of that equation is independent of the $y_{k'}$. We can then set $y_{k'} = \bar{y}_{k'}$ in the right-hand side, and when we consider (85), we will finally have:

$$\bar{H} = H(t, x_i, y_i) ,$$

i.e., precisely the formula that we would like to prove.

34. The formulas of the theory for the problems in parametric representation. – Our previous results can now be applied, without any new calculations, to the problem in parametric representation, whose meaning for the calculus of variations was emphasized by WEIERSTRASS especially. That will show that the treatment of the variational problem in parametric representation will only become more complicated when one has not previously developed the required formulas of the usual theory.

We shall then assume that our functions L and $G_{k'}$ do not depend upon t, and that L is homogeneous of degree one in the $\dot{x}_i$, while the $G_{k'}$ are homogeneous of an arbitrary degree $\rho_{k'}$. Therefore, for every *positive* value of a parameter λ, the following formulas should be true:

(86) $$L(x_i, \lambda \dot{x}_i) = \lambda L(x_i, \dot{x}_i) ,$$

(87) $$G_{k'}(x_i, \lambda \dot{x}_i) = \lambda^{\rho_{k'}} G_{k'}(x_i, \dot{x}_i)$$

when one sets:

$$M = L + \sum_{k'} \mu_{k'} G_{k'} ,$$

using (24), so:

(88) $$M(x_i, \lambda \dot{x}_i, \lambda^{1-\rho_{k'}} \mu_{k'}) = \lambda M(x_i, \dot{x}_i, \mu_{k'}) .$$

When this last equation is differentiated with respect to $\dot{x}_j$, that will yield the condition:

(89) $$M_{\dot{x}_j}(x_i, \lambda \dot{x}_i, \lambda^{1-\rho_{k'}} \mu_{k'}) = M_{\dot{x}_j}(x_i, \dot{x}_i, \mu_{k'}) .$$

When one partially differentiates (87), (88), and (89) with respect to λ and subsequently sets λ equal to unity, one will further get:

(90) $$\sum_i \frac{\partial G_{k'}}{\partial x_i} \dot{x}_i = \rho_{k'} G_{k'} ,$$

(91) $$\sum_i M_{\dot{x}_i} \dot{x}_i + \sum_{k'} (1 - \rho_{k'}) \mu_{k'} G_{k'} = M ,$$

(92) $$\sum_i M_{\dot{x}_i \dot{x}_j} \dot{x}_i + \sum_{k'} (1 - \rho_{k'}) \mu_{k'} \frac{\partial G_{k'}}{\partial \dot{x}_j} = 0 .$$

35. – It follows immediately from equations (90) and (92) that the determinant (53) is *identically* zero, such that our entire theory up to now would seem to be untenable. However, we remark that due to the homogeneity conditions (86) and (87), the value of our line integral depends upon only the form of the curve in the space of x_i, but not upon the choice of the parameter t. The variational problem will then remain the same when we choose, e.g., the length of the curve to be the parameter, or what amounts to the same thing, when we add the further condition that:

(93) $$\sum_i \dot{x}_i^2 - 1 = 0$$

to the auxiliary conditions $G_{k'} = 0$.

If we now introduce the function:

(94) $$M^* = M + \frac{\sigma}{2}\left(\sum \dot{x}_i^2 - 1\right) ,$$

along with the canonical coordinates:

(95) $$y_i = M^*_{\dot{x}_i} = M_{\dot{x}_i} + \sigma \dot{x}_i ,$$

then we can calculate the $\dot{x}_i$, $\mu_{k'}$, and σ as functions of x_i, y_i from equations (95), (93), and $G_{k'}$ = 0. If we let $\bar{H}(x_i, y_i)$ denote the associated Hamiltonian function, which is defined by the equation:

$$\bar{H} = - M^* + \sum \dot{x}_i M^*_{\dot{x}_i} ,$$

from (62), and we consider all of the last equations, as well as equation (91), then we will get:

$$\bar{H}(x_i, y_i) = \sigma . \tag{96}$$

36. – We could be satisfied with that result. However, we would like to further extend the condition (93), which have introduced into our problem artificially. In order to do that, we remark that when we calculate the $\dot{x}_i$ by the equations:

$$\dot{x}_i = \bar{H}_{y_i} ,$$

as usual, all of our auxiliary conditions must be satisfied, and in particular, equation (93). In that way, we will get the identity:

$$\sum_i \bar{H}^2_{y_i} = 1 , \tag{97}$$

and in addition, upon differentiating with respect to y_j, the equations:

