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## On the integration of certain undetermined systems of differential equations

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In an article that appeared recently in this Journal (t. **143**, pp. 300), *Zervos* generalized a theorem of *Hilbert* on the impossibility of solving the equation:

$$\frac{dz}{dx} = \left(\frac{d^2 y}{dx^2}\right)^2$$

by formulas of the form:

(1)  $x = f(t, w, w_1, ..., w_r), \qquad y = \varphi(t, w, w_1, ..., w_r), \qquad z = \psi(t, w, w_1, ..., w_r),$ 

in which w denotes an arbitrary function of t and in which  $w_1, ..., w_r$  denote its successive derivatives.

It is easy to find the necessary and sufficient condition that a system of h differential equations in h + 1 unknown functions must satisfy in order for one to be able to put its general solution, which is assumed to depend upon an arbitrary function of one argument (\*), into the form that was indicated above, while the right-hand sides can even contain a finite number of *arbitrary constants*.

Indeed, any system of the indicated nature depends upon a system S of *n* Pfaff equations in n + 2 variables, n + 1 of which are dependent, and one of which is independent.

Let

(2) 
$$\omega_i = 0$$
  $(i = 1, 2, ..., n)$ 

be that *Pfaff* system, and let  $\omega_{n+1}$ ,  $\omega_{n+2}$  be two independent *Pfaff* expressions that are also independent of the left-hand sides  $\omega_1, \omega_2, ..., \omega_n$  of the equations of the system.

Agree to let  $\omega'$  denote the bilinear covariant of a *Pfaff* expression  $\omega$ . For an arbitrary *Pfaff* system  $\Sigma$ :

<sup>(\*)</sup> The method applies, moreover, to any differential system whose general integral depends upon an arbitrary function of one argument.

$$\Omega_1 = 0, \qquad \Omega_2 = 0, \ldots, \quad \Omega_N = 0,$$

consider the set of equations of the form:

(4) 
$$l_1 \Omega_1 + l_2 \Omega_2 + \ldots + l_N \Omega_N = 0$$
,

which enjoys the property that is defined by the congruence:

$$l_1 \Omega'_1 + l_2 \Omega'_2 + \dots + l_N \Omega'_N \equiv 0 \qquad (\text{mod } \Omega_1, \Omega_2, \dots, \Omega_n).$$

Agree to say *derived system* of the system (3) to mean the *Pfaff* system  $\Sigma'$  that is defined by all equations like (4). One can similarly consider the system  $\Sigma''$  that is the derived system of  $\Sigma'$ , and so on.

In the case of (2), since one has (\*):

$$\omega'_i \equiv c_i [\omega_{n+1} \ \omega_{n+2}] \pmod{\omega_1, \omega_2, \ldots, \omega_n},$$

either the derived system S' will be identical to the system S for which all of the  $c_i$  are zero or else it will be composed of n - 1 independent equations. In the former case, the system S will be completely integrable; we then discard that case.

We can then suppose that the derived system S' is composed of the first n - 1 equations:

$$\omega_1 = 0, \ \omega_2 = 0, \ \dots, \ \omega_{n-1} = 0$$
.

**2.** – Let  $u_1, u_2, ..., u_s$  denote arbitrary constants that enter into the general solution for the system (2) and set:

$$\chi_i = du_i$$
 (*i* = 1, 2, ..., *s*).

Likewise, set:

(\*) I shall let  $[\omega_{n+1} \ \omega_{n+2}]$  denote the bilinear expression:

$$\omega_{n+1}^d \,\omega_{n+2}^\delta - \omega_{n+1}^\delta \,\omega_{n+1}^d,$$

to abbreviate. Furthermore, the congruence means that the equality will be true when one has:

$$\omega_1^d = \ldots = \omega_n^d = \omega_1^\delta = \ldots = \omega_n^\delta = 0.$$

It results from the hypotheses that were made that if one takes into account the formulas that express the idea that the variables of the system are given in terms of the  $u_i$  and t and the w and  $w_i$  then one will have some identities of the form:

