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On envelopes of curves and surfaces

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O. Biermann has recently ([1]) treated the problem of envelopes from the viewpoint that the envelope is not, as is ordinarily assumed, given by equations between the coordinates and arbitrary constants, but the coordinates of its points are expressed explicitly in terms of auxiliary variables (i.e., parametrically). In so doing, he made consistent use of the principle of the last intersection. That process makes it necessary to prove the relationship between the equations that characterize the envelope and the envelopes in each special case once one has exhibited those equations.

We shall take up the problem anew under the same analytical assumptions and follow through on its solution by a different method that can have the advantage that it makes the relationship between the envelope and what is being enveloped immediately recognizable by means of its line of reasoning, along with the advantage of its intuitive geometric character. In that way, singular points of the latter structure will then be ignored, which should be remarked here at the outset.

1. Simply-infinite system of plane curves.

α) Let the system be represented by the equations:

$$x = \varphi(u, a), \quad y = \psi(u, a). \tag{1}$$

For a fixed a, those equations will determine a curve (a) and for fixed u, they will determine a curve (u); those systems of curves might be denoted by A, U, resp. Let A be the system whose envelope is being addressed.

A certain point M on the curve (a) is given by the pair of values (u, a); the curve (u) also goes through it. If one varies u continuously then M will move along (a), and it will begin its motion in the direction:

$$\frac{d_u y}{d_u x} = \frac{\dfrac{\partial \psi}{\partial u}}{\dfrac{\partial \varphi}{\partial u}}.$$

([1]) Festschrift der k. k. Technischen Hochschule in Brünn zur Feier ihres fünfzigjährigen Bestehen, Brünn, 1899.

If one varies only *a* continuously then *M* will vary along (*u*) and begin its motion in the direction:

$$\frac{d_a y}{d_a x} = \frac{\dfrac{\partial\psi}{\partial a}}{\dfrac{\partial\varphi}{\partial a}} .$$

In that way, two directions of motion belong to each point of the curve (*a*). Those points at which those directions of motion coincide are points of the envelope. Namely, the condition for the equality of the directions, i.e.:

$$\text{(2)} \qquad \begin{vmatrix} \dfrac{\partial\varphi}{\partial u} & \dfrac{\partial\psi}{\partial u} \\ \dfrac{\partial\varphi}{\partial a} & \dfrac{\partial\psi}{\partial a} \end{vmatrix} \equiv \frac{\partial(\varphi,\psi)}{\partial(u,a)} = 0 ,$$

determines the *u* of that point on (*a*) that always moves in the direction of the relevant tangent to (*a*) when *a* varies and under the transformation of (*a*) that it requires. Those points describe the envelope, and on the basis of that argument, one can already see that it contacts the curves of the system *A* at the points that one imagines. Its equations are obtained by eliminating *a* from (1) with the help of (2).

The elimination of *u* yields the envelope of the system *U*, which can be inferred directly by deduction. In some situations, the envelope of *A* can be a special curve of the system *U*.

Example. – The system *A* of circles whose diameters can be described as chords of a parabola (of semi-parameter *p*) that are perpendicular to its axis can be represented by the equations:

$$x = a + \sqrt{2pa}\cos u , \qquad y = \sqrt{2pa}\sin u .$$

Eliminating *a* from them will produce the equation of the system *U* :

$$x = \tan u \cdot y + \frac{y^2}{2p\sin^2 u}$$

which then consists of parabolas that go through the vertex of the base parabola and have the same axis direction as it.

Equation (2) in the general development reads:

$$\cos u + \sqrt{\frac{p}{2a}} = 0$$

here, and eliminating *u* and *a* from it and the pair of equations above will lead to:

$$y^2 = 2\,p\,x + p^2 .$$

That parabola, whose axis and semi-parameter are common to the given one, and whose focus is the vertex of the given one, envelopes the circles *A*, as well as the parabolas *U*.

