"Ricerche sulle congruenze di curve in una varietà qualunque a tre dimensioni," Atti della Reale Istituto Veneto di scienze, lettere ed arti **59** (1899-1900), 245-352.

Study of congruences of curves in an arbitrary three-dimensional manifold ([1])

COMMUNICATION

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When one is given a normal congruence of curves in an arbitrary space S, the system of its orthogonal trajectories lies in a family of surfaces that stratify space in a certain way. If one considers (as is always possible) the manifold S to be immersed in a flat space Σ of a sufficient number of dimensions then that surface will admit a tangent plane (in Σ) that is tangent to the lines of the system and a normal (also in Σ) that is tangent to S and normal to those lines. Among the infinitude of lines on that surface, one has the asymptotic lines, the lines of curvature, and corresponding to the latter, the principal, mean, and total curvatures, etc.

However, in general, if the congruence is not normal then its orthogonal trajectories will not lie on the surface but will again striate the space S in a particular well-defined way. Assume that any point has a common tangent plane (in Σ) and a common normal that are tangent to S.

Since a single-valued correspondence exists between the given congruence and the system of its orthogonal trajectories, one can assume (as is known for normal congruences) that there is a coordinate system for one and the other. One can make analogous constructions between the lines of the *system of trajectories*, which I call an *orthogonal complex* (or more simply, when no ambiguity is possible, a *complex*), the asymptotic lines, and the lines of curvature, and extend the concepts of curvature that are known in the case of normal congruences.

That way of grouping the congruences gives one an elegant geometrical interpretation of those three-index invariants (γ_{hkl}) to which Ricci had given kinematical interpretations.

$-\gamma_{ppp}$ ($p = 1, 2$) ([2]) represents the curvature of the projection of λ_p onto the tangent plane to the complex λ, or the *tangential curvature* of λ_p on the complex.

([1]) For the notations that are used in this note and the elements of the *absolute differential calculus,* see the work of Ricci, and in particular: *Dei sistemi di congruenze ortogonali in una varietà qualunque*. – Memorie dei Lincei (Classe di scienze fisiche, mat. e nat.), series V, vol. II.

([2]) Here and in what follows, we shall define expressions relative to the complex λ. The ones that relate to λ_1 and λ_2 are obtained from it by setting $\lambda \equiv \lambda_3$, and performing a convenient rotation of the indices.

– γ_{3pp} is the curvature of the projection of λ_p onto the plane that is tangent to it and normal to the complex λ ; i.e., the *normal curvature*.

One can then infer from:

$$\gamma_{311} + \gamma_{322} = \sum_{r,s} a^{(rs)} \lambda_{rs}$$

that:

The sum of the normal curvatures to two orthogonal congruences over a complex is constant around a point.

Now call the congruences of a complex for which the first variation of the integral:

$$s_h = \int_{t_0}^{t_1} \sqrt{\sum_{r,s} a_{rs} \frac{dx_r}{dt} \frac{dx_s}{dt}}\, dt$$

(in which s_h is the arc length of the λ_h that belongs to the complex λ) is zero the *geodetics* of a complex. One then has that:

The geodetics of a complex are the lines whose tangential curvature is zero.

Consider the invariant ([1]):

$$A = \tfrac{1}{2}(\gamma_{312} - \gamma_{321}) = \tfrac{1}{2} \sum_{r,s,t} \lambda_r \lambda_{st} \varepsilon^{(rst)},$$

whose vanishing characterizes the normal congruences, which is an invariant to which one gives the name of the *abnormality* of the congruence or the complex that is orthogonal to it. One can then interpret the equation:

$$\gamma_{312} + \gamma_{321} = 0$$

that Ricci gave for the canonical orthogonal congruences by saying that:

The canonical orthogonal congruences have equal abnormalities.

One also has that:

([1]) The system $\varepsilon^{(rst)}$ is defined by the relations:

$$\varepsilon^{(rst)} + \varepsilon^{(srt)} = 0,$$
$$\varepsilon^{(rst)} + \varepsilon^{(rts)} = 0,$$
$$\varepsilon^{(r\,r+1\,r+2)} = \frac{1}{\sqrt{a}}.$$

The normal curvatures over a complex are maxima or minima along the canonical orthogonal lines and are equal along their bisectors.