(4) 
$$\begin{cases} \omega_{1} = \beta_{11} \chi_{1} + \dots + \beta_{1s} \chi_{s} + \alpha_{11} \varpi_{1} + \dots + \alpha_{1r} \varpi_{r}, \\ \dots \\ \omega_{n} = \beta_{1n} \chi_{1} + \dots + \beta_{ns} \chi_{s} + \alpha_{n1} \varpi_{1} + \dots + \alpha_{nr} \varpi_{r}. \end{cases}$$

The right-hand sides of those formulas contain the expressions  $\varpi_i$  up to a certain order, namely,  $\varpi_r$ . That does not preclude the possibility that the coefficients  $\alpha_{ik}$  and  $\beta_{ik}$  might depend upon  $\omega_r$ ,  $\omega_{r+1}$ ,  $\omega_{r+2}$ , etc. In any case, the coefficients  $\alpha_{1r}$ , ...,  $\alpha_{nr}$  are not all zero (\*).

If we form the bilinear covariants of the two sides of the relations (4) then we will easily get:

(5) 
$$\omega'_i \equiv \alpha_{ir} [dt \ \overline{\omega}_{r+1}] \qquad (\text{mod } \chi_1, ..., \chi_s, \ \overline{\omega}_1, ..., \ \overline{\omega}_r) \,.$$

On the other hand, we have, by hypothesis:

(6) 
$$\begin{cases} \omega_i' \equiv 0, \\ \omega_n' \equiv c [\omega_{n+1} \, \omega_{n+2}] \end{cases} \pmod{\omega_1, \, \omega_2, \, \omega_r}.$$

The congruences (6) are true *a fortiori* with respect to the moduli  $\chi_1, ..., \chi_s, \varpi_1, ..., \varpi_r$ . It results from a comparison of the formulas (5) and (6) that:

$$\alpha_{1r} = \alpha_{2r} = \dots = \alpha_{n-1,r} = 0 ,$$
  
$$\alpha_{nr} [dt \ \varpi_{r+1}] \equiv c \ [\omega_{n+1} \ \omega_{n+2}] \qquad (\text{mod } \chi_1, \dots, \chi_s, \ \varpi_1, \dots, \ \varpi_r) .$$

Since  $\omega_{n+1}$  and  $\omega_{n+2}$  are defined only up to a linear substitution, and since  $\alpha_{nr}$  is non-zero, one can then deduces that:

$$\begin{aligned} \omega_{n+1} &\equiv dt \\ \omega_{n+2} &\equiv \overline{\omega}_{r+1} \end{aligned} \quad (\text{mod } \chi_1, \, \dots, \, \chi_s, \, \overline{\omega}_1, \, \dots, \, \overline{\omega}_r) \,, \end{aligned}$$

or rather:

(7)  $\begin{aligned} \omega_{n+1} &\equiv dt + \lambda \ \omega_n \\ \omega_{n+2} &\equiv \overline{\omega}_{r+1} + \mu \ \omega_n \end{aligned} (\text{mod } \chi_1, \dots, \chi_s, \ \overline{\omega}_1, \dots, \ \overline{\omega}_{r-1}). \end{aligned}$ 

**3.** – Now consider the system S' whose equations are:

<sup>(\*)</sup> It is impossible for at least one of the expressions  $\varpi$  to not enter into the right-hand sides. Otherwise, one will see with no difficulty that the general integral of *S* will depend upon only arbitrary constants.

$$\omega_1=0$$
, ...,  $\omega_{r-1}=0$ .

The left-hand sides of those equations are linear combination of:

$$\chi_1, \ldots, \chi_s, \varpi_1, \ldots, \varpi_{r-1}$$

Now, from (4), one has:

(8)  $\omega'_i \equiv \alpha_{i,r-1} [dt \ \overline{\omega}_r] \pmod{\chi_1, \ldots, \chi_s, \overline{\omega}_1, \ldots, \overline{\omega}_{r-1}}$   $(i = 1, 2, \ldots, n-1).$ 

On the other hand, let:

(9) 
$$\omega'_{i} \equiv h_{i} [\omega_{n} \ \omega_{n+1}] + k_{i} [\omega_{n} \ \omega_{n+1}] \qquad (\text{mod } \omega_{1}, \ \omega_{2}, \dots, \ \omega_{n-1}) .$$