β) Consider the case in which the system of curves is given by the equations:

(1) $$x = \varphi(u, a, b) , \qquad y = \psi(u, a, b) ,$$

and the parameter equation:

(2) $$\omega(a, b) = 0 ,$$

in which the derivatives $\left(\frac{\partial\varphi}{\partial a}\right)$, $\left(\frac{\partial\psi}{\partial a}\right)$ are constructed by regarding *b* as something that depends upon *a* by way of (2), so:

$$\left(\frac{\partial\varphi}{\partial a}\right) = \frac{\partial\varphi}{\partial a} + \frac{\partial\varphi}{\partial b}\frac{\partial b}{\partial a} = \frac{\partial\varphi}{\partial a} - \frac{\partial\varphi}{\partial b}\frac{\dfrac{\partial\omega}{\partial a}}{\dfrac{\partial\omega}{\partial b}} , \qquad \text{etc.}$$

In that way, the equation above goes to:

$$\begin{vmatrix} \dfrac{\partial\varphi}{\partial u} & \dfrac{\partial\psi}{\partial u} \\ \dfrac{\partial(\varphi,\psi)}{\partial(a,b)} & \dfrac{\partial(\psi,\omega)}{\partial(a,b)} \end{vmatrix} = 0 ,$$

which can also be described as:

(3) $$\begin{vmatrix} \dfrac{\partial\varphi}{\partial u} & \dfrac{\partial\psi}{\partial u} & 0 \\ \dfrac{\partial\varphi}{\partial a} & \dfrac{\partial\psi}{\partial a} & \dfrac{\partial\omega}{\partial a} \\ \dfrac{\partial\varphi}{\partial b} & \dfrac{\partial\psi}{\partial b} & \dfrac{\partial\omega}{\partial b} \end{vmatrix} \equiv \frac{\partial(\varphi,\psi,\omega)}{\partial(u,a,b)} = 0 .$$

The envelope is determined by (1), (2), (3).

2. Simply-infinite system of space curves. – It is given by:

(1) $$x = \varphi(u, a) , \quad y = \psi(u, a) , \quad z = \chi(u, a) .$$

A fixed a characterizes a curve (a) of the system A. In addition, there are curves (u) that are each characterized by a fixed u; the system of all of them is called U.

The point M (u, a) on (a) is put into motion by merely changing u along that curve that begins in the direction:

$$d_u\, x : d_u\, y : d_u\, z = \frac{\partial \varphi}{\partial u} : \frac{\partial \psi}{\partial u} : \frac{\partial \chi}{\partial u}$$

of the tangent to (a). Those points of (a) at which the two directions coincide, so they move in the direction of the relevant tangent to (a) when (a) runs continuously through the system (A), describe the envelope of that system. However, since the equality of the directions requires the vanishing of the two-rowed determinants of the matrix:

$$(2) \qquad \left\| \begin{matrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \psi}{\partial u} & \frac{\partial \chi}{\partial u} \\ \frac{\partial \varphi}{\partial a} & \frac{\partial \psi}{\partial a} & \frac{\partial \chi}{\partial a} \end{matrix} \right\| ,$$

so the simultaneous existence of a excessive number of equations, an envelope will exist only when those equations reduce to one.

The envelope that possibly exists also envelops the system U when it is not a special example of the curves in that system.

Example. – The system A that is represented by the equations:

$$x = r\,(\cos a - u \sin a)\,, \qquad y = r\,(\sin a + u \cos a)\,, \qquad z = h\,(a + u)\,,$$

in which r, h mean given constants, is a system of lines, while the system U is a system of transcendental space curves. The associated matrix (2) reads:

$$\left\| \begin{matrix} -r\sin a & r\cos a & h \\ -r\,(\sin a + u\cos a) & r\,(\cos a - u\sin a) & h \end{matrix} \right\| .$$

Two of its two-rowed determinants vanish identically, while the third one leads to the equation:

$$u = 0\,.$$

Hence, in the present case, the envelope of A is a special U-curve, namely, the helix:

$$x = r\cos a\,, \quad y = r\sin a\,, \quad z = h\,a\,.$$

The *U*-curves are the intersections of the tangent surfaces to that helix with the cylinders $x^2 + y^2 = r^2(1+u^2)$.