$$\sum_i \bar{H}_{y_i y_j} \bar{H}_{y_i} = 0 . \tag{98}$$

It now follows from our previous theory that the determinant $\left| \bar{H}_{y_i y_j} \right|$ has rank $(n - p - 1)$, since we must consider $(p + 1)$ auxiliary conditions in all, including (93). It will further follow from this that the determinant:

$$\begin{vmatrix} \bar{H}_{y_i y_j} & 0 & \bar{H}_{y_i} \\ \bar{H}_{y_j} & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

has rank $(n - p + 1)$. If we now first multiply the first n columns by $\bar{H}_{y_j}$ and add then to the last one, and secondly add the sum of the first n rows, each multiplied by $\bar{H}_{y_i}$, to the last row then the rank will not change. However, due to (97) and (98), we will get:

$$\begin{vmatrix} \bar{H}_{y_i y_j} & 0 & \bar{H}_{y_i} \\ \bar{H}_{y_j} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

We can now drop the rows and columns that consist of nothing but zeroes from this last formula without reducing the rank and see that the determinant:

$$\begin{vmatrix} \bar{H}_{y_i y_j} & \bar{H}_{y_i} \\ \bar{H}_{y_j} & 0 \end{vmatrix} \tag{99}$$

must have rank $(n - p + 1)$. Since we can also perform the inverses of all those operations, we conclude that when the determinant (99) has rank $(n - p + 1)$, due to the existence of equations (97) and (98), the determinant $\left| \bar{H}_{y_i y_j} \right|$ must have rank $(n - p - 1)$.

37. – Since the parameter t for our problem must be only an auxiliary variable, it will suffice to consider only those families of geodetically-equidistant surfaces that are independent of t, i.e., that have the form $S(x_1, \ldots, x_n) = \lambda$. We must then have $S_t = 0$, and when that is substituted into the last equation in (72), that will show that we need to consider only those complete line elements for which the canonical coordinates fulfill the condition:

$$\bar{H}(x_i, y_i) = 0 . \tag{100}$$

We would now like to show that the condition (100) alone, and not, say, the function $\bar{H}$ itself, is already characteristic of the homogeneous variational problem of § **34**.

Namely, let $H(x_i, y_i)$ be an arbitrary function that vanishes at the same time as $\bar{H}$ without all H_{y_i} vanishing on the surface $\bar{H} = 0$. From § **3**, we will then have:

$$\bar{H}(x_i, y_i) = A(x_i, y_i) \cdot H(x_i, y_i) \tag{101}$$

identically, in which A is $\neq 0$ at all points where H vanishes (cf., § **4**). For all points at which (100) is true, one will further have, upon differentiating (101), that:

$$\begin{cases} \bar{H}_{y_i} = A \cdot H_{y_i}, \quad \bar{H}_{x_i} = A \cdot H_{x_i}, \\ \bar{H}_{y_i y_j} = A \cdot H_{y_i y_j} + A_{y_i} \cdot H_{y_j} + A_{y_j} \cdot H_{y_i}. \end{cases} \tag{102}$$

We next point out that the partial differential equation geodetically-equidistant surface $\bar{H}(x_i, S_{x_i})$ = 0 can always be replaced by the equation $H(x_i, S_{x_i}) = 0$.

The canonical differential equations (73) will represent the same curves in the space of x_i even when one replaces $\bar{H}$ with H *and the sign of* H *is chosen in such a way that* $A > 0$, *in addition.* This last equation, which is also significant to the Legendre condition, shall always be considered in what follows. The transition from $\bar{H}$ to H will then mean simply that one no longer takes the length of the curve to the parameter, but employs another parameter that naturally depends upon the choice of A.

38. – We would now like to speak briefly about the properties of the functions H. The relation (102), in conjunction with $A \neq 0$, shows that the determinants (99) and:

$$\begin{vmatrix} H_{y_i y_j} & H_{y_i} \\ H_{y_j} & 0 \end{vmatrix} \tag{103}$$

must have the same rank $(n + 1 - p)$. Conversely, if H is a function of x_i, y_i for which the determinant (103) has rank $(n + 1 - p)$ then we can regard it as the Hamiltonian function of a *homogeneous* variational problem with p differential equations as auxiliary conditions. We next determine the solution $\bar{H}(x_i, y_i)$ of the partial differential equation:

$$\sum_i \left(\frac{\partial U}{\partial y_i}\right)^2 = 1 ,$$

which vanishes at the same time as H, and its sign is chosen in such a way that one has $A > 0$ in equation (101), which must be true here ([1]). From the foregoing, the rank of the determinant (99) is then equal to $(n + 1 - p)$, and the rank of the determinant $\left|\bar{H}_{y_i y_j}\right|$ is equal to $(n - p - 1)$. From §§ **31-33**, the function $\bar{H}$ is the Hamiltonian function of a variational problem $\bar{L}(x_i, \dot{x}_i)$ with $(p + 1)$ auxiliary conditions that one can replace with p auxiliary conditions of the form $\bar{G}_{k'}(x_i, \dot{x}_i) = 0$, along with equation (93), because equation (97) is true here, by construction. Now, the equivalent *homogeneous* variational problem:

$$L = \bar{L}\left(x_i, \frac{\dot{x}_i}{\sqrt{\sum_j \dot{x}_j^2}}\right)\sqrt{\sum \dot{x}_i^2} ,$$

