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The scientific work of Ernest Vessiot

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It was Sophus Lie's theory of continuous groups, whether finite or infinite, that suggested most of Vessiot's work and gave a remarkable unity to it.

I. – Integration problems.

1. The reducibility of linear differential equations. – The reducibility of differential equations poses problems for the analysts that are most vast and profound. In the case of linear equations, Émile Picard had already extended the famous theory that Evariste Galois had applied to algebraic equations by introducing the *Galois group* or *rationality group* to the solution of those linear equations. Vessiot completed Picard's fertile idea and made it more profound by showing that one can deduce from it a complete theory of the reducibility of linear differential equations, in particular, the conditions for such an equation to be integrated algebraically or by quadratures. However, although all of the coefficients in the equations that Émile Picard had studied were rational functions of the independent variable, the same thing was not true for the equations that Vessiot studied. He began by defining a *domain of rationality* Δ that contained all real or complex constants, all rational functions of the independent variable, and all coefficients of the given equation. Furthermore, that domain Δ satisfied the condition that it had to contain, not only two functions, their sum, difference, product, and quotient, but also their derivatives. Vessiot then showed the existence of a group G of linear substitutions with constant coefficients that operated on a system of n independent solutions of the given equation (which is supposed to have order n) and enjoy the following property:

In order for a rational function (*i.e.*, *one with coefficients that belong to* Δ) *of n independent solutions of the given equation to be equal to a function of the independent variable that belongs to* Δ, it is necessary and sufficient that it should be invariant *under the group G*. That group G is the Galois group or the rationality group of the equation.

The structure of that group determines the operations that must be performed in order to integrate the equation, each of which has the goal of reducing the rationality group to one of its subgroups by adding conveniently-chosen auxiliary equations to the domain of rationality of the integral. One can then deduce that the equation is integrable by quadratures if the rationality group is integrable, i.e., if, assuming that it has order r, it admits an invariant subgroup of order $r - 1$, which admits an invariant subgroup of order $r - 2$, and so on. The equation is algebraically integrable if the rationality group is discontinuous. The general differential equation of order $n > 1$ is not integrable by quadratures.

The results that Vessiot presented in his thesis (1892) rapidly became classical and have stimulated numerous French and foreign works.

Recently, Vessiot extended his theory to an important class of linear differential equations in the case where the domain of rationality *does not contain all constants*, despite the difficulties that this hypothesis would imply for the proof of the existence of the rationality group.

2. Lie systems and their integration. – The linear equations are not the only ones that possess systems of fundamental integrals, i.e., that enjoy the property that their general integrals are expressed in terms of a certain number of particular integrals and integration constants by formulas that are independent of the choice of those particular integrals. The same thing is true for the auxiliary equation that Vessiot introduced into his theory in order to reduce the rationality group.

Vessiot was much-occupied with certain first-order differential systems that Sophus Lie encountered and to which he gave the name of *Lie systems*. A Lie system is associated with a finite and continuous group Γ that operates on the unknown functions $x_1, x_2, \ldots, x_n$ of the Lie system. If one knows the finite equations of the group Γ, which is supposed to depend upon r parameters $a_1, a_2, \ldots, a_r$, then one will get the general integral of the Lie system by subjecting an arbitrary point $x_1^0, x_2^0, \ldots, x_n^0$ to the continuous transformation of the group Γ whose parameters are $a_1(t), a_2(t), \ldots, a_r(t)$, in which the $a_i(t)$ are arbitrarily-given functions of the independent variable t. One sees immediately that if one applies a transformation of the group Γ whose parameters are $c_1, c_2, \ldots, c_r$ to a particular integral of the Lie system then one will again have an integral of the same Lie system that is obtained by applying the transformation of Γ that is obtained from the transformation whose parameters are $a_1, a_2, \ldots, a_r$, followed by the transformation whose parameters are c_i, to the point $x_1^0, x_2^0, \ldots, x_n^0$.

Vessiot showed that two Lie systems whose associated groups are isomorphic (i.e., they have the same structure) will enjoy the property that the integration of one of them will imply the integration of the other one. He then deduced from this that one can always reduce the integration of a Lie system to that of a linear system, which is possibly completed by quadratures. *One can always apply Galois theory to the integration of any Lie system then.*

In the course of his research in the integration of Lie systems, Vessiot was led to treat several problems in the theory of finite continuous groups. Thus, he devoted a paper to the determination of the finite equations of a finite continuous group when one knows the infinitesimal transformations. He showed that for the transitive groups, the search in question will depend upon only eliminations, quadratures, and the integration of linear differential equations.