3. Simply-infinite system of surfaces.

α) Let it be given by the equations:

(1) $$x = r \cos a , \quad y = r \sin a , \quad z = h\, a .$$

For a fixed *a* and variable *u*, *v*, the point *M* (*u*, *v*, *a*) moves on a surface (*a*) of the system *A* that comes under consideration.

For a fixed *u*, *v*, and variable *a*, it describes a curve (*w*) with the property that it pierces all surfaces of the system *A* at points of a fixed coupling of the values (*u*, *v*).

When only *a* changes, the point *M* will begin to move along the associated (*w*)-curve with the initial direction:

$$d_a x : d_a y : d_a z = \frac{\partial\varphi}{\partial a} : \frac{\partial\psi}{\partial a} : \frac{\partial\chi}{\partial a} .$$

If *u* and *v* vary, while *a* remains fixed, then *M* will begin to move in the tangent plane to (*a*) whose equation reads:

$$\frac{\partial(\psi,\chi)}{\partial(u,v)}(\xi - x) + \frac{\partial(\chi,\varphi)}{\partial(u,v)}(\eta - y) + \frac{\partial(\varphi,\psi)}{\partial(u,v)}(\zeta - z) = 0 .$$

Should that direction of motion fall in that tangent plane, then one would need to have:

$$\frac{\partial\varphi}{\partial a}\frac{\partial(\psi,\chi)}{\partial(u,v)} + \frac{\partial\psi}{\partial a}\frac{\partial(\chi,\varphi)}{\partial(u,v)} + \frac{\partial\chi}{\partial a}\frac{\partial(\varphi,\psi)}{\partial(u,v)} = 0 ,$$

i.e.:

(2) $$F(u, v, a) \equiv \begin{vmatrix} \dfrac{\partial\varphi}{\partial u} & \dfrac{\partial\psi}{\partial u} & \dfrac{\partial\chi}{\partial u} \\ \dfrac{\partial\varphi}{\partial v} & \dfrac{\partial\psi}{\partial v} & \dfrac{\partial\chi}{\partial v} \\ \dfrac{\partial\varphi}{\partial a} & \dfrac{\partial\psi}{\partial a} & \dfrac{\partial\chi}{\partial a} \end{vmatrix} = 0 .$$

If one regards *a* as constant in equation (2) then it will express a relation between *u*, *v* that determines a curve (*c*) on the surface (*a*). Each point of that curve will be described when one fixes, e.g., its *u* and varies *a*, and the transformation of (*a*) that is thus produced will give a curve (γ) that possesses the property that they contact all surfaces of the system *A*, and indeed at points of the associated (*c*)-curves – viz., the *characteristics*. The locus of curves (γ) is a surface *E* that then envelops the surfaces of the system *A*. However, one also recognizes that *E* is likewise the locus of curves (*c*).

The characteristics (*c*), which are determined by equations (1) and (2), in which *a* is regarded as the variable parameter, can have an envelope that lies on the envelope *E*, just like the system of (*c*), and which is called the *edge of regression*.

In order to arrive at this curve, one must apply the procedure in **2**. to the present case by analogy.

Therefore, those points on (*c*) that describe the envelope are coupled by equations (1), (2), and the relations:

$$\frac{\left(\frac{\partial\varphi}{\partial u}\right)}{\left(\frac{\partial\varphi}{\partial a}\right)} = \frac{\left(\frac{\partial\psi}{\partial u}\right)}{\left(\frac{\partial\psi}{\partial a}\right)} = \frac{\left(\frac{\partial\chi}{\partial u}\right)}{\left(\frac{\partial\chi}{\partial a}\right)} .$$

The latter can also be described in the form:

$$(3) \qquad \begin{cases} \left(\frac{\partial\psi}{\partial u}\right)\left(\frac{\partial\chi}{\partial a}\right) - \left(\frac{\partial\psi}{\partial a}\right)\left(\frac{\partial\chi}{\partial u}\right) = 0 , \\ \left(\frac{\partial\chi}{\partial u}\right)\left(\frac{\partial\varphi}{\partial a}\right) - \left(\frac{\partial\chi}{\partial a}\right)\left(\frac{\partial\varphi}{\partial u}\right) = 0 , \\ \left(\frac{\partial\varphi}{\partial u}\right)\left(\frac{\partial\psi}{\partial a}\right) - \left(\frac{\partial\varphi}{\partial a}\right)\left(\frac{\partial\psi}{\partial u}\right) = 0 . \end{cases}$$