([1]) The function $\bar{H}$ can be calculated with the help of mere eliminations, but with no integrations.

with the equivalent homogeneous auxiliary conditions:

$$G_{k'}(x_i, \dot{x}_i) = \bar{G}_{k'}\left(x_i, \frac{\dot{x}_i}{\sqrt{\sum_j \dot{x}_j^2}} \right) = 0\,,$$

and the auxiliary condition (93) have the same Hamiltonian function $\bar{H}$. As a result, the homogeneous variational problem with the function L and the auxiliary conditions $G_{k'} = 0$ has the given function $H = 0$ as its Hamiltonian function.

39. – In conclusion, we would like to calculate the Legendre and Weierstrass condition. From (64), for the problem with the Hamiltonian function $\bar{H}$, we can set:

$$D(\bar{\rho}) = D(0)\left| \delta_{ij} - \bar{\rho}\,\bar{H}_{y_i y_j} \right|.$$

We infer the formula:

$$D(\bar{\rho}) = -D(0)\begin{vmatrix} \delta_{ij} - \bar{\rho}\,\bar{H}_{y_i y_j} & 0 \\ \bar{H}_{y_j} & -1 \end{vmatrix}$$

from that and due to (97) and (98), it will follow from the latter that:

$$D(\bar{\rho}) = -D(0)\begin{vmatrix} \delta_{ij} - \bar{\rho}\,\bar{H}_{y_i y_j} & \bar{H}_{y_i} \\ \bar{H}_{y_j} & -1 \end{vmatrix}.$$

From (102), for the homogeneous variational problem with the Hamiltonian function $H = 0$, we then have:

$$D(\bar{\rho}) = -D(0)A^2\begin{vmatrix} \delta_{ij} - \bar{\rho}\,A H_{y_i y_j} & H_{y_i} \\ H_{y_j} & 0 \end{vmatrix}.$$

Now, since $A > 0$, from the foregoing, when we set $A\bar{\rho} = \rho$, we can formulate the Legendre condition in such a way that we demand that *all roots of the equation in* ρ:

(104) $$\begin{vmatrix} \delta_{ij} - \rho H_{y_i y_j} & H_{y_i} \\ H_{y_j} & 0 \end{vmatrix} = 0$$

should be positive.

From (66), the Weierstrass E-function for the problem with the Hamiltonian function $\bar{H}$ reads:

$$\bar{E} = \bar{H} - \bar{H}' - \sum_i \bar{H}'_{y'_i} (y_i - y'_i) \;.$$

If we then set $\bar{E} = A \cdot E$ then when we consider that we must have $H = H' = 0$, it will follow from equations (102) that:

(105) $$E = \sum_i H'_{y'_i} (y'_i - y_i) \;.$$

We can derive the Legendre condition that we presented above in a manner that is similar to the one in 13 when we start from (105) and consider the auxiliary conditions $H = H' = 0$.

40. – For a given homogeneous variational problem, one can calculate the Hamiltonian function H, or even better, the condition $H = 0$, without taking the detour to $\bar{H}$. In order to do that, it suffices to set the quantities $\sigma = 0$ in equations (95). One will then get the equations $y_i = M_{\dot{x}_i}$ and one will need only to eliminate the $\dot{x}_i$ and $\mu_{k'}$ from those equation and the conditions $G_{k'} = 0$. That is always possible, since the $M_{\dot{x}_i}$, as well as the $G_{k'}$, depend upon only the ratios of the $\dot{x}_i$.