The latter congruences will be true *a fortiori* when one takes the moduli to be  $\chi_1, ..., \chi_s, \varpi_1, ..., \varpi_{r-1}$ . From (7), they will then reduce to:

(10) 
$$\omega'_{i} \equiv h_{i} \alpha_{nr} [dt \ \overline{\omega}_{r}] + k_{i} \alpha_{nr} [\overline{\omega}_{r} \ \overline{\omega}_{r+1}] \qquad (\text{mod } \chi_{1}, ..., \chi_{s}, \overline{\omega}_{1}, ..., \overline{\omega}_{r-1})$$

Upon comparing (8) and (10), one will see that one has:

$$k_i = 0$$
,  $h_i = -\frac{\alpha_{i,r-1}}{\alpha_{nr}}$ .

Having posed that, if all the  $h_i$  are zero then the system S' will be its own derived system, and as a result, it will be completely integrable. If the  $h_i$  are not all zero then the system S'' will be defined by n - 2 equations. Moreover, since the covariants  $\omega'_1$ ,  $\omega'_2$ , ...,  $\omega'_{n-1}$  will depend upon only  $\omega_n$  and  $\omega_{n+1}$ , when one takes the equations of S' into account, and from a classical theorem, one knows that the equations:

$$\omega_1 = \omega_2 = \ldots = \omega_{n-1} = \omega_n = \omega_{n+1} = 0$$

are completely integrable. In other words, one can suppose (by a change of variables, if necessary) that *the system* S' is a system of n - 1 **Pfaff** equations in n + 1 variables that can be put into a form that is analogous to (4) with the coefficients  $\alpha_{i,r-1}$  not all being zero. One can then repeat the argument that was made for the system S with the system S'.

**4.** – The solution to the problem that was posed will result from this. In order to be able to put the general solution to the system S into the desired form, it is necessary that *if one forms the successive derived systems* S', S", ... *then each of those systems must contain exactly one equation less than the preceding one*, as long as one of the derived systems  $S^{(n-h)}$  is not completely integrable. In that case, it is obvious that the following derived systems will be identical to  $S^{(n-h)}$ 

. If  $S^{(i)}$  is not completely integrable then it can be put into the form of n - i equations in n - i + 2 variables.

*The condition obtained is sufficient.* First suppose that none of the derived systems is completely integrable. The system  $S^{(n-1)}$  will then contain just one equation, which one can assume to be:

 $\omega_1 = 0$ ,

while the system  $S^{(n-2)}$  will contain two, which one can assume to be:

$$\omega_1 = 0$$
,  $\omega_2 = 0$ ,

and so on.

The system  $S^{(n-2)}$  can be put into the form of an equation in three variables that is not completely integrable. It will then reduce to the form:

$$\omega_1 \equiv dy - y_1 \, dz = 0 \; .$$

Since the system  $S^{(n-1)}$  is derived from  $S^{(n-2)}$ , one will have:

$$\omega_1' = [dx \, dy_1] \equiv 0 \qquad (\text{mod } \omega_1, \, \omega_2).$$

As a result,  $\omega_2$  is linear in dx, dy, and dy<sub>1</sub>, and one can suppose that  $\omega_2$  has the form:

$$\omega_3 \equiv dy_2 - y_3 \, dx \, ,$$
$$\omega_n \equiv dy_{n-1} - y_n \, dx \, .$$

As a result, the general integral of the system *S* has the form:

$$x = t$$
,  $y = w$ ,  $y_1 = w_1$ , ...,  $y_{n-1} = w_{n-1}$ ,  $y_n = w_n$ .

Now suppose that the system  $S^{(n-h)}$  is completely integrable. It can then be reduced to the form:

$$dx_1=dx_2=\ldots=dx_h=0.$$

The system  $S^{(n-h+1)}$  is not completely integrable, and it has h + 3 variables. As a result, upon taking into account the equations of  $S^{(n-h)}$ , one can suppose that:

$$\omega_{h+1} \equiv dy - y_1 \, dx \; .$$

The argument then proceeds as before, and one will have:

$$\omega_{h+2} \equiv dy_1 - y_2 \, dx \,,$$
$$\ldots$$
$$\omega_n \equiv dy_{n-h-1} - y_{n-h} \, dx \,.$$

The general integral of *S* then has the form:

$$x_1 = a_1$$
,  $x_2 = a_2$ , ...,  $x_h = a_h$ ,  
 $x = t$ ,  $y = w$ ,  $y_1 = w_1$ , ...,  $y_{n-h} = w_{n-h}$ .