The parentheses shall suggest that one has introduced *v* as a function of *u*, *a* by means of equation (2) in (1). Therefore, one has:

$$\left(\frac{\partial\psi}{\partial u}\right) = \frac{\partial\psi}{\partial u} + \frac{\partial\psi}{\partial v}\frac{\partial v}{\partial u} = \frac{\partial\psi}{\partial u} - \frac{\partial\psi}{\partial v}\frac{\frac{\partial F}{\partial u}}{\frac{\partial F}{\partial v}} , \qquad \left(\frac{\partial\chi}{\partial u}\right) = \frac{\partial\chi}{\partial u} + \frac{\partial\chi}{\partial v}\frac{\partial v}{\partial u} = \frac{\partial\chi}{\partial u} - \frac{\partial\chi}{\partial v}\frac{\frac{\partial F}{\partial u}}{\frac{\partial F}{\partial v}} ,$$

$$\left(\frac{\partial\psi}{\partial a}\right) = \frac{\partial\psi}{\partial a} + \frac{\partial\psi}{\partial v}\frac{\partial v}{\partial a} = \frac{\partial\psi}{\partial a} - \frac{\partial\psi}{\partial v}\frac{\frac{\partial F}{\partial a}}{\frac{\partial F}{\partial v}} , \qquad \left(\frac{\partial\chi}{\partial a}\right) = \frac{\partial\chi}{\partial a} + \frac{\partial\chi}{\partial v}\frac{\partial v}{\partial a} = \frac{\partial\chi}{\partial a} - \frac{\partial\chi}{\partial v}\frac{\frac{\partial F}{\partial a}}{\frac{\partial F}{\partial v}} .$$

If one expresses the first of equations (3) using those expressions then that will give the corresponding reduction:

$$\frac{\partial F}{\partial u}\frac{\partial(\psi,\chi)}{\partial(v,a)} + \frac{\partial F}{\partial v}\frac{\partial(\psi,\chi)}{\partial(a,u)} + \frac{\partial F}{\partial a}\frac{\partial(\psi,\chi)}{\partial(u,v)} = 0 .$$

The other two are produced by the same procedure, as one can see with no further calculation:

$$\frac{\partial F}{\partial u}\frac{\partial(\chi,\varphi)}{\partial(v,a)}+\frac{\partial F}{\partial v}\frac{\partial(\chi,\varphi)}{\partial(a,u)}+\frac{\partial F}{\partial a}\frac{\partial(\chi,\varphi)}{\partial(u,v)}=0 ,$$

$$\frac{\partial F}{\partial u}\frac{\partial(\varphi,\psi)}{\partial(v,a)}+\frac{\partial F}{\partial v}\frac{\partial(\varphi,\psi)}{\partial(a,u)}+\frac{\partial F}{\partial a}\frac{\partial(\varphi,\psi)}{\partial(u,v)}=0 .$$

However, those three equations coincide, and as such, any one of them can be selected. That is because the coefficients of $\frac{\partial F}{\partial u}$, $\frac{\partial F}{\partial v}$, $\frac{\partial F}{\partial a}$ in the first one are the adjoints of the elements of the first column of the determinant in (2), while the coefficients in the second and third equations are the adjoints to the elements of the second and third columns, respectively. However, since that determinant vanishes for the points of (*c*), the adjoints of all three columns have the same relationship, and therefore the last three equations are, in fact, equivalent.