41. The Mayer problem. – When the function L in the homogeneous variational problem of § **34** has the form:

(106) $$L = \sum_i \xi_{x_i} \cdot \dot{x}_i \,,$$

we will have a *Mayer problem* before us for which the foregoing theory must be suitable modified somewhat. The ξ_{x_i} in it mean the partial differential quotients of a function $\xi(x_i)$ with respect to the variables x_i. Namely, equations (95) have the form:

(107) $$y_i = \xi_{x_i} + \sum_{k'} \mu_{k'} \frac{\partial G_{k'}}{\partial \dot{x}_i} + \sigma\,\dot{x}_i$$

here. If one now sets:

(108) $$\eta_i = y_i - \xi_{x_i}$$

and calculates the function $\bar{H} = \sigma$ as in § **35** then one will get:

(109) $$\bar{H}(x_i, y_i) = \bar{K}(x_i, \eta_i) \,.$$

In addition, it follows from (107) that the function $\bar{K}$ must satisfy the condition that:

(111)
$$\sum_i \bar{K}_{\eta_i} \cdot \eta_i = \bar{K}.$$

Now, if a function $K(x_i, \eta_i)$ is defined by the equation $\bar{K} = A$, in which one might once more have $A > 0$, then one will infer from (111) that:

(112)
$$\sum_i K_{\eta_i} \cdot \eta_i = 0$$

for all points in the space of (x_i, η_i) at which $\bar{K} = 0$.

42. – The partial differential equation $H(x_i, S_{x_i}) = 0$ for the geodetically-equidistant surface can be written here when one considers (108):

(113)
$$K(x_i, S_{x_i} - \xi_{x_i}) = 0.$$

If one sets $T(x_i) = S - \xi$ then one will get a partial differential equation:

(114)
$$K(x_i, T_{x_i}) = 0$$

in place of (113) *that is determined by the auxiliary conditions* $G_{k'} = 0$ *alone and is completely independent of the choice of function* ξ.

The canonical differential equations for the Cauchy characteristics of the partial differential equation (114) read:

(115)
$$\dot{x}_i = K_{\eta_i}, \qquad \dot{\eta}_i = -K_{x_i}.$$

They coincide with the extremals of our Mayer variational problem for an arbitrarily-prescribed ξ. Namely, one obtains them from (113) with the help of the equations:

$$\dot{x}_i = H_{y_i} = K_{\eta_i},$$
$$\dot{y}_i = -H_{x_i} = -K_{x_i} + \sum_j K_{\eta_j} \xi_{x_i x_j},$$

and the latter equations can be written:

$$\dot{y}_i - \frac{d}{dt}\xi_i = -K_{x_i}.$$

The extremals (115) of our variational problem lie on surfaces T = const. Namely, along those extremals, we have $\eta_i = T_{x_i}$, $\dot{x}_i = K_{\eta_i}$, such that equation (112) will assume the form:

$$\sum_i T_{x_i} \dot{x}_i = 0 ,$$

which proves our assertion.

43. – Now let P be any point on any one of the surfaces T = const., e.g., the surface $T = 0$. We let $S(x_1, \ldots, x_n) = 0$ denote a surface that cuts the surface $T = 0$ at the point P but does not contact it. If we now set $\xi = S - T$ then the surfaces $S = \lambda$ will be geodetically-equidistant for our problem. If the Legendre condition is now satisfied for our variational problem then it will follow from the argument in the first chapter that for all comparison curves that satisfy the auxiliary conditions $G_{k'} = 0$ and connect the point P with a point Q on a surface $S = S_2$, $\xi(Q)$ will have a greater value than it does at points of the extremal that lies on S_2 . However, since that extremal lies on $T = 0$, we can write:

$$\xi(Q) = S(Q) - T(Q) > S_2 ,$$

and since $S(Q) = S_2$, it will follow from this that $T(Q) < 0$. *The comparison curves always lie on one side of the surface* $T = 0$ *then, and one can employ the extremals* (115) *in order to determine the boundary of the region A that we mentioned in* § **7**.

44. – Any function $K(x_i, \eta_i)$ that is homogeneous of degree one in the η_i is characteristic of a Mayer variational problem. The number p of differential equations $G_{k'} = 0$ can, in fact, be calculated when one notes that the rank of the determinant:

$$\begin{vmatrix} K_{\eta_i \eta_j} & K_{\eta_i} \\ K_{\eta_j} & 0 \end{vmatrix}$$

must be equal to $(n + 1 - p)$, and the Legendre condition will be fulfilled when all roots of the equation in ρ :

$$\begin{vmatrix} \delta_{ij} - \rho K_{\eta_i \eta_j} & K_{\eta_i} \\ K_{\eta_j} & 0 \end{vmatrix} = 0$$

are positive.

45. – The extremals of a Mayer problem that start from a point P fill up at most an $(n - 1)$-dimensional manifold B, as they must. In fact, due to the homogeneity condition (110) for K, from which a similar condition for the K_{x_i} will follow, the solutions of the system of equations (115) will represent the same curve as when one fixes the initial values of the x_i and multiplies the initial values of the η_i by an arbitrary parameter value λ.

However, the dimension of the manifold B can be lower. That will happen, e.g., when the differential equations $G_{k'} = 0$ can be replaced with equivalent ones $\bar{G}_{k'} = 0$ in which one finds *complete differentials.*