3. Integrating systems that admit continuous groups of transformations. – Lie systems belong to that class, since any Lie system will admit the group that it is associated with. In that case, the group will be finite, and the integration will be performed by only quadratures and the transcendents that define the linear equations. In the general case, S. Lie indicated, *but only in some special cases*, a path to follow in order to decompose the desired integration into two successive stages:

1. The integration of a resolvent system (R) that admits no group of transformations.

2. The integration of a differential system (A) whose solutions are all deduced from any one of then by the transformations of a given group (G).

Vessiot reprised that decomposition of the problem by a new method whose success was based upon the possibility of obtaining *automorphic* systems for the systems (A). That is what one calls a system whose solutions are deduced from any one of them by the transformations of a group that are performed upon only the dependent variables. That group is said to be *associated* with the systems, which will again have fundamental solutions, in a sense.

He obtained the reduction of a given system to a resolvent system (R) and an automorphic system (A) that admits a group (G) for its associated group in a form that was direct and precise by means of a new and complete method that permitted one to form all of the differential invariants and invariant differential systems of an arbitrary group (G) when one starts from its defining equations. Lie had showed that, regardless of whether the group (G) was finite or infinite, that method will reduce to the successive integration of simple automorphic systems. Vessiot first concluded that the integration of the automorphic system reduces to the integration of ordinary differential systems, and he did that by assuming only the existence of the simple infinite groups that were known in 1902, which was the era in which he produced his article. However, the later results of Élie Cartan on the simple infinite groups in 1907 showed the existence of new *intransitive* simple infinite groups, which implied the possibility of integrating automorphic systems by utilizing systems of linear partial differential equations in a single unknown function of an arbitrary number of variables.

All of that extremely important research was naturally accompanied by work relating to the structure of infinite groups, about which one then has only the very fragmentary results that were due to S. Lie, F. Engel, and Medolaghi. It is curious that they still did not arrive at a precise definition of the isomorphism of two infinite groups. The definition that is now universally adopted is due to Vessiot, along with Élie Cartan. It is based upon the notion of *prolongation*.

Vessiot produced some other work on the theory of the similitude of infinite groups and the search for subgroups of an infinite group, in particular, invariant subgroups.

4. Generalized Lie systems. – Those are systems of ordinary differential equations in one independent variable t such that the formulas that give the general values x_1, x_2, …, x_n of the unknown functions as functions of their initial values x_1^0, x_2^0, …, x_n^0 are the equations of a family of transformations of an *infinite* group Γ that is said to be associated with the system when one considers the x_i^0 to be the coordinates of a point, while the x_i are those of its homologue, and one considers t to be a parameter. The notion of *isomorphic* systems extends from Lie systems to more general systems. One can then reduce their integration to that of the typical systems. The *canonical systems* of analytical mechanics constitute one particularly important class of generalized Lie systems.

5. The problem of the reducibility of more general systems of differential equations. – Vessiot's thesis explained the notion of the reducibility of systems of linear differential equations by generalizing Galois theory, and the method could be extended to certain automorphic systems of differential equations that were each associated with a finite group. In 1898, Jules Drach began to study an entirely-new problem by proposing to extend Galois theory to an absolutely-arbitrary system of ordinary differential equations that was not automorphic, in general. Drach's views were as original as they were fecund and extremely important, but his statements and proofs contained some serious gaps and sources of error. Drach took the unknowns of the problem to be n arbitrary independent first integrals $x_1, x_2, \ldots, x_n$, while supposing that the given system reduced to a system of first-order differential equations in n unknown functions of one independent variable, which is always possible. The *solutions* of the problem that were thus composed were deduced from each other by various point-wise transformations that were executed on the first integrals $x_1, x_2, \ldots, x_n$. If one is given a well-defined domain of rationality Δ then passing from one solution to another would not, unfortunately, be achieved by a *rational* transformation (in the sense that is conferred upon that term by being given the domain of rationality Δ), in general, in such a way that the existence of the *rationality group* would remain undetermined. It was then necessary to restate the question. That is what Vessiot succeeded in doing by constructing a remarkable perfectly-rigorous theory that won the Grand Prix des Sciences mathématiques in 1902. The committee that was charged with examining the papers that were presented to the competition showed its appreciation for Vessiot's paper by asserting that it constituted a coherent and very complete body of work that completely filled the gaps that persisted in the important question that Drach had raised.

Meanwhile, it would be entirely unjustified for one to fail to recognize the importance of Drach's work, since he made some remarkable applications of his theory, and at the very least, his discovery of the lines of curvature on the wave surface, which was a problem whose solution had eluded the efforts of the greatest geometers up till then.