The envelope of the characteristics, i.e., the edge of regression on *E*, is then determined by the system of equations:

$$x=\varphi(u, v, a) , \qquad y=\psi(u, v, a) , \qquad z=\chi(u, v, a) ,$$

$$F(u, v, a)\equiv\frac{\partial(\varphi,\psi,\chi)}{\partial(u,v,a)}=0 ,$$

$$\frac{\partial F}{\partial u}\frac{\partial(\psi,\chi)}{\partial(v,a)}+\frac{\partial F}{\partial v}\frac{\partial(\psi,\chi)}{\partial(a,u)}+\frac{\partial F}{\partial a}\frac{\partial(\psi,\chi)}{\partial(u,v)}=0 .$$

In order to represent them in the usual formulas, one can either express *u*, *v*, *a* using the first three equations and substitute them in the last two or determine *u*, *v* from the last two and substitute them in the first three.

The last of the equations above can also be written in the form:

(4) $$\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial \psi}{\partial u} & \frac{\partial \chi}{\partial u} \\ \frac{\partial F}{\partial v} & \frac{\partial \psi}{\partial v} & \frac{\partial \chi}{\partial v} \\ \frac{\partial F}{\partial a} & \frac{\partial \psi}{\partial a} & \frac{\partial \chi}{\partial a} \end{vmatrix}=0 .$$

β) If the system of surfaces is given by the equations:

(1) $$x=\varphi(u, v, a, b) , \qquad y=\psi(u, v, a, b) , \qquad z=\chi(u, v, a, b) ,$$

and the parameter equation:

(2) $$\omega\,(a, b) = 0$$

then the analytical procedure will experience the following alterations in the present case:

The elements of the third column in the determinant of equation (2) above will be replaced with:

$$\frac{\partial\varphi}{\partial a}-\frac{\partial\varphi}{\partial b}\frac{\frac{\partial\omega}{\partial a}}{\frac{\partial\omega}{\partial b}}\,,\qquad \frac{\partial\psi}{\partial a}-\frac{\partial\psi}{\partial b}\frac{\frac{\partial\omega}{\partial a}}{\frac{\partial\omega}{\partial b}}\,,\qquad \frac{\partial\chi}{\partial a}-\frac{\partial\chi}{\partial b}\frac{\frac{\partial\omega}{\partial a}}{\frac{\partial\omega}{\partial b}}\,.$$

In that way, that equation will be converted into:

$$F\,(u, v, a, b) \equiv \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\psi}{\partial u} & \frac{\partial\chi}{\partial u} \\ \frac{\partial\varphi}{\partial v} & \frac{\partial\psi}{\partial v} & \frac{\partial\chi}{\partial v} \\ \frac{\partial(\varphi,\omega)}{\partial(a,b)} & \frac{\partial(\psi,\omega)}{\partial(a,b)} & \frac{\partial(\chi,\omega)}{\partial(a,b)} \end{vmatrix} = 0\,.$$

However, the development of the foregoing determinant in the elements of the last row can be regarded as the development of the four-rowed determinant:

$$\begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\psi}{\partial u} & \frac{\partial\chi}{\partial u} & 0 \\ \frac{\partial\varphi}{\partial v} & \frac{\partial\psi}{\partial v} & \frac{\partial\chi}{\partial v} & 0 \\ \frac{\partial\varphi}{\partial a} & \frac{\partial\psi}{\partial a} & \frac{\partial\chi}{\partial a} & \frac{\partial\omega}{\partial a} \\ \frac{\partial\varphi}{\partial b} & \frac{\partial\psi}{\partial b} & \frac{\partial\chi}{\partial b} & \frac{\partial\omega}{\partial b} \end{vmatrix}$$

in the subdeterminants of the first two rows. Finally, one can also set:

(3) $$F\,(u, v, a, b) \equiv \frac{\partial(\varphi,\psi,\chi,\omega)}{\partial(u,v,a,b)} = 0\,.$$

In the same way, the elements of the last row in the determinant of equation (4) will be replaced with:

$$\frac{\partial F}{\partial a} - \frac{\partial F}{\partial b}\frac{\dfrac{\partial\omega}{\partial a}}{\dfrac{\partial\omega}{\partial b}}\,, \qquad \frac{\partial \psi}{\partial a} - \frac{\partial \psi}{\partial b}\frac{\dfrac{\partial\omega}{\partial a}}{\dfrac{\partial\omega}{\partial b}}\,, \qquad \frac{\partial \chi}{\partial a} - \frac{\partial \chi}{\partial b}\frac{\dfrac{\partial\omega}{\partial a}}{\dfrac{\partial\omega}{\partial b}}\,.$$