With that, the equation that is characteristic of the latter lines is:

$$\gamma_{311} - \gamma_{322} = 0 .$$

The interesting results of considering the *sheaves* of congruences over complexes (systems of congruences over a complex that meet a given congruence at a constant angle) are quite fruitful, as well as those of *stars* (double infinitudes of congruences in space that form constant arbitrary angles with two orthogonal congruences).

Above all, each sheaf corresponds to a unique simple system that is defined by:

$$\varphi_{12|r} = \sum_r \lambda_{1|rs}\, \lambda_2^{(r)},$$

which can be considered to be its coordinate system. As on surfaces, one deduces from the fact that:

$$\varphi_{12|r} = \varphi'_{12|r} + \alpha_r$$

that:

The difference between the coordinate systems of two sheaves is equal to the derivative of the angle with which two arbitrary congruences of the two sheaves intersect.

A consideration of the facts that:

$$\gamma'_{123} = \gamma_{123} + \frac{d\alpha}{ds}$$

and

$$\sum_r \varphi_{12|r}\, \varphi^{(r)}_{12} = \sum_p \gamma^2_{12p}$$

will imply that:

The sum of the squares of the tangential curvatures of two orthogonal congruences of a complex is constant around a point for the congruences of the same sheaf,

and from that:

All of the congruences of one sheaf are geodetics of the complex if any two of them are orthogonal to that complex.

What are more fruitful than anything else in regard to the interests of the theory of surfaces and in regard to the discovery of important new truths and interpretations for known expressions and formulas are the concepts of lines of curvature and asymptotic lines of complexes.

First of all, we say that a congruence is *associated* with λ_1 when its tangent at any point coincides with the intersection of the tangent planes to the complex at that point and in a neighborhood of that λ_1 . We then say that any congruence that with itself is *asymptotic*. However, it is necessary to note that there is no reciprocity in that association except in the case where λ is normal. The associated lines are then the conjugates to the orthogonal surfaces of λ .

Lines whose normals (which are lines of the complex) meet are called *lines of curvature*. The normal curvatures of the lines of curvature are called *principal curvatures*, and the *mean curvature:*

$$H = -\tfrac{1}{2}(\gamma_{311} + \gamma_{322}) = -\tfrac{1}{2}\sum_{r,s} a^{(rs)}\,\lambda_{rs}$$

is one-half their sum, while their product is the *total curvature*:

$$K = \gamma_{311}\,\gamma_{322} - \gamma_{312}\,\gamma_{321} = \tfrac{1}{2}\sum_{r,s} \Lambda^{(rs)}\,a_{rs}\,,$$

in which $a\,\Lambda^{(rs)}$ is the algebraic complement of the element λ_{rs} in the determinant ($\lambda_{11}\,\lambda_{22}\,\lambda_{33}$) ([1]).

Meanwhile, we have that:

Among the infinitude of complexes that belong to a congruence, the lines of curvature of one of them will also be lines of curvature for the one that is orthogonal to it. ([2])

In the case where λ_1 results from the lines of curvature for the complexes λ_2, λ, one will have:

$$A_1 = A_2 + A_3\,,$$

and one can deduce an important generalization of a theorem by Dupin that was completed by Darboux (Darboux, *Leçons sur la théorie Générale des surfaces*, vol. II, pp. 263):

If two complexes meet orthogonally and have the same congruence of lines of curvature then its abnormality will be equal to the sum of the abnormalities of the given complexes.

In regard to the asymptotic lines, one has:

Their normal curvature is zero.

They have the canonical orthogonal lines for bisectors. When one seeks the angle α that is formed between the asymptotes, one will find that:

$$\tan^2\alpha = \frac{A^2 - K}{H^2}\,,$$

and one will get that:

([1]) We point out that the expressions for A, H, and K are independent of the choice of λ_1 , λ_2 .

([2]) We can give the theorem the form: *If one passes a ruled surface through a curve whose generators are normal to the curve then it will or will not be developable, along with the orthogonal ruled surface, and similarly generated.*

The asymptotes of a complex are real and distinct, real and coincident, or imaginary according to whether the square of their abnormality is greater than, equal to, or less than the total curvature, resp.

One gives the name of *limit point* (as for a rectilinear congruence) to the extremes of the arc of λ long which they are real and the name of *limit surfaces* to the surfaces that they generate. Their equations are:

$$A \pm \sqrt{K} = 0 .$$

The asymptotes are coincident at the limit points and tangent to the line of one of the canonical orthogonal congruences. One can then observe that when one displaces from one limit point to the other, the asymptotic lines will cut the lines of one of the canonical orthogonal congruences until they approach and overlap along a line of the other one, and then move away from each other to an angle equal to π. The point at which the angular distance between the asymptotes is in the middle (= $\pi / 2$) is called the *midpoint*, and the locus of those points is the *middle surface*.