6. The rationality group and the specific group. – After publishing his 1902 paper, Vessiot brought some new ideas to his theory by introducing, along with the *rationality group*, what he called the *specific* group, which made the solution to the problem that Drach posed almost intuitive. The two groups are isomorphic. The specific group has rational defining equations; it acts upon the variables of the given differential system. The rationality group gives the following law by which the specific group permutes the first integrals of the system. Either of those two groups define the mode of reducibility of the system (for a given domain of rationality). Those two groups are finite. The integration of the system comes down to the integration of linear equations, or more simply to quadratures, if the two groups, which are isomorphic to each other, are integrable.

We should not abandon this subject without pointing out that Vessiot has quite recently given a new foundation for Galois theory itself, which relates to algebraic equations, by a recent evolution of his ideas.

7. Groups of linear functional transformations. – The Fredholm and Volterra integral equations of the second kind are associated with groups of linear functional transformations. Vessiot showed that the groups that relate to Fredholm integral equations can admit all of the possible structures of finite groups, whereas the groups that relate to Volterra integral equations are all Abelian. Vessiot proved the last result by proving that two functions that permute with a third, in the Volterra sense, will permute with each other, which is a theorem that Volterra himself had stated was likely. Those functional groups are associated with certain classes of integro-differential equations whose correspondents, like those of the Lie systems, correspond to point-wise groups of transformations.

8. The theory of integrating sheaves of infinitesimal transformations. – That is a new theory of the integration problems that appear in Élie Cartan's theory of Pfaff systems, and the two theories are coupled with each other by a type of duality. Cartan's systems in involution correspond to Vessiot's involutive sheaves. The existence of the characteristics of systems of partial differential equations and their classification depend upon the structure of the sheaf in involution. One thoroughly remarkable application that Vessiot made relates to the second-order partial differential equations $F = (x, y, z, p, q, r, s, t) = 0$ that are integrable by Darboux's method. He discovered the fundamental fact that the problem of studying those equations is entirely dominated by the theory of continuous groups of transformations. Each case of integrability is provided by one such group, and Vessiot gave a regular method of integration for each case.

9. Vessiot's geometric work. – Vessiot carried out some work that was concerned with various branches of geometry. In algebraic geometry, he gave two entirely-new solutions to the problem of reducing algebraic singularities. One of them was used by Émile Picard in order to reduce the singularities of algebraic surfaces. In the realm of Euclidian geometry, he developed the theory of invariants of minimal curves that is based upon the notion of a *pseudo-arclength*, and he showed how one could establish the theory of surfaces completely by considering only their null-length lines. He devoted several papers to conformal geometry, or as one can say, anallagmatic (i.e., elementary properties of figures that are composed of spheres and circumferences, differential invariants of curves, surfaces, and curves that are traced on a given surface, etc.)

Vessiot wrote a treatise on *Géométrie supérieure* that has had several successive editions and has found great success thanks to its precision and clarity, which distinguished all of his work.

II. – The propagation of waves. Celestial mechanics. Analytical mechanics. General relativity.

The theory of groups is not absent from this second part of our note devoted to the work of Vessiot. Whereas the first part was largely devoted to problems in analysis, this second part will be largely devoted to mechanics.

10. The propagation of waves. – Vessiot devoted several papers to the propagation of waves. He assumed that this propagation obeyed Huygens's principle of enveloping waves. The mode of propagation is then determined completely when one knows the wave surface that has each point of the medium for its origin. That propagation is realized step-by-step by a continuous *contact transformation*. Any point of the medium will describe a curve during its propagation that is nothing but the ray of propagation. Fermat's principle is valid, and it will be an automatic consequence of Huygens's principle. The general equation of wave surface, when written in tangential coordinates, can be interpreted as a first-order partial differential equation (E) in which time is the unknown function. Each solution of that equation will give the motion of a wave. The characteristics of that solution are the rays of propagation. Those conclusions apply to the *discontinuity* waves that are produced in an elastic medium or in an electromagnetic field. They also obey Huygens's principle and Fermat's principle.

11. Analytical mechanics and waves. – The motion of a material system that depends upon a finite number of parameters and whose *vis viva* does not depend upon time under the action of forces that are derived from a potential that is independent of time is governed by the principle of least action. Now, Vessiot proved that any problem of minimizing a simple definite integral corresponds to a propagation of waves. It then results that any motion of a material system under the conditions that were just specified can be likened to an undulatory motion. The trajectories of one are identical to the rays of the other. That remarkable result, which Vessiot stated in 1906, was of great interest much later in the creation of Louis de Broglie's undulatory mechanics. What corresponds to *time* in undulatory motion is the *action* of dynamical motion.