However, that will make the aforementioned equation go to:

$$\begin{vmatrix} \dfrac{\partial F}{\partial u} & \dfrac{\partial \psi}{\partial u} & \dfrac{\partial \chi}{\partial u} \\ \dfrac{\partial F}{\partial v} & \dfrac{\partial \psi}{\partial v} & \dfrac{\partial \chi}{\partial v} \\ \dfrac{\partial (F,\omega)}{\partial (a,b)} & \dfrac{\partial (\psi,\omega)}{\partial (a,b)} & \dfrac{\partial (\chi,\omega)}{\partial (a,b)} \end{vmatrix} = 0\,,$$

which can be written as:

(4)
$$\frac{\partial (F,\psi,\chi,\omega)}{\partial (u,v,a,b)} = 0$$

on similar grounds to the previous ones.

The result can be stated by saying that the envelope E of the system of surfaces is now determined by equations (1), (2), (3), and the edge of regression that might appear is now determined by equations (1), (2), (3), (4).

4. Twofold-infinite system of surfaces. – Let it be given by the equations:

(1)
$$x = \varphi(u, v, a, b)\,, \qquad y = \psi(u, v, a, b)\,, \qquad z = \chi(u, v, a, b)\,.$$

For fixed a and b and variable u, v, one gets a surface (a, b) of the system. The point $M(u, v)$ of that surface is put into motion by continuously varying a alone and the transformation of (a, b) that it requires and the initial direction of that motion is determined by:

$$d_a x : d_a y : d_a z = \frac{\partial\varphi}{\partial a} : \frac{\partial\psi}{\partial a} : \frac{\partial\chi}{\partial a}\,.$$

Those directions fall in the tangent plane:

$$\frac{\partial(\psi,\chi)}{\partial(u,v)}(\xi - x) + \frac{\partial(\chi,\varphi)}{\partial(u,v)}(\eta - y) + \frac{\partial(\varphi,\psi)}{\partial(u,v)}(\zeta - z) = 0$$

at (a, b) in M when, on the one hand:

$$\frac{\partial\varphi}{\partial a}\frac{\partial(\psi,\chi)}{\partial(u,v)} + \frac{\partial\psi}{\partial a}\frac{\partial(\chi,\varphi)}{\partial(u,v)} + \frac{\partial\chi}{\partial a}\frac{\partial(\varphi,\psi)}{\partial(u,v)} = 0$$

i.e.:

(2) $$\frac{\partial(\varphi, \psi, \chi)}{\partial(u, v, a)} = 0 ,$$

and one then has:

(3) $$\frac{\partial(\varphi, \psi, \chi)}{\partial(u, v, b)} = 0 ,$$

on the other hand.

Equations (1) and (2) determine a curve (*a*, *b*), and a second curve on that surface is determined by (1) and (3). The intersection point of both curves possesses the property that the two cited directions of motion will fall in the tangent plane for them. The locus of those two-fold infinite point manifolds is a surface *E* that envelops the system of surfaces (*a*, *b*).

One can also generate that surface by the motion of certain curves as follows: If one eliminates *v*, *b* from (1), (2), (3) then that will produce the equations:

$$x = \Phi_1(u, a) , \qquad y = \Psi_1(u, a) , \qquad z = X_1(u, a) ,$$

which represent a curve for fixed *a*, and the surface that it describes when *a* varies. If one eliminates *v*, *a* then that will yield equations:

$$x = \Phi_2(u, b) , \qquad y = \Psi_2(u, b) , \qquad z = X_2(u, b) ,$$

which determine a curve for fixed *b*, and the surface that it describes when *b* varies. However, those two systems of equations are equivalent to the one system (1), (2), (3), and represent the same surface, viz., *E*.

In order to obtain the envelope in one of the usual analytical forms of representation, one must either eliminate *u*, *v*, *a*, *b* from (1), (2), (3) or eliminate *a*, *b* from (1) with the help of (2) and (3).

Vienna, 24 January 1901.