One then has an interesting theorem:

At the midpoints of a congruence, the mean curvature of the complex that is orthogonal to the congruence is zero.

That theorem is very important because it gives one a way to give a simple and elegant form to the equation of the middle surface of a congruence, viz.:

$$-2H = \sum_{r,s} a^{(rs)} \lambda_{rs} = 0 .$$

In regard to the middle surfaces, one also has that the theory of such things is correlated with that of minimal surfaces by the theorem that Guichard ([1]) proved for a very special class of congruences:

If the middle surface of a congruence is orthogonal to the congruence then it will be a minimal surface.

The study of limit points of λ, when considered to be extremes of the segment along which the asymptotes are real, naturally induces one to seek the extremes of the arc of λ along which the lines of curvature are real: They are called *extreme points*. The surface that they generate:

$$H \pm \sqrt{K} = 0$$

is the *extreme surface*. Its lines of curvature and principal curvatures coincide.

([1]) Guichard, "Sur une classe particuliére de congruences des droites," C. R. Acad. Sci. Paris, June 1891.

The case of rectilinear congruences in which there exist no extreme points, but only extreme rays, is noteworthy, and their extreme surfaces will reduce to a ruled surface.

Returning to the stars of congruences, which are first defined by:

$$2A'_h = \sum_{r,s,t,k,l} \lambda_{k|r}\lambda_{l|st}\,\varepsilon^{(rst)}\,\alpha_{hk}\,\alpha_{hl}$$

(in which α_{pq} = const. is the cosine of the angle between λ'_p , λ_q), one has:

The sum of the abnormalities of three orthogonal congruences is constant for the same star.

That gives:

$$H'_h = \sum_k \alpha_{hk}\,H_k\,.$$

The sum of the squares of the mean curvatures of complexes that are orthogonal to three orthogonal congruences is constant for the same star.

One infers from this that:

If three orthogonal congruences have zero mean curvature for their orthogonal complex then it will also be zero for any congruence that lies in a star with it,

as well as:

If three orthogonal congruences are geodetic then any congruence that belongs to the star that it determines will have the mean curvature of its orthogonal complex, namely, zero.

In fact, one derives from:

$$\gamma_{hkh} = 0$$

that

$$H_h = 0\,.$$

Apropos of geodetic congruences, one gets from:

$$\frac{dA}{ds} + 2AH = 0$$

that:

If they admit a normal surface then they will be normal.

Let us move on to isotropic congruences. The isotropy conditions, which Levi-Civita ([1]) gave in the form:

$$\gamma_{312} + \gamma_{321} = 0\,,$$
$$\gamma_{311} - \gamma_{322} = 0\,,$$

can be written:

$$A^2 + H^2 - K = 0\,.$$

One can read this to mean:

The square of the abnormality is equal to the total curvature of the orthogonal complex, minus the square of the mean curvature.

One can generalize an important theorem that is due to Levi-Civita:

Any isotropic congruence can be regarded, in an infinitude of ways, as resulting from the intersections of two families of surfaces that meet at a constant angle.

If one further supposes that those congruences are also geodetic then one will have the theorem that was stated by Ribaucour ([2]) for the rectilinear ones in ordinary space:

The middle surface of an isotropic geodetic congruence corresponds to the hypersphere under orthogonality of elements.

We conclude this rapid summary with an interesting interpretation of the formula:

$$\frac{dH}{ds_h} = 0\,, \qquad h = 1, 2$$

that Ricci gave (Mem. cited on last page):

If a family of isothermal surfaces has geodetic orthogonal trajectories then they will prove to be surfaces of constant mean curvature.

These results of my studies have led me to a path that I believe to be new: Some of them seem truly important to me, but especially the equations of the middle surface, which present an extraordinary simplicity. It is a simplicity that would be difficult to reach along a different line of research and would even be impossible with methods that are not those of the absolute differential calculus.

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([1]) "Sulle congruenze di curve," Note by T. Levi-Civita. – Rendiconti dei Lincei, v. VIII, 1st sem., series 5, fasc. 5.

([2]) Ribaucour, "Étude des elassoïdes," Mémoires couronnés par l'Academie de Belgique **44** (1881).