12. The canonical equations and the series of celestial mechanics. – We have already pointed out (no. **4**) that a system of canonical equations realizes a generalized Lie system. Vessiot used that property in order to define the classical series of the theory of perturbations in celestial mechanics. In the first place, he showed that if the perturbing function was developed in integer powers of a parameter μ that was linked to the perturbing mass then one can calculate the elements of the orbits, which are developed in powers of μ, by simply differentiating functions that are known by quadratures. That method has the advantage of not necessitating successive changes of variables. It immediately exhibits the classical properties, such as the invariability of the major axes. In the second place, an analogous method will lead to a formal solution to the n-body problem by trigonometric series.

The nature of Vessiot's work in celestial mechanics led to some important developments.

13. The kinematics of continuous media. – One can thank Sophus Lie for using an infinitesimal transformation in three variables in order to represent the motion of a fluid in three-dimensional space, while Vessiot made a systematic use of that in order to simplify the kinematics of continuous media. He extended the theorem on the composition of velocities to the case of a point that displaced in a moving fluid. The theorem of the composition of accelerations will persist when one combines the Coriolis acceleration with a second complementary acceleration that involves the deformation that is experienced by the infinitesimal region of the fluid that surrounds the moving point. All of those theorems are involved with the delicate question that Vessiot likewise studied of the propagation of a fluid that moves in another (i.e., the theorems of Hugoniot and Hadamard). Finally, Vessiot, with Lie, used the representation of the continuous motion of a fluid by an infinitesimal transformation that acted upon the coordinates of space and time in order to exhibit the kinematical origin of the fundamental theorems of hydrodynamics.

14. General relativity. – Vessiot made a very personal study of general relativity. He showed the possibility of interpreting its principles without abandoning the usual notions of space and time. One knows that Einstein introduced a four-dimensional Riemannian space with a ds^2 that was a quadratic differential form in the differentials of four variables x_i that decomposed into four squares, three of which were positive, and one of which was negative, of four independent linear forms in dx_1, dx_2, dx_3, dx_4 . Any isolated material point that is subject to only the action of gravitation will describe a geodesic. Light rays are geodesics of null radius. The propagation of light takes place by ellipsoidal waves according to Huygens's principle. If one regards the laws of propagation of light as known then the laws of gravitation will be known by the supplementary given of the square root $\sqrt{-g}$ of the discriminant of the quadratic form ds^2. If one supposes the existence of an electromagnetic field then the laws of propagation of light, the gravitational field, and the electromagnetic field will be the same. The electromagnetic theory of light is thus preserved. We add that Vessiot did some very interesting research in the geometry of Riemannian space by applying the theory of sheaves of infinitesimal transformations, which permitted him to easily envision the notion of Levi-Civita parallelism.

15. Ballistics. – Vessiot arrived at a very important result in regard to a question in ballistics that showed a rare degree of insight. It amounted to determining a correct formula for the air resistance to the motion of an ogival projectile of revolution of caliber a and mass m that had its center of gravity along the axis. The formula that was adopted at the beginning of the war in 1914 for the daily corrections for shooting cannons disagreed with experience for very low or very high temperatures. Vessiot remarked that the cause of that was due to the fact that one was not taking the density of air into account in the formula, which was contrary to the principle of the homogeneity of dimensions. He proposed the formula $F = \frac{mv^2}{a}\Phi\left(\frac{a^3\rho}{m}\right)$, in which ρ is the specific mass of the resisting medium. If the air pressure p enters into consideration then one will have $F = mcv^2\Phi\left(\frac{cv^2}{p}\right)$, in which $c = ia^2\rho/m$, in which i depends upon the form of the projectile. One could obtain different formulas, depending upon the hypotheses that were made in regard to the law and compressibility of air, one of which, which was proposed by Darrieus, was studied theoretically by Paul Langevin. Vessiot successfully applied some analogous methods in order to determine the correction coefficients for the shooting of cannons (viz., range and drift).

As one can see from this very brief analysis, the work of Vessiot was considerable. The qualities that particularly distinguished it are its clarity and a rare elegance that is combined with a constant concern for the generality of the methods. Vessiot is one of the living mathematicians who have penetrated the notion of group most deeply, as well as its consequences and complexity, and who knows how to take the best advantage of all of the work of his master Sophus Lie, and even in some topics where S. Lie could only give a sketch of his research. That permitted him to shed some new light on some topics in mathematics that seemed to have only a distant relationship with the notion of group.

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