The theorem has been proved completely then.

It should be pointed out that, in the general case, the solution of the system S in the desired form demands the integration of a completely-integrable system of h equations, followed by the reduction of a *Pfaff* equation to the canonical form:

$$dy - y' dx = 0,$$

which will demand two operations of orders 3 and 1, as one knows.

5. – Let us apply the preceding general theorem to the differential equation:

(11) 
$$\frac{dz}{dx} = f\left(x, y, z, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right).$$

That equation is equivalent to the *Pfaff* system in five variables  $S(^*)$ :

(12)  
$$\begin{cases} \omega_1 \equiv dy - y' \, dx = 0, \\ \omega_2 \equiv dy' - y'' \, dx = 0, \\ \omega_3 \equiv dz - f(x, y, z, y', y'') \, dx = 0. \end{cases}$$

In order to determine S', calculate  $\omega'_1$ ,  $\omega'_2$ ,  $\omega'_3$ . We have:

$$\begin{split} \omega_1' &\equiv 0 ,\\ \omega_2' &\equiv [dx \, dy''] \qquad (\text{mod } \omega_1, \, \omega_2, \, \omega_3),\\ \omega_3' &\equiv f_{y'}' [dx \, dy'']. \end{split}$$

<sup>(\*)</sup> The study of *Pfaff* systems in five variables was the subject of a detailed treatise that I published in Ann. Éc. Norm. (3) **27** (1910), pp. 109.

The equations of the system S' are then:

(13) 
$$\begin{cases} \varpi_1 \equiv \omega_1 \equiv dy - y' \, dx = 0, \\ \varpi_2 \equiv \omega_3 - f'_{y'}, \, \omega_2 = dz - f'_{y'} \, dy' - (f - y'' \, f'_{y'}) \, dx = 0 \end{cases}$$

Now determine the derived system S''. One has:

$$\begin{aligned} &\varpi_1' \equiv [dx \, dy'], \\ &\varpi_2' \equiv f_{y'^2}' [dy' \, dy''] + y'' f_{y'^2}' [dx \, dy''] + \lambda [dx \, dy'] \end{aligned} \qquad (\text{mod } \varpi_1, \ \varpi_2). \end{aligned}$$

In order for the system S'' to be composed of one equation, it is necessary and sufficient for one to have:

$$f''_{y''^2} \equiv y'' f''_{y''^2} = 0$$

i.e., that f should be linear in y''.

Conversely, if *f* has the form:

$$f = y'' A(x, y, z, y') + B(x, y, z, y')$$

then the system S'' will be composed of one equation, and the general solution of equation (11) will have the desired form. In order to obtain it in that form, it will suffice to reduce the equation that represents S'' to its canonical form. Now, after all calculations have been done, that equation will be:

$$dz - A \, dy' - \left( A \frac{\partial B}{\partial z} - B \frac{\partial A}{\partial z} + \frac{\partial B}{\partial y'} - \frac{\partial A}{\partial x} - y' \frac{\partial A}{\partial y} \right) dy + \left[ y' \left( A \frac{\partial B}{\partial z} - B \frac{\partial A}{\partial z} + \frac{\partial B}{\partial y'} - \frac{\partial A}{\partial x} \right) - y'^2 \frac{\partial A}{\partial y} - B \right] dx = 0.$$

Another interesting consequence of this is the following one: The differential system that determines the skew curves with given constant torsion reduces to a system of three *Pfaff* equations in five variables. The calculation shows immediately that it does not satisfy the conditions that were stated above. It is therefore impossible to express the coordinates of a point on the most general curve of constant torsion in terms of expressions that depend upon one parameter t, an arbitrary function w of t, and its derivatives up to a certain order in a well-defined manner.